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On the Vacuum Structure of the Schwinger Model

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Abstract

We discuss the generators σ of the vacuum structure of the Schwinger model. In the Lowenstein-Swieca gauge, we identify these operators with intertwining operators between different charge sectors of a free, massless spinor field ψ and use them as approximations for new operators σ_∞ on an enlarged ψ-Fock space. The σ_∞'s generate states, which are vacua with respect to the ψ-Hamiltonian and their construction shows the connection with boundary conditions widely discussed in non-abelian gauge theories.

(+) On leave of absence from the University of Würzburg, Germany.

1) Introduction

One of the most puzzling problems of contemporary hadron physics is the confinement of quarks [1] in gauge theories like quantum chromodynamics. At least since the discovery of instantons [2] the vacuum structure of such theories plays a central rôle in the discussion of possible confinement mechanisms [3], which can be studied in detail and very explicitly with help of two-dimensional, superrenormalizable field-theoretical models.

Especially the exactly soluble Schwinger model [4], i.e. quantum electrodynamics of massless fermions in two-dimensional spacetime (QED_2) and its massive version show many expected dynamical properties of quantum chromodynamics [5]. The peculiar vacuum degeneracy of QED_2 reflects the spontaneous breaking of the gauge and chiral symmetry of the Lagrangean. Whereas instantons and "c instantons" [6] (merons [7]) are treated with help of functional integration methods, the explicit solution of QED_2 in terms of local operators is well-suited for the discussion of the vacuum structure. In this language, the vacuum space of the model is generated by zero-dimensional operators σ_+, σ_- ("spurions" [6]), which can be related to topological inequivalent gauge transformations [8].

In this paper, we are interested in the precise construction of these operators and their physical interpretation, especially in connection with the infrared problem of QED_2. For this, we consider the operator solution of QED_2, as it was given by Lowenstein and Swieca [9], and restrict it to the physical state space. On this space, the σ-operators are defined with help of a free, massless, canonical fermion field $\psi(x)$, only. In their construction, however, the current potentials of the vector and axial vector current of ψ are involved. These are free, massless, scalar fields in two dimensions, the definition of which is ambiguous because of infrared singularities [10].

Therefore, we replace the current potentials by well-defined charge creation operators and their exponentials by unitary charge displacement transformations. Due to their construction, the σ-operators then have a clear physical interpretation: they are the net-effect of a point charge, created by the ψ -field and displaced to infinity. Our treatment therefore illustrates the classical argument of Coleman [11], who described the vacuum structure of the Schwinger model as the result of a constant background field generated by "charges at infinity".

In section 2), we reconsider the σ-construction of ref. 9 and generalize Klaiber's regularization method [12] to any cut-off prescription $\mathcal{J}$. The thus reached $\sigma_{\mathcal{J}}$ -operators are interpreted as intertwining operators between different charge sectors of the ψ -Fock space $\mathcal{F}$, which generate states of definite charge distribution $\mathcal{J}'$ out of the Fock vacuum Ω_0 (eq. (4)). These states are again vacua of the Schwinger model, since the $\sigma_{\mathcal{J}}$'s do commute with the energy-momentum operator of that theory. They are, however, translational non-invariant states with respect to the Hamiltonian of the free ψ -field.

In order to remove this disadvantage, one has to enlarge the ψ -Fock space by a "vacuum subspace" $\mathcal{H}_0$: $\mathcal{H} = \mathcal{H}_0 \otimes \mathcal{F}$ [13]. In section 3) we therefore extend the ψ -field algebra by unitary operators σ_∞ , which

we characterize by their algebraic properties, and give an irreducible representation of the extended algebra on the (positive definite) state space $\mathcal{H}$. The vacuum sub-space $\mathcal{H}_0$ is generated cyclically by the σ_∞ -operators in this representation.

In section 4) we show, how the vacua in $\mathcal{H}_0$ can be approximated by states on the original Fock space. This we do by constructing a sequence of $\sigma_{\mathcal{J}}$ -fields on $\mathcal{F}$, the Wightman distributions of which are proved to converge towards those of the algebraically defined σ_∞ -spurions of section 3). The convergence is the strongest, one can expect.

2) Currents of a Free Spinor Field and their Exponentials

We consider a canonically quantized, free, massless spinor field $\psi(x)$ in one space dimension, which obeys the Dirac equation $i\gamma^\mu\partial_\mu\psi = 0$. According to our γ -matrix convention

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \gamma^5 = \gamma^0\gamma^1,$$

this equation requires, that the upper (lower) component ψ_- (ψ_+) of ψ does only depend on $x^+ = x^0 + x^1$ ($x^- = x^0 - x^1$). Hence, the chiral components of the currents of ψ

$$j_-(x^+) = :\psi_-(x^+)^\dagger \psi_-(x^+): \quad , \quad j_+(x^-) = :\psi_+(x^-)^\dagger \psi_+(x^-):$$

are free, massless, hermitean fields. They generate an algebra via the commutation relations

$$[j_\pm(x^\mp), j_\pm(y^\mp)] = \frac{i}{2\pi}\,\delta'(x^\mp - y^\mp), \quad [j_\pm(x^\mp), j_\mp(y^\pm)] = 0. \tag{1}$$

Because of the complete decoupling of left- and right-moving fields, we frequently omit the indizes, which distinguish both species.

