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CUTOFF DEPENDENCE IN LATTICE ϕ_4^4 THEORY

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CUTOFF DEPENDENCE IN LATTICE ϕ^4_4 THEORY

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0. INTRODUCTION

This seminar is logically the continuation of the one [1] I gave here three years ago. The stimulus to the present work came from recent results [2] on lattice ϕ^4_4 theory, obtained by high-temperature expansions, which seem to require to study the Λ^{-2} corrections (Λ^{-1} = a = lattice constant) to the formulae of [1] which dealt in detail only with ln Λ terms. We shall see that the numerical ϕ^4_4 results offer a test of the merely technical (!) idea of "asymptotic freedom". This is the link between the present topic and nonabelian gauge theory.

The tool to analyse the $\Lambda^{-2}(\ln\Lambda)^{\bullet}$ corrections is, as for the $(\ln\Lambda)^{\bullet}$ terms in [1], the renormalization group. While we could work entirely in four dimensions (as we shall indicate at the end of sect. 2) it is simpler to work initially in 4 + ε dimensions, with ε generic, and to set ε = 0 only later. That a 4 +ε-dimensional lattice offers no difficulty in perturbation theory (and this suffices for our purpose) we shall show in sect. 1. This allows us to write in sect. 2 a simple effective Lagrangean for small-a dependence, and to analyse its consequences, using the renormalization group, in sect. 3. The results in terms of formulae that can directly be compared with numerical data are given in sect. 4. Final remarks are offered in sect. 5.

1. LATTICE ϕ^4 THEORY IN 4 +ε DIMENSIONS

The Euclidean Lagrangean of hypercubic lattice ϕ^4 theory in D = 4 +ε dimensions is

$$(1.1)\quad L = a^D \sum_i \Big[-\frac{1}{2} \sum_\mu a^{-2} (\phi_i - \phi_{i+\hat\mu})^2 - \frac{1}{2} \Delta m_B^2 \phi_i^2 -$$
$$- \frac{1}{4!} g_B \phi_i^4 - \frac{1}{2} m_{B0}^2 \phi_i^2 \Big]$$

where $i \in \mathbb{Z}^D$, $i+\hat\mu$ is the lattice point next to i in the positive μ-direction, and the μ-sum is from 1 to D. $m_{B0}^2 = a^{-2} f(g_B a^{-\varepsilon}, \varepsilon)$ is the bare-mass-squared of the massless (critical) theory. For $\Delta m_B^2 > 0$, and we shall only consider this case, we are in the symmetric $(\langle\phi\rangle = 0)$ phase. $\langle \cdots \rangle$ is the "true" vacuum expectation value, i.e. the Gibbs-ensemble expectation in the thermodynamic limit.

We define the theory formally by perturbation expansion in g_B, i.e. by Feynman graphs with bare propagators

$$(1.2)\quad \langle \phi_i \, \phi_j \rangle_0 =$$
$$= (2\pi)^{-D} \prod_\mu \Big(\int_{-\pi/a}^{\pi/a} dk_\mu \Big) \exp\Big[i a \sum_\mu (i-j)_\mu k_\mu \Big] \cdot$$
$$\cdot \Big[\Delta m_B^2 + \sum_\mu 4 a^{-2} \sin^2 \big(\tfrac{1}{2} k_\mu a\big) \Big]^{-1} =$$
$$= \frac{1}{2} a^{2-D} \int_0^\infty dt \exp\Big[-Dt - \frac{1}{2} a^2 \Delta m_B^2 t \Big] \cdot$$
$$\cdot \prod_\mu I_{(i-j)_\mu}(t).$$

Noting that

$$a^D \sum_i \exp[i a \, i_\mu k_\mu] = (2\pi)^D \sum_\ell \delta^D (k_\mu - 2\pi \ell_\mu a^{-1})$$

where $\ell \in \mathbb{Z}^D$ one finds that summation over lattice points gives results completely analogous to continuum Feynman graphs: momentum conservation holds modulo $2\pi/a$, and all loop momenta are integrated over one Brillouin zone the precise location of which is immaterial due to the $2\pi/a$ periodicity of all factors. E.g., for a one-loop graph with external momenta $p_1 \ldots p_n$ (in order 1 ... n along the loop) one finds

$$(1.3)\quad I(p_1 \ldots p_n) = (2\pi)^D \sum_{l} \delta^D\Big(\sum_m p_m - 2\pi l a^{-1}\Big) \cdot$$

$$\cdot\, a^{-D} \prod_{m=1}^{n} \Big(\tfrac{1}{2} a^2 \int_0^\infty dt_m\Big) \exp\Big[-\big(D + \tfrac{1}{2}\Delta m_B^2 a^2\big) \sum_m t_m\Big] \cdot$$

$$\cdot \prod_{\mu} I_0\Big(\Big[\big(\sum_m t_m\big)^2 - 4 \sum_{l<m} t_l t_m \sin^2\big(\tfrac{1}{2} a \sum_{k=l+1}^{m} (p_k)_\mu\big)\Big]^{\frac{1}{2}}\Big)$$

where the t_m play the rôle of the usual Feynman parameters.

