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by

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Abstract: Using the correspondence between the σ-Model and the sine-Gordon field theory, those solutions of the σ-Model are calculated which correspond to the n-soliton solutions of the sine-Gordon equation.

1. Introduction

Recently there has been some progress concerning the nonlinear σ-model. All solutions of the two dimensional Minkowskian and Euclidean O(3) nonlinear σ-model have been found that are symmetric under a SO(2) subgroup of the conformal group [1]. An infinite set of conserved charges is given in [2]. In [3] a set of S-matrices is presented which are supposed to belong to the quantized O(n) models.

As for the Euclidean version, the instanton solutions have been found and it was pointed out that the σ-model might be a training area for the non Abelian gauge theories [4]. These instantons are the only solutions with finite action [5]. The structure of the instanton plasma in the σ-model has been investigated in [6] .

Here we present all solutions of the two dimensional Minkowskian O(3) model that correspond to the n-soliton solutions of the sine-Gordon (s.-G.) equation.

The O(3) field q^a (a=1,2,3) can be reduced to a normalized one n^a by a conformal transformation of the light cone coordinates. This normalized field has constant energy density $\frac{1}{2}$ and momentum density vanishing [7]. The field n^a and its time derivative n_t^a and space derivative n_x^a are an orthogonal dreibein field. Therefore, the field n^a can be represented by a corresponding field $R^b{}_c$ of rotation matrices. The normalized fields are, modulo constant rotations, in one-to-one correspondence to the solutions of the s.-G. equation [7].

Let us represent the s.-G. field α as the 4-fold argument of

the polar representation of a two dimensional vector field ψ. Applying a Bäcklund transformation yields another field $\hat{\psi}$, and $\hat{\psi}$ is determined by two linear, ordinary differential equations $\hat{\psi}_\xi = A\hat{\psi}$, $\hat{\psi}_\eta = B\hat{\psi}$. A and B are known explicitly and are essentially functionals of α. The two equations are compatible iff α obeys the s.-G. equation. The solution can be described by a Wronski matrix W. [8]

The Bäcklund transformation depends on a parameter a. For a = = i ($i^2 = -1$) the above Wronski matrix W is intimately related to the normalized field n^a corresponding to the s.-G. field α. Modulo some trivial factors, W is just the spin $\frac{1}{2}$ representation of the rotation R belonging to the field n^a, as described above. This situation is sketched in fig. 1.1 :

$$\begin{tikzcd}
\text{s.-G. field } \alpha \arrow[rr, leftrightarrow, "\text{Bäcklund transf.}", "\text{with parameter i}"'] \arrow[dd, leftrightarrow] & & \text{Wronski matrix W} \\
& & {} \arrow[u, "\text{spin } \frac{1}{2}\ \text{repres.}"'] \arrow[d, "\text{spin 1}"] \\
\text{normalized solution } n^a \text{ of the } \sigma\text{-model} \arrow[rr, leftrightarrow, "\text{orthogonality}", "\text{of } n^a, n_t^a, n_x^a"'] & & \text{rotation R}
\end{tikzcd}$$

fig. 1.1

Therefore, if the effect of this Bäcklund transformation on the s.-G. field α is known explicitly then so is the normalized field n^a, too. As the so called n-soliton solutions of the s.-G. model are given explicitely [9] and as Bäcklund transformations interpolate between n-and (n+1)-soliton solutions [10], any normalized field of the σ-model corresponding to a n-soliton solution of the s.-G. equation can be calculated.

Section 2 is on the transition from the s.-G. field to the normalized field n^a and the associated field R of rotations. Section 3 is on Bäcklund transformations of the s.-G. field. In section 4 the normalized field n^a corresponding to any n-soliton solution of the s.-G. equation is constructed.

