DEUTSCHES ELEKTRONEN-SYNCHROTRON DESY

DESY 80/03
January 1980

PROPERTIES OF LATTICE GAUGE THEORY MODELS AT LOW TEMPERATURES

by

Gerhard Mack

II. Institut für Theoretische Physik der Universität Hamburg

NOTKESTRASSE 85 · 2 HAMBURG 52

DESY 80/03
January 1980

Properties of lattice gauge theory models at low temperatures

Gerhard Mack
II.Institut für Theoretische Physik
der Universität Hamburg, Germany

Abstract: The Z(N) theory of quark confinement is discussed and how fluctuations of Z(N) gauge fields may continue to be important in the continuum limit. Existence of a model in four dimensions is pointed out in which confinement of (scalar) quarks can be shown to persist in the continuum limit. This article is based on the author's Cargèse lectures 1979. Some of its results are published here for the first time.

to appear in: "Recent progress in gauge theories", G.'t Hooft et al., eds., Plenum Press, New York (1980).

PROPERTIES OF LATTICE GAUGE THEORY MODELS AT LOW TEMPERATURES

Gerhard Mack
II. Institut für Theoretische Physik
Universität Hamburg, Germany.

INTRODUCTION

In quark confinement physics, the center of the gauge group plays a crucial role.[1] This can be seen from a proper formulation of the problem. One would like to explain

(1) Quark confinement: There are no physical states with the flavor quantum numbers of a quark = all physical particles have integral baryon number etc.,

(2) Saturation of forces: The known physical hadrons are made of three quarks, or a quark and an antiquark, but not six or nine quarks etc..

The center $\Gamma = Z(3)$ of the gauge group $G = SU(3)$ of quantum chromodynamics consists of matrices γ^n, $n = 0, 1, 2$, with

$$\gamma = \exp 2\pi i \lambda_8/\sqrt{3} = e^{2\pi i/3}\mathbf{1} \tag{1.1}$$

Quark and gluon field transform under γ according to

$$q(x) \longrightarrow e^{2\pi i/3} q(x) \quad ; \quad A_\mu(x) \longrightarrow A_\mu(x) \tag{1.2}$$

The U(1) symmetry group generated by baryon number B consists of elements $\exp i\varphi B$ with $0 \leq \varphi < 6\pi$. They act on quark and gluon fields according to

$$q(x) \longrightarrow e^{i\varphi/3} q(x) \quad ; \quad A_\mu(x) \longrightarrow A_\mu(x) . \tag{1.3}$$

By comparing with (1.2) we see that

$$\gamma = e^{2\pi i B} \quad , \tag{1.4}$$

since the action of both sides on quark and gluon fields, and therefore also on all physical states, is identical. In conclusion, the combined symmetry group is

$$(SU(3)_{local} \times U(1)_{global})/Z(3)_{global} \tag{1.5}$$

and quark confinement in quantum chromodynamics as formulated above means that all physical states transform trivially under global (x-independent) transformations in the center of the gauge group.

It is instructive to imagine a world with quarks that transform as octets, decuplets, or any other representation of the gauge group SU(3) that restricts to a trivial representation of the center. Such quarks would come free, i.e. physical states with their flavor quantum numbers would exist. From the work of Kogut and Susskind[2] we know that this is true even in the strong coupling limit of a lattice theory, where quark triplets are confined[3] (cp. Glimms lectures at this school[4]). The reason is simple. A string between such hypothetical quarks can break by creation of gluon pairs.

In these lectures we will study pure Yang Mills theories on a Euclidean lattice in three and four dimensions. Often we will take $G = SU(2)$ as our gauge group. Its center $\Gamma = Z(2)$. The Euclidean lattice formulation makes methods of classical statistical mechanics applicable. Choosing an SU(2) gauge group brings about some technical simplifications. Omission of dynamical quarks is a possibly serious mutilation though. It has not been proven that dynamical quarks cannot play an important role in their own confinement. There are models[5] in which charged scalar fields help in stabilizing the confining "high temperature phase" of the model. One of them will be considered briefly in the next section. In quantum chromodynamics one could think of Cooper pair formation to get scalars[6,7]. But saturation of forces is left unexplained in such a scheme. One hopes therefore that gluons, which are charged in a nonabelian theory, do the stabilizing job themselves. One hopes also that understanding of the phase structure and universality and stability properties of a pure Yang Mills theory will provide the tools to justify perturbative treatment of dynamical quarks, at least if they are not massless.

In view of what was said earlier, the idea emerges naturally to look at the SU(N) theory as a kind of Z(N) gauge theory. Such "reduction of the gauge group" will be an important theme in these lectures. A Z(N) theory of quark confinement provides a natural qualitative explanation of saturation of forces because, roughly speaking, only objects with nonzero N-ality can be confined by Z(N) gauge quanta[8]. It is not so obvious at first sight how fluctuations

of Z(N) variables may continue to be important and confine static quarks in the continuum limit, though. (Indeed, in the standard Z(2) model on a lattice these variables freeze and this leads to a phase transition to a nonconfining phase.) An answer to this question will be proposed in section 10. It will be seen there that the values of the relevant Z(2) variables in a SU(2) theory are determined by topological properties of a SO(3) gauge field in this limit. ($SO(3) = SU(2)/Z(2)$).

2. A HIGGS MODEL WITH PERSISTENT CONFINEMENT OF THE (SCALAR) QUARKS

On a lattice, quarks which transform nontrivially under the center of the gauge group are confined at high temperatures, i.e. for small values of $\beta = 4/g^2$, g = bare coupling constant. One hopes for a smooth transition with persistent confinement to a continuum limit. The continuum limit is supposed to be reached when $\beta \to \infty$. There exists a model[6] in four dimensions in which this hope can be shown to materialize. It is one of the models that were investigated by Fradkin and Shenker[5] and others [9].

The model differs from quantum chromodynamics (QCD) in that the matter fields are scalar rather that fermion fields, and there is a buildt-in mass scale (other than the lattice spacing). The gauge group is SU(2). In addition there is a global SU(2) symmetry which we call isospin. It takes the place of the baryon U(1) in QCD. The combined symmetry group is

$$(SU(2)_{local} \times SU(2)_{global})/Z(2)_{global} \tag{2.1}$$

in place of (1.5). The scalar quarks transform nontrivially under the center Z(2) of the gauge group. They carry isospin $\frac{1}{2}$. By quark confinement we mean again absence of physical states with the flavor quantum numbers of the scalar quarks. This will be true if all physical states have integral isospin. This is again equivalent to the requirement that all physical states transform trivially under the center of the gauge group.

The model possesses a continuum limit in which quark confinement in this sense persists. It describes an isotriplet of free massive vector mesons.

The Euclidean action $L(U, \varphi)$ of the model is a function of string bit variables $U(b) \in SU(2)$ which are attached to links b of the 4-dimensional lattice, and of a doublet of complex scalar fields

$$\varphi(x) = \begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \end{pmatrix} \quad ; \quad |\varphi_1(x)|^2 + |\varphi_2(x)|^2 = 1 \tag{2.2}$$

It splits in the usual way,

$$L(U, \varphi) = L_G(U) + L_M(U, \varphi) \quad . \tag{2.3}$$

The kinetic term L_G for the gauge field has the usual form[3] for an SU(2) lattice gauge theory. The boundary $\dot{p} = \partial p$ of a plaquette consists of four links $b_1 \dots b_4$, and we write

$$U(\dot{p}) = U(b_4)\dots U(b_1) \tag{2.4}$$

In this notation (tr = trace)

$$L_G(U) = \frac{\beta}{2} \sum_p \mathrm{tr}(\, U(\dot{p}) - 1\,) \tag{2.5}$$

Sum is over all unoriented plaquettes of the lattice.

The matter part L_M takes its most convenient form when expressed in terms of SU(2) matrices

$$\Phi(x) = \begin{pmatrix} \varphi_1(x) & -\bar{\varphi}_2(x) \\ \varphi_2(x) & \bar{\varphi}_1(x) \end{pmatrix} \tag{2.6}$$

In this notation

$$L_M(U, \varphi) = \varkappa \sum \mathrm{tr}(\Phi(x)^* U(xy)\Phi(y) - 1) \quad . \tag{2.7}$$

The parameter $\varkappa > 0$ introduces a mass scale into the theory.

Under a gauge transformation $V_1(x)$,

$$\Phi(x) \longrightarrow V_1(x)\Phi(x) \tag{2.8a}$$

$$U(xy) \longrightarrow V_1(x)U(xy)V_1(y)^{-1} \tag{2.8b}$$

(2.8a) says that the matter field $\varphi(x)$ transforms as a doublet under gauge transformations.

From Eq.(2.7) we see that the model has in addition another global SU(2) symmetry. We call it isospin. Isospin rotations act according to

$$\Phi(x) \longrightarrow \Phi(x)V_2^{-1} \tag{2.9a}$$

$$U(xy) \longrightarrow U(xy) \tag{2.9b}$$

for $V_2 \in SU(2)$ independent of x.

If $V_2 = -1$ then it commutes with all SU(2) matrices. Its action on Φ, U agrees therefore with the action of a global gauge transformation $V_1(x) \equiv -1$ in the center Z(2) of the gauge group. The combined symmetry group is therefore as indicated in (2.1).

To visualize the action of the symmetry transformations, it is convenient to regard $\varphi(x)$ as a real unit 4-vector whose components are real and imaginary parts of $\varphi_1(x)$ and $\varphi_2(x)$. Both the gauge transformations (2.8a) and the isospin rotations (2.9a) act as rotations on this 4-vector. They are different unless $V_1 = V_2 = -1$. The rotations are specified by the isomorphism

$$(SU(2) \times SU(2))/Z(2) \approx SO(4) \tag{2.10}$$

4-vectors transform according to the representation $(\frac{1}{2},\frac{1}{2})$ of SO(4). It restricts to the real representation $(\frac{1}{2})\oplus(\frac{1}{2})$ on each of the two SU(2) ideals. Thus our matter field $\varphi(x)$ carries isospin $\frac{1}{2}$. The gluon field carries no isospin according to (2.9b).

It was shown by Fradkin and Shenker that there exist values $\beta_o > 0$ and $k_o < \infty$ such that the model admits convergent cluster expansions whenever either

$$0 \leq \beta < \beta_o \quad \text{or} \quad \frac{\varkappa}{\beta} > k_o \quad . \tag{2.11}$$

The two areas overlap. Convergence of the cluster expansions ensures analyticity of the free energy and correlation functions, and uniqueness and invariance of the ground state under isospin rotations. For small β and $\varkappa$ one can also find a complete set of eigenstates of the transfer matrix (Hamiltonian) which retain finite energy in the infinite volume limit. They all carry integral isospin, therefore quarks are confined in the sense of our definition. It is reasonable to believe (and ought to be proven) that this remains valid throughout the domain (2.11).

The cluster expansions are obtained[5] by first transforming to gauge invariant variables W(b) given by

$$W(xy) = \Phi(x)^* U(xy)\Phi(y) \quad , \quad \text{whence} \quad \mathrm{tr}U(\dot{p}) = \mathrm{tr}W(\dot{p}) \quad . \tag{2.12}$$

Subsequently one writes

$$\exp\frac{\beta}{2}\mathrm{tr}(W(\dot{p}) - 1) = 1 + f_p(W) \tag{2.13}$$

and expands in products of f's. Results are expressed in terms of cluster integrals A(P),

$$A(P) = \int \prod_b d\nu(W(b)) \prod_{p\in P} f_p(W) \tag{2.14}$$

$$d\nu(W) = z^{-1} dW \exp\varkappa \mathrm{tr}W \quad ; \quad z = \int dW\, e^{\varkappa \mathrm{tr}\, W}$$

dW is normalized Haar measure on SU(2). P are connected collections of plaquettes.

Given a function A of P, one defines the ξ-norms for $\xi > 1$ by[10,11]

$$\|A\|_\xi = \max_b \frac{1}{\xi} \left(1 + \sum_{\substack{P \\ b \in P}} |A(P)| \, \xi^{|P|}\right) \tag{2.15}$$

$b \in P$ means $b \in \dot{p}$ for a plaquette p in P. $|P|$ is the number of plaquettes in P. It is well known[11] that cluster expansions for free energy, correlation functions etc. converge if there is $\xi > 1$ such that $\|A\|_\xi < 1$. This is fulfilled in the range (2.11) of parameters; ξ can be chosen to depend only on $k = \varkappa/\beta$. To prove this, one estimates A(P) by use of the inequality

$$\mathrm{tr}\,(W(\dot{p}) - 1) \geq 4 \sum_{b \in \dot{p}} \mathrm{tr}\,(W(b) - 1) \tag{2.16}$$

Summation is over all four links b in the boundary $\dot{p}$ of a plaquette p.

A continuum limit of the model can be approached in two steps. First one lets

$$\beta \to \infty \, , \, \varkappa \to \infty \quad \text{with} \quad \frac{\varkappa}{\beta} = k \quad \text{fixed and} > k_o \, . \tag{2.17}$$

In this way one stays inside the domain (2.11) of validity of the cluster expansions. For fixed $k > k_o$, their convergence is uniform in β. In this limit one obtains a free field theory on the lattice, with a mass determined by k. Convergence to a free field theory can be proven by showing that

$$\|A(P) - A_o(P)\|_\xi \to 0 \quad \text{as} \quad \beta \to \infty \, . \tag{2.18}$$

where $A_o(P)$ are the cluster integrals for a free field theory. This implies convergence of the free energy and of all correlation functions.