The use of momentum vectors with $4+\varepsilon$ components will, as in continuum dimensional regularization [3], not give rise to problems. E.g., we could think of external momenta with only the first four components different from zero. Moreover, the coefficients in the effective Lagrangean, to be described in the next section, can be expressed in closed form in terms of functions on the lattice at zero external momenta or derivatives there, see (2.3) below. – We do not know, however, whether the dimensional interpolation used here is the same as the one obtained from high-temperature expansions [4].

We shall always work with vertex functions (VFs), i.e. the full-propagator-amputated one-particle-irreducible connected parts of Green's functions. The Fourier transform of the VF to $g^{-l}\langle \phi_{i_1} \cdots \phi_{i_{2n}} \phi^2_{j_1} \cdots \phi^2_{j_l} \rangle$, with $(2\pi)^D \cdot$ momentum conserving delta function, as in (1.3), omitted we denote as $\Gamma_B(p_1 \cdots p_{2n}; q_1 \cdots q_l; g_B, \Delta m_B^2, \varepsilon, a)$, or $\Gamma_B((2n),(l))$ for short. We shall suppose these functions to have in perturbation theory the small-a-expansions

$$(1.4)\quad \Gamma_B((2n),(l); \varepsilon, a) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a^{2j-\varepsilon k} f_{jk}((2n),(l); \varepsilon).$$

The f_{jk} are finite for generic ε but have singularities at positive rational ε in such a fashion that the r.h.s. of (1.4) stays finite at all ε (if $\Delta m_B^2 = 0$, at all $\varepsilon \geq 0$ for nonexceptional momenta) due to the l.h.s. being finite. At $\varepsilon = 0$, (1.4) reduces to

$$(1.5)\quad \Gamma_B((2n),(l); 0, a) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a^{2j} (\ln a)^k \bar{f}_{jk}((2n),(l)).$$

The argument for (1.4-5) is given in the next section.

2. EFFECTIVE LAGRANGEAN FOR LARGE-CUTOFF BEHAVIOUR

The expansion (1.4) arises from the following effective Lagrangean:

$$\begin{aligned}(2.1)\quad L_{eff} = &-\tfrac{1}{2}\partial_\mu\phi_B\,\partial_\mu\phi_B - \tfrac{1}{2}\Delta m_B^2\,\phi_B^2 - \\ &-\tfrac{1}{4!}g_B\,\phi_B^4 + \sum_{n=2}^{\infty}[(2n)!]^{-1}a^{2n-2}\sum_\mu \phi_B\,\partial_\mu^{2n}\phi_B + \\ &+\sum_{r=0}^{\infty}\sum_{s=1}^{\infty}\sum_{t=0}^{\infty}\sum_{\nu=1}^{n_{rs}}(\text{"}D^{2r}\phi_B^{2s}\text{"})_\nu\,\Delta m_B^{2t}\,g_B^{s-1}\,\cdot \\ &\cdot f_{rst\nu}(g_B a^{-\varepsilon};\varepsilon)\,a^{-4+2r+2s+2t}\,.\end{aligned}$$

The $(\text{"}D^{2r}\phi_B^{2s}\text{"})_\nu$ are all monomials of order 2r in derivatives and order 2s in ϕ_B, having the lattice symmetry and being linearly independent at zero momentum. (For $r + s \geq 3$, $n_{rs} > 1$ in general.) Computations with L_{eff} are meant by use of continuum Feynman rules with bare propagator and vertex as obtained from the first three terms; all other vertices, including the two-vertices, are to be treated as insertion into graphs, with dimensional integration rules [3] to be used throughout.

All coefficients $f_{rst\nu}$ can be expressed by the regularized Γ_B, with $\Delta m_B^2 = 0$, at zero momenta and derivatives there: We define, with $\Box = \sum_\mu \partial_\mu^2$,

(2.2)

$$\bar{O}_1 = -\tfrac{1}{4!}g_B\,\phi_B^4 \qquad \bar{O}_2 = -\tfrac{1}{2}\Delta m_B^2\,\phi_B^2$$

$$\bar{O}_3 = -\tfrac{1}{2}\sum_\mu \partial_\mu\phi_B\,\partial_\mu\phi_B \qquad \bar{O}_4 = \tfrac{1}{2}\sum_\mu \phi_B\,\partial_\mu^4\phi_B$$

$$\bar{O}_5 = \frac{1}{2}\phi_B \Box^2 \phi_B \qquad \bar{O}_6 = \frac{1}{3!} g_B \phi_B^3 \Box \phi_B$$

$$\bar{O}_7 = \frac{1}{6!} g_B^2 \phi_B^6 \qquad \bar{O}_8 = \Delta m_B^2 \bar{O}_3$$

$$\bar{O}_9 = \Delta m_B^2 \bar{O}_1 \qquad \bar{O}_{10} = \Delta m_B^2 \bar{O}_2$$