2. Correspondence between s.-G. and normalized fields and their associated rotations

The normalized field $n^a(\xi,\eta)$ obeys the equation of motion:

(2.1.a) $$n^a_{\xi\eta} + (n^b_\xi n^b_\eta)\, n^a = 0, \quad n^a n^a = 1$$

and the normalization condition:

(2.1.b) $$n^a_\xi\, n^a_\xi = n^a_\eta\, n^a_\eta = 1$$

or, in space-time coordinates $t = \xi + \eta$, $x = \xi - \eta$:

(2.2.a) $$n^a_{tt} - n^a_{xx} + (n^a_t\, n^a_t - n^a_x\, n^a_x)\, n^a = 0, \quad n^a\, n^a = 1$$

(2.2.b) $$n^a_t\, n^a_t + n^a_x\, n^a_x = 1, \quad n^a_t\, n^a_x = 0$$

Any solution $q^a(\xi,\eta)$ of the equation of motion (2.1.a) can be written as

(2.3) $$q^a(\xi,\eta) = n^a(a(\xi), b(\eta))$$

where n^a is a normalized solution and $a(\xi)$ and $b(\eta)$ are suitably chosen functions. The energy E of q^a is equal to

(2.4) $$E = \frac{1}{2}\int_{-\infty}^{+\infty} d\tau\, (a'^2(\tau) + b'^2(\tau))$$

Therefore, if the set of normalized solutions n^a is known then so is the set of all solutions q^a, via (2.3). [6]

To any normalized field n^a belongs a field α:

(2.5) $$\cos\alpha = n^a_\xi\, n^a_\eta\,, \quad \sin\alpha = \varepsilon^{abc}\, n^a (n^b_\xi\, n^c_\eta)$$

which obeys the s.-G. equation:

(2.6) $$\alpha_{\xi\eta} + \sin\alpha = 0$$

Conversely, to any solution α of (2.6) a field n^a exists such that (2.5) and (2.1) hold. [7]

The dreibein field

(2.7) $$R^1{}_a = n^a, \quad R^2{}_a = \frac{n^a_t}{\cos\frac{\alpha}{2}}\,, \quad R^3{}_a = \frac{n^a_x}{\sin\frac{\alpha}{2}}$$

is an orthonormal system, because of (2.2) and (2.5). Hence, $R^a{}_b$ is a field of rotation matrices. It obeys the two ordinary, linear differential equations:

(2.8.a) $$R_\xi = \left(-I_1 \frac{\alpha_\xi}{2} + I_2 \sin\frac{\alpha}{2} - I_3 \cos\frac{\alpha}{2} \right) R$$

(2.8.b) $$R_\eta = \left(I_1 \frac{\alpha_\eta}{2} - I_2 \sin\frac{\alpha}{2} - I_3 \cos\frac{\alpha}{2} \right) R$$

where I_1, I_2, I_3 denote the 3-dimensional generators of rotation with $[I_1, I_2] = I_3, \dots$. The equations (2.8.a,b) are compatible iff α obeys the s.-G. equation [11].

3. Bäcklund transformations and polar representation

In this section we give a brief review of that part in [8] which is relevant for us. Our notation corresponds to that in [8], 1502, as follows:

Our notation :	ξ,	η,	α,	ψ^1,	ψ^2,	a,	" ^ "
Notation in [8]:	x,	-t,	φ,	f ,	g ,	-a,	" ′ "

The polar representation of a column vector $(\psi^1,\psi^2)^T$ (" T " denotes transposition and ψ^1 and ψ^2 are complex numbers) is determined by

$$(3.1) \qquad (\psi^1,\psi^2)^T \longleftrightarrow \psi^1 + \psi^2\,(-i\sigma_2) = \rho \exp\left(\frac{\alpha}{4}\,(-i\sigma_2)\right)$$

where σ_2 is the second Pauli matrix. In the following we shall use the symbol ψ for the column vector $(\psi^1,\psi^2)^T$ and the associated matrix $\psi^1 + \psi^2\,(-i\sigma_2)$, as well.

Let $\psi(\xi,\eta)$ and $\hat{\psi}(\xi,\eta)$ be two fields with a polar decomposition as described in (3.1). Then, for any complex constant a, the following identities hold:

$$\frac{1}{\hat{\psi}\,\psi^T}\left(\hat{\psi}_\xi\,\psi^T - \hat{\psi}\,\psi^T_\xi + \frac{a^{-1}}{2}\hat{\psi}^T\psi\right) \equiv$$

$$(3.2.a) \qquad \left(\ln\frac{\hat{\rho}}{\rho}\right)_\xi + \frac{a^{-1}}{2}\cos\left(\frac{\hat{\alpha}-\alpha}{2}\right) + {}$$
$$+ \frac{(-i\sigma_2)}{2}\left(\left(\frac{\hat{\alpha}+\alpha}{2}\right)_\xi - a^{-1}\sin\left(\frac{\hat{\alpha}-\alpha}{2}\right)\right),$$