Subsequently one lets $k \to 0$, so that the correlation length in units of the lattice spacing becomes infinite. Such a continuum limit of a free field theory presents no problems. It suffices to inspect the two point functions. As a result one obtains a theory of free massive vector mesons. They are described by a triplet $B^c_\mu(x)$ of vector fields which are related to the W-variables on the lattice by

$$W(b) = \exp -igaB(b) \quad ; \quad B(b) = \sum_c B^c_\mu(x) \frac{\tau^c}{2} \tag{2.19}$$

τ^c are Pauli matrices, $g = (4/\beta)^{1/2}$, $a = 1$ in units of lattice spacing, and b is the link which leaves x in the μ-direction. It follows that these fields carry isospin one since they transform under isospin rotations according to

$$B^c_\mu(x) \to R(V_2)^c{}_d B^d_\mu(x) \tag{2.20}$$

$R(V_2)$ is the SO(3) rotation associated with the element V_2 of SU(2). In conclusion, there are no physical states with the flavor quantum numbers of the <u>quarks</u> in the continuum limit - they <u>remain confined</u>.

3. DESCRIPTION OF SU(2) MODELS

From now on we restrict our attention to pure Yang Mills theories without matter fields, with a gauge group G that possesses a nontrivial center Γ. The action depends on variables $U(b) \in G$ that are attached to links b, $U(b) \to U(b)^{-1}$ under reversal of the direction of the link b. If C is a path consisting of links $b_1 \ldots b_n$ one defines the parallel transporter U(C) along C by

$$U(C) = U(b_n) \ldots U(b_1) \tag{3.1}$$

In the special case (2.4), $C = \dot{p}$ is the boundary of a plaquette p. The action will be taken to be of the form

$$L(U) = \sum_p \mathcal{L}(U(\dot{p})) \tag{3.2}$$

Sum is over all unoriented plaquettes. $\mathcal{L}$ is supposed to be bounded above, real , and it must satisfy $\mathcal{L}(V) = \mathcal{L}(V^{-1}) = \mathcal{L}(V_1 V V_1^{-1})$ to ensure gauge invariance.

In the standard SU(2) model of Wilson[3]

$$\mathcal{L}(V) = \frac{\beta}{2}\,\mathrm{tr}\,(V - 1) \tag{3.3}$$

and the path measure is given by

$$d\mu(U) = \frac{1}{Z} e^{L(U)} \prod_b dU(b) \quad . \tag{3.4}$$

dU(b) is normalized Haar measure on SU(2).

We will also consider a <u>modified model</u> proposed by Petkova and the author[12]. It has the same Lagrangean (3.3), but the admissible configurations U are restricted by the constraint

$$\prod_{p \in \dot{c}} \mathrm{tr}\, U(\dot{p}) > 0 \quad \text{for every elementary cube c.} \tag{3.5}$$

Product is over the six plaquettes in the boundary $\dot{c}$ of a 3-dimensional cube. As a result, the path measure takes the form

$$d\mu(U) = \frac{1}{Z} e^{L(U)} \prod_c \Theta\Big(\prod_{p \in \dot{c}} \mathrm{tr}\, U(\dot{p})\Big) \prod_b dU(b) \tag{3.6}$$

For either model, the expectation value of an observable $F(U)$ is given by

$$\langle F \rangle = \int d\mu(U)\ F(U) \qquad (3.7a)$$

We are particularly interested in the expectation value of the Wilson loop observable

$$\langle \mathrm{tr}\ U(C) \rangle = \int d\mu(U)\ \mathrm{tr}\ U(C) \qquad (3.7b)$$

where C is a rectangular path enclosing an area of $L\cdot T$ plaquettes. The partition function Z is always defined by the requirement that $\langle 1 \rangle = 1$.

In the modified model, it is most convenient to impose cyclic boundary conditions on the cosets $\bar{U}(b) = U(b)\Gamma \in SO(3)$ only.

The modified model has formally the same continuum limit as the standard model. In the continuum limit $\beta \to \infty$ one has in either model

$$U(\dot{p}) \to 1 \quad \text{as} \quad \beta \to \infty \qquad (3.8)$$

It follows that constraint (3.5) is almost always fulfilled in the standard model for any particular cube c, in the limit. The limit behavior (3.8) is to be understood <u>in the following probabilistic sense</u>.

Let P be any collection of n distinct plaquettes $p_1 \dots p_n$. Then the probability $p(\xi_1,\dots,\xi_n)$ that $\mathrm{tr}(U(\dot{p}_j)-1) \leq -\xi_j$ for all $j=1\dots n$ is bounded by

$$p(\xi_1,\dots,\xi_n) \leq \prod_j D(\beta, \xi_j) \qquad (3.9)$$

$$D(\beta, \xi) = \mathrm{const}\cdot\beta\ e^{-\beta\xi/48} \to 0 \quad \text{as} \quad \beta \to \infty \quad \text{if } \xi \neq 0.$$

This implies (3.8) because $\mathrm{tr}(U(\dot{p})-1) \leq 0$, and $= 0$ only if $U(\dot{p}) = 1$. Inequalities (3.8) derive from chessboard estimates[13,12]. The factor 1/48 in the exponent is probably far from optimal.

In a later section, we shall want to compare the SU(2) models with the standard Z(2) gauge theory model[14,15]. It has variables $\sigma(b) = \pm 1$ and action

$$L(\sigma) = \sum_p \beta(\sigma(\dot{p}) - 1) \qquad (3.10)$$

It is obtained from the SU(2) models with the same value of β by restricting variables $U(b)$ to the center of the gauge group. The constraint (3.5) is then automatically fulfilled.

4. THE 'T HOOFT DISORDER PARAMETER

In a pure Yang Mills theory on a lattice, the quantum field theoretic Hilbert space of physical states consists of wave functions $\Psi(U)$. They depend on variables $U(b)$ that are attached to links b in the (Euclidean) time t=0 hyperplane Σ. The scalar product of two such wave functions Ψ_1 and Ψ_2 is of the form

$$(\Psi_1,\Psi_2) = \int d\rho(U)\ \bar{\Psi}_1(U)\ \Psi_2(U) \qquad (4.1a)$$

One may take $d\rho(U) = \prod dU(b)$ (product over all links b in Σ). For our purposes it is however more convenient to extract a common multiplicative factor from all wave functions so that

$$d\rho(U) = \frac{1}{Z} \prod_{b\in\Sigma} dU(b)\ \exp \sum_{p\in\Sigma} \mathcal{L}(U(\dot{p})) \qquad (4.1b)$$

The vacuum state is given by

$$\Omega(U) = \int \prod_{b>0} dU(b)\ \exp \sum_{p>0} \mathcal{L}(U(\dot{p})) \qquad (4.2)$$

$b > 0$ resp. $p > 0$ are all links resp. plaquettes in the half space $t > 0$, excluding those in Σ. Note that $U(\dot{p})$ may involve variables $U(b)$ with $b\in\Sigma$, cp. (2.4). These are not integrated over, instead Ω depends on them.

Observables $F(U)$ which depend only on variables $U(b)$ with $b\in\Sigma$ act on wave functions Ψ as multiplication operators. One has in this case

$$\langle F \rangle = (\Omega, F\Omega) \qquad (4.3)$$

They form a complete set of commuting operators.

The formulae given above are appropriate for the standard SU(2) model, etc. For the modified model, step functions must be included which restrict integration to configurations which satisfy constraint (3.5).

Let S be any set of links b in the time t=0 hyperplane Σ, and σ an element of the center Γ of the gauge group G. The 't Hooft operator $B_\sigma(S)$ is defined by its action on wave functions[16]

$$(B_\sigma(S)\Psi)(U) = \Psi(U_\sigma) \qquad (4.4)$$

with

$$U_\sigma(b) = \begin{cases} U(b)\sigma^{-1} & \text{if } b \in S \\ U(b) & \text{otherwise} \end{cases}$$

For $\Gamma = Z(2)$ there is only one nontrivial element $\sigma = -1$ of Γ, and we write $B(S)$ in place of $B_{-1}(S)$ in this case. The 't Hooft disorder parameter is the expectation value of $B_\sigma(S)$,

$$\langle B_\sigma(S)\rangle = (\Omega, B_\sigma(S)\Omega) \quad . \tag{4.5}$$

It is convenient to use the coboundary operator $\hat{\partial}$ (= boundary operator on the dual lattice)[12,17]. Let us restrict attention to the ν-1 dimensional lattice Σ for a moment, and suppose that $\Gamma = Z(2)$ from now on. If S is a set of links in Σ then $\hat{\partial}S$ consists of those plaquettes p in Σ which have an odd number of links $b\varepsilon S$ in their boundaries. One is mainly interested in S, $\hat{\partial}S$ of the form shown in figure 1. Because of gauge invariance, $\langle B_\sigma(S)\rangle$ turns out to depend on S only through $\hat{\partial}S$.

As a consequence of its definition, the 't Hooft operator satisfies the following commutation relations with the multiplication operator tr U(C) for closed loops C in Σ ('t Hooft algebra[16])

$$B(S)\ \mathrm{tr}\ U(C) = \xi\ \mathrm{tr}\ U(C)\ B(S) \quad , \quad \xi = \pm 1 \quad \text{if} \quad \Gamma = Z(2) \, .$$

Let C be the boundary of a surface Ξ in Σ. Then $\xi = -1$ if Ξ contains an odd number of plaquettes in $\hat{\partial}S$, and $\xi = +1$ otherwise. ξ counts thus how many times (mod 2) $\hat{\partial}S$ winds around C.

It follows from the definitions (4.4), (4.5) and the explicit formula (4.2) for the vacuum wave function that[12]

$$\langle B(S)\rangle = \langle \exp \sum_{b\varepsilon S} \{ \mathcal{L}(-U(\dot{p}_b)) - \mathcal{L}(U(\dot{p}_b)) \} \rangle \quad \geq 0 \, . \tag{4.6}$$

p_b is the plaquette which protrudes from link $b\varepsilon\Sigma$ in positive time direction.

A duality transformation reveals[12,42] that $\langle B(S)\rangle$ may be interpreted as expectation value of a Wilson loop operator for a small static Z(2) monopole (of size 1 lattice spacing cubed) in ν=4 dimensions (cp. section 8). The loop is $\hat{\partial}S$, this is indeed a closed loop on the dual lattice of Σ (see figure 1). It will be seen in section 10 that both the modified and the standard SU(2) model may be interpreted as Z(2) gauge theories without dynamical Z(2) monopoles. Therefore the asymptotic behaviour of $\langle B(S)\rangle$ for large loops $\hat{\partial}S$ will determine whether small static Z(2) monopoles are confined or not.

All the formulae of this section are also valid for a theory with gauge group $G = \Gamma = Z(2)$, in particular for the standard Z(2) model of section 3.

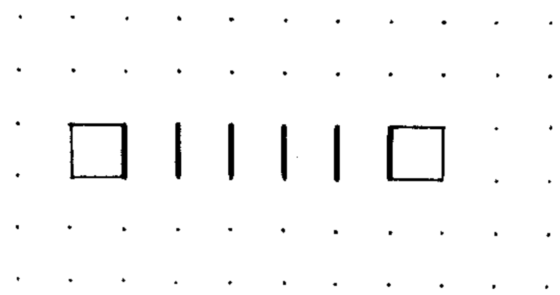

Fig.1a

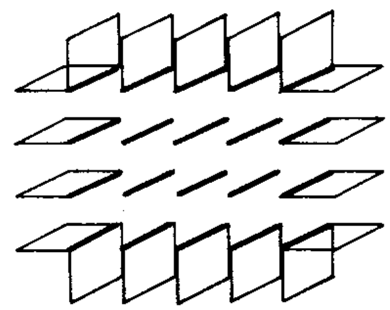

Fig.1b

Fig.1. Argument S of the 't Hooft operator B(S) in the time zero hyperplane Σ. Sets S of links (heavy lines), and plaquettes in $\hat{\partial}S$ (squares).
(a) for a $\nu-1 = 2$ dimensional lattice Σ.
(b) for a $\nu-1 = 3$ dimensional lattice Σ.

5. INEQUALITIES RELATING THEORIES WITH GAUGE GROUPS SU(2) AND Z(2)

We consider the SU(2) models described in section 3 and compare them with the standard Z(2) model, with the same value of β in the two Lagrangeans (3.3) and (3.10). The expectation values of the Wilson loop observables in these SU(2) and Z(2) models are related by the inequality

$$|\langle \mathrm{tr}\, U(C)\rangle_{SU(2)}| \leq 2\langle \sigma(C)\rangle_{Z(2)} \quad (5.1)$$

The factor of 2 comes about because $\mathrm{tr}\, 1_2 = 2$. This inequality was first derived by Petkova and the author[12] for the modified SU(2) model. It was subsequently generalized to arbitrary SU(N) models by Fröhlich[18]. It follows from (5.1) that the Wilson loop expectation value shows an area law decay in the SU(2) theory whenever the same is true in the Z(2) theory. According to Wilson[3], an area law decay implies that static quarks are confined.

There is also an inequality for the 't Hooft disorder parameter. It goes in the opposite direction.

$$\langle B(S)\rangle_{SU(2)} \geq \langle B(S)\rangle_{Z(2)} \geq 0 \quad (5.2)$$

This inequality was also first derived for the modified SU(2) model by Petkova and the author[12]. The proof was subsequently generalized to the standard SU(2) model by Korthals Altes[19].

To prove (5.1) one divides the configurations U into classes[20]. Configurations U_1 and U_2 will be in the same class if there exist $\sigma(b) = \pm 1$ for each link b such that $U_1(b)$ and $U_2(b)\sigma(b)$ are related by a gauge transformation. We may in some way select a representative W in each class, and label the classes by W. We imagine that the expectation value $\langle \mathrm{tr}\, U(C)\rangle$ is computed by first computing the average $\langle \mathrm{tr}\, U(C)\rangle_W$ in each class, and subsequently averaging over W also. In probability theory the result of the first step is called a <u>conditional expectation value</u> (see e.g. ref. 41).

