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EXPLICIT PARAMETRIZATION OF THE GENERAL SU(2) INSTANTONS

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Abstract

The 8k-3 parameters of the general SU(2) Yang Mills instanton of topological number k are identified. 8k-4 of the parameters are certain "angles" of the quotient group SU(2k+2)/SU(2k)xSU(2) and one is an overall scale.

I. Introduction

An understanding of the role played by instantons in the path integral of non abelian gauge theories requires a representation of the general instanton field in terms of a set of independent collective coordinates. The self-dual fields constructed in refs. (1-4) involves these parameters in an implicit way and its complete specification requires the solution of a non linear matrix equation.

In what follows, we shall show that the solutions of the equation (for the SU(2) case) may be parametrized uniquely in terms of the (8k-4) (k is the topological number) parameters of a certain class of (2k+2)x(2k+2) SU(2k+2)/SU(2k)xSU(2) unitary matrices and one overall scale parameters. An explicit solution however still seems difficult to obtain, although we conjecture that this may be possible by using the Lie algebra of U(k). While the considerations below refer explicitly to SU(2) instantons, it seems probable that a similar parametrization in terms of SU(2k+N)/SU(2k)xSU(N) exists for the general SU(N) self dual field.

II. Description of the Self Dual SU(2) Gauge Fields

The self dual gauge fields are best described in terms of quaternionic matrices. We thus first briefly review the necessary definitions and properties of quaternions.

A quaternion is a general 2x2 complex matrix and may be represented in terms of four complex numbers and a matrix basis:

$$q = z_\mu e_\mu \tag{1}$$

where z_μ is an arbitrary complex four-vector and e_μ are defined in terms of the unit matrix and the standard Pauli matrices:

$$e_\mu = (1, i\vec{\sigma}) \quad . \tag{2}$$

A quaternion is "real" when the four-vector z_μ is real:

$$r = x_\mu e_\mu : \quad x_\mu = x_\mu^* \quad . \tag{3}$$

Real quaternions possess an important property namely, they are real multiples of an SU(2) matrix. In particular, the product of a real quaternion and its hermitian conjugate is a positive real multiple of the unit matrix. More generally, any real hermitian quaternion is a real multiple of the unit matrix ($i\vec{\sigma}\cdot\vec{x}$ is anti-hermitian for $\vec{x} = \vec{x}^*$). The general SU(2) instanton field of topological number k belongs to a class of gauge fields defined by picking out an x -dependent quaternionic vector in a (k+1)-dimensional space (henceforth all vectors, matrices, spaces etc. will be understood to be quaternionic):

$$A_\mu(x) = p^\dagger(x)\partial_\mu p(x) \tag{4}$$

where the (k+1) dimensional vector $p(x)$ is normalized to unity thus insuring the anti-hermiticity of the quaternion $A_\mu(x)$:

$$p^+p \equiv p_o^+p_o + p_a^+p_a = 1 : a = 1,\ldots,k. \tag{5}$$

All gauge invariant quantities are uniquely determined by the projection operator onto the two-dimensional (one dimensional quaternionic) subspace spanned by $p(x)$. In particular, the trace of any gauge field loop is given by

$$tr\left(e^{-\oint_C dx_\mu A_\mu(x)}\right)_+ =$$

$$= Tr\left(\prod_{x(C)} P(x)\right)_+ = Tr\left(P(x_0)\exp\oint_C dx_\mu \partial_\mu P(x)\right)_+ \quad . \tag{6}$$

In eq. (6) $(\ )_+$ is the path ordering symbol, tr is performed on quaternions while Tr refers to (k+1)x(k+1) quaternionic matrices. The projection operator $P(x)$ is:

$$P(x) = p(x)p^\dagger(x) \tag{7}$$

or

$$P_{ij}(x) = p_i(x)p_j^\dagger(x) : i,j = 0,1,\ldots,k \quad . \tag{8}$$

A gauge field constructed from a vector $p(x)$ is self dual when the orthogonal complement of $p(x)$ can be represented in the following form:

$$p^\dagger(x)\Delta_a(x) = 0 \; : \; a = 1,\ldots,k \quad . \tag{9}$$

The k vectors Δ_a are linear functions of x_μ :

$$\Delta_a(x) = \Delta_a + b_a x \tag{10}$$

where:

$$x = x_\mu e_\mu \tag{11}$$

and b_a are the unit vectors

$$b_{oa} = 0 \ ; \ b_{ba} = \delta_{ba} \tag{12}$$

The k constant vectors Δ_a satisfy a condition of symmetry and reality:

$$\Delta_{ab} = \Delta_{ba} = \text{real quaternion } (a,b = 1,\dots,k) \tag{13}$$