$$\frac{1}{\hat{\psi}\,\psi}\left(\hat{\psi}_\eta\,\psi - \hat{\psi}\,\psi_\eta - \frac{a}{2}\hat{\psi}^T\psi^T\right) \equiv$$

$$(3.2.b) \qquad \left(\ln\frac{\hat{\rho}}{\rho}\right)_\eta - \frac{a}{2}\cos\left(\frac{\hat{\alpha}+\alpha}{2}\right) + {}$$
$$+ \frac{(-i\sigma_2)}{2}\left(\left(\frac{\hat{\alpha}-\alpha}{2}\right)_\eta + a\sin\left(\frac{\hat{\alpha}+\alpha}{2}\right)\right)$$

The fields α and $\hat{\alpha}$ are connected by the Bäcklund transformation with parameter a iff:

$$(3.3.a) \qquad \left(\frac{\hat{\alpha}+\alpha}{2}\right)_\xi - a^{-1}\sin\left(\frac{\hat{\alpha}-\alpha}{2}\right) = 0$$

$$(3.3.b) \qquad \left(\frac{\hat{\alpha}-\alpha}{2}\right)_\eta + a\sin\left(\frac{\hat{\alpha}+\alpha}{2}\right) = 0$$

This is obviously the case if, and only if, there exists a complex valued field $f(\xi,\eta)$ such that

$$(3.4.a) \qquad \frac{1}{\hat{\psi}\,\psi^T}\left(\hat{\psi}_\xi\,\psi^T - \hat{\psi}\,\psi^T_\xi + \frac{a^{-1}}{2}\hat{\psi}^T\psi\right) = f_\xi$$

$$(3.4.b) \qquad \frac{1}{\hat{\psi}\,\psi}\left(\hat{\psi}_\eta\,\psi - \hat{\psi}\,\psi_\eta - \frac{a}{2}\hat{\psi}^T\psi^T\right) = f_\eta$$

Thus, (3.4) describes the Bäcklund transformations in the new ψ-language. One can look upon (3.4) as a system of ordinary, linear differential equations for the column vector $\hat{\psi}$:

$$(3.5.a) \qquad \hat{\psi}_\xi = \left((f + \ln\rho)_\xi + \left(\frac{-\alpha_\xi}{2}\right)\left(\frac{\sigma_2}{2i}\right) + \frac{i}{a}\sin\left(\frac{\alpha}{2}\right)\left(\frac{-\sigma_1}{2i}\right)\right.$$
$$\left. - \frac{i}{a}\cos\left(\frac{\alpha}{2}\right)\left(\frac{\sigma_3}{2i}\right)\right)\hat{\psi},$$

$$\hat{\psi}_\eta = ((f + \ln\rho)_\eta + \frac{\alpha_\eta}{2}(\frac{\sigma_2}{2i}) + ia \sin(\frac{\alpha}{2})(\frac{-\sigma_1}{2i}) + ia \cos(\frac{\alpha}{2})(\frac{\sigma_3}{2i}))\hat{\psi} \tag{3.5.b}$$

The compatibility of this system is equivalent to the statement that α obeys the s.-G. equation. The solution of this system then can be described by a Wronski matrix $W(\xi,\eta)$.

To transform W into a suitable form we introduce

$$Q = \begin{bmatrix} i & 0 \\ 0 & -1 \end{bmatrix} \tag{3.6.}$$

and consider $\frac{1}{\rho}\exp(-f)\, Q^+ W Q$. It is easy to show that

$$(\frac{1}{\rho}\exp(-f)\, Q^+WQ)_\xi = (-\frac{\alpha_\xi}{2}\frac{\sigma_1}{2i} + \frac{i}{a}\sin(\frac{\alpha}{2})\frac{\sigma_2}{2i} - \frac{i}{a}\cos(\frac{\alpha}{2})\frac{\sigma_3}{2i})(\frac{1}{\rho}\exp(-f)\, Q^+WQ), \tag{3.7.a}$$

$$(\frac{1}{\rho}\exp(-f)\, Q^+WQ)_\eta = (\frac{\alpha_\eta}{2}\frac{\sigma_1}{2i} + ia\sin(\frac{\alpha}{2})\frac{\sigma_2}{2i} + ia\cos(\frac{\alpha}{2})\frac{\sigma_3}{2i})(\frac{1}{\rho}\exp(-f)\, Q^+WQ) \tag{3.7.b}$$

4. Construction of the normalized field n^a corresponding to any n-soliton solution of the s.-G. equation

Notice that in the case a = i the matrix $\frac{1}{\rho}\exp(-f)\, Q^+WQ$ can be chosen to be the spin $\frac{1}{2}$ representation of R, as can be seen by comparing (3.7) and (2.8).