The conditional expectation values $\langle\cdot\rangle_W$ are computed by averaging over the variables σ which relate configurations in the same class. (The average over the gauge group is trivial.) Explicitly,

$$\langle \mathrm{tr}\, U(C)\rangle_W = \mathrm{tr}\, W(C)\left\{Z_W^{-1}\int \prod_b d\sigma(b)\; \sigma(C)\, \exp \sum_p K_p(W)\sigma(\dot{p})\right\} \quad (5.3)$$

with

$$K_p(W) = \frac{\beta}{2}\, \mathrm{tr}\, W(\dot{p}) \quad (5.4)$$

$d\sigma$ is normalized Haar measure on Z(2), viz.

$$\int d\sigma\, (\ldots) = \frac{1}{2}\sum_{\sigma=\pm 1} (\ldots)\ , \quad (5.5)$$

and

$$Z_W = \int \prod_b d\sigma(b)\, \exp \sum_p K_p(W)\sigma(\dot{p})\ . \quad (5.6)$$

The expression in { } in (5.3) is expectation value of the Wilson loop in a Z(2) gauge theory with space time dependent coupling constants $K_p(W)$. Because of Eq.(5.4) they are bounded by

$$|K_p(W)| \leq \beta \quad (5.7)$$

In the modified SU(2) model one can choose W so that $K_p(W) \geq 0$. It follows then from the second Griffiths inequality[21] that the expectation value $\langle\sigma(C)\rangle$ is decreased when one replaces coupling constants K_p by β. As a result

$$|\langle \mathrm{tr}\, U(C)\rangle_W| \leq 2\langle\sigma(C)\rangle_{Z(2)} \quad \text{for all } W\ . \quad (5.8)$$

There are generalized Griffiths inequalities[22] which assert that (5.8) follows from (5.7) even if K_p are not necessarily nonnegative. The result generalizes therefore to the standard SU(2) model[18,19].

Finally one can average over the classes to obtain $\langle \mathrm{tr}\, U(C)\rangle$. Inequalities (5.8) imply (5.1) because

$$|\langle \mathrm{tr}\, U(C)\rangle| \leq \sup_W |\langle \mathrm{tr}\, U(C)\rangle_W| \quad (5.9)$$

The other inequalities (5.2) were derived in ref. 12 by first performing a duality transformation on the Z(2) variables σ, followed by a similar argument as above. The direction of the inequality (5.2) is the reverse of that in (5.1) because the duality transformation takes small coupling constants into large ones and vice versa.

6. PHASE TRANSITION IN THE MODIFIED SU(2) MODEL

It was shown by Petkova and the author[12] that the <u>modified SU(2) model</u> described in section 3 possesses a high temperature phase (β small) and a low temperature phase (β large) that are distinguished by a qualitatively different behavior of the 't Hooft disorder parameter. This is true both in ν=3 and 4 dimensions.

Let S be a set of $|S|$ links in the time zero hyperplane as shown in figure 1, and denote by $|\hat{\partial} S|$ the number of plaquettes in its coboundary. One finds that

$$\langle B(S)\rangle \geq \text{const}\cdot e^{-\alpha|\hat{\partial}S|} \qquad \text{for small } \beta \qquad (6.1)$$

whereas

$$\langle B(S)\rangle \leq \text{const}\cdot e^{-\alpha'|S|} \qquad \text{for large } \beta\ . \qquad (6.2)$$

In four dimensions, (6.1) implies a perimeter law, whereas (6.2) is an area law decay. It follows according to the discussion at the end of section 4 that small Z(2) monopoles are confined at low temperatures β^{-1}. This does however not imply monopole confinement in the sense in which this term is used in 't Hooft's lectures at this school[23], and one cannot conclude that quarks are not confined in the low temperature phase of our modified model.

The bound (6.1) follows from inequality (5.2) and known properties of the standard Z(2) lattice gauge theory model. (In four dimensions, a perimeter law decay (6.1) in the Z(2) model for small β is equivalent, by virtue of a duality transformation, to a perimeter law decay of the Wilson loop expectation value at large β - i.e. the nonconfinement of static quarks at low temperatures.)[14,15,26,34]

The result (6.2) is derived in several steps.

1st step. One regards the model as a Z(2) gauge theory model with fluctuating coupling constants K_p in the same manner as described in section 5. That is, one expresses $\langle B(S)\rangle$ in terms of conditional expectation values $\langle B(S)\rangle_W$. Starting from Eq. (4.6), one finds in place of Eq. (5.3)

$$\langle B(S)\rangle_W = Z_W^{-1}\int \Pi\, d\sigma(b) \left\{\exp -2\sum_{b\in S} K_{p_b}(W)\sigma(\dot{p}_b)\right\} \cdot \exp \sum_p K_p(W)\sigma(\dot{p}) \qquad (6,3)$$

Z_W and $K_p(W)$ are the same as in section 5. It is crucial for the following second step that the representatives W of the classes can be chosen so that always

$$K_p(W) \geq 0 \qquad \text{(in the modified model)} \qquad (6.4)$$

This follows from validity of the constraint (3.5) in the modified model.

It follows from standard properties of conditional expectation values that

$$\langle B(S)\rangle = \int d\mu(U)\ \langle B(S)\rangle_{W(U)} = Z^{-1}\int \prod_b dU(b)\ Z_{W(U)}\langle B(S)\rangle_{W(U)}\ . \qquad (6.5)$$

μ is the path measure defined in section 3, and W(U) is the representative of the class of configurations to which U belongs (cp. section 5).

2nd step. We observe that expression (6.3) is an expectation value in a Z(2) gauge theory with space time dependent coupling constants K_p. One performs a Kramers Wannier duality transformation[15]. In four dimensions, one obtains another Z(2) gauge theory with new coupling constants $\hat{K}_p$ that are related to the old one by

$$\hat{K}_p(W) = \frac{1}{2}\ln\coth K_p(W) \geq 0\ .$$

These new coupling constants come out real because the old coupling constants were positive by (6.4). If it were otherwise, the new model would not be statistical mechanics. The new model has variables $\omega(c) = \pm 1$ attached to cubes c (= links of the dual lattice). It is convenient to use the notation

$$\int d\omega\,(\ldots) = \sum_{\omega=\pm 1}(\ldots)$$

Expression (6.3) becomes

$$\langle B(S)\rangle_W = \hat{Z}_W^{-1}\int \prod_c d\omega(c)\ \omega(C_S)\ \exp\sum_p K_p(W)\omega(\hat{\partial}p) \qquad (6.6)$$

$\hat{\partial}p$ consists of the cubes c that have p in their boundary $\dot{c}$, and

$$\omega(\hat{\partial}p) = \prod_{p\in\dot{c}}\omega(c) \quad , \quad \text{while} \quad \omega(C_S) = \prod_{p\in\hat{\partial}S}\omega(c_p)\ . \qquad (6.7)$$

c_p is the cube protruding from plaquette p in positive time direction. $\hat{Z}_W$ is the partition function of the new model with coupling constants $\hat{K}_p$. It differs from Z_W by a certain W-dependent factor. C_S is a closed loop on the dual lattice, and expression (6.6) is therefore expectation value of a Z(2) Wilson loop in the new Z(2) model.

Remark. Under the duality transformation, Z(2) monopoles go into Z(2) quarks and vice versa[42,43]. The Wilson loop interpretation of Eq. (6.6) explains thus the remark about monopoles at the end of section 4, at least for the modified model.

3rd step. Inequalities (3.9) and expression (5.4) for $K_p(W)$ tell us that $K_p(W)$ tend to be large, of order β, for large β. It follows that $\hat{K}_p(W)$ are small, except for a set of classes (labelled by W) of small measure. This suggests to use high temperature expansions. Because of the exceptional W's, a little care is needed, though. One writes down a cluster expansion for (6.3) on a finite lattice where it is a finite sum (and converges therefore). Then one inserts into Eq. (6.5). One uses chessboard estimates to derive estimates on the individual terms of the expansion which results by integrating term by term. They are uniform in the size of the lattice. They show that the expansion continues to converge in the infinite volume limit.

4th step. One identifies the leading term of the expansion. Because of the Wilson loop interpretation of expression (6.6), the result (6.2) follows from the standard proof of the area law decay of the Wilson loop expectation value at high temperatures[3].

7. THIN VORTICES

For the sake of comparison, let us first recall some properties of the two dimensional Ising ferromagnet. Its spin variables $\sigma(x) = \pm 1$ are attached to vertices x of the lattice. A link b of the lattice has a boundary $\dot{b}$ which consists of two vertices x and y, and we write

$$\sigma(\dot{b}) = \sigma(x)\sigma(y) \tag{7.1}$$

in analogy with (2.4). Links b with $\sigma(\dot{b}) = -1$ form closed paths on the dual lattice. They separate regions with spins up from regions with spins down. These domain walls are the famous Peierls contours[24]. A configuration σ is determined by its Peierls contours up to a global spin rotation $\sigma \to -\sigma$. The two point correlation function $\langle \sigma(x)\sigma(z) \rangle$ is determined by the probability distribution of Peierls contours that wind around either x or z. Let p_N be the probability that there are N of them. Then

$$\langle \sigma(x)\sigma(z) \rangle = \sum_N (-)^N p_N \quad . \tag{7.2}$$

See figure 2a. Absence of spontaneous magnetization and exponential falloff of the correlation function obtains if long Peierls contours are abundant. This is true at high temperatures. Conversely, at low temperatures β^{-1}, long Peierls contours are very rare since they cost energy proportional to their length, and the entropy S, which is also proportional to length, cannot make up for this since its contribution to the free energy $F = E - TS$ of a contour is suppressed by a factor $T = \beta^{-1}$. Short contours winding around x cannot see the other point z in $\langle \sigma(x)\sigma(z) \rangle$. They can therefore not produce a falloff as $z \to \infty$.

The situation in the standard Z(2) lattice gauge theory model in $\nu = 3$ and 4 dimensions is quite analogous. The variables $\sigma(b) = \pm 1$ are now attached to links b. A plaquette p of the lattice has a boundary $\dot{p}$ which consists of four links $b_1 \ldots b_4$, and we write

$$\sigma(\dot{p}) = \sigma(b_4)\ldots\sigma(b_1) \tag{7.3}$$

in place of (7.1). As a consequence of this definition, for every cube c

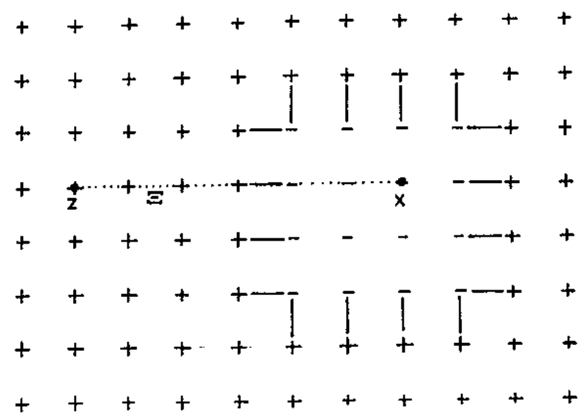

Fig.2a

Fig. 2a. Peierls contour in a two dimensional Ising ferromagnet. The contour consists of the black links. It winds around x or z if it contains an odd number of links in Ξ. (Ξ is any line from x to z.) Formula (7.2) obtains because $\sigma(z)\sigma(x) = \prod_{b \in \Xi} \sigma(\dot{b})$; $\sigma(\dot{b}) = -1$ on the contour by definition.

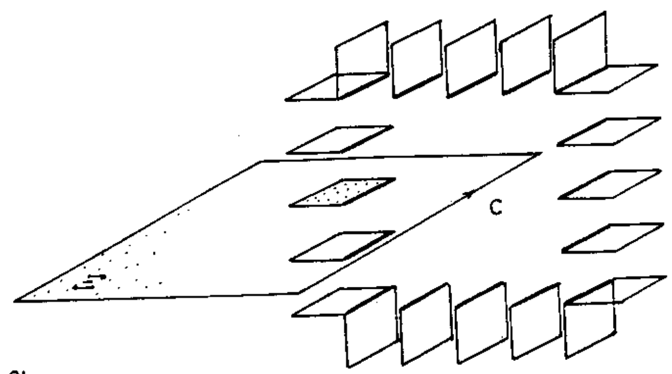

Fig.2b

Fig. 2b. Vortex in a Z(2) lattice gauge theory. Drawing for three dimensions. The vortex consists of the plaquettes shown as squares in the figure. It winds around C if it contains an odd number of plaquettes in Ξ. (Ξ is any surface with boundary C). Formula (7.5) obtains because $\sigma(C) = \prod_{p \in \Xi} \sigma(\dot{p})$; $\sigma(\dot{p}) = -1$ on the vortex by definition.

$$\prod_{p\epsilon\check{c}} \sigma(\dot{p}) = 1 \qquad (7.4)$$

Plaquettes p with $\sigma(\dot{p}) = -1$ form closed paths (ν=3) resp. surfaces (ν=4) T on the dual lattice. See figure 2b. They are called vortices. A configuration σ is determined by its vortices up to a gauge transformation.

Let C be a rectangular closed path consisting of links $b_1 \ldots b_n$, and $\sigma(C) = \sigma(b_n)\ldots\sigma(b_1)$. The Wilson loop expectation value $\langle \sigma(C) \rangle$ is determined by the probability distribution of vortices that wind around the path C. Let p_N be the probability that there are N of them. Then

$$\langle \sigma(C) \rangle = \sum_N (-)^N p_N \quad . \qquad (7.5)$$

At high temperatures, large vortices are abundant and lead to an area law decay of $\langle \sigma(C) \rangle$. At low temperatures, however, large vortices are very rare, for the same reasons as discussed after Eq. (7.2) for the Ising ferromagnet[25]. Therefore the only vortices that are relevant to (7.5) wind tightly around the path C. They are only able to produce a perimeter law decay

$$\langle \sigma(C) \rangle \geqslant \text{const}\cdot e^{-\alpha|C|} \qquad \text{for large } \beta. \qquad (7.6)$$

A proof of (7.6) along these lines has been worked out by Göpfert[26].