Then the (total) coefficients in L_{eff} of these operators are

(2.3)

$$\bar{O}_1: \; -g_B^{-1}\Gamma_B(0000,;g_B,0,\varepsilon,\alpha) \equiv \bar{Z}_1$$

$$\bar{O}_2: \; \Gamma_B(00,0;g_B,0,\varepsilon,\alpha) \equiv \bar{Z}_2$$

$$\bar{O}_3: \; -[\partial/\partial p^2]\Gamma_B(p(-p),;g_B,0,\varepsilon,\alpha)\Big|_{p=0} \equiv \bar{Z}_3$$

$$\bar{O}_4: \; \frac{1}{24} D^{-1}(D-1)^{-1}\Big[(D+2)\sum_\mu(\partial/\partial p_\mu)^4 - 3\big(\sum_\mu(\partial/\partial p_\mu)^2\big)^2\Big]\cdot$$
$$\cdot\Gamma_B(p(-p),;g_B,0,\varepsilon,\alpha)\Big|_{p=0} \equiv a^2\bar{Z}_4$$

$$\bar{O}_5: \; -\frac{1}{8} D^{-1}(D-1)^{-1}\Big[\sum_\mu(\partial/\partial p_\mu)^4 - \big(\sum_\mu(\partial/\partial p_\mu)^2\big)^2\Big]\cdot$$
$$\cdot\Gamma_B(p(-p),;g_B,0,\varepsilon,\alpha)\Big|_{p=0} \equiv a^2\bar{Z}_5$$

$$\bar{O}_6: \; -\frac{1}{12} D^{-1}\sum_{i=1}^{3}\sum_\mu(\partial/\partial p_{i\mu})^2 \,\cdot$$
$$\cdot\Gamma_B(p_1 p_2 p_3(-p_1-p_2-p_3),;g_B,0,\varepsilon,\alpha)\Big|_{p=0} \equiv a^2\bar{Z}_6$$

$$\bar{O}_7: \; g_B^{-2}\Gamma_B(000000,;g_B,0,\varepsilon,a) \equiv a^2\bar{Z}_7$$

$$\bar{O}_8: \; [\partial/\partial p^2]\,\Gamma_B(p(-p),0;g_B,0,\varepsilon,\alpha)\Big|_{p=0} \equiv a^2\bar{Z}_8$$

$$\bar{O}_9: \quad g_B^{-1}\, \Gamma_B^{\pi}(0000,0;\, g_B, 0, \varepsilon, a) \equiv a^2 \bar{Z}_9$$

$$\bar{O}_{10}: \quad \Gamma_B^{\pi}(00,00;\, g_B, 0, \varepsilon, a) \equiv a^2 \bar{Z}_{10}\,.$$

Hereby $\bar{Z}_1$, $\bar{Z}_2$, $\bar{Z}_3$ are defined by analytic continuation from $\varepsilon > 0$ being IR singular at (rational) $\varepsilon \leq 0$, and $\bar{Z}_4 \ldots \bar{Z}_{10}$ are defined by analytic continuation from $\varepsilon > 2$, being IR singular at (rational) $\varepsilon \leq 2$. The rôle of these singularities is seen by writing (2.1-3) as

$$\mathcal{L}_{eff} = \sum_{i=1}^{3} \bar{Z}_i\, \bar{O}_i + a^2 \sum_{i=4}^{10} \bar{Z}_i\, \bar{O}_i + a^4 \ldots = \tag{2.4}$$

$$= \mathcal{L}_{eff(0)} + \sum_{K=1}^{\infty} a^{2K} \mathcal{L}_{eff(K)}.$$

$\mathcal{L}_{eff(0)}$ yields finite results to ℓ-loop order for (not only generic) $\varepsilon < 2/\ell$. $\mathcal{L}_{eff(0)} + a^2 \mathcal{L}_{eff(1)}$ does so for $\varepsilon < 4/\ell$, whereby the IR singularities of the $\bar{Z}_i$ (i = 4 ... 10) act as final subtractions for graphs computed from $\mathcal{L}_{eff(0)}$ and no longer convergent for $\varepsilon \geq 2/\ell$ (and for graphs with lower-order $\mathcal{L}_{eff(1)}$ insertions), etc. For $\varepsilon \sim 0$, $\mathcal{L}_{eff(0)}$ is a dimensionally regularized Lagrangean with counter terms, however, not precisely of the 't Hooft [5] but of the Zinn-Justin form [6]: $\bar{Z}_i - 1$ (i = 1,2,3) are not pure pole terms in ε but have also regular parts, determined by the regularization (2.4) comes from and characterized by having in their perturbation theoretical expansion only powers of $a^{-\varepsilon}$ or, for $\varepsilon = 0$, $\ln a$ appearing. If we only consider $\varepsilon \to 0$, then $\mathcal{L}_{eff(1)}$ is an altogether finite operator insertion, at zero momentum, of dimension six, and so on for higher insertions. Hereby the a^2 explicit in (2.4) can be chosen different from "normalization length" a that plays the rôle of μ^{-1} in 't Hooft's approach [5].

The reason [7] for validity of (2.3) is that at the momenta indicated, all non-Born contributions from $\mathcal{L}_{eff}$ with $\Delta m_B^2 = 0$ vanish for dimensional reasons as far as $\bar{Z}_1 \ldots \bar{Z}_7$ are concerned, and for $\bar{Z}_8 \ldots \bar{Z}_{10}$ one uses that for Γ_B, $\phi_B^2/2$ insertion at zero momentum is equivalent to differentiation w.r.t. $-\Delta m_B^2$.