In this section the spin $\frac{1}{2}$ representation of R and the normalized field n^a belonging to a n-soliton field α are given explicitly , via the Bäcklund transformation with parameter a = i.

A two dimensional vector representation $\psi = (\psi^1,\psi^2)^T$ of any n-soliton solution $\alpha = 4 \arctan(\psi^2/\psi^1)$ is available [9], 1046. The notation in [9] corresponds to ours as follows:

Our notation : α, m_k, λ_k

Notation in [9] : $-u$, $-4im_k$, $-4i\,\zeta_k$

The vector representation $\psi(\xi,\eta;n,m_1\dots m_n,\lambda_1\dots\lambda_n)$ of the n-soliton solution α with the parameters $m_1\dots m_n,\lambda_1\dots\lambda_n$ is:

$$\begin{bmatrix} \psi^1 \\ \psi^2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \det(1-iML\exp(\xi\Lambda-\frac{\eta}{\Lambda})) + \det(1+iML\exp(\xi\Lambda-\frac{\eta}{\Lambda})) \\ \frac{1}{i}\det(1-iML\exp(\xi\Lambda-\frac{\eta}{\Lambda})) - \frac{1}{i}\det(1+iML\exp(\xi\Lambda-\frac{\eta}{\Lambda})) \end{bmatrix} \tag{4.1}$$

M, L, Λ are n x n-matrices:

$$M = \mathrm{diag}(m_1,\dots m_n), \tag{4.2.a}$$

$$\Lambda = \mathrm{diag}(\lambda_1,\dots\lambda_n), \tag{4.2.b}$$

$$L_{ik} = \frac{1}{\lambda_i+\lambda_k}, \quad (i,k = 1,\dots n) \tag{4.2.c}$$

The reality of α is guaranteed by the requirement that m_i and λ_i are either real or can be combined to conjugate complex pairs.

The Bäcklund transform $\hat{\psi} = \psi(\xi,\eta;\hat{n},\hat{m}_1,\dots\hat{m}_{\hat{n}},\hat{\lambda}_1,\dots\hat{\lambda}_{\hat{n}})$, as determined by (3.4), can be described by the new parameters (see appendix):

(4.3.a) $\hat{n} = n + 1$,

(4.3.b) $\hat{m}_i = m_i \left(\frac{a^{-1} + \lambda_i}{a^{-1} - \lambda_i} \right)$, $(i = 1, \ldots n)$, $\hat{m}_{n+1}$ arbitrary,

(4.3.c) $\hat{\lambda}_i = \lambda_i$, $(i = 1, \ldots n)$, $\hat{\lambda}_{n+1} = a^{-1}$

where the function f in (3.4) is chosen to be $f = \frac{a^{-1}}{2}\xi - \frac{a}{2}\eta$.

A Wronski matrix of (3.5) can be found by choosing the two linear independent $\hat{\psi}$-s:

(4.4) $W = \frac{1}{\sqrt{2}} \left(\hat{\psi} \Big|_{\hat{m}_{n+1}=2} , \hat{\psi} \Big|_{\hat{m}_{n+1}=-2} \right)$

To simplify the formulas we use the abbreviations:

(4.5.a)
$$u = \frac{1}{\sqrt{2}} \det\left(1 - iM\left(\frac{-i+\Lambda}{-i-\Lambda}\right) L \exp\left(\xi\Lambda - \frac{\eta}{\Lambda}\right)\right) \cdot$$
$$\cdot \left| \det^{-1}\left(1 - iML \exp\left(\xi\Lambda - \frac{\eta}{\Lambda}\right)\right) \right| ,$$

(4.5.b)
$$v = \frac{1}{\sqrt{2}} \det\left(1 + iM\left(\frac{-i+\Lambda}{-i-\Lambda}\right) L \exp\left(\xi\Lambda - \frac{\eta}{\Lambda}\right)\right) \cdot$$
$$\cdot \left| \det^{-1}\left(1 - iML \exp\left(\xi\Lambda - \frac{\eta}{\Lambda}\right)\right) \right|$$