In order to get more familiar looking formulae it is convenient to use additive notation. One introduces vector potentials $A_\mu(x)$ which take their values in the field $\mathbb{F}_2 = \{0, 1\}$, etc. Let e_μ be the lattice vector in μ-direction, and write

$$\sigma(b) = \exp i\pi A_\mu(x) \quad , \quad \text{for} \quad b = (x, x+e_\mu) \ . \qquad (7.7a)$$

Then $\sigma(\dot{p})$ is expressed in terms of the field strength,

$$\sigma(\dot{p}) = \exp i\pi F_{\mu\nu}(x) \quad , \; F_{\mu\nu}(x) = \Delta_\mu A_\nu(x) - \Delta_\nu A_\mu(x) \qquad (7.7b)$$

if $p=p_{\mu\nu}(x)$ is the plaquette with corner points $x, x+e_\mu, x+e_\mu+e_\nu, x+e_\nu$. Δ_μ is the difference operator on the lattice. In this notation, (7.4) takes the familiar form (2nd Maxwell equation)

$$\Delta_\mu F_{\nu\lambda} + \Delta_\lambda F_{\mu\nu} + \Delta_\nu F_{\lambda\mu} = 0 \ . \qquad (7.8)$$

Consider now the intersection $\hat{\partial}S$ of a vortex with the time zero plane Σ in four dimensions. (It is closed in the dual lattice of Σ and is therefore indeed coboundary $\hat{\partial}S$ of some set S of links in Σ, compare figures 1a, 2b for the three dimensional case). By definition,

$\hat{\partial}S$ consists of spacelike plaquettes $p = p_{ij}(x)$ with $F_{ij}(x) = 1$. There is thus a quantum of magnetic flux passing through p, and $\hat{\partial}S$ is a magnetic flux loop.

A vortex is (Euclidean) world sheet of such a flux loop. From Eq. (4.4) one sees that the 't Hooft operator (for the Z(2) theory) creates a magnetic flux loop on $\hat{\partial}S$, since $U_{-1}(\dot{p}) = -U(\dot{p})$ for $p \in \hat{\partial}S$. Because of the conservation law (7.4) or (7.8) it cannot just disappear but must evolve in time into a vortex. At low temperatures large vortices are very rare, this leads to an area law decay of $\langle B(S) \rangle$ in the standard Z(2) model at low temperatures. The vortices in a Z(2) theory have a thickness of only one lattice spacing. They will therefore be called thin vortices.

Let us now turn to SU(2) theories. Thin vortices also exist in such theories[37]. We may (for instance) define a thin vortex to consist of a coclosed set T of plaquettes p such that sign tr $U(\dot{p}) = -1$ for every p in T. In the modified model, the set of all plaquettes p on the lattice with sign tr $U(\dot{p}) = -1$ is always coclosed (= closed on the dual lattice). In the standard model this is not so.

The 't Hooft operator creates a thin magnetic flux loop on $\hat{\partial}S$. In the modified model, this magnetic flux loop must evolve into a thin vortex much as in the standard Z(2) model, due to the constraint (3.5) which replaces (7.4). But large thin vortices are still very rare at low temperatures because of the large cost of energy associated with them. This leads to the area law decay (6.2) of the 't Hooft disorder parameter in the modified model.

In the standard model, the conservation law (3.5) need not hold. Therefore the flux loop created by the 't Hooft operator need not evolve into a thin vortex. This does not mean that the excitation created by the 't Hooft operator may disappear from one instant of time to the next. It still has to evolve into a more general type of vortex, with a "vortex soul" that is coclosed. This can be seen from the formalism of section 10 below. But in the standard model, a thin flux loop may spread out as time evolves. As a consequence, the scarcity of thin vorticies at low temperatures need not necessarily lead to an area law of the 't Hooft disorder parameter in this case.

In contrast with the Ising model, the domain walls in ferromagnets with continuous symmetry can spread. The spins may rotate gently across such a domain wall so that no two neighboring ones make a large angle. In two dimensions there is no spontaneous magnetization because a domain wall which surrounds a cluster of n aligned spins can lower its free energy to a value independent of n by spreading. This physical explanation has been known for a long time[27]. A mathematical proof that it works has been given by Dobrushin and Shlosman[28].

This suggests that one should consider <u>thick vortices</u> in gauge theories with continuous gauge group. In such a thick vortex, $U(\dot{p}) \approx 1$ is possible for all plaquettes p. In the next section we will derive a sufficient condition for confinement of static quarks by condensation of such thick vortices.

8. SUFFICIENT CONDITION FOR CONFINEMENT OF STATIC QUARKS

Let D be a unitary representation of the gauge group G and $\chi(\cdot) = \mathrm{tr}\, D(\cdot)$ the corresponding character. If C is a rectangular path enclosing T·L plaquettes, with $T \gg L$, then

$$\langle \chi(U(C)) \rangle \sim \mathrm{const}\cdot e^{-TV(L)} \qquad \text{for} \quad T \gg L\ , \tag{8.1}$$

in the limit of large T. According to Wilson[3], static quarks which transform according to representation D of G will be confined if $V(L) \to \infty$ as $L \to \infty$. In this section we will discuss an inequality for V(L). A sufficient condition for confinement of static quarks follows from it. This result was first obtained by Petkova and the author.[29] A similar condition was independently proposed by 't Hooft[23,30]. The condition is valid for pure Yang Mills theories. It generalizes to theories with fundamental scalar fields, provided these scalar fields transform trivially under the center Γ of the gauge group G (triplets, quintuplets etc., if G = SU(2)).

The representation D of G determines a character ω of a 1-dimensional representation of Γ.

$$D(U\gamma) = D(U)\omega(\gamma) \quad ; \tag{8.2}$$

$\omega(\gamma)$ are complex numbers of modulus 1.

Let us choose sublattices Λ_i of our total lattice Λ which wind around the path C as shown in figure 3. They will be called vortex containers. Different vortex containers are allowed to touch each other, but they may not intersect each other or the path C. We imagine that variables U(b) are fixed for the links b on the boundary $\partial\Lambda_i$ of Λ_i. We consider the <u>partition functions</u> $Z(\Lambda_i, U)$ <u>as functions of these boundary conditions</u>. In particular we consider their change when a singular gauge transformation[31,32] $U \to U_\sigma$ is applied to these boundary conditions ($\sigma \in \Gamma$).

A <u>singular gauge transformation</u> by σ acts as follows. One chooses a set T_i of links in $\partial\Lambda_i$ which is coclosed in $\partial\Lambda_i$ and winds once around C as shown in figure 3. Then

$$U(b) \to U_\sigma(b) = \begin{cases} U(b)\sigma^{-1} & \text{if } b \in T_i \\ U(b) & \text{otherwise} \end{cases} \tag{8.3}$$

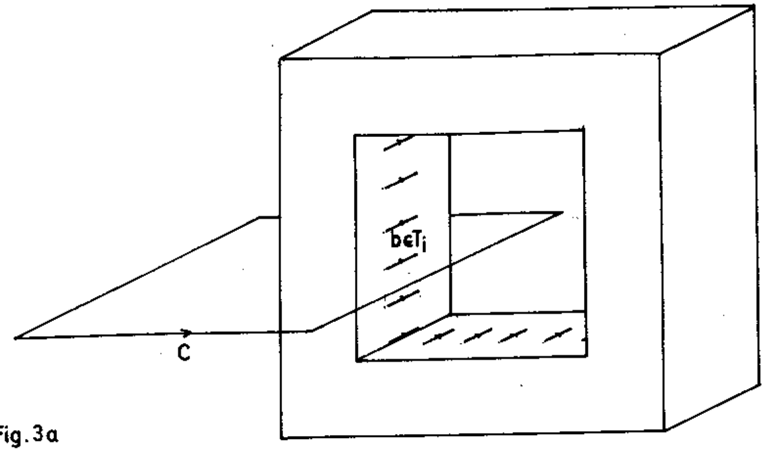

Fig.3a

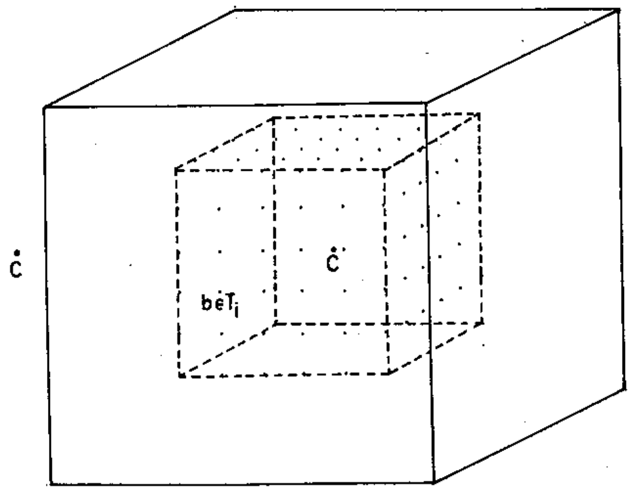

Fig.3b

<u>Fig. 3</u>. Vortex container Λ_i winding around the Wilson loop C, and the links $b \in T_i$ that appear in definition (8.3) of a singular gauge transformation; (a) in 3 dimensions, (a),(b) in 4 dimensions (intersections with the hyperplanes $x^4 = 0$, resp. $x^1 = 0$).

The precise choice of T_i is immaterial since different choices lead to transformations that differ only by ordinary gauge transformations. A singular gauge transformation on $\partial\Lambda_i$ agrees with some ordinary gauge transformation on each topologically trivial piece of $\partial\Lambda_i$, but not on all of it.

In ν dimensions, vortex containers have the topology of $I\times I\times S^{\nu-2}$ (I = interval, S^n = unit sphere in n+1 dimensions), therefore $\partial\Lambda_i$ has the topology of $S^1\times S^{\nu-2}$. (This differs from 't Hooft's choice in ≥ 4 dimensions.)[23]. T_i has the topology of $S^{\nu-2}$ (when considered as a cell complex in the dual lattice of $\partial\Lambda_i$).

Our inequalities are as follows. Consider the following normalized probability distributions on Γ, ($d\sigma$ = normalized Haar measure on Γ)

$$p_{\Lambda_i,U}(\sigma) = Z(\Lambda_i, U_\sigma)/\int d\sigma'\, Z(\Lambda_i, U_{\sigma'}) \qquad (8.4)$$

Let $\hat{p}$ be their Fourier transforms on Γ,

$$\hat{p}_{\Lambda_i,U}(\omega) = \int d\sigma\, \omega(\sigma)\, p_{\Lambda_i,U}(\sigma) \qquad (8.5)$$

Then

$$|\langle \chi(U(C))\rangle| \leq \chi(1) \prod_i \left\{\max_U |\hat{p}_{\Lambda_i,U}(\omega)|\right\} \qquad (8.6)$$

ω is the character of Γ defined by (8.2). Product is over all the nonintersecting vortex containers that are fitted around C as described above.

Let us spell out this result in more detail for G = SU(2), D(U) = tr U. In this case

$$\hat{p}_{\Lambda_i,U}(\omega) = \left(1 - \frac{Z(\Lambda_i, U_\sigma)}{Z(\Lambda_i, U)}\right)\left(1 + \frac{Z(\Lambda_i, U_\sigma)}{Z(\Lambda_i, U)}\right)^{-1} \qquad (8.7)$$

with $\sigma = -1$.

The ratio of partition functions in (8.7) can also be expressed as a ratio of expectation values of a certain observable $\tilde{B}(S_i)$ in the two Gibbs states of the system in Λ_i which are specified by the two boundary conditions U and U_σ on Λ_i.

$$\ln\left[Z(\Lambda_i, U_\sigma)/Z(\Lambda_i, U)\right] = -\frac{1}{2}\ln\left\{\langle \tilde{B}(S)\rangle_{\Lambda_i,U_\sigma}/\langle \tilde{B}(S)\rangle_{\Lambda_i,U}\right\} \qquad (8.8)$$

S is a set of plaquettes which is such that it could be location of a hypothetical thin vortex that winds around C inside Λ_i, and

$$\langle \tilde{B}(S)\rangle = \langle \exp \sum_{p\in S}\left\{\mathcal{L}(-U(\dot{p})) - \mathcal{L}(U(\dot{p}))\right\}\rangle \qquad (8.9)$$

Eq. (8.8) follows from the following relation (8.8') and $(U_\sigma)_\sigma = U$.

$$Z(\Lambda_i, U_\sigma) = Z(\Lambda_i, U)\langle B(S)\rangle_{\Lambda_i,U} \qquad (8.8')$$

Let us now illustrate the possible uses of inequality (8.6) with an example. Suppose that the vortex container Λ_i has thickness d_i and extension $|T_i|$ (= number of links in $T_i \approx$ number of plaquettes in S). We must expect that the change of free energy ln Z will be proportional $|T_i|$. S can be so chosen that it has distance $\approx d/2$ from the boundary $\partial\Lambda_i$. Suppose that the influence of the boundary conditions on the expectation value $\langle B(S)\rangle$ decays exponentially with this distance so that $|\ln Z(\Lambda_i,U_\sigma)/Z(\Lambda_i,U)| \leq |T_i| \exp -md/2$. We may choose vortex containers of suitable thickness, so that this is less than $\xi < 1$ for all of them. By counting the number of them that can be fitted around C we find from inequality (8.6) and (8.1) that $V(L) \geq \text{const}\cdot L\,(\ln L)^{-2}$, i.e. an at least almost linear increase with L which ensures confinement.