Now, we define positive and negative frequency parts $j^{(\pm)}$ of j with help of their Fourier decomposition:

$$\begin{aligned} &j_\pm(x^\mp) = j_\pm(x^\mp)^{(+)} + j_\pm(x^\mp)^{(-)}, \quad j_\pm(x^\mp)^{(+)} = (j_\pm(x^\mp)^{(-)})^+, \\ &j_\pm(x^\mp)^{(-)} = (2\pi i)^{-1}\int_0^\infty dk\,\sqrt{k}\, c(\pm k)\, e^{-ikx^\mp}, \\ &[j_\pm(x^\mp)^{(-)}, j_\pm(y^\mp)^{(+)}] = (2\pi i(x^\mp - y^\mp - i0))^{-2}, \\ &[c(k), c(k')^+] = \delta(k-k'), \quad [c(k), c(k')] = 0, \text{ etc.} \end{aligned} \tag{2}$$

Since the currents are operator-valued distributions, we use the relations (1) and (2) also in their smeared-out version with real-valued test functions Λ, for which we adopt the convention

$$\Lambda(x) = \Lambda(x)^{(+)} + \Lambda(x)^{(-)} = (2\pi)^{-1}\int_{-\infty}^{\infty} dk\,\hat{\Lambda}(k)e^{-ikx}, \quad \tilde{\Lambda}(k)^{(\pm)} = \theta(\pm k)\tilde{\Lambda}(k).$$

Irreducible representations of the current commutation relations (1) are characterized by the total charges, which are the eigenvalues of the charge operators [14] $Q_\pm = \int dx^\mp j_\pm(x^\mp)$ on that representation. Now, it is interesting to consider automorphisms of the current algebra, which change the charge density by c-number functions Λ :

$$j(x) \longmapsto j(x) - d\Lambda(x)/dx. \tag{3}$$

As long as the total charge remains unchanged, i.e. $\Delta Q = \Lambda(-\infty) - \Lambda(\infty) = 0$, $\exp(-2\pi i j(\Lambda))$ is a unitary operator, which makes the mapping (3) an inner automorphism. This follows from the integration of the current commutator with Schwinger term (1)

$$\exp(-2\pi i j(\Lambda))\, j(x)\, \exp(2\pi i j(\Lambda)) = j(x) - d\Lambda(x)/dx.$$

For $\Delta Q \neq 0$, the problem arises, how to define precisely objects "$\exp(-2\pi i j(\Lambda))$", which transform between representations with different charges. It is shown below, that this problem is intimately connected with the definition of vacuum generating σ-operators in the Schwinger model.

Since the fermion field $\psi(x)$ intertwines different charge sectors with $\Delta Q = 1$, too, it is a natural question to study the relation between ψ and the charge changing objects of type "$\exp(-2\pi i j(\Lambda))$". This second problem is the question of bosonizing fermions and the inverse one of constructing fermions out of observables.

We take the rigorous mathematical definition of the objects "$\exp(-2\pi i j(\Lambda))$" with help of intertwining operators from the work of R.F. Streater and I.F. Wilde [15], who based their analysis on the investigations of S. Doplicher, R. Haag and J.E. Roberts [16]. For this, we consider a Fock-type representation of the current algebra (1) on a Hilbert space H_0 with Fock vacuum Ω_0, i.e. $c(k)\Omega_0 = 0$ for all k. The discussion can be standardized by putting $\Lambda(\infty) = 0$. Then, for $\Delta Q = \Lambda(-\infty) = \alpha \neq 0$, the operators

$$E(\Lambda) = :\exp(-2\pi i j(\Lambda)): = \exp(-2\pi i j(\Lambda)^{(+)})\exp(-2\pi i j(\Lambda)^{(-)})$$

would contain infrared divergences, because $\tilde{\Lambda}(k)$ behaves like αk^{-1} for small momenta k. Therefore, we choose a standard charge distribution J with $J(-\infty) = 1$, $J(\infty) = 0$, which defines a splitting of the whole test function space according to $\Lambda(x) = \alpha J(x) + (\Lambda(x) - \alpha J(x))$.

In order to give an operator realization $\sigma_{\alpha J}$ of the outer automorphism $(j(x) \longmapsto j(x) - d(\alpha J)(x)/dx)$, we take two Fock spaces H_0 and $H_\alpha \cong H_0$ with Fock vacua Ω_0 and Ω_α, respectively. On H_α, we change the representation by the requirement, that the current $j(x)$ should act in the displaced form $j(x) - d(\alpha J)(x)/dx$. $\sigma_{\alpha J}$ intertwines between the - now inequivalent - representations on H_0 and H_α, if we define

$$\begin{aligned} &\sigma_{\alpha J}\,\Omega_0 = \Omega_\alpha\,, \\ &[\sigma_{\alpha J}, j(x)^{(\pm)}] = -(d(\alpha J)(x)/dx)^{(\mp)}\,\sigma_{\alpha J}\,. \end{aligned} \tag{4}$$

This construction can be generalized to arbitrary real charges α on the positive definite state space $H = \bigoplus_\alpha H_\alpha$, which is the direct integral of charge sectors H_α. The action of $\sigma_{\alpha J}: H_\beta \longrightarrow H_{\alpha+\beta}$ is given by

$$\sigma_{\alpha J}\,\sigma_{\beta J} = \sigma_{(\alpha+\beta)J}\,,\quad \sigma_{\alpha J}^{\,+} = \sigma_{-\alpha J}\,,\quad \sigma_{0J} = 1\,,$$

and the orthogonality of the sum $\bigoplus_\alpha H_\alpha$ can be expressed in the form

$\langle \Omega_\alpha / \Omega_\beta \rangle = \delta(\alpha - \beta)$ or equivalently by

$$\langle \Omega_0 / \sigma_{\alpha_1 J} \ldots \sigma_{\alpha_n J}\,\Omega_0 \rangle = \delta(\Sigma \alpha_i)\,. \tag{5}$$

The δ-function in the vacuum expectation value of the σ's reflects just charge conservation.