When α is real the relation holds:

(4.6) $|u^2| + |v^2| = 1$

Using (4.4), $f = \frac{a^{-1}}{2}\xi - \frac{a}{2}\eta$ and $\varrho = \left| \det\left(1 - iML \exp\left(\xi\Lambda - \frac{\eta}{\Lambda}\right)\right) \right|$ a spin $\frac{1}{2}$ representation of R in (2.8) is equal to:

(4.7)
$$\frac{1}{\varrho} \exp(-f)\, Q^{+} W Q \begin{bmatrix} \frac{1}{\sqrt{2}} , & \frac{-1}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} , & \frac{-i}{\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} (u+v) \exp(i\frac{t}{2}), & (\bar{u}-\bar{v}) \exp(-i\frac{t}{2}) \\ (-u+v) \exp(i\frac{t}{2}), & (\bar{u}+\bar{v}) \exp(-i\frac{t}{2}) \end{bmatrix}$$

The normalized field n^a is just the first row of the corresponding spin 1 representation:

(4.8.a) $n^1 = (uv \exp(it) + \bar{u}\bar{v} \exp(-it))$,

(4.8.b) $n^2 = \frac{1}{i} (uv \exp(it) - \bar{u}\bar{v} \exp(-it))$,

(4.8.c) $n^3 = 1 - 2 |u^2|$

To **summarize**: Let $\alpha = 4 \arctan(\psi^2/\psi^1)$ be a real n-soliton solution of the s.-G. equation where ψ^1 and ψ^2 are given by (4.1). Then the associated spin $\frac{1}{2}$ representation of R in (2.8) and the corresponding normalized field n^a are given by (4.7) and (4.8) where u and v are the functions in (4.5). Solutions $q^a(\xi,\eta)$ of the equation of motion (2.1.a) (n^a replaced by q^a) with energy finite can be obtained from n^a in (4.8) by the substitution (2.3) where $\int_{-\infty}^{+\infty} dx\, (a'^2(x) + b'^2(x)) < \infty$.

Acknowledgement: I would like to thank H. Lehmann for bringing this problem to my attention.

Appendix

ψ and $\hat{\psi}$ as determined by (4.1), (4.2) and (4.3) shall be shown to be connected by (3.4) where f is equal to $f = \frac{a^{-1}}{2}\xi - \frac{a}{2}\eta$:

(A.1.a) $$\hat{\psi}_\xi \psi^T - \hat{\psi}\psi_\xi^T + \frac{a^{-1}}{2}(\hat{\psi}^T\psi - \hat{\psi}\psi^T) = 0 \ ,$$

(A.1.b) $$\hat{\psi}_\eta \psi - \hat{\psi}\psi_\eta - \frac{a}{2}(\hat{\psi}^T\psi^T - \hat{\psi}\psi) = 0$$

Using the determinant formula [12]

(A.2) $$\det\left(\frac{1}{\lambda_i+\lambda_k}\right) = \prod_i \frac{1}{2\lambda_i} \cdot \prod_{i<k}\left(\frac{\lambda_i-\lambda_k}{\lambda_i+\lambda_k}\right)^2$$

we find the following expressions for $\psi = \psi^1 - i\sigma_2\psi^2$ and $\hat{\psi}$:

(A.3.a) $$\psi = \sum_{\mathbb{I} \subseteq \{1,\cdots n\}} (i\sigma_2)^{|\mathbb{I}|} \cdot \prod_{i\in\mathbb{I}} \frac{m_i}{2\lambda_i} \cdot \prod_{\substack{i<k \\ i,k\in\mathbb{I}}} \left(\frac{\lambda_i-\lambda_k}{\lambda_i+\lambda_k}\right)^2 \cdot$$
$$\cdot \exp\left[\sum_{i\in\mathbb{I}}\left(\xi\lambda_i - \frac{\eta}{\lambda_i}\right)\right] ,$$