We will call

$$\frac{1}{|T_i|}\ln Z(\Lambda_i, U_\sigma)/Z(\Lambda_i, U) = \beta\cdot\text{vortex free energy} \qquad (8.10)$$

per unit area ($\nu=4$) resp. length ($\nu=3$) of the vortex. It depends on boundary conditions U on $\partial\Lambda_i$. Our results show that confinement is true if the vortex free energy decreases sufficiently fast with the thickness d (cross section) of the vortex container.

Exactly the opposite behavior corresponds to what 't Hooft calls a Higgs phase. There

$$\frac{1}{|T_i|}\ln p_{\Lambda_i,U}(\sigma) = \frac{1}{|T_i|}\ln\left[Z(\Lambda_i, U_\sigma)/\int d\sigma' Z(\Lambda_i,U_{\sigma'})\right] \not\to 0$$

$$\text{when } d\to\infty, \quad \text{with } T_i \gg d, \qquad (8.11)$$

for some boundary conditions U, for instance U(b) = 1. The limit $f(\sigma)$ is a gauge theory analog of the surface tension[46] in ferromagnets. The limit $|T_i|\to\infty$ is taken first. (This eliminates the "translational" entropy associated with the penetration point of the vortex soul (cp. section 10) in some fixed surface Ξ across Λ_i).

Münster has shown[33] that lattice Higgs models with sufficiently many fundamental scalars (with trivial transformation law under Γ) possess such a Higgs phase. Göpfert has shown[26] that the low temperature phase of the standard Z(2) model is also a Higgs phase in this sense. Proofs are based on a Peierls argument. We shall show in the next section that pure Abelian Yang Mills theories with continuous gauge group have no Higgs phase. All this holds in $\nu \geq 3$ dimensions.

We emphasize that 't Hoofts notion of Higgs phase is much more restrictive than what used to be called a "Higgs mechanism". In particular, it is customary to speak of a Higgs mechanism also when the fundamental scalars do not transform trivially under any nontrivial subgroup of Γ. The model of section 2 is an example.

We wish to emphasize also that the inequality (8.6) and the resulting sufficient condition for confinement as well as 't Hoofts variant of it are results of a kinematical nature. They do not depend on the details of the dynamics of a pure Yang Mills theory but are also valid for theories with fundamental scalars.

The proof of inequality (8.6) uses only very basic properties of the theories - nearest neighbor interaction and gauge invariance. We sketch the proof for a theory of the general form (3.2), (3.4) of the action and path measure. The result is also valid for our modified SU(2) model.

Let Λ^c be the complement of the interior of all vortex containers Λ_i. Then $\Lambda^c \cap \Lambda_i = \partial\Lambda_i$. From Eq. (3.7) and the definition of the partition functions $Z(\Lambda_i, U)$ one obtains

$$\langle \chi(U(C)) \rangle = \frac{1}{Z} \int \prod_{b\in\Lambda^c} dU(b)\ \chi(U(C))\ \exp \sum_{p\in\Lambda^c} \mathcal{L}(U(\dot{p})) \prod_i Z(\Lambda_i, U) \tag{8.12}$$

$Z(\Lambda_i, U)$ depends on the restriction of the configuration U to $\partial\Lambda_i$.

One performs a variable substitution $U(b) \to U(b)\sigma(b)$, $\sigma(b) \in \Gamma$, with the following properties.

i) For all plaquettes $p \in \Lambda^c$, $U(\dot{p})$ stays invariant.

ii) On $\partial\Lambda_i$, the transformation is gauge equivalent to a singular gauge transformation (8.3),

$$U(b) \to U(b)_{\gamma_i} = \begin{cases} U(b)\gamma_i^{-1} & \text{if } b \in T_i \\ U(b) & \text{otherwise.} \end{cases}$$

iii) On the Wilson loop

$$U(C) \to U(C) \prod_i \gamma_i$$

The transformation is determined by the values of the γ_i's. One

averages over all γ_i's, using normalized Haar measure on Γ(e.g. (5. 5) if $\Gamma = Z(2)$). Since the partition functions are gauge invariant functions of U, this gives

$$\begin{aligned} \langle \chi(U(C)) \rangle &= \frac{1}{Z} \int \prod dU(b)\ \chi(U(C))\ \exp \sum \mathcal{L}(U(\dot{p}))\ \cdot \\ &\quad \cdot \prod_i \int \left\{ d\gamma_i\ \omega(\gamma_i)\ Z(\Lambda_i, U_{\gamma_i}) \right\} \\ &= \frac{1}{Z} \int \prod_{b\in\Lambda^c} dU(b) \left[\exp \sum_{p\in\Lambda^c} \mathcal{L}(U(\dot{p})) \right] \chi(U(C)) \\ &\quad \cdot \prod_i \left\{ \hat{P}_{\Lambda_i, U}(\omega) \int d\sigma_i\ Z(\Lambda_i, U_{\sigma_i}) \right\} \end{aligned} \tag{8.13}$$

In the first equation we have used that $\chi(U(C)\prod\gamma_i) = \chi(U(C))\prod\omega(\gamma_i)$. In the second equation we have inserted the definitions (8.4) and (8.5).

By the same procedure one finds that

$$1 = \frac{1}{Z} \int \prod_{b\in\Lambda^c} dU(b)\ \exp \sum_{p\in\Lambda^c} \mathcal{L}(U(\dot{p})) \prod_i \left\{ \int d\sigma_i\ Z(\Lambda_i, U_{\sigma_i}) \right\} \tag{8.14}$$

Since $|\chi(U(C))| \leq \chi(1)$, inequality (8.6) follows from (8.13) and (8.14).

9. COMPUTATIONS OF VORTEX FREE ENERGIES

High Temperature Expansions.

For small values of β, the vortex free energy can be computed by high temperature expansion in the standard models. A set P of plaquettes will be called a polymer[11] if it is connected, i.e. cannot be split into two sets P_1 and P_2 such that no plaquette in P_1 touches a plaquette in P_2. For polymers P one defines activities

$$A(P) = \int \prod dU(b) \prod_{p\in P} \left(e^{\mathcal{L}(U(\dot{p}))} - 1 \right) \tag{9.1}$$

We are interested in a finite lattice Λ_i with prescribed boundary conditions $U(b)$ for links b in $\partial\Lambda_i$. A(P) will depend on these boundary conditions if P touches the boundary. We will write A(P, U) to exhibit this dependence.

The free energy possesses a high temperature expansion of the form[11,34]

$$\ln Z(\Lambda_i, U) = \sum_Q a(Q) \prod_{P\in Q} A(P, U) \qquad (9.2)$$

Summation is over all linked clusters Q of polymers P. They are collections of not necessarily distinct polymers which are linked in the following sense. Draw a graph with a vertex for each polymer, and a link between two vertices if the two polymers share a plaquette, or a plaquette in one of them touches a plaquette in the other. This graph must be connected. a(Q) are combinatorial factors. a(Q) = 1 if Q consists of a single polymer.

From (9.2) one obtains an expansion for the vortex free enery, viz. $\ln Z(\Lambda_i, U) - \ln Z(\Lambda_i, U_\sigma)$. All those terms in the sum drop out for which $A(P, U_\sigma) = A(P, U)$ for all polymers P in Q. Since A(P, U) is a gauge invariant function of U, this equality will hold unless the intersection of at least one polymer P in Q with $\partial\Lambda_i$ is a topologically nontrivial part of $\partial\Lambda_i$ - i.e. it should contain a closed path C which winds around Λ_i. The requirement that $A(P, U_\sigma)$ or $A(P, U) \neq 0$ leads to the further requirement that P must contain all plaquettes in a whole surface Ξ which cuts through Λ_i and has C as its boundary. But for such P, $A(P, U) = O(\beta^{|\Xi|})$.

For the standard SU(2) and Z(2) models one finds to leading orders in β in ν=3 and 4 dimensions (σ=-1)

$$\frac{1}{|T|}\ln Z(\Lambda_i, U_\sigma)/Z(\Lambda_i, U) = -4c(U)\, e^{-\alpha|\Xi|} \qquad (9.3)$$

with $\qquad |c(U)| \leq 2$.

$|T|$ is the extension of the vortex container, and $|\Xi|$ its cross section. $|\Xi| = d^2$ if the container has thickness d. The exponent

$$\alpha = -\ln\frac{\beta}{4} + \frac{2}{3}\left(\frac{\beta}{4}\right)^2 + \ldots \qquad \text{for } G = SU(2) \qquad (9.4a)$$

$$\alpha = -\ln\beta + \frac{1}{3}\beta^2 + \ldots \qquad \text{for } G = Z(2) \qquad (9.4b)$$

and

$$c(U) = \frac{1}{|T|}\sum_{\Xi} \text{tr}\, U(\partial\Xi) + \ldots \qquad (9.5)$$

Summation in (9.5) is over all surfaces Ξ of minimal area $|\Xi|$ that cut through Λ_i. There are $|T|$ of them for containers of regular shape. Some higher order corrections to α have been computed by Münster[35]. See also Yaffe[36] (Appendix). In order β^4 a difference appears between 3 and 4 dimensions.

We see that our sufficient condition for confinement is ful-

filled at high temperatures β^{-1}. Eq. (9.3) is an area law. It is interesting that a less rapid falloff, e.g. a perimeter law, would still suffice to produce confinement; see the discussion in the last section.

Low Temperature Behavior

For a vortex container of fixed cross section $|\Xi|$, the limiting behavior of the vortex free energy as $\beta\rightarrow\infty$ can be found. It is a fact that

$$\lim_{\beta\rightarrow\infty}\left(\int d\mu\, e^{\beta S}\right)^{1/\beta} = \|e^S\|_\infty = \text{ess. sup. } e^S \qquad (9.6)$$

for any normalized measure μ. For the path measures of section 3 and $S = L/\beta$ this gives

$$\frac{1}{\beta}\ln Z(\Lambda_i, U) \rightarrow \max_{U'} L(U') \qquad (9.7)$$

The maximum is with respect to all configurations U' in Λ_i which agree with U on the boundary $\partial\Lambda_i$.

We restrict our attention to special boundary conditions U. L assumes its absolute maximum for U' = 1. This takes care of boundary conditions U = 1 (i.e. U(b) = 1 for all links b in $\partial\Lambda_i$).

Consider now boundary conditions 1_σ ($1_\sigma(b) = \sigma$ if $b \in T_i$, and = 1 otherwise). For abelian theories and large containers, the maximum in (9.7) is easy to determine. It corresponds to a uniform distribution of magnetic flux, if G is continuous. One finds

$$-\frac{1}{\beta|T|}\ln Z(\Lambda_i, U_\sigma)/Z(\Lambda_i, U) \rightarrow \begin{cases} 2 & \text{if } G = Z(2) \qquad (9.8a) \\ \frac{\text{const}}{|\Xi|} & \begin{cases}\text{if } G = U(1) \\ \text{or } G = SU(2)\end{cases} \qquad (9.8b)\end{cases}$$

σ can be any element of the center Γ of G. The proportionality contant depends on $\sigma \in \Gamma = G$ if G = U(1). For G = Z(2) or SU(2) there is only one choice $\sigma = -1$.

The result (9.8) for the nonabelian case is obtained by noting that Abelian configurations $U(b) \in H = U(1) \subset SU(2)$ bound the maximum in (9.7) from below by the result for the U(1) case. An upper bound of the same form is obtained from the inequality[43]

$$0 \geq \text{tr}\,(U(\partial\Xi) - 1) \geq |\Xi| \sum_{p\in\Xi} \text{tr}\,(U(\dot{p}) - 1) \qquad (9.9)$$

This is valid for any rectangular surface of $|\Xi| = 2^{N_1} \times 2^{N_2}$ plaquettes, and arbitrary U.

The interpretation of results (9.8) is obvious. Since the continuum action involves the square of the magnetic field, a factor of $1/|\Xi|$ in the action can be gained by spreading a given magnetic flux over an area $|\Xi|$, for continuous gauge group G in four dimensions. In a Z(2) theory, the flux cannot be spread, since the flux through any plaquette must take values 0 or 1.

Result (9.8) is of very limited use. It cannot be used to obtain the asymptotic behavior as $|\Xi| \to \infty$ for any finite ß: As $ß \to \infty$, the correlation length ξ becomes infinite, if not before, and a vortex container of given thickness d in units of lattice spacing becomes very thin in physical units. In contrast, $|\Xi| \to \infty$ for fixed ß would mean $d/\xi \to \infty$.

Numerical Calculations and Effective Z(2) Coupling Constants

No computer calculation of vortex free energies has yet been performed. Such computations would however appear feasible[38] and desirable, for gauge group SU(2). The dependence on vortex thickness would be most interesting - and also the actual numerical values, for the following somewhat more speculative reasons.

Imagine that a lattice of block cells of $d \times \ldots \times d = d^{\nu}$ elementary hypercubes is superimposed on the original lattice. A vortex container of thickness d may be composed of such blocks. It looks like a thin vortex in the new lattice. If P is a plaquette in the new lattice, perpendicular to the vortex container, then $U(\partial P) \to -U(\partial P)$ under a singular gauge transformation (8.3). There are matching conditions between the configurations on the boundaries of neighboring block cells. They imply, for instance, that the soul of a vortex (to be defined in section 10) has to enter a neighboring block cell where it leaves the old one. These matching conditions affect the entropy in the block cells, and therefore in a vortex container composed from them. We may hope that this entropy remains approximately the same if we replace the matching conditions by periodic boundary conditions (p.b.c.) for each cell. This suggests to regard

$$-\frac{1}{2} \ln Z(\text{block cell}, (\text{p.b.c.})_{\sigma})/Z(\text{block cell, p.b.c.}) = ß_{eff}(d) \qquad (9.10)$$

as effective coupling constant of a Z(2) theory on the block lattice. $(\text{p.b.c.})_{\sigma}$ are boundary conditions obtained from periodic ones by a singular gauge transformation $U \to U_{\sigma}$, $\sigma = -1$, cp. (8.3).

