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SO(2)-Symmetric Solutions of the Two Dimensional Minkowskian and Euclidean O(3) Nonlinear σ-Model

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Abstract: All solutions of the two dimensional Minkowskian and Euclidean nonlinear σ-model are found that are symmetric under an SO(2)-subgroup of the conformal group. They have finite energy and action. The connection of Minkowskian and Euclidean solutions via analytic continuation is pointed out.

The two dimensional O(3) nonlinear σ-model in Minkowski space is completely integrable, because it can be reduced to the sine-Gordon equation by conformal transformations [1]. However, the physical relevance of this statement is obscured by the fact that the conformal transformations depend on the initial data. Conformal symmetry has been used with success to construct solutions of the four dimensional Minkowskian Yang-Mills field theory with simple physical properties as finite energy and action [2], [3],[8]. In particular, Lüscher [3] has obtained all solutions symmetric with respect to an SO(4)-subgroup of the conformal group in four dimensional Minkowski space. It seems worthwhile to look for similar solutions of the two dimensional Minkowskian O(3) nonlinear σ-model and to discuss their relation to the sine-Gordon equation.

To do so, we have to restrict ourselves to the subgroup isomorphic to O(2,2) of the two dimensional conformal group and to search for solutions symmetric under an SO(2) subgroup of O(2,2). The solutions we find have finite energy and action as in the Yang-Mills case. The corresponding solutions in two dimensional Euclidean space are instanton configurations symmetric under an SO(2) subgroup of O(3,1). They interpolate between zero energy solutions in Minkowski space, an indication of quantum tunneling in the sense of Jackiw and Rebbi [4].

We parameterize the unit vector $\vec{n} \in S^2$ of the O(3) nonlinear σ-model by stereographic coordinates $(\pi^1, \pi^2) \in \mathbb{R}^2$. The mapping $\mathbb{R}^2 \cup \{\infty\} \to S^2$ is one-to-one onto and shall map the unit circle onto the set $\mathbb{R}^2 \cap S^2$. The action is

$$S = \int d^2x \, \frac{2\,\partial^\mu \pi \, \partial_\mu \pi^*}{(1+|\pi|^2)^2} \quad , \tag{1}$$

where $\pi = \pi^1 + i\pi^2$ and $g_{00} = 1$, $g_{11} = -1(+1)$ in Minkowskian (Euclidean) space respectively. We shall denote these spaces from now by M(2) and E(2). The action is invariant under the conformal mappings

$$\begin{aligned} M(2)&: \quad x^1 + x^0 \mapsto f(x^1+x^0) \quad , \quad x^1 - x^0 \mapsto g(x^1 - x^0) \\ E(2)&: \quad x^1 + i\,x^0 \mapsto F(x^1 + i\,x^0) \; (\text{or} \quad F(x^1 - i\,x^0)) \quad , \end{aligned} \tag{2}$$

where f and g are arbitrary real valued functions and F is an arbitrary holomorphic function.

Let us now consider the subgroups O(2,2) and O(3,1) respectively. As usual [5] we identify the point $x = (x^0, x^1)$ with a ray in the cones

$$M(2):\quad C_{2,2} = \{\xi^A, A = 0,1,4,5 \mid (\xi^0)^2 - (\xi^1)^2 - (\xi^4)^2 + (\xi^5)^2 = 0\}$$
$$E(2):\quad C_{3,1} = \{\xi^A, A = 0,1,4,5 \mid -(\xi^0)^2 - (\xi^1)^2 - (\xi^4)^2 + (\xi^5)^2 = 0\} \tag{3}$$

and introduce projective coordinates

$$x^\mu = \frac{\xi^\mu}{\xi^4 + \xi^5} \;, \qquad \mu = 0,1 \,. \tag{4}$$

The SO(2) subgroup we want to discuss in either case is the group of rotations in the (ξ^1,ξ^4)-plane. Its action is linear on (ξ^1,ξ^4) but nonlinear on (x^0,x^1). In order to represent the latter in a simple fashion we introduce new coordinates

$$M(2):\quad \frac{2x^1}{1+x^2} = \operatorname{tg} u \;,\quad \frac{2x^0}{1-x^2} = \operatorname{tg}\tau \;,\quad x^0 = \frac{\sin\tau}{\cos\tau + \cos u} \;,\quad x^1 = \frac{\sin u}{\cos\tau + \cos u} \tag{5}$$