For test functions with $\Lambda(-\infty) = \alpha$, the desired quantity $E(\Lambda)$ can now be taken to be

$$E(\Lambda) := \exp(-2\pi i j(\Lambda - \alpha J)^{(+)})\,\sigma_{\alpha J}\,\exp(-2\pi i j(\Lambda - \alpha J)^{(-)})\,.$$

This is a well-defined operator on H with vacuum expectation values

$$\begin{aligned} &\langle \Omega_0 / E(\Lambda_1) \ldots E(\Lambda_n)\,\Omega_0 \rangle = \delta(\Sigma \alpha_i)\,\exp(2\pi i \sum_{i<k} F_\Lambda\,(x_i, x_k))\,, \\ &F_\Lambda\,(x_i, x_k) = \int dx \left\{ \left(\frac{d(\alpha_i J)}{dx}\right)^{(-)}(x)\,(\Lambda_k - \alpha_k J)^{(+)}(x) - (\Lambda_i - \alpha_i J)^{(-)}(x) \left(\frac{d\Lambda_k}{dx}\right)^{(+)}(x) \right\}. \end{aligned} \tag{6}$$

Of course, the whole construction must be doupled. For left- and right-moving fields we have different operators $\sigma_{J,+}$ and $\sigma_{J,-}$, which we assume to anticommute $\{\sigma_{J_+,+}, \sigma_{J_-,-}\} = 0$, such that they correspond to the non-Klein transformed σ-operators of ref. 9.

We now consider the special, but important case of a representation space $\mathcal{F}$ with only integer charges: $\mathcal{F} = \bigoplus_{\alpha = n} H_\alpha$, $n = 0, \pm 1, \pm 2, \ldots$. On this space, we change the relation (5) to be

$$\langle \Omega_0 / (\sigma_J)^n (\sigma_J^{+})^m\,\Omega_0 \rangle = \delta_{n,m}\,,\quad \alpha = 1\,, \tag{5'}$$

according to the now descrete measure on the charge space. It is well-known [15,17], that $\mathcal{F} = \bigoplus_n \mathcal{H}_n$ corresponds to the charge sector decomposition of the Fock space $\mathcal{F}$ of a massless, free spinor field $\psi(x)$. Hence, it is interesting to relate the fermion field ψ to the charge creation operators $\sigma_{\alpha J}$ with $\alpha = 1$. As we show below, this can be done by the bosonization formula

$$\psi(x) = N_J^{-1} \exp(-2\pi i\, j(\theta_x - J)^{(+)})\, \sigma_J \, \exp(-2\pi i\, j(\theta_x - J)^{(-)}). \qquad (7)$$

Eq. (7) should be interpreted such that, given a sequence $\Lambda(z)$, which converges towards the step function $\theta_x(z) \equiv \theta(x-z)$, the sequence of fields $N_J^{-1} \exp(-2\pi i\, j(\Lambda - J)^{(+)})\, \sigma_J \exp(-2\pi i\, j(\Lambda - J)^{(-)})$ approximates the ψ-field (7) with respect to the strong operator topology on $\mathcal{F}$ [17].

An explicit choice for smoothed step functions is for instance given by [18]

$$\theta_x^{\varepsilon}(z) = (2\pi i)^{-1} \log \frac{-(x-z)+i\varepsilon}{-(x-z)-i\varepsilon} = \frac{1}{\pi}\left(\operatorname{arctg} \frac{x-z}{\varepsilon} + \frac{\pi}{2}\right),$$
$$\lim_{\varepsilon \to 0} \theta_x^{\varepsilon}(z) = \theta_x(z) \equiv \theta(x-z), \qquad (8)$$

and we assume the described smoothening procedure, whenever we use step functions below.

Although the cut-off prescription J appears on the right hand side of eq. (7), the resulting ψ is J-independent, since the charge distribution $-dJ(x)/dx$ generated by σ_J is transformed into a point-like one by $\exp(-2\pi i\, j(\theta_x - J))$.

The normalization constant N_J^{-1} has to appear in (7) already for dimensional reasons and can be determined from the two-point function $\langle \Omega_0 / \psi(x)\psi(y)^{+} \Omega_0 \rangle$. The general n-point function is calculated with help of eqs. (2), (4) and (7). The result coincides with what one expects from a free spinor field in one space dimension:

$$\langle \Omega_0 / \psi(x_1) \dots \psi(x_m) \psi(y_1)^{+} \dots \psi(y_n)^{+} \Omega_0 \rangle =$$
$$= \delta_{mn} (2\pi i)^{-n} \Big(\prod_{j<k} (x_j - x_k)\Big)\Big(\prod_{j<k} (y_j - y_k)\Big)\Big(\prod_{j,k} (x_j - y_k - i0)^{-1}\Big). \qquad (9)$$

The equations of motion for ψ can be read off from (7). Besides the Dirac equation, we have the relation

$$i \frac{d}{dx} \psi(x) = 2\pi \left(j(x)^{(+)} \psi(x) + \psi(x) j(x)^{(-)} \right),$$

which is well-known in the literature [17,19].

