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RECENT RESULTS ON THE INSTANTON APPROXIMATION

by

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Abstract

Recent results on the two-dimensional $\mathbb{C}P^{N-1}$ models suggest that the QCD instanton gas is dense and infrared finite. These results and the computation of quantum fluctuations of quarks in a multi-instanton background are reviewed.

I. Introduction

In the Euclidean region the action of QCD is defined by

$$S = \int d^4x \, \mathcal{L}$$

with

$$\mathcal{L} = -\frac{1}{2g^2} \cdot \mathrm{tr}\,(F_{\mu\nu} F_{\mu\nu}) - i\bar{\Psi}\,(\mathcal{D}_\mu \gamma_\mu - M)\Psi$$

The gauge fields A_μ are antihermitean 3 x 3 matrices which belong to the adjoint representation of the SU(3) color group and

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu].$$

+)Seminar talk delivered at the Kaiserslautern Conference on "Field Theoretical Methods in Elementary Particle Physics", Kaiserslautern, 13.8.-20.8.1979.

The quark fields Ψ are in the fundamental representation of the color group. Color as well as an unknown number $(\gtrsim 4)$ of flavor indices are supressed. $\mathcal{D}_\mu = \partial_\mu + A_\mu$ is the covariant derivative, M the mass matrix. We shall deal with massless quarks and admit a SU(n) color group.

Since 1975 one knows about instantons /1,2/, i.e. finite action

$$S = -\frac{1}{2g^2} \int d^4x \, \mathrm{tr}\,(F_{\mu\nu} F_{\mu\nu})$$

solution of the Yang-Mills (Y.M.) equations. The instantons carry a topological charge K. For a SU(2) gauge group the 1-instanton solution /2/ (K=1) is given by

$$A_\mu(x) = \frac{1}{2}\bar{\sigma}_{\mu\nu}\,\partial_\nu \ln \rho(x)$$

with

$$\rho(x) = 1 + \frac{\lambda}{(x-y)^2}$$

Here $\bar{\sigma}_{\mu\nu} = \bar{\eta}^a_{\mu\nu}\, i\sigma^a$ where $\bar{\eta}^a_{\mu\nu}$ is the anti-selfdual 't Hooft /3/ tensor and σ^a are the Pauli matrices. λ is the scale size and y the position of the instanton. For higher SU(n) gauge groups the 1-instanton solution can be simply obtained by imbedding the SU(2) instanton.

The quantum theory is obtained from the classical action by means of functional integration. In the saddle point approximation to the functional integral one has to sum over all finite action solution in the Euclidean region and to calculate the quantum fluctuations around these. Therefore for this approximation the relevance of instantons is evident. Taking into account that their action is $S[\text{inst.}] > 0$, we see that instanton effects are of order $\exp\{-\frac{1}{2g^2}\}$ i.e. exponentially small in the coupling. This is the reason why instantons could be responsible for physical effects which cannot be seen in ordinary perturbation theory.

Important physical question which have been discussed in connection with instantons are:

1. Is there a dynamical mass gap generation in the Y.M. theory?
2. Quark confinement.
3. The U(1) problem.

Quantum fluctuations around the 1-instanton solution have been calculated in Ref./3/. It turns out that the integration over the instanton scale size is infrared divergent like

$$\int_0^\infty \frac{d\lambda}{\lambda^5} (\lambda m)^{\frac{11\cdot n}{3}} \quad . \qquad (1)$$

A popular approximation for the computation of instanton effects is the dilute gas approximation.
In this approximation one replaces the instanton gas by an ensemble of well seperated 1-instanton configurations. This exponentiates essentially the infrared divergence of the 1-instanton contribution (1). Introducing by hand a phenomenological parameter λ_c the integration over the instanton scale size is cut off at a size much smaller then the assumed mean instanton seperation. At the end of the calculation the limit $\lambda_c \to \infty$ does not exist, in other words the dilute instanton gas does not behave thermodynamically. Within this approximation instanton effects cannot account for a mass gap or quark confinement. Concerning the U(1) problem instantons have given important hints how this problem may be resolved. They give, however, not a complete understanding of the chiral symmetry breaking /4/ and in contrary to recent results /5/ based on the 1/N expansion they cannot account for the mass of the η'.

So far the dilute gas approximation. Recently, however, it has turned out that the Y.M. instanton gas is presumably dense and infrared finite. If this is correct instantons could eventually give an answer to the raised physical questions.

The suggestion that the Y.M. instanton gas is dense and infrared finite comes from recent investigations /6,7/ of the $\mathbb{C}P^{N-1}$ models in two dimensions. This is reported in Section II. Some speculations concerning the Y.M. case (based on Ref./8/) and the computation of quantum fluctuation of quarks in an arbitrary Y.M. instanton background /9/ are reported in Section III.