In a fully perturbative construction, in (2.1)

$$f_{rstv}(g_B a^{-\varepsilon}, \varepsilon) = \sum_{\ell=1}^{\infty} f_{rstv\ell}(\varepsilon)(g_B a^{-\varepsilon})^{\ell}$$

with meromorphic $f_{rstu\mathcal{L}}(\varepsilon)$ holds, where $\mathcal{L}$ is the number of loops in the VFs in (2.3). In this way, (1.4) is obtained and in the $\varepsilon \searrow 0$ limit (1.5). However, formulae (2.3) are meant "exactly", i.e. computing from (1.1) to arbitrary order in g_B. In this sense, L_{eff} is not mere perturbation theory.

The argument for validity of (2.1) (with only covariant operators needed) for Pauli-Villars regularization and $\Delta m_B^2 = 0$, using oversubtraction, is given elsewhere [7]. The extension to $\Delta m_B \neq 0$ is straight-forward, whereby one uses that differentiation with respect to the bare mass, all other bare parameters fixed, reduces UV divergence degree by two. (A related argument within merely dimensional regularization is used in [8]).

The present point is that (2.1) seems to hold also for lattice regularization at least to order a^2. Consider the one-loop graph contribution to the four-point VF. Using (1.2) we can write in obvious notation

$$(2.4) \quad I(a,p) = \prod_\mu \Big(\int_{-\pi/a}^{\pi/a} dk_\mu \Big) N(a,k)^{-1} N(a,k+p)^{-1}.$$

Then

$$(2.5) \quad [\partial/\partial a^2]\, I(a,p) =$$

$$= -\pi a^{-3} \sum_\mu \prod_{\nu\neq\mu}{}' \Big(\int_{-\pi/a}^{\pi/a} dk_\nu \Big) N(a,k)^{-1} N(a,k+p)^{-1} \Big|_{k_\mu = \frac{\pi}{a}} -$$

$$- \prod_\mu \Big(\int_{-\pi/a}^{\pi/a} dk_\mu \Big) \Big[N(a,k)^{-1} \frac{\partial}{\partial a^2} N(a,k) \cdot N(a,k)^{-1} N(a,k+p)^{-1} + \dots \Big]$$

$\equiv$ Bound. term + Insert. term.

Due to $[\partial/\partial a^2]\, N(a,k) = -\frac{1}{12} \sum_\mu k_\mu^4 + O(a^2 k_\mu^6)$ the insertion term is a regularized form of the graphs with dimension-six insertions in the two lines. As $a \searrow 0$, this term would be quadratically divergent. Thus, with $T_{2n}(p)$ the Taylor operator around zero momentum, according to BPHZ [9] in the identity

Ins. term = $[1 - T_2(p)]$ Ins. term + $T_2(p)$ Ins. term

the first term on the r.h.s. has a limit as $a \searrow 0$, which is the square bracket applied to the unregularized insertion term. In order to split the square bracket up, we need to give an interpretation to the unregularized unsubtracted insertion term, which we do by the dimensional-integration rule [3]. It gives always a finite result for generic ε , and we denote its use by underlining. Thus,

$$(2.6)\ \text{Ins. term} = \underline{\text{Ins.term }(a=0)} + \\ + T_2(p)\left[\text{Ins.term} - \underline{\text{Ins.term}(a=0)}\right] + O(a^2).$$

The boundary term in (2.5) is a sum of $3+\varepsilon$ -dimensional graphs with mass $\Delta m_B^2 + 4a^{-2}$ and has a Taylor expansion in pa with finite convergence radius. Thus,

$$(2.7)\ \text{Bound.term} = T_2(p)\ \text{Bound.term} + O(a^{2-\varepsilon})$$

Using (2.6-7) in (2.5) and integrating yields

$$(2.8)\ I(a,p) = a^2\ \underline{\text{Ins.term }(a=0)} + R(p) + \\ + T_2(p)\left[I(a,p) - a^2\ \underline{\text{Ins.term }(a=0)}\right] + O(a^{4-\varepsilon})$$

whereby $T_2(p)R(p) = 0$, with the interpretation: The first term on the r.h.s. is the contribution from the insertion of the a^2 part of the fourth term in (2.1). The third term is the sum of the f_{0201}, $\hat{f}_{0211}$ and f_{1201} contribution, plus part of the ordinary one-loop graph the remainder of which is in R(p). All integrations are to be done dimensionally. If $\Delta m_B^2 \to 0$, the Taylor operator should be taken around some nonzero a-independent momentum. (2.8) as well as agreement of the constants therein with (2.3) can also be verified directly on the basis of (1.3). In a similar fashion, correctness of (2.1) for all one-loop graphs is easily shown.

For graphs with ≥ 2 loops, the BPHZ method requires to subtract all subdivergences before performing the final subtraction whereupon one proceeds as before. For the second-order self-energy part, I have verified that (2.1) does describe it to order a^2 correctly. Hereby the subdivergences are cancelled by the contributions from the $\bar{O}_{1,9}$ and $\bar{O}_6$ counter terms in (2.1) to the order stated. The total effect of the boundary terms is again absorbed by the constants in (2.1). On the basis of this, we here take (2.1) as valid at least to order a^2, the one we are interested in.