(A.3.b) $$\hat{\psi} = \sum_{\hat{\mathbb{I}} \in \{1,\cdots(n+1)\}} (i\sigma_2)^{|\hat{\mathbb{I}}|} \cdot \prod_{i\in\hat{\mathbb{I}}} \frac{\hat{m}_i}{2\hat{\lambda}_i} \cdot \prod_{\substack{i<k \\ i,k\in\hat{\mathbb{I}}}} \left(\frac{\hat{\lambda}_i-\hat{\lambda}_k}{\hat{\lambda}_i+\hat{\lambda}_k}\right)^2 \cdot$$
$$\cdot \exp\left[\sum_{i\in\hat{\mathbb{I}}}\left(\xi\hat{\lambda}_i - \frac{\eta}{\hat{\lambda}_i}\right)\right]$$

where the $\hat{m}_i$, $\hat{\lambda}_i$ are given by (4.3). Insertion of (A.3) into (A.1) yields:

$$\hat{\psi}_\xi \psi^T - \hat{\psi}\psi_\xi^T + \frac{a^{-1}}{2}(\hat{\psi}^T\psi - \hat{\psi}\psi^T) =$$
$$= \sum_{\substack{\hat{\mathbb{I}} \subseteq \{1,\cdots(n+1)\} \\ \mathbb{I} \subseteq \{1,\cdots n\}}} (-1)^{|\mathbb{I}|} \left[\sum_{i\in\hat{\mathbb{I}}}\lambda_i - \sum_{i\in\mathbb{I}}\lambda_i - a^{-1} \cdot \frac{1-(-1)^{|\mathbb{I}|+|\hat{\mathbb{I}}|}}{2}\right] \cdot$$

$$\cdot \prod_{i\in\mathbb{I}-\mathbb{I}\cap\hat{\mathbb{I}}} (a^{-1}-\lambda_i) \cdot \prod_{i\in\hat{\mathbb{I}}-\mathbb{I}\cap\hat{\mathbb{I}}} (a^{-1}+\lambda_i) \cdot \prod_{\substack{i<k \\ i,k\in\hat{\mathbb{I}}-\mathbb{I}\cap\hat{\mathbb{I}}}} (\lambda_i-\lambda_k)^2 \cdot$$

$$\cdot \prod_{\substack{i<k \\ i,k\in\mathbb{I}-\mathbb{I}\cap\hat{\mathbb{I}}}} (\lambda_i-\lambda_k)^2 \cdot \prod_{\substack{i\in\mathbb{I}-\mathbb{I}\cap\hat{\mathbb{I}} \\ k\in\hat{\mathbb{I}}-\mathbb{I}\cap\hat{\mathbb{I}}}} (\lambda_i+\lambda_k)^2 \cdot$$

(A.4.a) $$\cdot (i\sigma_2)^{|\mathbb{I}|+|\hat{\mathbb{I}}|} \cdot \prod_{i\in\mathbb{I}} \frac{\hat{m}_i}{2\lambda_i(a^{-1}+\lambda_i)} \exp\left(\xi\lambda_i - \frac{\eta}{\lambda_i}\right) \cdot$$

$$\cdot \prod_{i\in\hat{\mathbb{I}}} \frac{\hat{m}_i}{2\lambda_i(a^{-1}+\lambda_i)} \exp\left(\xi\lambda_i - \frac{\eta}{\lambda_i}\right) \cdot$$

$$\cdot \prod_{i\in\mathbb{I}\cap\hat{\mathbb{I}}} (a^{-1}+\lambda_i)(a^{-1}-\lambda_i) \cdot \prod_{\substack{i<k \\ i,k\in\mathbb{I}\cap\hat{\mathbb{I}}}} \left(\frac{\lambda_i-\lambda_k}{\lambda_i+\lambda_k}\right)^4 \cdot$$

$$\cdot \prod_{\substack{i<k \\ i,k\in\mathbb{I}\cup\hat{\mathbb{I}}-\mathbb{I}\cap\hat{\mathbb{I}}}} (\lambda_i+\lambda_k)^{-2} \cdot \prod_{\substack{i\in\mathbb{I}\cup\hat{\mathbb{I}}-\mathbb{I}\cap\hat{\mathbb{I}} \\ k\in\mathbb{I}\cap\hat{\mathbb{I}}}} \left(\frac{\lambda_i-\lambda_k}{\lambda_i+\lambda_k}\right)^2 ,$$

$$\hat{\psi}_\eta \psi - \hat{\psi}\psi_\eta - \frac{a}{2}\left(\hat{\psi}^T\psi^T - \hat{\psi}\psi\right) \equiv$$