II. The $\mathbb{C}P^{N-1}$ Instanton Gas

The $\mathbb{C}P^{N-1}$ model in Euclidean space describes n complex fields $(n = N)$

$$z_\alpha(x) \qquad (\alpha = 1, \dots, n; \; x = (x_1, x_2))$$

which are subject to the constraint $\bar{z}_\alpha z_\alpha = 1$.
Fields related by $z'_\alpha(x) = e^{i\Lambda(x)} z_\alpha(x)$
are considered to be gauge equivalent. The Euclidean action is given by

$$S = \frac{n}{2f} \int d^2x \, \overline{\mathcal{D}_\mu z} \, \mathcal{D}_\mu z$$

where

$$\mathcal{D}_\mu = \partial_\mu + A_\mu \,, \quad A_\mu = -\bar{z}\partial_\mu z \,, \quad (\mu = 1,2)$$

is the covariant derivative. The $\mathbb{C}P^{N-1}$ models have a non-trivial topology and the topological charge is

$$Q = \frac{-i}{2\pi} \int d^2x \, \varepsilon_{\mu\nu} \partial_\mu A_\nu \,; \qquad \varepsilon_{12} = 1 \quad .$$

The instanton solutions of the $\mathbb{C}P^{N-1}$ model are known /10/ and the multi-instanton z^0 with topological charge Q = K and action $S[z^0]$ can (up to solutions which are of measure zero) be given by

$$z^0_\alpha = \frac{p_\alpha}{|p|} \,, \quad p_\alpha = c_\alpha \prod_{j=1}^{K} (s - a^j_\alpha), \; (\alpha = 1, \dots, n), \; c_n = 1 \,, \; s = x_1 - i x_2 \; . \quad (2)$$

The n(K+1)-1 complex parameters c_α, a^j_α will be labelled by λ_i $(i = 1 \dots n(K+1)-1)$.

The functional integral for the expectation of a (gauge invariant) observable $\mathcal{O}$ is

$$\langle \mathcal{O} \rangle = Z^{-1} \int \mathcal{D}[z] \, \mathcal{O} \, e^{-S}$$

where the partition function Z normalizes the expectations: $\langle 1 \rangle = 1$. The functional integral may be approximated+) by summing over all saddle points and taking into account the quantum fluctuation around each saddle point. The pure (anti-) instanton gas arises by taking as saddle points all the (anti-) instanton solutions.

Let z(x) be a $\mathbb{C}P^{N-1}$ field with topological charge K. It can be written as

$$z_\alpha = e^{i\Lambda} \{ (1 - |\eta|^2)^{\frac{1}{2}} z^0_\alpha + \eta_\alpha \}$$

$$\bar{z}^0 \eta = 0 \qquad \text{("background gauge")}$$

For fluctuations around z^0 we neglect terms of order $\mathcal{O}(\eta^3)$ and obtain the Gaussian approximation to the action

$$S = \frac{n\pi}{f} K + \frac{2n\pi}{f} (\eta, \Delta\eta) \quad .$$

+) This is an approximation to function space and not a systematic expansion in some coupling constant.

Here we have introduced the scalar product

$$(\varphi, \psi) = \frac{2n\pi}{f} \int d^2x\, \bar{\varphi} \psi$$

and Δ is the fluctuation operator defined by this expansion of the action.

Now I like to proceed in a formal way. Suppose that

$$e_\alpha^i(x, \lambda) \qquad (i = 1, 2, \ldots)$$

denotes a complete set of orthonormal eigenvectors with eigenvalues $E_i \neq 0$ of the fluctuation operator Δ.+)

Then by expanding $\quad \eta_\alpha = \sum_{i=1}^{\infty} \xi_i e_\alpha^i(x, \lambda)$

the Gaussian approximation to the action becomes

$$S = \frac{n\pi}{f} K + \frac{2n\pi}{f} \sum_{i=1}^{\infty} E_i |\xi_i|^2 .$$

Approximating furthermore $\mathcal{O}$ by its value $\mathcal{O}(\lambda)$ for the instanton $z^0(x, \lambda)$, the instanton gas expectation of $\mathcal{O}$ becomes

$$\langle \mathcal{O} \rangle_{inst} = Z^{-1} \sum_{K=0}^{\infty} (K!)^{-n} \int \prod_{j,i} d^2\lambda_j\, d^2\xi_i\, J(\lambda) \mathcal{O}(\lambda) \cdot \exp - \left\{ \frac{n\pi}{f} K + \frac{2n\pi}{f} \sum_{i=1}^{\infty} E_i |\xi_i|^2 \right\} \tag{3}$$

where $J(\lambda)$ is the Jacobian defined by

$$\mathcal{D}[z^0] = \prod_j d^2\lambda_j\, J(\lambda) .$$

The factor $(K!)^{-n}$ in (3) is obtained by recognizing that the multi-instanton solution as given by (2) remains the same under interchange of the parameters a_α^i $(i = 1, \ldots, K)$ within a fixed component α. Therefore the factor $(K!)^{-n}$ ensures that each multi-instanton solution is counted only once.