Working in four dimensions throughout would require to write, for reproducing (1.5), to use an effective Lagrangean with normal products in the sense of Zimmermann [10], which imply an elaborate subtraction prescription, in particular in the massless theory. The coefficients prop. a^2 in such effective Lagrangean would be proportional to the difference between VFs computed with regularization and computed from the a^0 terms alone, with ln a occurring in the finite renormalizations. This difference-taking is replaced in our case by the need of continuing analytically $\bar{Z}_4 \ldots \bar{Z}_{10}$ in (2.3) from $\varepsilon > 2$. (Effective Lagrangeans of the Zimmermann type, for large-mass rather than large-cutoff behaviour, have recently been used by C.K. Lee [11] and Kazama and Yao [12].)

As pointed out before [1], there is no local L_{eff} describing large-cutoff behaviour beyond the order $\Lambda^0(\ln\Lambda)^\cdot$ for e.g. sharp cutoff, i.e. using propagators $\theta(\Lambda^2-p^2)\,[p^2+m^2]^{-1}$ in momentum space. The origin of this is that in this regularization, the cutoff-change effect is carried by boundary terms only. In contrast, in lattice regularization, the effect is mainly carried by insertion terms which are essentially local but depend on the choice of the Brillouin zone of the integration momenta; the main function of the boundary terms is to restore the Brillouin-zone independence, as in the transition from (2.6) to (2.8) above.

3. RENORMALIZATION GROUP PROPERTIES OF OPERATOR INSERTIONS

To obtain the consequences of (2.4), for convenience we temporarily use 't Hooft's Lagrangean [5]

$$(3.1)\qquad L_{\,'tH.} = \sum_{i=1}^{3} Z_i\, O_i$$

and define

$$(3.2)\qquad \hat{O}_i = \sum_j Z_{ij} O_j \quad (i,j = 4 \ldots 10).$$

The operators O_i are obtained from the $\bar{O}_i$ of (2.2) by the replacements

$$(3.3)\qquad \phi_B \to \phi,\ \Delta m_B^2 \to m^2,\ g_B \to g\mu^{-\varepsilon}$$

and we may identify μ with a^{-1}. The coefficients

$$(3.4)\qquad Z_{ij} = \delta_{ij} + \sum_{k=1}^{\infty} \varepsilon^{-k} f_{ijk}(g)$$

are so chosen, uniquely, as to give, as $\varepsilon \searrow 0$, finite operator insertions $\hat{O}_i$ of dimension six.

With

$$(3.5a)\qquad \beta(g,\varepsilon) = \varepsilon\left([\partial/\partial g]\ln(g Z_1 Z_3^{-2})\right)^{-1} = \\ = \varepsilon g + b_0 g^2 + b_1 g^3 + \cdots$$

where

(3.5b) $$b_0 = 3\,(16\pi^2)^{-1}, \quad b_1 = -17\,(2^8 \cdot 3\pi^4)^{-1},$$

(3.6) $$\gamma(g) = \beta(g,\varepsilon)\,[\partial/\partial g]\ln Z_3 = c_0\, g^2 + \dots,$$

(3.7) $$\eta(g) = -\beta(g,\varepsilon)\,[\partial/\partial g]\ln(Z_2 Z_3^{-1}) = \tfrac{1}{3} b_0 g + \dots,$$

the VFs to (3.1) obey

(3.8a) $$Op_{2n}\,\Gamma((2n);\mu, m, g, \varepsilon) = 0$$

where

(3.8b) $$Op_{2n} = [\mu/\partial\mu] + \beta(g,\varepsilon)[\partial/\partial g] - 2n\gamma(g) + \eta(g)\, m^2[\partial/\partial m^2].$$

Then for the VFs with $\hat{0}_i$ inserted at zero momentum

(3.9) $$Op_{2n}\,\hat{\Gamma}_i((2n)) = \gamma_{ij}\,\hat{\Gamma}_j((2n))$$

holds, with an upper-right (4/567/89/10) block-triangular mixing matrix

(3.10) $$\gamma_{ij}(g) = b_0\, q_{ij}\, g + O(g^2)$$

with

(3.11) $$q_{..} = \left(\begin{array}{c|ccc|cc|c} 0 & 0 & 0 & -15 & 0 & 3 & 1/2 \\ \hline & 0 & 0 & -30 & 0 & 6 & 1 \\ & 0 & 0 & -60 & -2/3 & 8 & -2/3 \\ & 0 & 0 & -3 & 0 & 1/3 & 0 \\ \hline & & & & 1/3 & 2 & 2/3 \\ & & & & 0 & -2/3 & -1/3 \\ \hline & & & & & & 1/3 \end{array}\right)$$

obtained from one-loop calculations. The $\gamma_{ij}(g)$ with i,j = 8, 9, 10 can be expressed, using the method of [13], in terms of β, γ, η and their derivatives alone.

From now on we consider $\varepsilon = 0$ only. We set

(3.12) $$\hat{\Gamma}_i = W_{ij}(g)\,\tilde{\Gamma}_j$$

and demand

$$\mathrm{Op}_{2n}\ \tilde{\Gamma}_i((2n)) = 0 \tag{3.13}$$

and invertibility of the matrix W. Then we must solve

$$g[\partial/g]\,W_{ij}(g) = [q_{ik} + u_{ik}(g)]\,W_{kj}(g) \tag{3.14}$$

where $u_{..}(g) = O(g)$. The eigenvalues λ_α of q are its diagonal elements, and one finds that it can be diagonalized by a similarity transformation. Consequently, (3.14) can be solved by power series $g^{2\alpha}(X_0^\alpha + g X_1^\alpha + \ldots)$, with X_0^α the corresponding eigenvectors, except possibly lng appearing in

$$W_{..} = g^{-3}\,(X_0^{-3} + g X_1^{-3} + \ldots) + \ln g\,(Y_0 + g Y_1 + \ldots) \tag{3.15}$$

due to integer-spacing with the threefold eigenvalue zero. To determine whether the logarithm does appear or not requires to compute γ_{ik} and β to four loops. There is no $g^{1/3}\ln g$ in the solution to eigenvalue -2/3 as one finds using the known [13] higher terms in $\gamma_{ik}(g)$.

