



DEUTSCHES ELEKTRONEN-SYNCHROTRON **DESY**

DESY 77/48
July 1977

Continuity Equations for the Classical Euclidean
Two-Dimensional Non-Linear σ - Models

by

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The two-dimensional non-linear chiral O_3 model is the closest known tractable analogue model for the four-dimensional pure Yang-Mills theory¹⁾. Like the latter it defines an asymptotically free theory and enjoys instanton solutions^{2,3)}. Because of this analogy, the model - which under the name "confining isotropic planar spin one Heisenberg ferromagnet" always had attracted considerable interest in mathematical physics - came under particularly intense examination recently.

The classical non-linear chiral O_n models in two-dimensional Minkowski space possess an infinite number of conservation laws⁴⁾. In this note we want to point out that the corresponding classical models in two-dimensional Euclidean space (entering the Euclidean functional integrals of the corresponding quantized models via the extrema of the Euclidean actions) possess an infinite number of continuity equations.

We shall follow the same line of arguments as given in reference⁴⁾ with the only deviation that at an intermediate stage we shall also consider complex solutions.

Towards the end of this note we shall present an alternative, more direct derivation of the continuity equations for the instanton solutions of the O_3 -model.

Abstract:

We derive an infinite set of independent covariant local non-polynomial continuity equations for the classical non-linear chiral O_n -models in two-dimensional Euclidean space.

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The two-dimensional non-linear chiral O_n model involves a real n-component unit vector field $\varphi = \varphi(x, y) = (\varphi_1(x, y), \dots, \varphi_n(x, y))$.

$$S = \int_{R^2} dx dy (\varphi_x^2 + \varphi_y^2)$$

Here we use the short hand notation

$$\varphi_x^2 = \sum_j \rho_j \varphi_j^2, \quad \varphi^2 = \sum_i \rho_i^2, \quad \varphi_x = \frac{\partial}{\partial x} \varphi, \quad \varphi_y = \frac{\partial}{\partial y} \varphi, \dots$$

The action density is invariant under global internal O_n transformations and under conformal transformations.

We introduce the complex-plane notation

$$z = \frac{x + iy}{2}, \quad \bar{z} = \frac{x - iy}{2}$$

$$\varphi_x = \frac{\partial}{\partial x} \varphi = \frac{\partial}{\partial z} \varphi + \frac{\partial}{\partial \bar{z}} \varphi$$

$$\varphi_y = \frac{\partial}{\partial y} \varphi = i \left(\frac{\partial}{\partial z} \varphi - \frac{\partial}{\partial \bar{z}} \varphi \right)$$

Complex conjugation is denoted by a bar.

The extrema of the Euclidean action satisfy the equations

$$q_2 \bar{z} + (q_2 \cdot q_2) q = 0, \quad q^2 = 1.$$

(Moreover, $q_2^2 = 0$ and $q_2^2 = 0$ are valid for the instanton solutions of the O_3 -model.)

In the following we assume that $n \geq 3$, the cases $n = 1$ and 2 being trivial and explicitly soluble, respectively.

Lemma 1: Along with every (not necessarily real) vector field q , a whole two-parameter family of vector fields $q^{(s)}$, $s \in \mathcal{A}$ with

$$q_2^{(s)2} = s^{-2} q_2^2, \quad q_2^{(s)2} = s^2 q_2^2, \quad (q_2^{(s)}, q_2^{(s)}) = (q_2, q_2)$$

satisfies the extremum equations

$$q_2 \bar{z} + (q_2 \cdot q_2) q = 0, \quad q^2 = 1.$$

The vector field $q^{(s)}$ is obtained from the vector field q with the help of a $SL(n, \mathcal{A})$ matrix $\mathcal{R}^{(s)} = \mathcal{R}^{(s)}(z, \bar{z}; q)$ with

$$\begin{aligned} \mathcal{R}^{(s)} \mathcal{R}^{(s)tr} &= \mathcal{R}^{(s)tr} \mathcal{R}^{(s)} = 1, \quad \mathcal{R}^{(s)} = 1: \\ q^{(s)} &= \mathcal{R}^{(s)} q, \quad q_2^{(s)} = s^{-1} \mathcal{R}^{(s)} q_2, \quad q_2^{(s)} = s \mathcal{R}^{(s)} q_2. \end{aligned}$$

Real solution vectors q go over into real solution vectors $q^{(s)}$ if and only if $|s| = 1$.

Proof: The first part of the lemma follows from the compatibility of the equations

$$\mathcal{R}^{(s)} = (1 - s^{-1}) \mathcal{R}^{(s)} (q \otimes q_2 - q_2 \otimes q)$$

$$\mathcal{R}^{(s)} = (1 - s) \mathcal{R}^{(s)} (q \otimes q_2 - q_2 \otimes q)$$

and

$$\mathcal{R}^{(s)} \mathcal{R}^{(s)tr} = \mathcal{R}^{(s)tr} \mathcal{R}^{(s)} = 1.$$

The second part of the lemma follows from the fact that for a real solution vector \bar{q} and for $|s|=1$ along with $\mathcal{R}^{(s)}$, also $\overline{\mathcal{R}^{(s)}}$ satisfies the last three equations.