After these general considerations, which hold true for any cut-off J, we specialize the J-function such that it corresponds to Klaiber's partie finie-prescription [12], namely

$$J(z) \equiv K(z) = \frac{1}{2} - \frac{1}{\pi} \int_0^{\kappa z} dt \, \frac{\sin t}{t} =$$
$$= (2\pi i)^{-1} \int_{-\infty}^{\infty} dp \, e^{-ipz} \theta(\kappa - |p|)\left(\text{p.v.} \frac{1}{p} + i\pi \delta(p)\right). \qquad (10)$$

From eq. (2), we derive

$$j_{\pm}(\theta_x - K)^{(-)} = \frac{1}{2\pi} \int_0^{\infty} \frac{dk}{\sqrt{k}} \, c(\pm k) \left(e^{-ikx} - \theta(\kappa - k)\right) \qquad (11)$$

and thus end up with the positive frequency parts of the current potentials of familiar Schwinger model calculations. Hence, eq. (7) gives us exactly the σ -operators of Lowenstein and Swieca [9]

$$\sigma_{\kappa} = N_{\kappa} \exp(2\pi i j(\theta_x - \kappa)^{(+)}) \psi(x) \exp(2\pi i j(\theta_x - \kappa)^{(-)}), \tag{12}$$

once we have determined the normalization constant $N_{\kappa} = (2\pi/\mu)^{1/2}$, $\mu = \kappa e^{\gamma}$, from the two point function of ψ.

Due to our general analysis, the infrared regularized current potentials have a clear physical interpretation as generators of charge displacement transformations, and therefore σ_{κ} creates a charged state with charge distribution $dK(z)/dz = -\sin \kappa z / \pi z$ out of the ψ-Fock vacuum (eq. (4)). Since σ_{κ} commutes with the Hamiltonian of the Schwinger model, this charge carrying state is again a vacuum of the model.

The disadvantage of the cut-off choice κ is, that the charge distribution dK/dz suggests no further physical interpretation. Therefore, we replace κ by another cut-off function

$$\begin{aligned} J(z) \equiv R(z) &= \frac{1}{2}\left(\theta^a_R(z) + \theta^a_{-R}(z)\right) = \\ &= (2\pi i)^{-1} \int_{-\infty}^{\infty} dp\, e^{-ipz} e^{-|p|a} \cos pR \cdot \left(p.v. \frac{1}{p} + i\pi \delta(p)\right). \end{aligned} \tag{13}$$

In this case, we have

$$j_{\pm}(\theta_x - R)^{(-)} = \frac{1}{2\pi} \int_0^{\infty} \frac{dk}{\sqrt{k}}\, c(\pm k)\left(e^{-ikx} - e^{-ka} \cos kR\right),$$

and

$$\begin{aligned} \sigma_R &= N_R \exp(2\pi i j(\theta_x - R)^{(+)}) \psi(x) \exp(2\pi i j(\theta_x - R)^{(-)}), \\ N_R &= 2\left(\pi^4 a^2 (a^2 + R^2)\right)^{1/8} \end{aligned} \tag{14}$$

creates two charge clouds of total charge $1/2$ around the points $+R$ and $-R$.

Our aim during the rest of the paper is, to take the limit $R \to \infty$. We can do this such that $a \cdot R$ remains constant and hence, $R(z)$ becomes the superposition of two step functions at $+\infty$ and $-\infty$. In this limit, $j(\theta_x - R)$ is the difference of the current potential at x and at $\pm\infty$. If we proceed in this heuristic way, $\sigma_\infty = \text{"}\lim_{R \to \infty} \sigma_R\text{"}$ can be interpreted as the net-result of exponentials of current potentials at infinity and therefore as a boundary value effect.

Before we go into the details of the limiting procedure, we would like to mention, that we gain the covariant Wightman distributions of exponentials of a massless, free scalar field, if we introduce step functions $\Lambda_i = \alpha_i \cdot \theta_{x_i}$ in eq. (6). For Klaiber's cut-off (10) the result is [18]

$$\langle \Omega_0 | E(\alpha_1 \theta_{x_1}) \dots E(\alpha_n \theta_{x_n}) \Omega_0 \rangle = \delta(\Sigma \alpha_i) \prod_{j<k} \left(\frac{\mu}{i}\left(-(x_j - x_k) + i0\right)\right)^{\alpha_j \cdot \alpha_k}. \tag{15}$$

In all other cases only the length parameter $1/\mu$ is different, for instance $1/\mu$ must be replaced by $(16 a^2 (a^2 + R^2))^{1/4}$ in our example (13).

The result (15) coincides with what one obtains as a limiting case from exponentials of a massive, free, scalar field [20], if we combine left- and right-moving components.