It is tempting to speculate that confinement will prevail if

$$ß_{eff}(d) < ß_c \qquad (9.11)$$

for sufficiently large side length d of the block cells, $ß_c$ = the critical value of the coupling constant for the standard Z(2) model: $ß_c = 0.44$ in 4 dimensions, and $ß_c = 0.76$ in 3 dimensions, see Balian et al.[15]

The effective Z(2) coupling constant could in principle be computed by the MonteCarlo method[38], since the ratio of partition functions on the left hand side is expectation value of a positive observable according to Eq. (8.8').

At high temperatures, $ß_{eff}(d)$ behaves as

$$ß_{eff}(d) \sim 2d^{\nu-2} e^{-\alpha d^2} \quad \text{for small } ß, \qquad (9.12)$$

with α from (9.4a).

For fixed block size d, the bound (9.11) will of course be violated for sufficiently large ß where $ß_{eff}$ will behave like

$$ß_{eff}(d) \propto ß \cdot d^{\nu-4} \quad \text{for fixed d, and } ß \to \infty \qquad (9.13)$$

in ν dimensions. This obtains in the same way as (9.8b).

For finite ß, the dependence on block size d will be changed from the (zeroth order perturbation theory) behavior (9.13) by renormalization effects. It may be useful to distinguish* between not necessarily nonperturbative renormalization effects which will hopefully bring $ß_{eff}(d)$ below $ß_c$ in 4 dimensions** for large enough d,

---

* The standard Z(2) model in four dimensions appears to have a first order phase transition at $ß = ß_c$. Therefore, even at temperatures above but close to $ß_c^{-1}$, there should exist a metastable phase without vortex condensation and confinement. Making the distinction in the text amounts to talking about a similar hypothetical metastable phase in the SU(2) theory. It is supposedly unstable against condensation of vortices of sufficiently large thickness d (depending on ß) at any ß. Nevertheless it may be useful to imagine its existence.

** The factor $d^{\nu-4}$ in (9.13) suggests that in $\nu > 4$ dimensions it will be otherwise.

and the vortex condensation mechanism which is triggered by this. The latter is a nonperturbative effect, but is familiar from the standard Z(2) gauge theory model. It would lead to quark confinement, and would also lower $\beta_{eff}(d)$ still further for d larger than the critical value $d = d_c(\beta)$ at which (9.11) begins to be satisfied.

Currently, the behavior of the 4 - dimensional standard SU(2) model around $\beta = 2$ appears to be of considerable interest[38]. Is the bound (9.11) fulfilled there, for a reasonable sized block? This question could be studied by computer calculation. Sufficiency of (9.11) could also be studied by computations for Higgs models[33].

Abelian Theories

If the gauge group G is a continuous Abelian group, its center $\Gamma = G$ is the whole group, and the elements $\sigma \in \Gamma$ which parametrize singular gauge transformations (8.3) may take values arbitrarily close to 1. As $\sigma \to 1$, the vortex free energy tends to zero. As a consequence, at any nonzero temperature, thin vortices with arbitrarily large extension but σ sufficiently close to 1 will survive. Thick vortices with σ away from 1 can be composed from thin vortices with σ close to 1.

Consider a vortex container Λ' of cross section $|\Xi|$, and regard it as composed from $|\Xi|$ ("thin") vortex containers Λ_i of cross section 1. The composition law for the partition functions $Z(\cdot, U_\sigma)$ leads to the following composition law for the corresponding normalized probability distributions (8.4) on Γ. ($*$ is the convolution product on Γ.)

$$p_{\Lambda',U}(\sigma) = \int d\nu(U) \mathop{*}_{i} p_{\Lambda_i,U}(\sigma) \tag{9.14}$$

$d\nu$ is a normalized measure. Integration is over variables U(b) attached to the links b in the interior of Λ'; the result depends on the remaining variables U(b) with b in $\partial\Lambda'$. Explicitly,

$$d\nu(U) = \bar{Z}(\Lambda', U)^{-1} \left\{ \prod_i \bar{Z}(\Lambda_i, U) \right\} \prod_b dU(b) \quad ,$$
$$\bar{Z}(\cdot, U) = \int d\sigma \, Z(\cdot, U_\sigma) \tag{9.15}$$

One can derive estimates[29] for thin vortex containers Λ_i. It suffices to consider $G = U(1) = \Gamma$ with elements $\sigma = \exp i\varphi$, $\varphi = 0, \ldots 2\pi$, and Haar measure $d\sigma = d\varphi/2\pi$. We assume that the Lagrangean is of the form

$$\mathcal{L}(e^{i\vartheta}) = \beta \cos\vartheta \tag{9.16}$$

Characters are of the form

$$\omega_q(e^{i\varphi}) = e^{iq\varphi} \quad , \quad q = 0, \pm 1, \pm 2, \ldots \tag{9.17}$$

The partition function of a thin vortex container is

$$Z(\Lambda_i, U_\gamma) = \exp \Sigma \, \mathcal{L}(U(\dot{p})\gamma) \tag{9.18}$$

Summation is over all plaquettes that are not on the boundary of Λ_i. Each of them contains a link of T_i, and is supposed to have orientation inherited from that link. There are $|T_i|$ such plaquettes. There are no integrations involved, all variables U(b) in (9.18) are fixed by the boundary conditions U on $\partial\Lambda_i$.

Inserting the form (9.16) of the action, one obtains for $\gamma = e^{i\varphi}$

$$Z(\Lambda_i, U_\gamma) = \exp \beta \, \Sigma \cos(\vartheta_p + \varphi) = \exp \beta |T_i| \varkappa(U) \cos(\bar{\vartheta} + \varphi)$$

with $0 \leq \varkappa(U) \leq 1$. Therefore

$$\hat{P}_{\Lambda_i,U}(\omega_q) = I_q(\beta|T_i|\varkappa(U))/I_o(\beta|T_i|\varkappa(U)) \leq \exp - \frac{q^2}{2\beta}|T_i|^{-1} \tag{9.19}$$

for large $|T_i|$.

From these bounds one may draw two conclusions for pure Yang Mills theories with continuous Abelian gauge group U(1).

(1) They possess no Higgs phase in any number of dimensions (in the sense of definition (8.11), with Λ_i a box).

(2) In three dimensions, static quarks of any charge $q \neq 0$ are confined by a potential

$$V(L) \geq \frac{q^2}{16\beta} (\ln L + \text{const.}) \tag{9.20}$$

Property (1) obtains by inserting (9.18) and definition (8.4) of $p_{..}(\cdot)$ into composition law (9.14). The large $|T_i|$-behavior of the $*$-product can be estimated by the saddle point method. This produces a bound on $p_{\Lambda',U}(\cdot)$ because the measure ν is normalized. Finally, one compares with the definition (8.11) of a Higgs phase.

To learn as much as possible from our derivation of property (2), the reader should study Fisher's domain wall argument[27] and compare with it.

Property (2) is obtained by inserting bound (9.19) into our basic inequality (8.6). Let C be a rectangular path which encloses an area of $L\cdot T$ plaquettes, $T \gg L$. Divide the lattice into T-1 slices perpendicular to the long legs of C, each of them one lattice spacing thick. Into each slice pack the maximum number L of (quadratic) thin vortex containers that can be wound around one penetration point of C. They have extension (length) $4(2j+1)$, $j = 0 \ldots L-1$. Therefore, inequality (8.6) gives

$$\langle \omega_1(C) \rangle < \exp - T\left[\frac{g^2}{16\beta}\sum_{j=0}^{L-1}(j+\tfrac{1}{2})^{-1}\right] \tag{9.21}$$

For large L, the sum behaves like ln L + const.. Comparing with (8.1) gives the bound (9.20).

The Coulomb law (9.20) as a lower bound on V(L) was derived before by Glimm and Jaffe by another method.[39] The derivation given here serves to illustrate the use of our basic inequality (8.6). It may be viewed as a generalization of the Dobrushin Shlosman argument[28] for two dimensional ferromagnets.

TOPOLOGICALLY DETERMINED Z(2) VARIABLES IN SU(2) THEORIES

Preliminary Considerations: Abelian Higgs Model

To fix ideas, let us first consider a Higgs model with U(1) gauge group on three dimensional continuous space time. There will be a charged complex field $\varphi(x)$ which carries charge 1 (i.e. transforms according to the faithful fundamental representation of U(1)), and the space time components $A_\mu(x)$ of the vector potential take real values.

Consider a smooth field configuration with a (Nielsen Olesen[40]) vortex of winding number $k > 0$ along the x^3- axis. Near the x^3-axis, the fields will behave like

$$\varphi(x) = a(x^1 + ix^2)^k + \ldots$$
$$A_\mu(x) = A_\mu(0) + \ldots \tag{10.1}$$

if they are invariant under translations in the x^3- direction.

Let us make the transition to the unitary gauge. It is defined by the requirement that $\varphi(x) > 0$. The unitary gauge is a special case of what the author calls a "local gauge"[32]. It is convenient to use cylinder coordinates

$$x^1 = \rho\cos\vartheta \quad , \quad x^2 = \rho\sin\vartheta$$

The gauge transformation to the unitary gauge involves $S(x) = e^{-ik\vartheta} \in U(1)$. Scalar fields $\hat{\varphi}(x)$ and vector potential $\hat{A}_\mu(x)$ in the unitary gauge are given by

$$\hat{\varphi}(x) = a\,\rho^k + \ldots$$
$$\hat{A}_3(x) = A_3(0) + \ldots$$
$$\hat{A}_1(x) = -\frac{k}{\rho}\sin\vartheta + A_1(0) + \ldots \tag{10.2}$$
$$\hat{A}_2(x) = \frac{k}{\rho}\cos\vartheta + A_2(0) + \ldots$$

We see that the vector potential $\hat{A}_\mu(x)$ in the local gauge is singular on the x^3- axis although we start from a perfectly smooth vector potential. This is an instance of what the author calls a "local gauge singularity"[32]. Consider now a path C of infinitesimal length which winds once around the x^3- axis. Then the contour integral

$$\frac{1}{2\pi}\oint_C \hat{A}_\mu\, dx^\mu = k \ . \tag{10.3}$$

It counts vorticity (winding number k) which need not be zero even for a perfectly smooth field configuration φ, A_μ. The "soul of the vortex" is here located at the zero of the scalar field.

The local gauge used here is a complete gauge; it leaves no freedom of gauge transformations whatsoever. Had we chosen a charged field of charge two then any local gauge would still leave the freedom of Z(2) gauge transformations (rotations by π). On a lattice, these could depend on space and time.

In the following we will introduce variables $\bar{W}(b)$ for the SU(2) theory. They will be exponentials of the vector potential in a certain <u>local gauge</u>. This gauge (and any other conceivable local gauge in a pure SU(2) Yang Mills theory) leaves the freedom of Z(2) gauge transformations. This is why we will be left with Z(2) gauge variables $\sigma(b)$ in addition to the local gauge invariants $\bar{W}(b)$.

The New Variables for SU(2) Theories

Given the matrix U(b) in G = SU(2), let

$$\bar{U}(b) = U(b)\Gamma \in G/\Gamma \ . \tag{10.4}$$

$\bar{U}(b)$ may be regarded as a real 3×3 matrix $(\bar{U}(b)^{ac})$ in SO(3). Let p be the plaquette with corner points x, $x+e_i$, $x+e_i+e_j$, $x+e_j$ (e_μ =

unit vector in μ-direction.), and ijk = 123 or cyclic permutation. We define 3×3 matrices

$$B(x) = (B^a{}_k(x)) , \quad B^a{}_k(x) = \frac{1}{2}\varepsilon^{abc}\,\bar{U}(\dot{p})^{bc} \qquad (10.5)$$

If $U(b) = \exp A^a{}_\mu(x)\tau^a/2$ with small $A^{\cdot}{}_{\cdot}$, then $B^a{}_k$ are the components of the magnetic field associated with the vector potential A.

The configurations U which are such that for a fixed x on the lattice one has det $B(x) = 0$ form a set of (Gibbs) measure zero. This remains true in the infinite volume limit. This follows from the Markov property by a standard argument that has been described in Glimms lectures at this school[4].

Every nonsingular real 3×3 matrix admits a unique decomposition $B = OP_+$, where O is orthogonal and P_+ is a positive matrix. Therefore we may decompose

$$B(x) = \bar{S}(x)P(x) \quad \text{with} \quad \bar{S}(x) \in SO(3) \qquad (10.6)$$

and $P(x)$ is either positive or negative definite, depending on sign det $B(x)$. $\bar{S}(x)$ is uniquely defined if det $B(x) \neq 0$.

We define

$$\bar{W}(b) = \bar{S}(x)^{-1}\bar{U}(b)\bar{S}(y) \in SO(3) \quad \text{for} \quad b = (x,y) . \qquad (10.7)$$

It is easily verified that $\bar{W}(b)$ is gauge invariant. It is also local in the sense that it depends only on gauge fields $U(b')$ attached to links within a neighborhood of one lattice spacing of b. One lattice spacing becomes infinitesimally small in the continuum limit.

Now we turn to the definition of the Z(2) variables $\sigma(b)$. The group $G = SU(2)$ is a twofold covering of $G/\Gamma = SO(3)$. To every matrix $\bar{W}$ resp. $\bar{S}$ of SO(3) there correspond two matrices $\pm W$ resp. $\pm S$ in SU(2). We select the one with positive trace. Thus, $W(b)$ and $S(b)$ will be defined by the requirements that

$$W(b)\Gamma = \bar{W}(b) \quad , \ \mathrm{tr}\, W(b) \geq 0 , \qquad (10.8a)$$

and

$$S(x)\Gamma = \bar{S}(x) \quad , \ \mathrm{tr}\, S(x) \geq 0 . \qquad (10.8b)$$

The variables $\sigma(b)$ can now be defined by the formula

$$W(b)\sigma(b) = S(x)^{-1}U(b)S(y) \qquad \text{for} \quad b = (x,y). \qquad (10.9)$$

Taking Γ-cosets of both sides of this equation we see by comparison with (10.7) that

$$\sigma(b) = \pm 1 \quad .$$

We leave it to the reader to verify that an SU(2) gauge transformation $V(\cdot)$ of variables $U(\cdot)$ induces a Z(2) gauge transformation of variables $\sigma(b)$, viz.

$$\sigma(b) \rightarrow \upsilon(x)\sigma(b)\upsilon(y)^{-1} \qquad \text{for} \quad b = (x,y) , \qquad (10.10)$$

with $\upsilon(\cdot) = \pm 1$. It follows that $\sigma(\dot{p})$ is gauge invariant. It is also local in the same sense as $\bar{W}(b)$. For any closed path C, in particular for the boundary $C = \dot{p}$ of a plaquette, $\sigma(C)$ is defined as in Eq. (3.1).