Lemma 2: Along with every (not necessarily real) vector field q , the vector fields q^+ and q^- defined (up to some coordinate independent rotations) by the four complex compatible equations

$$(q^{\pm})_2 q^{\pm} = \pm \frac{(q^{\pm} \cdot q_2) - (q^{\pm} \cdot q)}{2} (q^{\pm}, \pm) q$$

$$(q^{\pm})_2 q^{\pm} = \pm \frac{(q^{\pm} \cdot q) - (q^{\pm} \cdot q_2)}{2} (q^{\pm}, \pm) q$$

$$(q^{\pm})^2 = 1, \quad (q^{\pm} \cdot q) = 0$$

satisfy the extremum equations

$$q_2 \bar{z} + (q_2 \cdot q_2) q = 0, \quad q^2 = 1.$$

Proof: The relevant compatibility equations are just

$$q_2 \bar{z} + (q_2 \cdot q_2) q^{\pm} = 0 \quad \text{and} \quad q_2 \bar{z} + (q_2 \cdot q_2) q = 0.$$

Lemma 3: For every $s \in \mathcal{A}$, the following continuity equations are valid:

$$\begin{aligned} 0 &= \frac{\partial}{\partial z} (q^{(s)2} \cdot q^{(s)})_z + \frac{\partial}{\partial \bar{z}} (q^{(s)2} \cdot q^{(s)})_{\bar{z}} \\ &= \frac{\partial}{\partial z} \left\{ (q^{(s)2} \cdot q_2^{(s)}) + (q^{(s)2} \cdot q_2^{(s)}) \right\}_z - \frac{\partial}{\partial \bar{z}} \left\{ (q^{(s)2} \cdot q_2^{(s)}) - (q^{(s)2} \cdot q_2^{(s)}) \right\}_{\bar{z}} \end{aligned}$$

The lemma is proved by carrying out the differentiations of the brackets and by inserting the defining equations of $q^{(s)}$.

Next we expand $q^{(s)}$ near the asymptote of $(q_2^2)^{-1/2} \mathcal{R}^{(s^{-1})} q_2$ for $s \sim 0$ into a formal power series in the asymptote of $(q_2^2)^{-1/2} \mathcal{R}^{(s^{-1})} q_2$ and set the resulting coefficients of the various terms of the same order in s and set the resulting coefficients of the various powers of s separately equal to zero. In this way we obtain an infinite number of independent covariant local non-polynomial continuity equations.

The first three pairs of complex continuity equations

$$U^{(s)} + V^{(s)} = 0, \quad \tilde{U}^{(s)} + \tilde{V}^{(s)} = 0$$

are given explicitly by

$$\begin{aligned}
 U^{(1)} &= \frac{1}{2} q_2^2, & V^{(1)} &= 0, \\
 U^{(2)} &= \frac{1}{2 \sqrt{q_2^2}} \left[\left(\frac{q_2}{\sqrt{q_2^2}} \right)^2 \right]_z, & V^{(2)} &= - \frac{(q_2 \cdot q_2)}{\sqrt{q_2^2}}, \\
 U^{(3)} &= \frac{1}{2 \sqrt{q_2^2}} \left[\left(\frac{q_2}{\sqrt{q_2^2}} \right)^2 \right]_z - \frac{5}{8 \sqrt{q_2^2}} \left[\left(\frac{q_2}{\sqrt{q_2^2}} \right)^2 \right]_z, & V^{(3)} &= \frac{(q_2 \cdot q_2)}{2 \sqrt{q_2^2}} \left[\left(\frac{q_2}{\sqrt{q_2^2}} \right)^2 \right]_z \\
 & & (U^{(i)}, \tilde{V}^{(i)}) &= (U^{(i)}, V^{(i)}) \quad (z \leftrightarrow \bar{z}).
 \end{aligned}$$

Now we specialize to real solution vectors for which the second set of continuity equations apart from complex conjugation is identical with the first one. The first three pairs of real continuity equations are explicitly given by

$$\begin{aligned}
 \left\{ \operatorname{Re} (U^{(i)} + V^{(i)}) \right\}_x + \left\{ \operatorname{Im} (V^{(i)} - U^{(i)}) \right\}_y &= 0 \\
 \left\{ \operatorname{Im} (U^{(i)} + V^{(i)}) \right\}_x + \left\{ \operatorname{Re} (U^{(i)} - V^{(i)}) \right\}_y &= 0
 \end{aligned}$$

with the above expressions for $U^{(i)}, V^{(i)}, i = 1, 2, 3, \dots$

It is possible, though cumbersome, to derive the continuity equations for the instanton solutions of the non-linear chiral O_3 model, the only finite extrema of the corresponding Euclidean action⁵⁾, by a limiting procedure ($q_2^2 \approx 0 \approx q_2^2$) from the above continuity equations. Instead, we shall present a simpler and more direct approach.