3) Idealized σ -Operators

The operators $\sigma_{\mathcal{J}}$, defined in the last section, depend explicitly on the cut-off function $\mathcal{J}$. Practical calculations are rendered more difficult by this fact, although the $\mathcal{J}$ -dependence cancels out in many final results. In order to define operators σ_{∞} , which do not show this disadvantage, we enlarge the field algebra $\mathcal{A}_{\psi}$ of the ψ -field by space-time independent unitary elements $\sigma_{\infty,\pm}$ and $\mathcal{U}_{\pm}(\varphi) = \exp(i\varphi Q_{\pm}),\ 0 \le \varphi < 2\pi$. Together with $\mathcal{A}_{\psi}$, they generate a $*$ -algebra $\mathcal{A}_{\psi,Q,\sigma}$, the structure of which is described by the following commutation relations:

$$\psi:\ \{\psi_+(x^-), \psi_+(y^-)^{\dagger}\} = \delta(x^- - y^-),\ \{\psi_+(x^-), \psi_-(y^+)^{\#}\} = 0,\ \{\psi_-(x^+), \psi_-(y^+)^{\dagger}\} = \delta(x^+ - y^+)$$

$$\sigma_{\infty}:\ [\sigma_{\infty,+}, \sigma_{\infty,+}^{\#}] = 0, \qquad \{\sigma_{\infty,+}, \sigma_{\infty,-}^{\#}\} = 0,\ [\sigma_{\infty,-}, \sigma_{\infty,-}^{\#}] = 0$$

$$[\sigma_{\infty,+}, \psi_+(x^-)^{\#}] = 0, \quad \{\sigma_{\infty,\pm}, \psi_{\mp}(x^{\pm})^{\#}\} = 0,\ [\sigma_{\infty,-}, \psi_-(x^+)^{\#}] = 0 \tag{16}$$

$$Q:\ \mathcal{U}_+(\varphi)\psi_+(x^-)\mathcal{U}_+(\varphi)^{-1} = e^{-i\varphi}\psi_+(x^-), \quad \mathcal{U}_-(\varphi)\psi_-(x^+)\mathcal{U}_-(\varphi)^{-1} = e^{-i\varphi}\psi_-(x^+)$$

$$[\mathcal{U}_{\pm}(\varphi), \psi_{\mp}(x^{\pm})] = 0$$

$$[\mathcal{U}_+(\varphi), \mathcal{U}_-(\varphi')] = 0$$

$$\mathcal{U}_+(\varphi)\sigma_{\infty,+}\mathcal{U}_+(\varphi)^{-1} = e^{-i\varphi}\sigma_{\infty,+}, \quad \mathcal{U}_-(\varphi)\sigma_{\infty,-}\mathcal{U}_-(\varphi)^{-1} = e^{-i\varphi}\sigma_{\infty,-}$$

$$[\mathcal{U}_{\pm}(\varphi), \sigma_{\infty,\mp}] = 0$$

($\psi^{\#} = \psi$ or $\psi^{\dagger}$, ditto for σ).

Now, we construct a representation of these commutation relations. Since the σ_{∞} 's commute with the energy momentum operator

$$P^{\mu} = i\int dx^1 :(\psi_-(x^+)^{\dagger}\partial^{\mu}\psi_-(x^+) + \psi_+(x^-)^{\dagger}\partial^{\mu}\psi_+(x^-)):, \quad i\partial^{\mu}\psi_{\pm} = [\psi_{\pm}, P^{\mu}], \tag{17}$$

of the underlying ψ -field, they generate a non-trivial vacuum structure and we begin with the description of the vacuum sub-space:
For this, we consider the unitary groups $\mathcal{U}_{\pm}$, generated by $\Sigma_{\infty,\pm} = \mathcal{U}_{\mp}(\pm\frac{\pi}{2})\sigma_{\infty,\pm}$ and $\mathcal{U}_{\pm}(\varphi), 0 \le \varphi < 2\pi$. $\sigma_{\infty,\pm} \longmapsto \Sigma_{\infty,\pm}$ is a Klein transformation, which yields $[\Sigma_{\infty,+}, \Sigma_{\infty,-}] = 0$ compared to $\{\sigma_{\infty,+}, \sigma_{\infty,-}\} = 0$ in eq. (16).

A unitary, irreducible, faithful, continuous, linear representation of $\mathcal{U}_{\pm}$ is given on a Hilbert space $\mathcal{H}_{0,\pm}$ with orthonormal basis $\{|n_{\pm}\rangle,\ n_{\pm} = 0, 1, -1, 2, -2, \ldots\}$, if we define

$$(\Sigma_{\infty,\pm})^{m_{\pm}}|n_{\pm}\rangle = |n_{\pm} + m_{\pm}\rangle,$$

$$\mathcal{U}_{\pm}(\varphi)|n_{\pm} = 0\rangle = |n_{\pm} = 0\rangle \text{ for all } \varphi. \tag{18}$$

The action of the other group elements on the basis vectors $|n_{\pm}\rangle$ is then fixed by the commutation relations, and on all other vectors of $\mathcal{H}_{0,\pm}$ by linear extension, f.i.: $\mathcal{U}_{\pm}(\varphi)|n_{\pm}\rangle = \mathcal{U}_{\pm}(\varphi)(\Sigma_{\infty,\pm})^{n_{\pm}}|n_{\pm} = 0\rangle =$

$$= e^{-i\varphi n_{\pm}}(\Sigma_{\infty,\pm})^{n_{\pm}}\mathcal{U}_{\pm}(\varphi)|n_{\pm} = 0\rangle \overset{(18)}{=} e^{-i\varphi n_{\pm}}|n_{\pm}\rangle.$$