$$E(2):\quad \frac{2x^1}{1-x^2} = \operatorname{tg} u \;,\quad \frac{2x^0}{1+x^2} = \operatorname{tgh}\sigma \;;\quad x^0 = \frac{\sinh\sigma}{\cosh\sigma + \cos u} \;,\quad x^1 = \frac{\sin u}{\cosh\sigma + \cos u} \;, \tag{6}$$

where $x^2 = g_{\mu\nu}x^\mu x^\nu$. The coordinates (σ,u) are bipolar coordinates. A rotation with angle χ simply acts on u according to $u \mapsto u + \chi$. The mappings

$$\begin{aligned} M(2):\quad & x^0 \mapsto \tau = \operatorname{arc\,tg}(x^1+x^0) - \operatorname{arc\,tg}(x^1-x^0) \;, \\ & x^1 \mapsto u = \operatorname{arc\,tg}(x^1+x^0) + \operatorname{arc\,tg}(x^1-x^0) \;, \\ & \mathbb{R}^2 \to \{(\tau,u) \mid |\tau| < \pi \;,\; |u| < \pi,\; \cos\tau + \cos u > 0\} \end{aligned} \tag{7}$$

$$\begin{aligned} E(2):\quad & z = x^1 + i x^0 \mapsto \sigma - iu = \log\frac{1-iz}{1+iz} \;, \\ & \mathbb{R}^2 - L \to \{(\sigma,u) \mid |\sigma| < \infty \;,\; |u| < \pi\} \;, \\ & L = \{(x^1,x^0) \in \mathbb{R}^2 \mid x^1 = 0,\; |x^0| \geq 1\} \;, \end{aligned} \tag{8}$$

are conformal and one-to-one onto.

SO(2)-symmetric solutions are most easily obtained, if we introduce polar coordinates (θ,ϕ) on the sphere S^2,

$$\pi = \pi^1 + i\pi^2 = \operatorname{tg}\frac{\theta}{2}\exp i\phi \;. \tag{9}$$

In terms of (θ,ϕ) the action (17) reads

$$S = \frac{1}{2}\int d^2x\,(\partial^\mu\theta\,\partial_\mu\theta + \sin^2\theta\,\partial^\mu\phi\,\partial_\mu\phi) \;. \tag{10}$$

The Ansatz

$$\theta = \theta(\tau)\;\; (\theta = \theta(\sigma)) \;,\quad \phi = \pm m u \;,\quad m = 1,2,\ldots \tag{11}$$

leads to

$$M(2):\quad S = 2\pi\int_{-\pi}^{+\pi} d\tau\,\frac{1}{2}(\dot\theta^2 - m^2\sin^2\theta) \;,\quad \dot\theta = d\theta | d\tau \tag{12}$$

$$E(2):\quad S = 2\pi\int_{-\infty}^{+\infty} d\sigma\,\frac{1}{2}(\dot\theta^2 + m^2\sin^2\theta) \;,\quad \dot\theta = d\theta | d\sigma \,. \tag{13}$$

In the Euclidean case we obtain a finite action only, if

$$\varepsilon = \frac{\dot\theta^2}{2} - \frac{m^2}{2}\sin^2\theta = 0 \;. \tag{14}$$

The solutions

$$\theta(\sigma) = 2\operatorname{arc\,tg}\exp \pm m\sigma \;,\quad \phi = -mu \tag{15}$$

correspond to the SO(2)-symmetric instanton configurations [6]

$$\pi = \exp m(\sigma - iu) = \left(\frac{1-iz}{1+iz}\right)^m \;,\quad \pi = \exp -m(\sigma + iu) = \left(\frac{1-iz^*}{1+iz^*}\right)^m \;,\quad m = 1,2,\ldots \tag{16}$$

with action $S = 4\pi m$, while the index of the mapping $\pi: \mathbb{C} \cup \{\infty\} \to S^2$ is $m(-m)$ respectively.