For the $\mathbb{C}P^1$ /6/ and $\mathbb{C}P^{N-1}$ /7/ models the instanton gas can be computed rigorously. The result is

+) This makes of course only sense if the spectrum of the fluctuation operator is discrete. This is the case if one works on a compact space. Because of conformal invariance a sphere with radius R is appropriate.

$$\langle \mathcal{O} \rangle_{inst} = Z^{-1} \sum_{K=0}^{\infty} (K!)^{-n} z^{nK} \cdot \int \frac{\prod_{\alpha=1}^{n-1} d^2c_\alpha}{(\bar{c}_\gamma c_\gamma)^n} \prod_{\beta=1}^{n} \prod_{j=1}^{K} d^2a_\beta^j\, \mathcal{O}(c_\alpha, a_\beta^j) \exp\{-U(c_\alpha, a_\beta^j)\} \tag{4}$$

Here

$$z = m^{\frac{2n}{f}} e^{-\frac{n+2}{2n}}$$

is the fugacity and $\quad m = \mu \exp -\frac{\pi}{f_R(\mu)}$

is the renormalization group invariant mass.

By choosing an appropriate mass-scale the fugacity can be scaled to $z = 1$ and therefore the physics of the instanton gas does not depend essentially on z.

The many body potential

$$U = \frac{n}{2\pi} \int d^2x\, \ln|p|^2 \partial_{\bar{s}} \partial_s \ln|p|^2 + \frac{1}{2} nK (\ln(\bar{c}_\alpha c_\alpha) - 1) - \sum_{\alpha=1}^{n} \left[K \ln|c_\alpha|^2 + \sum_{i<j} \ln|a_\alpha^i - a_\alpha^j|^2 \right] \tag{5}$$

accounts for the interaction (p is defined by (2)).

For the $\mathbb{C}P^{N-1}$ model the potential U describes the interaction of n different types of particles, which I will call "instanton constituents". In the sector K there are $n \cdot K$ particles, K particles of each consituent type. Their position is described by the parameters a_α^j $(j = 1, \ldots, K;\ \alpha = 1, \ldots, n)$; α counts the different constituents. For $N = 2$ the gas simplifies to the Coulomb gas at temperatur $T = 1$ (cf. Ref. /6,7/).

The instanton gas (4) is dense and infrared finite, whereas the dilute gas approximation for the $\mathbb{C}P^{N-1}$ models is infrared divergent. I like to clarify this point further.

What we are computing is the grand canonical ensemble. Let Z be its partition function, then the pressure is proportional to P,

$$P = \lim_{V \to \infty} \frac{1}{V} \ln Z .$$

Here V is a (box) volume. Infrared finite means that the partition function Z behaves like $\exp P\cdot V$ for $V\to\infty$, but not worse then that. It is instructive to investigate this first for the free gas, i.e. let us set $U \equiv 0$ for the moment. Introducing a temperature parameter via

$$U \to \frac{1}{T} U$$

this corresponds to the high temperature limit $T = \infty$.
Now we easily read off from (4) that Z behaves like

$$\sum \frac{V^{nk}}{(k!)^n} \quad , \quad V \to \infty$$

and the simple estimate

$$\exp\{V\} < \sum \frac{V^{nk}}{(k!)^n} < \exp\{nV\}$$

proves the infrared finiteness. This is the kinematical aspect of the problem. The dynamical aspect amounts to including the potential U. This is non-trivial. For the Coulomb gas in two dimensions Fröhlich /11/ has proven that the thermodynamic limit exists for $T > 1$ and for $T = 1$ it becomes equivalent to the massive Dirac field. Therefore the $\mathbb{C}P^1$ instanton gas is well-defined. The similar behaviour of the potential U for the seperation of all instanton constituents large suggest that the thermodynamic limit of the instanton gas exists also for $N \geq 3$.

Let us compare this with the dilute gas approximation. Introducing the scale size

$$\lambda^2 = \frac{1}{\bar{c}_\gamma c_\gamma} \sum_{i<j} |c_i|^2 \, |c_j|^2 \, |a_i - a_j|^2$$

of a single instanton, we can calculate from (4) the 1-instanton contribution, which is proportional to

$$V \int_0^{\lambda_c} d\lambda \, \lambda^{n-3} \qquad . \qquad (6)$$

Here λ_c cuts off the scale size integration at a finite value. Assuming the instanton gas to be dilute we have to exponentiate the 1-instanton contribution

$$Z_{dilute} = \sum_k \frac{1}{k!} \left(V \int_0^{\lambda_c} d\lambda \, \lambda^{n-3} \right)^k$$

and after carrying out the infinite volume limit $V\to\infty$ we are left with a divergent scale size integration in the physical quantities, i.e. the limit does not exist.
The correct interpretation of (6) is that large instanton are more probable then small instanton and therefore the instanton gas is dense. In the exact instanton gas (4) dilute configurations are of such a neglegible weight that they cannot destroy the infrared finiteness. Concluding this section I like to remark that one can compute the energy of the instanton partition function for all N in the $T = \infty$ limit. The result has a nice 1/N expension

$$\lim_{V\to\infty} \frac{\mathcal{U}}{V} = \text{const.} \cdot n^2 \left(1 - \frac{1}{n} + \mathcal{O}\left(\frac{1}{n^2}\right) \right) .$$