Since all group elements of $\mathcal{U}_+$ commute with those of $\mathcal{U}_-$, we can form the direct product group $\mathcal{U}_+ \times \mathcal{U}_-$, which can be represented on

$\mathcal{H}_o = \mathcal{H}_{o,+} \otimes \mathcal{H}_{o,-}$ (basis vectors $|n_+ n_-\rangle \equiv |n_+\rangle \otimes |n_-\rangle$) in the usual way. This representation is readily extended to an irreducible, faithful, continuous and (with eq. (18)) unique $*$ -representation of the $*$ -algebra $\mathcal{A}_{Q,\sigma}$, generated by $\sigma_{\infty,\pm}$ and $\mathcal{U}_\pm(\varphi)$, $0 \le \varphi < 2\pi$.

In a second step, we extend the $\mathcal{A}_{Q,\sigma}$ -representation to a representation of the whole algebra $\mathcal{A}_{\psi,Q,\sigma}$. The representation space $\mathcal{H}$ is the tensor product of $\mathcal{H}_o$ with the Fock space $\mathcal{F}$: $\mathcal{H} = \mathcal{H}_o \otimes \mathcal{F}$. The ψ -field is defined to act on $\mathcal{H}$ as $1 \otimes \psi$ and the action of the $\mathcal{A}_{Q,\sigma}$ -algebra on the sub-space $\mathcal{H}_o \otimes \Omega_o$ is, with a self-explaining notation, characterized by $\mathcal{A}_{Q,\sigma} \otimes 1$. In order to extend this definition to the total representation space, it suffices to define $\mathcal{A}_{Q,\sigma}$ on improper vectors of type
$\psi_-(x_1^+)\dots\psi_-(x_\mu^+)\psi_-(y_1^+)^+\dots\psi_-(y_\nu^+)^+\psi_+(x_1^-)\dots\psi_+(x_m^-)\psi_+(y_1^-)^+\dots\psi_+(y_n^-)^+(|n_+ n_-\rangle \otimes \Omega_o)$,
$\mu, \nu, m, n = 0, 1, 2, \dots$; $n_+, n_- = 0, \pm 1, \pm 2, \dots$, which are dense in $\mathcal{H}$ after smearing with test functions and taking the linear hull. On these vectors, however, the action of $\mathcal{A}_{Q,\sigma}$ is well-defined via the commutation relations (16) and the previous definitions, which thus give the representation of $\mathcal{A}_{\psi,Q,\sigma}$ unambiguously.

It follows from Schur's lemma and the irreducibility of the two components of the $\mathcal{A}_{\psi,Q,\sigma}$ -representation, that this representation is again irreducible. The representation space contains an infinite dimensional vacuum sub-space $\mathcal{H}_o \otimes \Omega_o$ and this space is generated out of the cyclic vacuum $\Omega' \equiv |n_+ = 0, n_- = 0\rangle \otimes \Omega_o$ by application of the operators $\sigma_{\infty,\pm}$ (eq. (18)), which commute with the energy-momentum operator (17). The different vacua $|n_+ n_-\rangle \otimes \Omega_o$ carry charge, because $\sigma_{\infty,\pm}$ does (eq.'s (16)).

The Wightman distributions of the algebra $\mathcal{A}_{\psi,Q,\sigma}$ in the representation above are readily calculated to be

$$\langle \Omega' | \psi(x_1)\dots\psi(x_m)\psi(y_1)^+\dots\psi(y_n)^+(\sigma_\infty)^r(\sigma_\infty^+)^s\,\Omega' \rangle = \qquad (19)$$
$$= \delta_{rs}\ \langle \Omega' | \psi(x_1)\dots\psi(x_m)\psi(y_1)^+\dots\psi(y_n)^+\,\Omega' \rangle ,$$

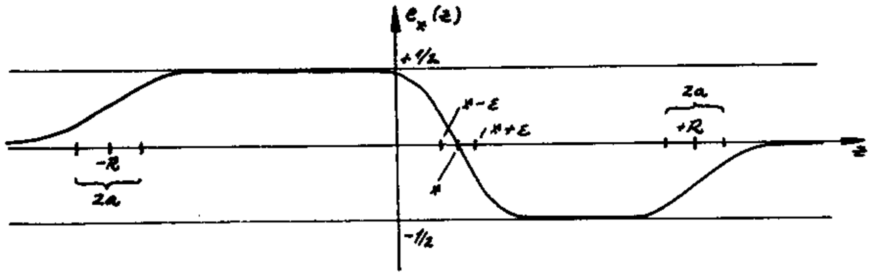
where the right hand side is given by eq. (9). With help of the commutation relations, we arrive at different orders of the operators involved in eq. (19), and the expectation values containing the finite global gauge transformations $\mathcal{U}_\pm(\varphi)$ differ from those given above only by a phase, which follows from (16) and (18). Therefore, we need not discuss them separately. Finally, since the left- and right-moving fields decouple completely, the Wightman distributions (19) are already the most general ones to be considered.