If C is boundary of a surface Ξ consisting of plaquettes p then

$$\sigma(C) = \prod_{p\in\Xi} \sigma(\dot{p}) \qquad (C = \partial\Xi) \qquad (10.11)$$

The local gauge invariants $\bar{W}(b)$ and $\sigma(\dot{p})$ determine a configuration U up to the action of a gauge transformation.

If b are spacelike links in the t=0 hyperplane Σ, $\bar{W}(b)$ can act as multiplication operators on wave functions Ψ in the Hilbert space (4.1) of physical states. They commute (at t=0). These operators are thus gauge invariant local SO(3)-valued field operators whose action does not lead out of the physical Hilbert space. In contrast, Z(2) gauge fields $\sigma(b)$ can never share these properties, only their field strengths $\sigma(\dot{p})$ do.

The reader may find it interesting to compare Eq.(10.9) with work of Glimm and Jaffe[39] on vortices in Abelian theories.

Wilson Loop and Action in Terms of the New Variables

Let C be a closed loop consisting of links $b_1..b_n$ which is boundary of a surface Ξ. For instance, Ξ may be a rectangle of L·T plaquettes. It follows from Eqs. (10.9) and (10.11) that

$$\mathrm{tr}\, U(C) = \sigma(C)\ \mathrm{tr}\, W(C) = \Big(\prod_{p\in\Xi}\sigma(\dot{p})\Big)\mathrm{tr}\, W(C) \ . \qquad (10.12)$$

The last formula expresses the Wilson loop observable tr U(C) as a sum of products of local gauge invariants. This is so because both $\sigma(\dot{p})$ and $W_{\alpha\beta}(b)$ are local gauge invariants, and

$$\mathrm{tr}\, W(C) = \sum_{(\alpha_1\ldots\alpha_n)} W_{\alpha_1\alpha_n}(b_n)\ldots W_{\alpha_3\alpha_2}(b_2)W_{\alpha_2\alpha_1}(b_1) \ . \qquad (10.13)$$

More generally, let χ_l be the character of the $2l+1$ dimensional irreducible representation of SU(2). If l is integer, it is at the same time a character of SO(3). From Eqs.(10.9) and (10.11) one finds that

$$\chi_l(U(C)) = \begin{cases} \chi_l(W(C)) \prod_{\dot p \in \Xi} \sigma(\dot p) & \text{if } l = \frac{1}{2}, \frac{3}{2}, \ldots \quad (10.14a) \\ \chi_l(W(C)) & \text{if } l = 0, 1, 2, \ldots \quad (10.14b) \end{cases}$$

There is reason to believe that the fluctuations of the Z(2) variables $\sigma(\dot p)$ are crucial in producing an area law decay of the Wilson loop observable $\langle(10.12)\rangle$, whereas the factor $\chi(W(C))$ (which is a sum of products of local gauge invariants that are localized near the path) is relatively unimportant. It is known[2] that $\langle \chi_l(U(C)) \rangle$ will not have an area law decay if l is integer (see the introduction). This is in agreement with the absence of the factor $\Pi\, \sigma(\dot p)$ in (10.14b).

Next, we rewrite the action. Specializing Eq. (10.12) to a single plaquette, we obtain

$$\operatorname{tr} U(\dot p) = \operatorname{tr} W(\dot p)\, \sigma(\dot p) \tag{10.15}$$

Therefore

$$L(U) \equiv L(\bar W, \sigma) = \sum_p K_p(W)\sigma(\dot p) + \text{const.} \tag{10.16}$$

The fluctuating Z(2) coupling constants K_p are given by

$$K_p(W) = \frac{\beta}{2} \operatorname{tr} W(\dot p) \,. \tag{10.17}$$

Fluctuations of Z(2) Variables and Topological Interpretation

At high temperatures β^{-1}, fluctuations of the Z(2) variables $\sigma(\cdot)$ will confine static quarks according to the result of section 5. (The variables W(b) defined by Eqs.(10.7), (10.8) are special representatives of the classes of configurations that were considered in section 5.)

At low temperatures, we have instead the limiting behavior

$$\sigma(\dot p) \longrightarrow \operatorname{sign} K_p(W) \quad \text{as} \quad \beta \longrightarrow \infty \tag{10.18}$$

This follows from Eq.(10.15) and (3.8). It means that the probability that the right hand side is different from the left hand side tends to zero as $\beta \to \infty$, for any fixed plaquette p.

We see that the Z(2) field strengths freeze to values that are determined by the SO(3) variables $\bar W(b)$. (W is obtained from $\bar W$ as explained before (10.8)). The Z(2) coupling constants $K_p(W)$ are not positive definite (neither in the standard model nor in the modified model) but their sign is determined by topological properties of the SO(3) gauge field. This will be explained below. As a result, the Z(2) variables $\sigma(\dot p)$ will continue to fluctuate at large β, as a consequence of the fluctuations of the topological properties of the SO(3) gauge field. Such fluctuations can in principle confine static quarks because of the factor $\Pi\, \sigma(\dot p)$ in (10.12).

We turn now to the topological interpretation of our Z(2) variables. Consider continuous space time R^4 and imagine that a lattice of arbitrarily small lattice spacing is superimposed on it. Let $A_\mu(x) \in su(2)$ be a smooth vector potential. Parallel transporters along paths C are then defined by

$$U(C) = T \exp \int_C A_\mu(x)dx^\mu \quad ; T = \text{path ordering}. \tag{10.19}$$

This holds in particular for links b of the lattice.

The magnetic field is

$$\partial_i A_j(x) - \partial_j A_i(x) + [A_i(x), A_j(x)] = i \sum B^a{}_k(x)\tau^a/2 \,. \tag{10.20}$$

This defines the 3×3 matrix $B(x) = (B^a{}_k(x))$.

Let $\mathcal{J}$ be the set of points x where $\det B(x) = 0$. For x not in $\mathcal{J}$ we may define $\bar S(x)$ in SO(3) by the unique decomposition (10.6). A local gauge is defined by the requirement $\bar S(x) = 1$. The vector potential $\hat A_\mu$ in the local gauge is given in terms of A_μ by

$$\hat A_\mu(x) = \bar S(x)^{-1} A_\mu(x) \bar S(x) + \bar S(x)^{-1}\partial_\mu \bar S(x) \tag{10.21}$$

This formula makes sense since the Lie algebra of SU(2) and of SO(3) is the same. $\hat A_\mu$ may be singular on $\mathcal{J}$ even if A_μ is smooth.

Consider now a closed path C and the homotopy class $[\bar S]_C$ of the map

$$\bar S: C \longrightarrow SO(3) \tag{10.22}$$

$[\bar S]_C = \pm 1$ since the first homotopy group $\pi_1(SO(3)) \equiv Z(2)$. Moreover, $[\bar S]_C = -1$ is only possible when C winds an odd number of times around the set $\mathcal{J}$ of local gauge singularities. Let Ξ be a surface whose boundary is C. If $[\bar S]_C = -1$ we say that the soul of a vortex passes through Ξ. Souls of vortices are counted modulo 2 (for SU(2)).

Let us write

$$W(C) = T \exp \int_C \hat{A}_\mu(x)dx^\mu \qquad (10.23)$$

For paths C of small length*

$$U(C) \approx 1 \quad \text{and} \quad W(C) \approx [\bar{S}]_C = \pm 1 . \qquad (10.24)$$

Let us now restrict attention to paths which consist of links of the superimposed lattice. In the limit of small lattice spacing, $U(b) \to 1$, and the magnetic field defined by Eq.(10.20) agrees with (10.5). We assume that the lattice spacing is sufficiently small on a physical scale so that (10.24) holds, yet C stays sufficiently many lattice spacings away from $\mathcal{J}$ so that $\bar{S}(x)$ varies little along a link in C. Then W(C) as defined in (10.23) also agrees with the quantity W(C) that was introduced previously.

If we insert Eqs.(10.24) into (10.12) we obtain

$$\prod_{p\in\Xi} \sigma(\dot{p}) = [\bar{S}]_{C=\partial\Xi} = \pm 1 . \qquad (10.25)$$

We see that the Z(2) variables $\sigma(\dot{p})$ count the number of souls of vortices that pass through the surface Ξ. According to Eq. (10.25) they have the meaning of a topological density (similar to the Pontrjagin density that is used in instanton physics). Because of (10.18) this interpretation carries over to the sign of the coupling constants K_p.

The reader can see now that it is absolutely crucial that the coupling constants $K_p(W)$ of the Z(2) theory with variables $\sigma(b)$ as defined here are not positive definite. This is also the reason why the dynamics of an SU(2) theory (in contrast with its kinematics, cp. section 8 and 't Hoofts lectures[23]) is not symmetric under a Kramers Wannier duality transformation when it is interpreted as a Z(2) theory in the manner of this section. If one applies a Kramers Wannier duality transformation to a Z(2) gauge theory with coupling constants that are not positive, one obtains something that is no longer statistical mechanics (the action is not real). Therefore the existence of a confining high temperature phase does not imply existence of a nonconfining low temperature Higgs phase.

To avoid misunderstanding, let us point out that the modified SU(2) model may also be interpreted as a Z(2) gauge theory in another way, with positive coupling constants and different Z(2) variables

* In a lattice theory with large β, $U(C) \approx 1$ follows for such C from chessboard estimates via (3.8) and (9.9)

$\gamma(\dot{p})$ that freeze to 1 as $\beta \to \infty$. A Kramers Wannier duality transformation can be applied to this Z(2) theory and establishes the existence of two distinct phases as was shown in section 6. But the relation (10.12) for the Wilson loop involves the variables $\sigma(\dot{p})$ and not $\gamma(\dot{p})$ $(= \sigma(\dot{p}) \operatorname{sign} K_p(W))$. Therefore nothing can be learned from this duality transformation about the behavior of the Wilson loop observable in the low temperature phase. We know of no reason to believe that it is nonconfining in the modified model[12,36].

Example of a Field Configuration with a Vortex Soul

We shall exhibit an analog of the field configuration (10.1) for an SU(2) theory in four dimensions. It will have a vortex soul located in the x^3x^4 plane. In the neighborhood of this plane let

$$B(x) = \mu \begin{pmatrix} x^1 & x^2 & 0 \\ -x^2 & x^1 & 0 \\ 0 & 0 & \lambda/\mu \end{pmatrix} + \ldots \qquad (10.26)$$

This can be obtained from a vector potential $A_\mu = i \sum A^a_\mu \tau^a/2$ of the form

$$\begin{aligned} A^1_3 &= \mu x^1 x^2 + \ldots , \quad A^2_3 = \tfrac{1}{2}\mu[(x^1)^2 - (x^2)^2] + \ldots \\ A^3_2 &= \lambda x^1 + \ldots , \quad A^3_1 = -\lambda x^2 + \ldots , \quad \text{others} = 0 . \end{aligned} \qquad (10.27)$$

In cylinder coordinates $x^1 = \rho\cos\vartheta$, $x^2 = \rho\sin\vartheta$, this gives

$$\bar{S}(x) \approx \begin{pmatrix} \cos\vartheta & \sin\vartheta & 0 \\ -\sin\vartheta & \cos\vartheta & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (10.28)$$

If C is a path that winds once around the vortex soul at $x^1 = x^2 = 0$ then the map $\bar{S}: C \to SO(3)$ belongs to the nontrivial homotopy class (since the image $\bar{S}(C)$ is projection to SO(3) of a path in SU(2) that links antipodes).

Note also that for $(x^1,x^2) \neq (0,0)$, $B(x)$ belongs to the class of elements of GL(3,R) with one real and two complex conjugate eigenvalues.

In the Abelian Higgs model with a potential $V(\varphi(x))$ of the usual double well shape, presence of a vortex soul costs energy proportional to length, since $V(\varphi(x)=0) > \min V$. No similar concentration of energy is associated with a vortex soul in a pure SU(2) gauge field theory. This is in accordance with the fact that the energy of a quantum of magnetic flux can be lowered by spreading the flux, as was discussed in section 9 (Eqs. (9.8b)).

Some solutions of the Yang Mills field equations with nonvanishing vorticity were found by Vinciarelli[45].

11. RELATION BETWEEN THEORIES WITH GAUGE GROUPS SU(2) AND SO(3)

In a pure Yang Mills theory on continuous space time, the dynamical variables are vector potentials $A_\mu(x)$. They take values in the Lie algebra $\mathfrak{g}$ which is the same for gauge groups G = SU(2) and SO(3). There is therefore no visible difference between pure Yang Mills theories in the continuum with either gauge group. (If one wants to add matter fields, they should transform according to a representation of the gauge group, though).

In lattice gauge theories there is a difference to begin with. The continuum limit is supposed to be approached when $\beta \to \infty$. One should therefore expect that it will be possible to express expectation values of the SU(2) theory in terms of expectation values of an SO(3) theory, at least for large values of β. This is indeed the case. In the present section we will describe formulae which achieve this.