By this heuristic argument we expect instanton effects in the $\mathbb{C}P^{N-1}$ models to be neither exponentially small nor exponentially large in 1/N.

III. Status of the QCD Instanton Gas

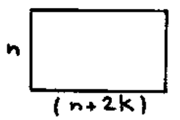
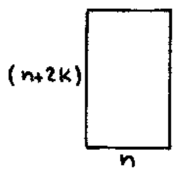
Relying on the results for the $\mathbb{C}P^{N-1}$ models the infrared divergence of the 1-instanton contribution (1) suggests that the QCD instanton gas may be dense and infrared finite. I will further comment on this in Section III.2. In Section III.1 the Atiyah, Hitchin, Drinfeld, Manin /12/ (AHDM) construction of the multi-instanton solution is given for SU(n). For establishing explicitly the QCD instanton gas one has to calculate the quantum fluctuations of quarks and gluons in a multi-instanton background. For the determinant of the corresponding Dirac operator quite explicit formulas were derived recently /9/. This is reported in Section III.3.

III.1 The Yang-Mills Instantons

We consider the multi-instanton solution of AHDM /12/ for the case of SU(n) in the notation relying on Ref./13/.
The gauge potential of the multi-instanton solution is given by

$$A_\mu = V^+(x)\, \partial_\mu V(x) \qquad (7)$$

where V is an $(n + 2K)\times n$ complex matrix

$V^+ =$  $\quad V =$ 

K denotes the instanton number, viz.

$$K = -\frac{1}{16\pi^2} \int d^4x \, \mathrm{Tr} \left(F_{\mu\nu} F^*_{\mu\nu} \right)$$

where

$$F^*_{\mu\nu} = \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} F_{\rho\sigma} \quad , \qquad \varepsilon_{0123} = 1$$

in the dual field.

V is normalized to $V^+(x)\,V(x) = 1_n$ and behaves as $V(x) \to V(x)g(x)$, $g(x) \in SU(n)$ under gauge transformations. V has to be determined as solution of a set of 2K complex linear equations

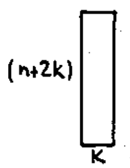
$$V^+ \Delta_{A'} = 0 \qquad (A' = 1,2) \tag{8}$$

with $\Delta_{A'}$ a $SU(2)$ spinor (cf. e.g. Ref./9/) of $(n+2K) \times K$ matrices depending linearly on x_μ :

$$\Delta_{A'} = a_{A'} + b^A x_{AA'} \quad ; \qquad x_{AA'} = x_\mu (e_\mu)_{AA'} \tag{9}$$

$e_\mu = (1, -i\sigma_j)$ and σ_j the Pauli matrices.

$a_{A'}$ and b_A are constant $(n+2K) \times K$ matrices



which parametrize the multi-instanton solution. The instanton parameter matrices are not unconstrained, however, but must be choosen such that

$$\Delta^+_{A'} \Delta_{B'} = \varepsilon_{A'B'} f^{-1} \qquad \text{for all } x \, . \tag{10}$$

Here Δ^+_A is the adjoint in the sense of spinors defined by

$$\Delta^+_1 = -\bar{\Delta}^T_2 \quad , \quad \Delta^+_2 = \bar{\Delta}^T_1$$

f^{-1} is a complex $K \times K$ matrix and the gauge potential (7) constructed via (8) is non-singular if f^{-1} is non-singular.
From (9) it follows that (10) holds if and only if

$$a^+_{A'} a_{B'} + a^+_{B'} a_{A'} = 0 \tag{11.a}$$

$$a^+_{A'} b_A + b^+_A a_{A'} = 0 \tag{11.b}$$

$$b^+_A b_B + b^+_B b_A = 0 \tag{11.c}$$

The most general transformations of the $a_{A'}$ and b_A parameter matrices which leave the gauge potential A_μ and the constraints (11) invariant[+)] are

$$a_{A'} \longmapsto U a_{A'} K \quad ; \qquad b_A \longmapsto U b_A K \quad . \tag{12}$$