and a norm condition:

$$\Delta_a^\dagger \Delta_b = Q_{ab} e_o \ : \ Q_{ab} = Q^*_{ab} \ . \tag{14}$$

Conditions (13,14) mean that the scalar products among the Δ's are real multiples of the 2x2 unit matrix. This property is crucial in proving the self duality of the field strength $F_{\mu\nu}$. The latter is readily shown to be [4]:

$$F_{\mu\nu} = p^\dagger b [e_\mu Q^{-1} e_\nu^\dagger - e_\nu Q^{-1} e_\mu^\dagger] b^\dagger p \ , \tag{15}$$

where b designates the (k+1)xk matrix built out of the matrix elements b_{ja} of eq. (9). Since Q^{-1} is numerical it can be transferred to the left of its adjacent e's and the quaternion $(e_\mu e_\nu^\dagger - e_\nu e_\mu^\dagger)$ is self-dual. The symmetry and reality of Δ_{ab} then imply that $\Delta_a(x)$ also satisfy the conditions (13,14) so that $F_{\mu\nu} = \tilde{F}_{\mu\nu}$ for all x.

III. The Instanton Parameters

As has been observed in sec. II, the instanton field is uniquely determined by the projection operator P(x). We thus first parametrize all such projection operators. Since the conditions (13,14) are imposed on the orthogonal complement of P, we shall concentrate on the latter.

At $x \to \infty$ we have:

$$x \to \infty: \ (1 - P(x))_{ja} \to b_{ja} \ ; \ (1 - P(x))_{jo} \to 0 \tag{16}$$

where as usual: a,b = 1,...,k. j = 0,1...,k. Designating 1-P by Q, we may parametrize Q in terms of a unitary matrix U:

$$Q = U Q_\infty U^\dagger \tag{17}$$

where the generators of U satisfy:

$$U = \exp \Phi \tag{18}$$

$$\Phi_{oo} = \Phi_{ab} = 0 \ ; \ \Phi_{oa} = -\Phi_{ao}^\dagger = \varphi_a \ . \tag{19}$$

Define now a unitary matrix F which brings Φ to its canonical form:

$$\varphi_a = h F_{a1}^\dagger \equiv h f_a^\dagger \ : \ h = h^\dagger \ . \tag{20}$$

$$F_{oa} = F_{ao} = 0 \ ; \ F_{oo} = 1 \ ; \ F_{cb}^\dagger F_{ca} = \delta_{ab} \ . \tag{21}$$

We then have:

$$\mathcal{Q}_{jl} = q_{ja}\, q^{\dagger}_{la} \qquad (22)$$

where the k orthonormal vectors q_a are given by:

$$q_{a1} = f_a c \qquad q_{o1} = s$$
$$b > 1:\ q_{ab} = F_{ab} \qquad q_{ob} = o \qquad (23)$$

where the quaternions (c, s) are:

$$(c,s) = (\cos, \sin)\, h \qquad (24)$$

We shall now restrict the allowed hermitian quaternions h to be <u>real</u>. In other words: h is a real multiple of the unit quaternion e_o so that (c,s) are also numerical multiples of e_o:

$$(c,s) = (\cos, \sin)\vartheta \quad . \qquad (25)$$

The projection operator $\mathcal{Q}$ is thus determined by the arbitrary orthonormal vector f_a and the real angle ϑ. The orthonormality condition on f is:

$$f_a^{+} f_a = e_o \qquad (26)$$

which leaves (8k-4) real parameters. Together with ϑ we thus have 8k-3 real parameters to fix the projection operator $\mathcal{Q}$.

In order to satisfy (13,14) we have to find a linear transformation K_{ab} which satisfies:

$$q_{jb} K_{ba} = \Delta_{ja} \quad . \qquad (27)$$

Define a matrix K as follows:

$$K_{1a} = c^{-1} f_b^{\dagger} \Delta_{ba}$$
$$b > 1:\ K_{ba} = F^{\dagger}_{bc} \Delta_{ca} \qquad (28)$$

where

$$\Delta_{ab} = \Delta_{ba} \quad = \text{real quaternions.} \qquad (29)$$

The subspace $\mathcal{Q}$ is now defined in terms of the (non orthogonal) vectors:

$$(\Delta_a)_b = \Delta_{ba}\ ;\ \Delta_{oa} = tg\theta f_c^{\dagger} \Delta_{ca} \quad . \qquad (30)$$