We find the following solution $(i = 4 \ldots 10)$

$$
\begin{aligned}
W_{i4} &= \{1,\ -,-,-,-,-,-\} \\
W_{i5} &= \{0,1,0,0,-,-,-\} \\
W_{i6} &= \{0,0,1,0,-,-,-\} \\
W_{i7} &= g^{-3}\,\{5,10,20,1,-,-,-\} + \ln g\,\{O(1),O(1),O(1),0,-,-,-\} \\
W_{i8} &= g^{1/3}\{0,0,-2,0,1,0,-\} \\
W_{i9} &= \tfrac{1}{7}\,g^{-2/3}\{-9,-18,-8,1,-14,7,-\} \\
W_{i10} &= \tfrac{1}{3}g^{1/3}\{0,0,-4,-\tfrac{1}{10},2,-1,3\}.
\end{aligned}
\tag{3.16}
$$

Herein, in the wavy bracket only the constant terms are written and to be amended by terms with higher integer powers of g, except for the barred entries which are empty. For the inverse matrix we have

$$
\begin{aligned}
(W^{-1})_{4i} &= \{1,0,0,-5,0,2,1/2\} + g^3\ln g.. \\
(W^{-1})_{5i} &= \{-,1,0,-10,0,4,1\} + g^3\ln g.. \\
(W^{-1})_{6i} &= \{-,0,1,-20,2,8,-2\} + g^3\ln g..
\end{aligned}
\tag{3.17}
$$

$$(W^{-1})_{7i} = \frac{1}{7}g^3\left\{-,0,0,7,0,-1,-\frac{1}{10}\right\}$$
$$(W^{-1})_{8i} = g^{-1/3}\{-,-,-,-,1,2,-\}$$
$$(W^{-1})_{9i} = g^{2/3}\{-,-,-,-,0,1,1/3\}$$
$$(W^{-1})_{10i} = g^{-1/3}\{-,-,-,-,-,-,1\}$$

where lng-terms appear if and only if they do in (3.16).

4. APPLICATION TO LATTICE VERTEX FUNCTIONS

We now can organize and resum the $a^2(\ln a)^K$ terms in (1.5) as it was done for the $(\ln a)^K$ terms in [1]. To this end we identify (2.4) with (3.1) amended by the appropriate linear combination of the $\hat{O}_i$. Returning for the moment to $\varepsilon > 0$, we set

(4.1)
$$a^{-1} = \mu$$
$$g_B a^{-\varepsilon} = \bar{g}$$
$$\phi_B = c(g,\varepsilon)\phi$$
$$\Delta m_B^2 = d(g,\varepsilon) m^2$$

Defining in analogy to (3.5a)

(4.2)
$$\bar{\beta}(\bar{g},\varepsilon) = \varepsilon\left([\partial/\partial\bar{g}]\ln(\bar{g}\bar{Z}_1\bar{Z}_3^{-2})\right)^{-1} =$$
$$= \varepsilon\bar{g} + \bar{b}_0(\varepsilon)\bar{g}^2 + \bar{b}_1(\varepsilon)\bar{g}^3 + \dots =$$
$$= \varepsilon\bar{g} + \bar{\beta}_0(\bar{g}) + \varepsilon\bar{\beta}_1(\bar{g}) + \dots$$

one finds that

(4.3) $$d\bar{g}/dg = \bar{\beta}(\bar{g},\varepsilon)/\beta(g,\varepsilon)$$

is uniquely solved by

(4.4) $$\bar{g} = f_0(g) + \varepsilon f_1(g) + \varepsilon^2 f_2(g) + \dots$$

with $f_0(g) = g + O(g^2)$, whereby

(4.5) $f_o(g) = \bar{\rho}^{-1}(\rho(g))$

with the definition

(4.6) $$\bar{\rho}(\bar{g}) = \int^{\bar{g}} d\bar{g}' \, \bar{\beta}_0(\bar{g}')^{-1},$$

$$\rho(g) = \int^{g} dg' \, \beta(g';0)^{-1}.$$

(The integration constant in (4.5) is determined via fitting (4.3) in order ε.) With $\bar{\gamma}$ and $\bar{\eta}$ defined in analogy to (3.6) and (3.7), one finds

(4.7a) $$c(g,\varepsilon) =$$
$$= \exp\left\{ 2 \int_0^{g} dg' \, \beta(g';\varepsilon)^{-1} \left[\gamma(g') - \bar{\gamma}(\bar{g}';\varepsilon) \right] \right\}, =$$
$$= 1 + O(g^2) \quad \text{for } \varepsilon = 0,$$