The invariance (12) can be used to transform b_A into its "normal form"

$$b^1 = \begin{array}{l} \left.\begin{pmatrix} \quad 0 \quad \end{pmatrix}\right\} n \\ \left.\begin{pmatrix} 1 & 0 & \\ 0 & 0 & \\ 0 & 1 & \\ 0 & 0 & \\ & & \ddots \\ & & 1 \\ & & 0 \end{pmatrix}\right\} 2K \end{array} \quad ; \qquad b^2 = \begin{array}{l} \left.\begin{pmatrix} \quad 0 \quad \end{pmatrix}\right\} n \\ \left.\begin{pmatrix} 0 & 0 & \\ 1 & 0 & \\ 0 & 0 & \\ 0 & 1 & \\ & & \ddots \\ & & 0 \\ & & 1 \end{pmatrix}\right\} 2K \end{array} \tag{13}$$

+) Up to gauge transformations of the potential of course

Now all instanton parameters are "sitting" in $a_{A'}$. They may be counted with the result that

$$\begin{aligned} &4nk - n^2 + 1 && \text{if} \quad k \geq \tfrac{1}{2} n \\ &4k^2 + 1 && \text{if} \quad k < \tfrac{1}{2} n \end{aligned}$$

in agreement with results obtained by applying /14/ the Atiyah Singer index theorem.
The self-duality

$$F^*_{\mu\nu} = F_{\mu\nu}$$

of the field strength can be proven by using some properties of the projection operator $P = VV^+ = 1 - \Delta^{A'} f \Delta^+_{A'}$ (cf. Ref./13/).

III.2 Is the QCD Instanton Gas Infrared Finite?

If the QCD instanton gas exists in the thermodynamic sense, then it is dense, because this means that the infrared divergent dilute configurations have a negligible statistical weight. In Section II we have seen that the kinematical aspect of the question of infrared finiteness amounts in counting the number of instanton parameters which can be interchanged without changing the instanton. The ADHM construction is, however, not as explicit as the $\mathbb{C}P^{N-1}$ instanton solution (2). The question is, how to introduce convenient parameters. An interesting proposal has recently been made by Belavin et al. /8/.

Let the gauge group be SU(2) and N be a (K + 1) dimensional quaternionic vector, i.e. having components of the form $n_i = n_i^\mu (e_\mu)$. Then Manin (unpublished) has proven that the equation

$$N^+ V(x) = 0$$

has K solutions if the number of solutions is finite at all. Therefore the authors of Ref./8/ suggest to fix two quaternionic vectors N_1, N_2 and to define instanton parameters $a_1 \dots a_K$ as the solutions of $N_1^+ V = 0$ and $b_1 \dots b_K$ as the solutions of $N_2^+ V = 0$. Clearly translations and dilatations of the instanton introduce translations and dilatations of these parameters. The hope is that a convenient choice of N_1 and N_2 can be done such that the parameters $a_1 .. a_K$ and $b_1 .. b_K$ describe positions of instanton constituents analog to the $\mathbb{C}P^1$ case. Of course there has to be some dependence between these parameters, because there are only 8K-3 independent. A heuristical intuition

can be gained by studying the 't Hooft solutions. For example the choice $N_1 = V(\infty)$ gives as parameters $a_1 .. a_K$ the centers of the instanton.

Within each set of solutions the instanton parameters can be interchanged without changing the instanton and, if the functional determinant coming from this parametrization does not depend on K,

$$Z = \sum_K \frac{1}{(K!)^2} \int z^{2K} \exp - U(a_1 \dots a_K, b_1 \dots b_K)\, d^4a_1 \dots d^4a_K\, d^4b_1 \dots d^4b_K .$$

The kinematical problem U = 0 is then analog to the $\mathbb{C}P^1$ instanton gas (cf. Section II).

To set up the complete instanton gas one major problem is to determine the potential U. This has not yet been carried out. However, the methods which enable /9/ the computation of quantum fluctuation of quarks in a general instanton background (cf. next section) can be generalized for treating the quantum fluctuation of gluons.

III. 3 Quantum Fluctuations of Quarks in an Arbitrary Yang-Mills Instanton Background

We like to compute

$$\Gamma = \ln \det{}' D \underset{\text{def}}{=} \tfrac{1}{2} \ln \det{}' D^2$$

Where D is the Dirac operator and the prime indicates that zero modes have to be omitted.
Explicitly:[+)]

$$D = 2i \gamma_\mu D_\mu = \begin{pmatrix} 0 & T^+ \\ T & 0 \end{pmatrix}$$

with

$$\gamma_\mu = \begin{pmatrix} 0 & e_\mu \\ e_\mu^+ & 0 \end{pmatrix}, \qquad D_\mu = \partial_\mu + A_\mu$$

and

$$T^+ = 2i e_\mu D_\mu , \qquad T = 2i e_\mu^+ D_\mu$$

+) For the actual calculation we have worked on the sphere S^4, here I give all formulas in the flat Euclidean space.