4) A σ_∞ -Approximation

In this section, we show, that the operators σ_∞ , which were characterized so far only algebraically, are indeed idealizations of the operators σ_J of section 2). For this, we reconsider the operators (14), which are well-defined on the ψ -Fock space, and we now indicate the necessary smoothening of all step functions explicitly:

$$\sigma_R^\varepsilon = N_R \exp(2\pi i j(\theta_x^\varepsilon - R)^{(+)})\,\psi(x)\exp(2\pi i j(\theta_x^\varepsilon - R)^{(-)}). \qquad (14')$$

The rough shape of $e_x(z) := \theta_x^\varepsilon(z) - R(z)$ is drawn in the figure below:



We show, that in the limit $R \to \infty$ and $\varepsilon \to 0$, a fixed, the Wightman distributions of the operators (14') approximate those of the σ_∞ -operators of the last section (eq. (19)) [21]:

Left- and right-moving fields can be considered independently:

$$\{\sigma^\varepsilon_{R,+}, \sigma^\varepsilon_{R,-}\} = 0,\ \{\sigma^\varepsilon_{R,\pm}, \psi_\mp(x^\pm)^\#\} = 0,\ [\mathcal{U}_\pm(\varphi), \sigma^\varepsilon_{R,\mp}] = 0,$$

and since $\mathcal{U}(\varphi)\sigma_R^\varepsilon\,\mathcal{U}(\varphi)^{-1} = e^{-i\varphi}\sigma_R^\varepsilon$ holds true, the treatment of expectation values involving $\mathcal{U}(\varphi)$-operators is trivial. We therefore start with the Wightman functions corresponding to those of eq. (19):

$$\langle \Omega_0 | \psi(x_1)\ldots\psi(x_m)\psi(y_1)^\dagger\ldots\psi(y_n)^\dagger\sigma_R^\varepsilon(v_1)\ldots\sigma_R^\varepsilon(v_r)(\sigma_R^\varepsilon(w_1))^\dagger\ldots(\sigma_R^\varepsilon(w_s))^\dagger\Omega_0\rangle =$$

$$= N_R^{r+s}\Big(\prod_{\substack{i,j=1\\i<j}}^{r} \exp G(v_i,v_j)\Big)\Big(\prod_{\substack{k,l=1\\k<l}}^{s} \exp G(w_k,w_l)\Big)\Big(\prod_{i=1}^{r}\prod_{k=1}^{s}\exp(-G(v_i,w_k))\Big)\cdot \tag{20}$$

$$\cdot\prod_{\rho=1}^{r}\Big(\prod_{\mu=1}^{m}\exp(2\pi i\, e_{v_\rho}(x_\mu)^{(+)})\prod_{\nu=1}^{n}\exp(-2\pi i\, e_{v_\rho}(y_\nu)^{(+)})\Big)\cdot$$

$$\cdot\prod_{\sigma=1}^{s}\Big(\prod_{\mu=1}^{m}\exp(-2\pi i\, e_{w_\sigma}(x_\mu)^{(+)})\prod_{\nu=1}^{n}\exp(2\pi i\, e_{w_\sigma}(y_\nu)^{(+)})\Big)\cdot$$

$$\cdot\langle\Omega_0|\psi(x_1)\ldots\psi(x_m)\psi(y_1)^\dagger\ldots\psi(y_n)^\dagger\psi(v_1)\ldots\psi(v_r)\psi(w_1)^\dagger\ldots\psi(w_s)^\dagger\Omega_0\rangle$$

where $G(x,y) = (2\pi i)^2[j(e_x)^{(-)}, j(e_y)^{(+)}] + 2\pi i\,(e_y(x)^{(+)} - e_x(y)^{(-)})$ (see eqs. (2), (4), (7) and (9)).

We use the same R and ε for all σ_R^ε -operators, the limits $R \to \infty$ and $\varepsilon \to 0$ are independent, however. From eq. (20) it follows, that

$$\lim_{\substack{\varepsilon\to 0\\R\to\infty}} \langle\Omega_0|\psi(x_1)\ldots\psi(x_m)\psi(y_1)^\dagger\ldots\psi(y_n)^\dagger\sigma_R^\varepsilon(v_1)\ldots\sigma_R^\varepsilon(v_r)(\sigma_R^\varepsilon(w_1))^\dagger\ldots(\sigma_R^\varepsilon(w_s))^\dagger\Omega_0\rangle =$$

$$= \delta_{rs}\,\delta_{mn}\,(2\pi i)^r\prod_{i,j=1}^{r}(v_i - w_j)\Big(\prod_{\substack{i,j=1\\i<j}}^{r}(v_i - v_j - i0)(w_i - w_j - i0)\Big)^{-1}\cdot$$

$$\cdot\prod_{\rho=1}^{r}\Big(\prod_{\mu=1}^{m}\frac{x_\mu - w_\rho}{x_\mu - v_\rho - i0}\prod_{\nu=1}^{n}\frac{y_\nu - v_\rho}{y_\nu - w_\rho - i0}\Big)\cdot$$

$$\cdot\langle\Omega_0|\psi(x_1)\ldots\psi(x_m)\psi(y_1)^\dagger\ldots\psi(y_n)^\dagger\psi(v_1)\ldots\psi(v_r)\psi(w_1)^\dagger\ldots\psi(w_s)^\dagger\Omega_0\rangle.$$