The Minkowskian action (12) on the other side is finite, if

$$0 < \varepsilon = \frac{\dot\theta^2}{2} + \frac{m^2}{2}\sin^2\theta < +\infty \;,$$

because the interval of integration is finite. The total energy of these solutions is

$$E = \int_{-\infty}^{+\infty} dx^1\, T_{oo} = \int_{-\pi}^{+\pi} du\, \frac{1}{2}(\dot{\theta}^2 + m^2 \sin^2\theta) = 2\pi\varepsilon \quad , \tag{18}$$

whereas the total momentum vanishes,

$$P_1 = \int_{-\infty}^{+\infty} dx^1\, T_{o1} = 0 \quad . \tag{19}$$

The solutions of the first order equation (17) correspond to the well known pendulum solutions. For $\varepsilon = 0$ we have $\theta(\tau) \in \{0,\pi\}$. For $\varepsilon > 0$ we state the solutions of the initial value problem $\theta(0) = 0$ as typical examples,

$$0 < \frac{2\varepsilon}{m^2} < 1 : \quad \sin\theta(\tau) = k\, \mathrm{sn}(m\tau;k) \quad , \qquad k = \sqrt{\frac{2\varepsilon}{m^2}} \quad , \tag{20}$$

$$1 < \frac{2\varepsilon}{m^2} < \infty : \quad \theta(\tau) = \mathrm{am}(\frac{\tau m}{k}\, ; k) \quad , \qquad k = \sqrt{\frac{m^2}{2\varepsilon}} \quad , \tag{21}$$

(For the definition of the elliptic functions sn and am see [7] e.g.). The solution is elemtary, if $2\varepsilon/m^2 = 1$,

$$\frac{2\varepsilon}{m^2} = 1 : \quad \theta(\tau) = -\frac{\pi}{2} + 2\, \mathrm{arc\,tg}\, \exp \pm m\tau \, . \tag{22}$$

It interpolates between two successive maxima of the "potential" $V(\theta) = m^2 \sin^2\theta$.

In general one can say that the initial values of the solutions

$$\pi(\tau,u) = \mathrm{tg}\, \frac{\theta(\tau)}{2} \, \exp \pm i m u \tag{23}$$

for $\tau = 0 \Rightarrow x_0 = 0$ describe a parallel of latitude on the sphere. The configuration in the limits $x^0 \to \pm\infty$ can be calculated from (23), if one observes that the set $\{(x^0,x^1) \in \mathbb{R}^2 \mid |x^1| < \infty\}$ is mapped onto the characteristics $\tau \pm u = \pi\ (-\pi)$ in the (τ,u)-plane for $x^0 \to +\infty\ (-\infty)$ respectively. The reduction of the O(3) nonlinear σ-model to the sine-Gordon equation [1] can be accomplished by putting

$$\frac{|\pi_\tau|^2 - |\pi_u|^2}{|\pi_\tau|^2 + |\pi_u|^2} = \cos\alpha \quad , \tag{24}$$

where the indices denote partial derivatives. After a conformal transformation $\tau \mapsto f(\tau)$, $u \mapsto u$ depending on the initial data the function $\alpha(f(\tau),u)$ obeys the sine-Gordon equation, if $\pi(\tau,u)$ is a solution of the field equation derived from the action (1). In case of the solutions (23) α depends only on the variable τ and, therefore, does not correspond to a solitary solution of the sine-Gordon equation.

Writing the Euclidean conformal mapping (8) in the form

$$i\sigma = \mathrm{arc\,tg}\, z - \mathrm{arc\,tg}\, z^* \quad , \quad u = \mathrm{arc\,tg}\, z + \mathrm{arc\,tg}\, z^* \quad , \tag{25}$$

we see that we arrive at the Minkowskian conformal mapping (7) by analytic continuation in the independent complex variables $z = x^1 + i x^0$ and $z^* = x^1 - i x^0$ from real values of x^0 to imaginary values $x^0 \to i x^0$, while x^1 is kept real. This corresponds to the substitution $\sigma \mapsto i\tau$. We can also analytically continue the instanton solutions (16),

$$\begin{aligned} \exp m(\sigma - iu) &\to \pi(\tau,u) = \exp i m(\tau - u) = \exp i m 2\, \mathrm{arc\,tg}(x^0 - x^1) \\ \exp -m(\sigma + iu) &\to \pi(\tau,u) = \exp -i m(\tau + u) = \exp -i m 2\, \mathrm{arc\,tg}(x^0 + x^1) \quad . \end{aligned} \tag{26}$$

The Minkowskian solutions obtained are of the special form

$$\pi(x^0,x^1) = \exp i f(x^0 \pm x^1) = \pi^1 + i\pi^2 \tag{27}$$

and have zero action. The stereographic coordinates (π^1,π^2) are real-valued but do not separate in τ and u. On the other side the instanton sulutions interpolate between two successive minima of the "potential" $V(\theta) = m^2 \sin^2\theta$. This may be considered as an indication of quantum tunneling in the sense of Jackiw and Rebbi [4].

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