The UV divergence of Γ can be regularized by adding four Pauli-Villars regulator field with large masses M_i and alternating "metric" e_i, such such that

$$\sum_{i=1}^{4} e_i = -1 \quad ; \quad \sum_{i=1}^{4} e_i M_i^{2p} = 0 \qquad (p = 1, 2, 3) \ .$$

The regularized determinant

$$\Gamma_{reg} = \frac{1}{2} \mathrm{Tr} \left\{ \ln(\mathcal{D}^2 + P_0) + \sum_{i=1}^{4} e_i \ln(\mathcal{D}^2 + M_i^2) \right\}$$

is finite and perfectly well-defined. (P_0 is the projector onto the zero modes of D).

Γ_{reg} is calculated by computing the variation $\delta\Gamma_{reg}$ with respect to the instanton parameters and integrating subsequently. The first part of the calculation can be traced back to the known /13,15/ Green's function of the Dirac operator. This part is rather lengthly but nevertheless quite straight forward. The result, which has been obtained independently by several authors /8,9,16/, is

$$\delta\Gamma_{reg} = \frac{1}{6\pi^2} \int d^4x \ \mathrm{Tr}\,(\delta A_\mu j_\mu) \ ^{+)} \tag{14}$$

where

$$\delta A_\mu = V^+ \{ d^{A'} f \partial_\mu \Delta_A^+ - \partial_\mu \Delta^A f d_{A'}^+ \} V , \quad d^{A'} \underset{def}{=} \delta a_{A'}$$

$$j_\mu = V^+ \{ b^A f b_A^+ \Delta^{A'} f \partial_\mu \Delta_{A'}^+ - \partial_\mu \Delta^{A'} f \Delta_{A'}^+ b^A f b_A^+ \} V$$

and the matrices b^A have been taken in their normal form (13).
The main difficulty was to integrate $\delta\Gamma_{reg}$, a step which involved a lot of guesswork. I like therefore to give some details how this was actually done. Integrating $\delta\Gamma_{reg}$ amounts to write $\mathrm{Tr}(\delta A_\mu j_\mu)$ as a sum of total variations and total derivatives:

$$\mathrm{Tr}(\delta A_\mu j_\mu) = \sum_i v_i \,\delta\, \mathrm{Tr}(V_i) + \sum_j x_j \,\partial_\mu \mathrm{Tr}(X_\mu^j) \ . \tag{15}$$

+) $\mathrm{Tr} = \mathrm{Tr}_{color}$

If we could classify all possible total variation $\delta\,\mathrm{Tr}(V_i)$ and all possible total derivatives $\mathrm{Tr}(X_\mu^j)$ and find a set of linear independent "basic traces" F_K such that we can expand the left-hand side and right-hand side of (15) in terms of basic traces, then we would end up with a set of linear equations for the coefficients v_i and x_j which has to be solved.
Classifying the total variation and total derviatives one is finally led (after trying some larger sets) to

$$\begin{aligned}
\mathrm{Tr}(V_1) &= \mathrm{Tr}\{ f\partial_\mu f^{-1} f\partial_\mu f^{-1} f\partial_\nu f^{-1} f\partial_\nu f^{-1} \} \\
\mathrm{Tr}(V_2) &= \tfrac{1}{4} \mathrm{Tr}\{ f b^{+A} b_A f b^{+B} b_B \} \\
\mathrm{Tr}(X_\mu^1) &= \mathrm{Tr}\{ f\delta f^{-1} f\partial_\mu f^{-1} f\partial_\nu f^{-1} f\partial_\nu f^{-1} \} \\
\mathrm{Tr}(X_\mu^2) &= \mathrm{Tr}\{ f\delta f^{-1} f\partial_\nu f^{-1} f\partial_\mu f^{-1} f\partial_\nu f^{-1} \} \\
\mathrm{Tr}(X_\mu^3) &= \mathrm{Tr}\{ f\partial_\mu\delta f^{-1} f\partial_\nu f^{-1} f\partial_\nu f^{-1} \} \\
\mathrm{Tr}(X_\mu^4) &= \mathrm{Tr}\{ f\partial_\nu\delta f^{-1} f\partial_\mu f^{-1} f\partial_\nu f^{-1} \} \\
\mathrm{Tr}(X_\mu^5) &= \tfrac{1}{2}\mathrm{Tr}\{ f\delta f^{-1} f bb^+ f\partial_\mu f^{-1} \} \\
\mathrm{Tr}(X_\mu^6) &= \tfrac{1}{2}\mathrm{Tr}\{ f bb^+ f\partial_\mu\delta f^{-1} \}
\end{aligned} \tag{16}$$

In the search for basic traces one first recognizes, that by the use of the matrix algebra of the AHDM construction all operators in Eq.(15) may be expanded into linear combinations of traces of products of the form

$$d_{A'} f b_A^+ \ , \quad b_A f d_A^+ \ , \quad \Delta_{A'} f d_{B'}^+ \ ;$$

$$\Delta_{A'} f \Delta_{B'}^+ \ , \quad \Delta_{A'} f b_A^+ \ , \quad b_A f \Delta_A^+ \ , \quad b_A f b_B^+ \ ,$$

where all spinor indices are contracted. All terms have to contain precisely one matrix $d_{A'}$ or $d_{A'}^+$ and four matrices b_A or b_A^+.