The Wightman distribution on the right hand side can be replaced by $(-1)^{r\cdot n}\langle\Omega_0|\psi(x_1)\ldots\psi(x_m)\psi(v_1)\ldots\psi(v_r)\psi(y_1)^\dagger\ldots\psi(y_n)^\dagger\psi(w_1)^\dagger\ldots\psi(w_s)^\dagger\Omega_0\rangle$, because it is multiplied by $\prod_\rho\prod_\nu(y_\nu - v_\rho)$. With help of eq. (9) we therefore get

$$\lim_{\substack{\varepsilon\to 0\\R\to\infty}}\langle\Omega_0|\psi(x_1)\ldots\psi(x_m)\psi(y_1)^\dagger\ldots\psi(y_n)^\dagger\sigma_R^\varepsilon(v_1)\ldots\sigma_R^\varepsilon(v_r)(\sigma_R^\varepsilon(w_1))^\dagger\ldots(\sigma_R^\varepsilon(w_s))^\dagger\Omega_0\rangle =$$

$$= \delta_{rs}\,\langle\Omega_0|\psi(x_1)\ldots\psi(x_m)\psi(y_1)^\dagger\ldots\psi(y_n)^\dagger\Omega_0\rangle,$$

in agreement with eq. (19).

In order to verify the commutation relations, we discuss the extra terms, which appear, if the operators in eq. (20) are interchanged. These terms are given by

$$\sigma_R^\varepsilon(x)\,\psi(y) = -\exp(-2\pi i\, e_x(y))\,\psi(y)\,\sigma_R^\varepsilon(x),$$

$$\sigma_R^\varepsilon(x)\,\psi(y)^+ = -\exp(2\pi i\, e_x(y))\,\psi(y)^+\sigma_R^\varepsilon(x) + N_R :\exp(2\pi i j(e_x)): \exp(2\pi i\, e_x(y)^{(-)})\,\delta(x-y),$$

$$\sigma_R^\varepsilon(x)\,\sigma_R^\varepsilon(y) = -\exp(G(x,y)-G(y,x))\,\sigma_R^\varepsilon(y)\,\sigma_R^\varepsilon(x),$$

$$\sigma_R^\varepsilon(x)\,\sigma_R^\varepsilon(y)^+ = -\exp(-(G(x,y)-G(y,x)))\,\sigma_R^\varepsilon(y)^+\sigma_R^\varepsilon(x) + N_R^2\exp(-G(x,y))\cdot\delta(x-y).$$

After perfoming the limiting procedure, the factors $\exp(2\pi i\, e_x(y))$ and $\exp(G(x,y)-G(y,x))$ become equal to -1 and $\lim\limits_{\substack{\varepsilon\to 0\\ R\to\infty}} N_R^2\exp(-G(x,y))\,\delta(x-y) = 0$, such that, with one exception, the commutation relations (16) are verified. The one relation, which requires closer inspection, is that involving $\sigma_R^\varepsilon(x)$ and $\psi(y)^+$ because of the term proportional to $:\exp(2\pi i j(e_x)):$. Although $N_R\cdot\exp(2\pi i\, e_x(y)^{(-)})$ behaves - up to constants - asymptotically like $(x-y)\,R^{-3/4}$, we nevertheless have to ensure, that a general matrix element of the type

$$\langle\Omega_0|\psi\ldots\psi^+\ldots\sigma_R^\varepsilon\ldots(\sigma_R^\varepsilon)^+ :\exp(2\pi i j(e_x)): \psi\ldots\psi^+\ldots\sigma_R^\varepsilon\ldots(\sigma_R^\varepsilon)^+\Omega_0\rangle$$

with an arbitrary number of fields ψ, ψ^+, σ_R^ε and $(\sigma_R^\varepsilon)^+$ in an arbitrary order does not grow faster than $R^{3/4}$ for $R\to\infty$. Indeed, such expectation values behave like $R^{\frac{1}{4}(1-(s-r-1)^2)}$, if r and s are the number of fields σ_R^ε and $(\sigma_R^\varepsilon)^+$, respectively. This completes the proof of our assertion about the σ_R^ε -fields.

On the original indefinite metric state space of the operator solution of the Schwinger model, gauge transformations with gauge functions

$\Lambda(x) = \Lambda_+(x^-) + \Lambda_-(x^+)$ can be performed with help of unitary operators [8]

$$T(\Lambda) = T_+(\Lambda_+)\cdot T_-(\Lambda_-),\quad T_\pm(\Lambda_\pm) = \exp(2\pi i\, \eta_\pm(\Lambda_\pm'))\exp(-2\pi i j_\pm(\Lambda_\pm)). \tag{21}$$

Here, $\eta(x) = \eta_+(x^-) + \eta_-(x^+)$ denotes the auxiliary free gauge field quantized with indefinite metric and $\Lambda_\pm'(z) \equiv d\Lambda_\pm(z)/dz$. If we try to define transformations, which lead to topological inequivalent gauges, i.e. with $\Lambda_\pm(-\infty) = 1$, $\Lambda_\pm(\infty) = 0$, this in unproblematic for the η -term in eq. (21), because of the special quantization procedure of this field. For the j -exponential, however, the described infrared regularization must be applied and σ-operators emerge, which cause the degeneracy of the vacuum. This gives the connection with the general discussion of the vacuum structure of non-abelian gauge theories and leads us back to the introductory remarks of this paper ...

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