Set the O_3 action density $\frac{1}{2}(q_2 \cdot q_2)$ equal to $\frac{1}{2} e^{\nu}$. It is an easy matter to show that $\tilde{\nu}$ satisfies the Liouville equation

$$\partial_{z\bar{z}} \tilde{\nu} + e^{\tilde{\nu}} = 0.$$

Real solutions of the Liouville equation are mapped into complex solutions of the same differential equation by the following family of transformations $\mathcal{T}_S, \mathcal{S} e^{\nu}$

$$\mathcal{T}_S: \left(\frac{\nu + \tilde{\nu}}{2} \right)_z = \mathcal{S}^{-1} \operatorname{sinh} \left(\frac{\nu - \tilde{\nu}}{2} \right), \quad \left(\frac{\nu - \tilde{\nu}}{2} \right)_{\bar{z}} = -\mathcal{S} e^{\left(\frac{\nu + \tilde{\nu}}{2} \right)}.$$

$$\left(\mathcal{T}_S: \left(\frac{\nu + \tilde{\nu}}{2} \right)_{\bar{z}} = \mathcal{S}^{-1} \operatorname{sinh} \left(\frac{\nu - \tilde{\nu}}{2} \right), \quad \left(\frac{\nu - \tilde{\nu}}{2} \right)_z = -\mathcal{S} e^{\left(\frac{\nu + \tilde{\nu}}{2} \right)} \right).$$

The compatibility equations are just

$$\partial_{z\bar{z}} \tilde{\nu} + e^{\tilde{\nu}} = 0, \quad \partial_{z\bar{z}} \nu + e^{\nu} = 0.$$

Along with the transformations \mathcal{T}_S go the continuity equations

$$\mathcal{S} \left\{ e^{\left(\frac{\nu - \tilde{\nu}}{2} \right)} \right\}_z + \mathcal{S}^{-1} \left\{ \operatorname{cosh} \left(\frac{\nu - \tilde{\nu}}{2} \right) \right\}_{\bar{z}} = 0.$$

We expand ν' near $\tilde{\nu} \approx 0$ into a formal power series in \mathcal{S} , insert this expansion into the last equation, collect all terms of the same order in \mathcal{S} and set the resulting coefficients separately equal to zero. Finally we express $\tilde{\nu}$ in terms of the Euclidean action density. In this way we obtain the desired infinite number of independent covariant non-polynomial local conservation laws for the instanton solutions

$$\begin{aligned}
 \left\{ \operatorname{Re} (U^{(i)} + V^{(i)}) \right\}_x + \left\{ \operatorname{Im} (V^{(i)} - U^{(i)}) \right\}_y &= 0 \\
 \left\{ \operatorname{Im} (U^{(i)} + V^{(i)}) \right\}_x + \left\{ \operatorname{Re} (U^{(i)} - V^{(i)}) \right\}_y &= 0.
 \end{aligned}$$

with

$$U^{(1)} = \frac{1}{2} \left(\frac{q_2 \cdot q_2}{q_2 \cdot q_2} \right)^2, \quad V^{(1)} = (q_2 \cdot q_2)$$

$$U^{(2)} = \frac{1}{2} \left[\left(\frac{q_2 \cdot q_2}{q_2 \cdot q_2} \right)^2 \right]_z - \frac{(q_2 \cdot q_2)^2}{(q_2 \cdot q_2)^2} + \frac{1}{8} \frac{(q_2 \cdot q_2)^4}{(q_2 \cdot q_2)^4}, \quad V^{(2)} = \frac{1}{2} \frac{(q_2 \cdot q_2)^2}{(q_2 \cdot q_2)^2}$$

$$U^{(3)} = \frac{1}{2} \left[\left(\frac{q_2 \cdot q_2}{q_2 \cdot q_2} \right)^2 \right]_z + \frac{5}{4} \left[\left(\frac{q_2 \cdot q_2}{q_2 \cdot q_2} \right)^2 \right]_z + \frac{1}{16} \frac{(q_2 \cdot q_2)^6}{(q_2 \cdot q_2)^6}$$

$$V^{(3)} = \frac{1}{2} \frac{(q_2 \cdot q_2)^2}{(q_2 \cdot q_2)^2} - \frac{1}{8} \frac{(q_2 \cdot q_2)^4}{(q_2 \cdot q_2)^4}.$$

In conclusion we remark that - similarly as in the hyperbolic case - the Euclidean extremum equations of the non-linear chiral O_3 model for solutions other than instanton solutions can be locally reduced to the following

equations for f_2^2 , f_2^2 and

$$u = \operatorname{arccosh} \left(\frac{f_2^2 \cdot f_2^2}{f_2^2 \cdot f_2^2} \right)$$

$$\left\{ f_2^2 \right\}_{\bar{z}} = 0 = \left\{ f_2^2 \right\}_z + \sqrt{f_2^2 \cdot f_2^2} \sinh u = 0$$

One of the authors (K.P.) would like to thank Professors H. Schopper, H. Joos, and G. Weber for the kind hospitality extended to him at DESY.

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