From (10) we note

$$f^{-1} = -\frac{1}{2} \underbrace{\Delta^+ \Delta} \tag{17}$$

where we have introduced the graphical notation

$$\underbrace{\Delta^+ \Delta}\ \stackrel{=}{\mathrm{def}}\ \Delta^{+A'} \Delta_{A'}$$

for spinor contractions. By (17) the maximum number of f's in each basic trace is limited to 5. Therefore the system of basic traces closes. Already using a reality property (see below) we end up with 66 operators F_K . E.G. the 5f-terms give 5·3·3 = 45 different operators from the following possible contractions:

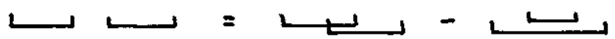
$$\underbrace{d f \overbrace{b^+}^{3}}_{5\cdot 3} \quad \Delta f b^+ \quad \Delta f b^+ \quad \Delta f b^+ \quad \Delta f \Delta^+$$

We have used an algebraic computer program to expand the l.h.s. of (15) and a set of total variations and total derivatives like (16) in terms of these basic traces, and found out that this overdetermined system of linear equations has no solution. This is not a big surprise, since the basic traces so far defined are <u>not</u> linearly independent.

In our paper /9/ we have used 3 types of linear relations between the basic traces. First the identity

$$\varepsilon_{A'B'}\, \varepsilon_{C'D'} = \varepsilon_{A'C'}\, \varepsilon_{B'D'} - \varepsilon_{A'D'}\, \varepsilon_{B'C'}$$

which reads graphically



Second the relations

$$d^+_A \Delta_{B'} + \Delta^+_{A'} d_{B'} + d^+_{B'} \Delta_{A'} + \Delta^+_{B'} d_{A'} = 0$$

$$d^+_{A'} b_A + b^+_A d_{A'} = 0$$

Finally reality of the traces which relied on the use of SP(r) as gauge group instead of SU(r). We were at the end led to a "minimal" basis of 10 operators given in the following:

$$T_1 = \varepsilon_{\mu\nu\rho\sigma} \mathrm{Tr}\{ f\delta f^{-1}\, f\partial_\mu f^{-1}\, f\partial_\nu f^{-1}\, f\partial_\rho f^{-1}\, f\partial_\sigma f^{-1} \}$$

$$T_2 = \mathrm{Tr}\{ f\delta f^{-1}\, f\partial_\mu f^{-1}\, f\partial_\mu f^{-1}\, f\partial_\nu f^{-1}\, f\partial_\nu f^{-1} \}$$

$$T_3 = \mathrm{Tr}\{ f\delta f^{-1}\, f\partial_\mu f^{-1}\, f\partial_\nu f^{-1}\, f\partial_\mu f^{-1}\, f\partial_\nu f^{-1} \}$$

$$T_4 = \mathrm{Tr}\{ f\delta f^{-1}\, f\partial_\mu f^{-1}\, f\partial_\nu f^{-1}\, f\partial_\nu f^{-1}\, f\partial_\mu f^{-1} \}$$

$$T_5 = \mathrm{Tr}\{ f\partial_\mu \delta f^{-1}\, f\partial_\mu f^{-1}\, f\partial_\nu f^{-1}\, f\partial_\nu f^{-1} \}$$

$$T_6 = \mathrm{Tr}\{ f\partial_\mu \delta f^{-1}\, f\partial_\nu f^{-1}\, f\partial_\mu f^{-1}\, f\partial_\nu f^{-1} \}$$

$$T_7 = \mathrm{Tr}\{ f\delta f^{-1}\, f\overbrace{b^+ b}\, f\partial_\mu f^{-1}\, f\partial_\mu f^{-1} \}$$

$$T_8 = \mathrm{Tr}\{ f\delta f^{-1}\, f\partial_\mu f^{-1}\, f\overbrace{b^+ b}\, f\partial_\mu f^{-1} \}$$

$$T_9 = \mathrm{Tr}\{ f\partial_\mu \delta f^{-1}\, f\partial_\mu f^{-1}\, f\overbrace{b^+ b} \}$$

$$T_{10} = \mathrm{Tr}\{ f\delta f^{-1}\, f\overbrace{b^+ b}\, f\overbrace{b^+ b} \}$$