(4.7b) $$d(g,\varepsilon) =$$
$$= \exp\left\{ - \int_0^{g} dg' \, \beta(g';\varepsilon)^{-1} \left[\eta(g') - \bar{\eta}(\bar{g}';\varepsilon) \right] \right\}, =$$
$$= 1 + O(g) \quad \text{for } \varepsilon = 0,$$

whereby (4.4) is to be used. One now sees that

(4.8) $$L_{eff}(\text{lattice}) = L_{\text{'t Hooft}} + a^2 \sum_{i=4}^{10} \bar{c}_i(\bar{g},\varepsilon) \, \hat{O}_i + \cdots$$

where the $\bar{c}_i(\bar{g},\varepsilon)$ are linear combinations of the $\bar{Z}_i(\bar{g},\varepsilon)$ ($i = 4 \ldots 10$) of (2.3) with coefficients, obtained from (4.1), (4.4), (4.7), and (3.2), such that the $\bar{c}_i(\bar{g},0)$ are finite since the l.h.s. of (4.8) is finite at $\varepsilon = 0$. Explicitly, one finds from (2.3)

(4.9) $$\bar{c}_i(\bar{g},0) = \begin{cases} \frac{1}{12} + O(\bar{g}^2), & i = 4 \\ O(\bar{g}) & , i = 6,7,9,10 \\ O(\bar{g}^2) & , i = 5,8. \end{cases}$$

(4.8) now gives at $\varepsilon = 0$

$$(4.10)\quad \Gamma_B((2n)_i;\, g_B, \Delta m_B^2, a) =$$
$$= c(g(g_B))^{-2n} \cdot$$
$$\cdot\Big[\Gamma + a^2 \sum_{i=4}^{10} \bar{c}_i(g_B)\, \hat{\Gamma}_i\Big]\big((2n)_i;\, a^{-1}, g(g_B), d(g(g_B))^{-1}\Delta m_B^2\big) +$$
$$+\, O(a^4 (\ln a)^{\cdot})$$

with the obvious insertions. The use herein of

$$(4.11)\quad \hat{\Gamma}_i(\ldots) = W_{ij}(g(g_B))\, \tilde{\Gamma}_j(\ldots)$$

from (3.12) yields, with (3.16,17), the desired reorganization of logarithms due to (3.13).

For application to the setting of [2], we define

$$(4.12a)\quad \tilde{g} = -\Big[\frac{\partial}{\partial p^2} \Gamma_B(p(-p))\Big|_{p=0}\Big]^{-2} \Gamma_B(000q),$$

$$(4.12b)\quad \tilde{m}^2 = \Big[\frac{\partial}{\partial p^2} \ln(-\Gamma_B(p(-p)))\Big|_{p=0}\Big]^{-1}.$$

Here $\tilde{m}^2$ is the, for Euclidean computations convenient, second-moment definition of mass, and $\tilde{g}$ is a normalization-independent renormalized coupling constant. Inserting (4.10) into (4.12) and omitting at first the a^2 parts, indicating this by a zero subscript, we find from (3.8) that

$$(4.13a)\quad \tilde{g}_o = A(C_1, C_2),$$

$$(4.13b)\quad \tilde{m}_o^{\,2} = B(C_1, C_2)$$

with

$$(4.14a)\quad C_1 = \bar{\rho}(g_B) + \ln a,$$

$$(4.14b)\quad C_2 = \bar{\sigma}(g_B) - \ln \Delta m_B^{\,2}$$

where

$$(4.15)\quad \bar{\sigma}(\bar{g}) = \int^{\bar{g}} d\bar{g}'\, \bar{\beta}_o(\bar{g}')^{-1}\, \bar{\eta}(\bar{g}';0).$$

Solving (4.13b) for C_2 and inserting in (4.13a) gives, for dimensional reasons,

$$(4.16)\qquad \tilde{g}_0 = \tilde{\rho}^{-1}\left(\bar{\rho}(g_B) + \ln(a\tilde{m}_0)\right)$$

It is not difficult to show that $\tilde{\rho}(\tilde{g}) = \int^{\tilde{g}} d\tilde{g}'\,\tilde{\beta}(\tilde{g}')^{-1}$, where $\tilde{\beta}$ is the β-function to "intermediate" renormalization specified by $\tilde{\Gamma}(00) = -\tilde{m}^2$, $[\partial/\partial\mu^2]\tilde{\Gamma}(\mu(-\mu))|_{\mu=0} = -1$, $\tilde{\Gamma}(0000) = -\tilde{g}$. Therefore, in (4.16) only the function $\bar{\rho}(g_B)$, or $g(g_B) = f_0^{-1}(g_B)$ of (4.5), depends on the regularization chosen.