Condition (14) now reads:

$$\Delta^{\dagger}_{cb}[\delta_{cd} + tg^2\vartheta f_c f_d^{\dagger}]\Delta_{da} = \mathcal{Q}_{ab} e_o \qquad (31)$$

where $\mathcal{Q}_{ab}$ is a real symmetric kxk numerical matrix. Separate now the vector f into its real and imaginary parts:

$$f_a = r_a + i s_a \qquad (32)$$

where r_a, s_a are real quaternions which satisfy:

$$r_a^\dagger s_a = s_a^\dagger r_a$$
$$r_a^\dagger r_a + s_a^\dagger s_a = e_0 \qquad (33)$$

The condition (31) now implies:

$$(r^\dagger \Delta_b)^\dagger (s^\dagger \Delta_a) = (s^\dagger \Delta_b)^\dagger (r^\dagger \Delta_a) \quad . \qquad (34)$$

Since (r, s) are linearly independent and Δ is a real quaternionic vector we infer:

$$s^\dagger \Delta_a = \lambda\, r^\dagger \Delta_a \qquad (35)$$

where λ is a real number. Now the k vectors Δ_a are linearly independent and may be expanded in terms of r,s and k-2 real vectors orthogonal to (r,s). The condition (35) thus leaves k(k-1) undetermined real quaternionic expansion coefficients and one real number. The requirement that Δ_{ab} be symmetric imposes $\frac{1}{2}k(k-1)$ linear relations and we are left with $\frac{1}{2}k(k-1)$ unknown real quaternions. The real part of condition (31) now supplies $\frac{1}{2}k(k-1)$ real quaternionic equations. (The requirement that the real part of the l.h.s. be symmetric is sufficient to insure [redacted] the proportionality to e_0 since $\Delta^\dagger \Delta$ is hermitian and real). Hence, equation (31) determines Δ_{ab} uniquely up to one arbitrary real number which may be fixed by imposing some overall normalization condition (say $T_r Q = 1$). Conversely, given $\Delta_{ab} = \Delta_{ba}$ = real and Δ_{oa} which satisfy conditions (13,14) we find:

$$(\Delta_0 \Delta^{-1})^\dagger_a (\Delta_0 \Delta^{-1})_b = (\Delta^{-1\dagger} Q \Delta^{-1})_{ab} - \delta_{ab} \qquad (36)$$

where Δ is the matrix Δ_{ab}. Summing over the diagonal elements and using the reality and symmetry conditions we find:

$$(\Delta_0 \Delta^{-1})^\dagger_a (\Delta_0 \Delta^{-1})_a = Q_{ab} (\Delta^\dagger \Delta)^{-1}_{ba} - k \quad . \qquad (37)$$

The r.h.s. of eq. (37) is a real hermitian quaternion which means that $(\Delta_0 \Delta^{-1})_a$ is a vector normalized to a real number. We have thus proved: The rectangular matrix Δ_{ja} is uniquely determined (up to one real overall normalization) by the 8k-3 parameters (ϑ, f_a) of the projection defined in eqs. (17-26).

Given Δ_{ja} , we may now compute $\Delta_{ja}(x)$ by adding $\rho^{-1} x\, b_{ja}$ where ρ is an arbitrary scale parameter and compute $Q(x)$ and P(x) by solving eq. 9. Moreover, if desired, P(x) may be expressed in terms of x- dependent parameters:

$$\vartheta\left(\frac{x}{\rho}\right) , \quad f_a\left(\frac{x}{\rho}\right) \quad . \qquad (38)$$

It would seem that we actually have one parameter too many since the scale ρ has been added to the 8k-3 parameters of the subspace P(o). Note, however, that all gauge invariant quantities are determined by eq. (6), which is invariant under:

$$p_0(x) \rightarrow p_0(x) e^{i\varphi}$$
$$p_a(x) \rightarrow p_a(x) \qquad (39)$$

This corresponds to:

$$f_a \rightarrow e^{i\varphi} f_a \qquad \Delta_{ba} \rightarrow \Delta_{ba} \qquad (40)$$

Hence, the overall phase of f is physically irrelevant although the identification of the subspace P(x) does depend on it. Thus, we are left with precisely 8k-3 parameters, of which one is a noncompact overall scale, and 8k-4 are "angles" of the unitary transformation U (eqs. 18, 19) subject to the restriction (25) and to one phase condition (eq. 40).

We end by a comment on a possible approach for obtaining the general solution of eq. (31). The antisymmetric part of $\Delta^\dagger\Delta$ may be expressed in terms of commutators and anticommutators of four real symmetric matrices Δ_μ . Hence, after substituting eq. (35) the condition (31) may be rewritten as an equation which determines four symmetric generators of the group U(k) by requiring that certain combinations of their (anti)commutators be zero. In particular, the equations are linearized in the case of an infinitesimal transformation so that a differential equation for Δ as a function of ϑ and the independent components of f might be obtained.

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