In this basis we obtained the equation:

$$\mathrm{Tr}(\delta A_\mu j_\mu) - 2\partial_\mu \mathrm{Tr}\{ d f \partial_\mu \Delta^+ \Delta f b^+ \Delta f b^+ \quad \underbrace{\Delta f \Delta^+} \} - \sum_{i=1}^{2} v_i\, \delta\, \mathrm{Tr}(V_i) - \sum_{j=1}^{6} x_j\, \partial_\mu \mathrm{Tr}(X^j_\mu) =$$

$$\begin{array}{llllllll}
 & & & & & & & \frac{1}{4} T_1 \\
+\{ -\frac{3}{4} & +\, 2V_1 & & +2X_1 & & & & \}\ T_2 \\
+\{ \frac{3}{4} & & & +\, X_1 & +\, 2X_2 & & & \}\ T_3 \\
+\{ \frac{1}{4} & +\, 2V_1 & & +\, X_1 & +\, 2X_2 & & & \}\ T_4 \\
+\{ 1 & -\, 4V_1 & & -\, X_1 & & +2X_3 + 2X_4 & & \}\ T_5 \\
+\{ -\frac{1}{2} & & & & -\, X_2 & +\, X_3 + X_4 & & \}\ T_6 \\
+\{ 2 & & & -10X_1 & -\, 4X_2 & & +2X_5 & \}\ T_7 \\
+\{ -2 & & & -\, 2X_1 & -\, 8X_2 & & +\, X_5 & \}\ T_8 \\
+\{ 6 & & & & & -4X_3 - 10X_4 & -\, X_5 + 2X_6 & \}\ T_9 \\
+\{ & & -2V_2 & & & & +8X_5 & \}\ T_{10}
\end{array}$$

One immediately recognizes that T_1 decouples completely from the V_i and x^j and therefore has to be itself a total divergence. In fact

$$T_1 = 5\delta \int_0^1 dt\, \varepsilon_{\mu\nu\rho\sigma} \, Tr\{K^{-1}\partial_t K K^{-1}\partial_\mu K \ldots K^{-1}\partial_\sigma K\} + \partial_\mu \Sigma_\mu$$

with

$$K = (1-t)(1+x^2)\,1 + t f^{-1} .$$

The other nine equations have a solution given by

$$V_1 = \frac{1}{4} \quad , \quad V_2 = -5$$

$$x_1 = \frac{1}{8} \; , \; x_2 = -\frac{7}{16} \; , \; x_3 = \frac{1}{16} \; , \; x_4 = 0 \; , \; x_5 = -\frac{5}{4} \; , \; x_6 = -\frac{7}{2} .$$

This solves the problem of integrating $\delta \, \Gamma_{YM}$.

We have made use of the gauge group SP(r). By imbedding SU(r) into SP(r) it is not difficult to proof that the final answer for Γ_{YM}, as given in the following forumula, is also true for SU(r), i.e. f defined by (10,17). The result is:

$$\Gamma_{YM} - 2r\,\Gamma^0_{YM} =$$

$$\alpha + \frac{1}{24\pi^2} \int d^4x \, I_1(x) + \frac{5}{24\pi^2} \int d^4 \int_0^1 dt \, I_2(t,x)$$

with

$$I_1 = Tr\{f\partial_\mu f^{-1} \, f\partial_\mu f^{-1} \, f\partial_\nu f^{-1} \, f\partial_\nu f^{-1}\}$$
$$- 2\partial \, Tr\{f^2\} + 4k(1+x^2)^{-2}$$

and

$$I_2 = \varepsilon_{\mu\nu\rho\sigma} \, Tr\{K^{-1}\partial_t K K^{-1}\partial_\mu K K^{-1}\partial_\nu K K^{-1}\partial_\rho K K^{-1}\partial_\sigma K\}$$

The determination of the integration constant α can be reduced to computing explicitly the 1-instanton case. Those calculations have been done previously /3,17/. The result for the integration constant is

$$\alpha = K\left\{\sum_{i=1}^{4} e_i \ln M_i - 4\zeta'(-1) - \ln 2 + \frac{5}{12}\right\} .$$

For the case of the 't Hooft solutions analog results were first obtained in the work of Brown and Creamer /18/.

Summary and Conclusions

The results show that even for a dense instanton gas calculations are still possible to a certain extent. This gives rise to some optimism concerning the future understanding of the QCD instanton gas.
On the other hand an incoherent sum of the pure instanton and anti-instanton contributions runs into difficulties with the cluster theorem and the Θ-vacua cannot incorporated. This can probably be repaired by taking into account instanton-antiinstanton approximate solutions. Also there are technical problems concerning the computation of correlation functions in a dense instanton gas. For example even for the $\mathbb{C}P^1$, alias O(3) non-linear σ-model, it has so far not been proved rigorously that the dynamically generated finite correlation length of the instanton gas gives rise to the physical mass of the σ-particles which we know to present in the O(3) σ - model.

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