Including also the a^2 corrections, one finds from (4.10) after an easy calculation that

$$(4.17)\qquad \tilde{g}(g_B, a\tilde{m}) = \tilde{\rho}^{-1}\left(\bar{\rho}(g_B) + \ln(a\tilde{m})\right) + (a\tilde{m})^2 \sum_{i=4}^{10} \bar{c}_i(g_B) \sum_{j=4}^{10} W_{ij}(g(g_B)) \cdot F_j\left(\tilde{\rho}^{-1}\left(\bar{\rho}(g_B) + \ln(a\tilde{m})\right)\right) + O\left((a\tilde{m})^4 (\ln a)^{\cdot}\right)$$

where, besides $g(g_B)$ or $\rho(g_B)$, only the $\bar{c}_i(g_B)$ depend on the regularization chosen. Now observe that, with $\tilde{m}a = \xi^{-1}$,

$$(4.18a)\qquad \tilde{\rho}^{-1}\left(\bar{\rho}(g_B) - \ln\xi\right) \equiv \tilde{g}_{\log} = b_0\,\overline{\ln\xi}^{\,-1} + b_0\,\overline{\ln\xi}^{\,-2}\left[\bar{\rho}(g_B) - b_0^{-2} b_1 \ln\left(b_0\,\overline{\ln\xi}\,(\ln\xi)^{-1}\right)\right] + O\left(\overline{\ln\xi}^{\,-3}\right)$$

where

$$(4.18b)\qquad \overline{\ln\xi} = \ln\xi + b_0^{-2} b_1 \ln\ln\xi,$$

such that it goes to zero as $a \searrow 0$. For the $\bar{c}_i$ and W^{-1}_{ij} in (4.17), only the power series in g_B are (in principle) available. From (4.9), (3.16), (3.17), and (4.12) we finally get

$$(4.19) \quad \tilde{g}(g_B,\xi) = \tilde{g}_{log} +$$

$$+ \xi^{-2}\left[\frac{1}{6}\tilde{g}_{log} - \frac{5}{84} g_B^{-3}\tilde{g}_{log}^{4} - \frac{3}{28} g_B^{-\frac{2}{3}}\tilde{g}_{log}^{5/3} + \ldots\right] +$$

$$+ O(\xi^{-4}(\ln\xi)^{\cdot})$$

where in the square bracket only terms with at least one more factor g_B or $\tilde{g}_{log}$ relative to a kept one are omitted. Since the estimate (4.18) is only meaningful for $\xi \gg 1$ such that $\tilde{g}_{log} < g_B$, (4.19) is well behaved.

The interest of these corrections formulae is: Assume that $\tilde{g}$ is inside the region next to the origin where $\beta(\tilde{g})$ is positive ($\tilde{g}$ is always positive due to the Lebowitz inequality). Then $\tilde{\rho}^{-1}$ is a monotonic function of its argument. Therefore, if in (4.18) $\tilde{g}_{log}$ has a maximum at g_B' for some fixed ξ, it has a maximum at g_B' for any fixed ξ provided $\tilde{g}$ stays in the mentioned region. If the observed $\tilde{g}(g_B,\xi)[2]$ does not behave so, this can only happen due to the correction terms in (4.17). (4.19) shows, however, that these corrections can be estimated quite well and are $O(\xi^{-2}(\ln\xi)^{-1})$ for $\xi \to \infty$.

If, on the other hand, $\tilde{g}$ is not monotonic in g_B and if this cannot be ascribed to corrections to $\tilde{g}_{log}$, then $\bar{\rho}(g_B) = \rho(g(g_B))$ is not monotonic in g_B. Due to

$$\bar{\beta}(g_B) = \left\{[\partial/\partial g_B]\,\bar{\rho}(g_B)\right\}^{-1} =$$

$$= \beta(g(g_B))\left[\frac{\partial}{\partial g_B} g(g_B)\right]^{-1}$$

this would mean, if the β-function is positive, that $\bar{\beta}(g_B)$ has at that g_B a pole of first (or higher odd) order if $\bar{\rho}(g_B)$ is differentiable there. Such behaviour was indeed already found by Wilson [14] on the basis of nine-terms high-temperature series, and this feature was confirmed by Baker and Kincaid [2] with ten-terms series. Neither authors found a Gell-Mann-Low eigenvalue g_B'' defined by $\lim_{\xi\to\infty} \tilde{g}(g_B'',\xi) = \tilde{g}_\infty > 0$, or $\bar{\rho}(g_B'') = +\infty$, and such g_B'' is needed for a nontrivial continuum ϕ_4^4 theory (or nontrivial continuum Ising$_4$ model, which would require $g_B'' = +\infty$ to be such fixed point) to exist, at least as a limit of the lattice-regularized theory (1.1). - However, Wilson ([14], and discussion remark at the lecture) has remarked that (1.1) may not be a suitable starting point for finding a fixed point $\tilde{g}(\xi = \infty)$ different from the vanishing Gaussian one.

5. DISCUSSION

Anyone familiar with the idea and practice of "asymptotic freedom" (AF) [15] will recognize that the assumptions that led to (4.17-19) are formally identical with the assumptions that underlie AF. AF of nonabelian gauge theory is not understood physically, however, but (so far) merely an outgrowth of the perturbation theoretical formalism. Thus, failure of (4.17-19) to account for non-perturbation theoretical results where these formulae should do so would indicate a serious flaw in the AF reasoning. Note, again, that (4.17-19) do not presuppose a nontrivial continuum ϕ^4_4 theory to exist.

The method of effective Lagrangeans applies directly to all renormalizable theories regularized on the lattice such as to have vertices and propagators. For Abelian gauge theory such form has been discussed by Sharatchandra [17]. An equivalent discussion for nonabelian gauge theory has not been given yet; however, the questions one is interested in there are not in the realm of perturbation theory, and the cutoff problem is supposed to be not acute due to AF.

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