$$\equiv \sum_{\substack{\hat{\mathbb{I}} \subseteq \{1, \cdots (n+1)\} \\ \mathbb{I} \subseteq \{1, \cdots n\}}} \left[- \sum_{i \in \hat{\mathbb{I}}} \frac{1}{\lambda_i} + \sum_{i \in \mathbb{I}} \frac{1}{\lambda_i} + a \; \frac{1-(-1)^{|\mathbb{I}|+|\hat{\mathbb{I}}|}}{2} \right] \cdot$$

$$\text{(A.4.b)} \quad \cdot \prod_{i \in \mathbb{I} - \mathbb{I} \cap \hat{\mathbb{I}}} (a^{-1} - \lambda_i) \cdot \prod_{i \in \hat{\mathbb{I}} - \mathbb{I} \cap \hat{\mathbb{I}}} (a^{-1} + \lambda_i) \cdot \prod_{\substack{i<k \\ i,k \in \hat{\mathbb{I}} - \mathbb{I} \cap \hat{\mathbb{I}}}} (\lambda_i - \lambda_k)^2 \cdot$$

$$\cdot \prod_{\substack{i<k \\ i,k \in \mathbb{I} - \mathbb{I} \cap \hat{\mathbb{I}}}} (\lambda_i - \lambda_k)^2 \cdot \prod_{\substack{i \in \mathbb{I} - \mathbb{I} \cap \hat{\mathbb{I}} \\ k \in \hat{\mathbb{I}} - \mathbb{I} \cap \hat{\mathbb{I}}}} (\lambda_i + \lambda_k)^2 \cdot$$

$\cdot \quad \cdot \cdot \cdot$(as in (A.4.a))

Collecting in (A.4.b) the terms for which $\mathbb{I} \cup \hat{\mathbb{I}}$ = const. and $\mathbb{I} \cap \hat{\mathbb{I}}$ = = const. leads to:

$$\text{(A.5)} \quad \sum_{\substack{\hat{\mathbb{I}} \subseteq \{1, \cdots (n+1)\} \\ \mathbb{I} \subseteq \{1, \cdots n\} \\ \mathbb{I} \cup \hat{\mathbb{I}} = \text{const.} \\ \mathbb{I} \cap \hat{\mathbb{I}} = \text{const.}}} \left[- \sum_{i \in \hat{\mathbb{I}}} \frac{1}{\lambda_i} + \sum_{i \in \mathbb{I}} \frac{1}{\lambda_i} + a \; \frac{1-(-1)^{|\mathbb{I}|+|\hat{\mathbb{I}}|}}{2} \right] \cdot$$

$$\cdot \prod_{i \in \mathbb{I} - \mathbb{I} \cap \hat{\mathbb{I}}} (a^{-1} - \lambda_i) \cdot \prod_{i \in \hat{\mathbb{I}} - \mathbb{I} \cap \hat{\mathbb{I}}} (a^{-1} + \lambda_i) \cdot \prod_{\substack{i<k \\ i,k \in \hat{\mathbb{I}} - \mathbb{I} \cap \hat{\mathbb{I}}}} (\lambda_i - \lambda_k)^2 \cdot$$

$$\cdot \prod_{\substack{i<k \\ i,k \in \mathbb{I} - \mathbb{I} \cap \hat{\mathbb{I}}}} (\lambda_i - \lambda_k)^2 \cdot \prod_{\substack{i \in \mathbb{I} - \mathbb{I} \cap \hat{\mathbb{I}} \\ k \in \hat{\mathbb{I}} - \mathbb{I} \cap \hat{\mathbb{I}}}} (\lambda_i + \lambda_k)^2 \cdot \cdots \overset{?}{\equiv} 0$$

and we have to show that (A.5) vanishes. Notice that the terms which are not explicitely written down are constant if $\mathbb{I} \cup \hat{\mathbb{I}}$ = const. and $\mathbb{I} \cap \hat{\mathbb{I}}$ =const.. (A.4.a) gives nothing new as can be seen by dividing (A.5) with $\prod_{i \in \mathbb{I} \cup \hat{\mathbb{I}} - \mathbb{I} \cap \hat{\mathbb{I}}} (\lambda_i a^{-1}) \cdot \prod_{\substack{i<k \\ i,k \in \mathbb{I} \cup \hat{\mathbb{I}} - \mathbb{I} \cap \hat{\mathbb{I}}}} (\lambda_i \lambda_k)^2$ and substituting $a \to a^{-1}$, $\lambda_i \to \lambda_i^{-1}$.

It is sufficient only to consider the cases $\mathbb{I} \cap \hat{\mathbb{I}} = \emptyset$ in (A.5). To write the formulas in a more symmetric form the projection operators

$$\text{(A.6)} \quad \mathcal{P}_i \; : \quad (\mathcal{P}_i f)(\lambda_1, \cdots \lambda_i, \cdots) = f(\lambda_1, \cdots -\lambda_i, \cdots)$$

are introduced and the condition (A.5) is equivalent to

$$\text{(A.7)} \quad \sum_{\substack{\hat{\mathbb{I}} \subseteq \{1, \cdots (n+1)\} \\ \mathbb{I} \subseteq \{1, \cdots n\} \\ \mathbb{I} \cup \hat{\mathbb{I}} = \text{const.} \\ \mathbb{I} \cap \hat{\mathbb{I}} = \emptyset}} \left. \prod_{i \in \mathbb{I}} \mathcal{P}_i \cdot \left[- \sum_{i \in \mathbb{I} \cup \hat{\mathbb{I}}} \frac{1}{\lambda_i} + a \; \frac{1-(-1)^{|\mathbb{I}|+|\hat{\mathbb{I}}|}}{2} \right] \cdot \right. \cdot \prod_{\substack{i<k \\ i,k \in \mathbb{I} \cup \hat{\mathbb{I}}}} (\lambda_i - \lambda_k)^2 \cdot \prod_{i \in \mathbb{I} \cup \hat{\mathbb{I}}} (a^{-1} + \lambda_i) \overset{?}{\equiv} 0$$

or

$$\text{(A.8)} \quad \prod_{i \in \text{const.} - \{n+1\}} (1 + \mathcal{P}_i) \cdot \left[- \sum_{i \in \text{const.}} \frac{1}{\lambda_i} + a \cdot \frac{1-(-1)^{|\text{const.}|}}{2} \right] \cdot$$

$$\cdot \prod_{i<k \in \text{const.}} (\lambda_i - \lambda_k)^2 \cdot \prod_{i \in \text{const.}} (a^{-1} + \lambda_i) \overset{?}{\equiv} 0$$

The cases $(n+1) \in$ const. and $(n+1) \notin$ const. must be distinguished but, in any case, (A.8) holds iff the following identity is true:

$$\text{(A.9)} \quad \prod_{i=1}^{m} (1 + \mathcal{P}_i) \cdot \left[- \sum_{i=1}^{m} \frac{1}{\lambda_i} + a \; \frac{1-(-1)^m}{2} \right] \cdot \prod_{i=1}^{m} (a^{-1} + \lambda_i) \cdot \prod_{i<k=1}^{m} (\lambda_i - \lambda_k)^2 \equiv 0$$

We prove (A.9) by induction. It is true for m = 1. Assume it holds for m-1. The left hand side of (A.9) obviously is a polynomial in a^{-1} of degree $\leq$ m-1. Therefore it suffices to show that (A.9) vanishes at the points $a^{-1} = \lambda_i$ (i = 1,... m), e.g. $a^{-1} = \lambda_m$:

$$\prod_{i=1}^{m-1} (1+\mathcal{P}_i)(1+\mathcal{P}_m) \cdot \left[-\sum_{i=1}^{m} \frac{1}{\lambda_i} + a \, \frac{1-(-1)^m}{2} \right] \cdot \prod_{i=1}^{m} (a^{-1}+\lambda_i) \cdot \prod_{i<k=1}^{m} (\lambda_i - \lambda_k)^2 =$$

$$= 2a^{-1} \cdot \prod_{i=1}^{m-1} (\lambda_i^2 - a^{-2}) \cdot \prod_{i=1}^{m-1} (1+\mathcal{P}_i) \cdot \left[-\sum_{i=1}^{m-1} \frac{1}{\lambda_i} - a \, \frac{1-(-1)^{m-1}}{2} \right] \cdot$$

$$\cdot \prod_{i=1}^{m-1} (\lambda_i - a^{-1}) \cdot \prod_{i<k=1}^{m-1} (\lambda_i - \lambda_k)^2 = 0$$

by assumption.

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