We will work on a finite lattice Λ throughout. The infinite volume limit $\Lambda \to \infty$ is supposed to be taken only at the end. We will consider only the modified SU(2) model. Results can be generalized to the standard model, but some of the formulae below would have a somewhat more complicated form there.

SU(2) is a twofold covering of SO(3), viz. $SO(3) \cong SU(2)/Z(2) = G/\Gamma$. In the last section we have "split" the variables U(b) of the SU(2) theory into gauge invariants $\bar{W}(b) \in SO(3)$, and Z(2) gauge fields $\sigma(b) = \pm 1$. Together they determine U up to a gauge transformation. $\bar{W}(b)$ depends only on the cosets $\bar{U}(b) = U(b)\Gamma \in SO(3)$. If we had started from a theory with gauge group SO(3), the variables σ(b) would not be needed, and the gauge invariants $\bar{W}(b)$ would determine the configuration $\bar{U}$ up to a gauge transformation. In the SU(2) theory, the Z(2) variables freeze to values determined by the SO(3) variables $\bar{W}(b)$ when $\beta \to \infty$, according to (10.18). This suggests to treat the fluctuations of $\gamma(\dot{p}) = \sigma(\dot{p})\,\mathrm{sign}\,K_p(W)$ at large but finite β as perturbations. The complete SO(3) action $\bar{L}$ is obtained by integrating out these fluctuations.

Consider first observables $F = F(\bar{U})$ of the SU(2) theory which depend on the configuration U only through cosets $\bar{U}(b) = U(b)\Gamma \in SO(3)$. They can be regarded as observables in a SO(3) theory with variables $\bar{U}(b)$, and it follows from our definition of the action $\bar{L}$ of the SO(3) theory that

$$\langle F \rangle_{SU(2)} = \langle F \rangle_{SO(3)} \quad \text{for} \quad F = F(\bar{U}) \tag{11.1}$$

For observables F of the SU(2) theory which are not SO(3) observables (= functions of an SO(3) configuration $\bar{U}$), the situation is not so simple. We limit our attention to the Wilson loop tr U(C). In this case the answer is suggested by Eqs. (10.12) and (10.18). One finds that

$$\langle \mathrm{tr}\, U(C) \rangle_{SU(2)} = \langle z_C(W)\, \mathrm{tr}\, W(C) \prod_{p \in \Xi} \mathrm{sign}\, K_p(W) \rangle_{SO(3)} \tag{11.2}$$

This is true for any surface Ξ whose boundary is C. W is determined by $\bar{W}$, and therefore by the configuration $\bar{U}$ of the SO(3) theory by Eqs. (10. 8a) and (10.7), (10.6), viz.

$$W = W(\bar{U})$$

According to the discussion in the last section, the topologically determined factor $\prod \mathrm{sign}\, K_p(W)$ will be most important in (11.2). The origin of the "wave function renormalization" factor $z_C(W)$ will be explained below. It depends essentially only on variables W(b') attached to links b' close to the path C if β is large. It satisfies

$$0 \leq z_C(W) \leq 1 .$$

The path measure of the SO(3) theory is of the form

$$d\mu(\bar{U}) = \frac{1}{Z} \prod d\bar{U}(b)\; e^{\bar{L}(\bar{U})} \cdot (\text{step functions}) \tag{11.4}$$

$d\bar{U}(b)$ is normalized Haar measure on SO(3). The step functions enforce the constraints (3.5). They are the same as in (3.6). Making a substitution $U(b) \to \pm U(b)$ will convince the reader that constraints (3.5) are in fact constraints on the cosets $\bar{U}(b)$.

The measure (11.4) is obtained from the path measure (3.6) by integration out the Z(2) variables σ(b). Thus

$$\bar{L}(\bar{U}) = \ln Z_{W(\bar{U})}/Z \tag{11.5}$$

with Z_W from Eq. (5.6). Due to the constraint (3.5), we may make a variable transformation $\sigma(b) \to \gamma(b) = \pm 1$ in (5.6) which is such that $\sigma(\dot{p}) = \gamma(\dot{p})\,\mathrm{sign}\, K_p(W)$. As a result,

$$Z_{W(\bar{U})} = \int \prod_b d\gamma(b)\; \exp \sum_p |K_p(W)|\, \gamma(\dot{p}) \tag{11.6}$$

Expression (11.6) is partition function of a Z(2) theory. We want to bring $\ln Z_W$ into a more useful form. We use low temperature expansions in order to exploit the fact that large thin vortices are rare and small ones dilute at low temperatures.

A configuration γ is determined up to a gauge transformation by its vortices, i.e. by the coclosed set T of plaquettes p where $\gamma(\dot{p}) = -1$ (see section 7). Two sets P_1 and P_2 of plaquettes will be called disjoint if no plaquette in P_1 coincides with or touches a plaquette in P_2 along a link. A coclosed set T of plaquettes will be called a contour if it cannot be split into two disjoint coclosed subsets P_1 and P_2. Any T as above can be decomposed uniquely into disjoint contours $P_1 \ldots P_k$.

For every contour P we define

$$A(P) = \prod_{p \in P} \exp -2|K_p(W)| \qquad (11.7)$$

Let X_Λ be the set of all contours in Λ. We convert the sum over γ in (11.6) into a sum over vortices. This gives

$$Z_{W(\tilde{U})} = 2^{-|\Lambda|} \varsigma(X_\Lambda) \exp \sum_p |K_p(W)| \qquad (11.8)$$

with

$$\varsigma(X) = 1 + \sum_{D \subset X}{}' \prod_{p \in D} A(P) \qquad (11.9)$$

The prime ' indicates that summation is only over nonempty subsets D of X which consist of dijoint contours $P_1 \ldots P_n$.

We take the logarithm of (11.8) and convert the result into a linked cluster expansion by use of a Möbius inversion[44]. Let X, Y,... be subsets of X consisting of n(X), n(Y),... contours respectively. We define

$$\vartheta_{YX} = \begin{cases} (-1)^{n(Y)-n(X)} & \text{if } X \subseteq Y \\ 0 & \text{otherwise} \end{cases} \qquad (11.10)$$

It satisfies

$$\sum_{\substack{Y \\ X \subseteq Y \subseteq X'}} \vartheta_{YX} = \begin{cases} 1 & \text{if } X = X' \\ 0 & \text{otherwise} \end{cases} \qquad (11.11)$$

We use this to define

$$\zeta_Y(W) = \sum_{X \subseteq Y} \vartheta_{YX} \ln \varsigma(X)$$

Explicitly

$$\zeta_Y(W) = \sum_{X \subseteq Y} \vartheta_{YX} \ln \Big(1 + \sum_{D \subset X}{}' \prod_{P \in D} A(P)\Big)$$

$$= \sum_{X \subseteq Y} \vartheta_{YX} \ln \Big(1 + \sum_{D \subset X}{}' \exp\Big[-2 \sum_{P \in D} \sum_{p \in P} |K_p(W)|\Big]\Big) \quad . \qquad (11.12)$$

Σ' is again summation over nonempty subsets D which consist of disjoint contours P.

It follow from (11.5), (11.8) and (11.11) that

$$\bar{L}(\tilde{U}) = \sum_p |K_p(W)| + \sum_{Y \subseteq X_\Lambda} \zeta_Y(W) + \text{const.} \qquad (11.13)$$

with const. $= -\ln(2^{|\Lambda|} Z)$

The sum over Y in (11.13) can be restricted to linked clusters Y of distinct contours in Λ, because $\zeta_Y = 0$ if Y is not linked. (This follows from Eq.(11.11) with X empty, and factorization properties of ς's.) "Linked" means that Y is not union of any two subsets Y_1 and Y_2 which are such that all contours in Y_1 are disjoint from contours in Y_2.

Eq. (11.13) together with (11.12) is our final result for the action $\bar{L}$ of the SO(3) theory.

Suppose that Y consists of contours $P_1 \ldots P_k$. By expanding the logarithm in (11.12) into a Taylor series one obtains

$$\zeta_Y(W) = \sum_{n_1 \geq 1} \ldots \sum_{n_k \geq 1} a(P_1^{n_1} \ldots P_k^{n_k})\, A(P_1)^{n_1} \ldots A(P_k)^{n_k} . \qquad (11.14)$$

These are essentially the low temperature expansions of Marra and Miracle-Solé[25]. a(Q) are certain combinatorial factors. They depend on the linked cluster Q of not necessarily distinct contours. (Contour P_j may occur $n_j \geq 1$ times.)

Series (11.14) need not (and will not) always converge, even on a finite lattice and for large ß. For any finite ß there is a set of configurations $\tilde{U}$ of small but finite path measure which has the property that $|K_p(W(\tilde{U}))|$ for $p \in P_j \in Y$ are not sufficiently large to make activities $A(P_j)$ small enough to ensure convergence of series (11.14).

To escape from this difficulty, we stick to Eqs. (11.12), (11.13) as our definition of the SO(3) action. We work on a finite lattice Λ. All expectation values are to be computed on this finite lattice, and the infinite volume limit is supposed to be taken only at the end. On a finite lattice, Eqs. (11.12) and (11.13) involve finite sums, so there is no convergence problem.

Expansions (11.14) are useful for estimates of the correction terms ζ_Y in the action, however. They, and the result of substituting them into (11.13), converge even on an infinite lattice if the ξ-norm $\|A\|_\xi$ of the activities A (11.7) is less than 1 for some $\xi > 1$. The ξ-norm is defined as in section 2.[11]

Starting from the bounds (3.9) and suitable estimates[26] of the combinatorial factors a(Q), one can deduce estimates of the probability that series (11.14) converges and sums to a result of modulus less than some small number. These estimates are uniform in the lattice size. They show that, generally speaking, the correction terms ζ_Y in $\bar{L}$ are small for large β, and decrease exponentially with β times the size of the cluster Y. In this sense, <u>the action $\bar{L}(\bar{U})$ is local</u> at large β.

To conclude that these "small" correction terms are actually unimportant, one would need information on the stability properties of the low temperature phase of the SU(2) theory. They have not been established. Therefore we must be content with the statement that our results are consistent with the idea of universality.

It remains to establish formula (11.2). We adapt the treatment[25,26] of the low temperature phase of the standard Z(2) model. We start from Eq.(5.3) and make a variable transformation $\sigma \to \gamma$ as before. This gives

$$\langle \mathrm{tr}\, U(C) \rangle_W = \mathrm{tr}\, W(C) \prod_{p\in\Xi} \mathrm{sign}\, K_p(W) \cdot \left\{ Z_W^{-1} \int \prod_b d\gamma(b) \left[\prod_{p\in\Xi} \gamma(\dot{p}) \right] \exp \sum_p |K_p(W)|\, \gamma(\dot{p}) \right\}$$

$$\equiv z_C(W)\, \mathrm{tr}\, W(C) \prod_{p\in\Xi} \mathrm{sign}\, K_p(W) \qquad (11.15)$$

The result (11.2) follows from this because of the formula

$$\langle \mathrm{tr}\, U(C) \rangle = \frac{1}{Z} \int \prod d\bar{U}(b)\, Z_{W(\bar{U})} \langle \mathrm{tr}\, U(C) \rangle_{W(\bar{U})}$$

To see this, substitute Eq.(11.5) for Z_W/Z into it. An analogous formula (6.5) was used before in section 6.

To analyze $z_C(W)$, we expand the integral in { } in (11.15) in contours and proceed as before. We write

$$z_C(W) = \exp - \left[\ln Z_W - \ln Z_W^- \right] . \qquad (11.16)$$

The linked cluster expansion for $\ln Z_W = \bar{L} + \mathrm{const.}$ was found before in Eq. (11.13). To obtain $\ln Z_W^-$ one substitutes

$$A^-(P) = (-1)^{n(P,C)} A(P) \quad , \; n(P,C) = |P \cap \Xi| \pmod 2 \qquad (11.17)$$

for A(P) in the correction terms ζ_Y in this expansion. n(P,C) = (number of plaquettes in $P \cap \Xi$) is odd only if the contour P winds an odd number of times around C. Therefore, in the difference $\ln Z_W - \ln Z_W^-$, all those terms in the linked cluster expansion cancel out that come from clusters Y which do not contain a contour that winds around C. In conclusion

$$z_C(W) = \exp - \sum_Y \left[\zeta_Y(W) - \zeta_Y^-(W) \right] . \qquad (11.18)$$

Summation is over linked clusters Y of distinct contours, with the property that at least one of these contours winds around C. ζ_Y^- is obtained from expression (11.12) for ζ_Y by substituting $A^-(P)$ for A(P).

$z_C(W)$ is positive by the first Griffith's inequality and ≤ 1, by its definition (11.15).

Because ζ_Y and ζ_Y^- decrease exponentially with the size of the cluster Y (times β) in the sense explained earlier, $z_C(W)$ depends essentially only on variables W(b') attached to links b' near the path C.

In conclusion, the factor $z_C(W)$ may be interpreted as a renormalization of the factor tr W(C) in (11.2). Otherwise, formula (11.2) looks entirely natural in view of the general discussion[32] of locality properties in gauge field theories.

REFERENCES

1. A.M. Polyakov, Phase transitions and quark confinement, internal ICTP-report IC/78/4, Trieste (Feb. 1978).
   G. 't Hooft, reference 16 below.
   G. Mack, references 6, 8 below.
2. J. Kogut and L. Susskind, <u>Phys. Rev</u>. D11:395 (1975).
3. K. Wilson, <u>Phys. Rev</u>. D10:2445 (1974).
   K. Osterwalder, E. Seiler, <u>Ann. Phys</u>. (N.Y.) 110:440 (1978), and references given there.
   M. Lüscher, <u>Commun. Math. Phys</u>. 54:283 (1977).
4. J. Glimm, lectures presented at this school.
   J. Glimm and A. Jaffe, <u>Nucl. Phys</u>. B149:49 (1979).
5. E.S. Fradkin and S.H. Shenker, <u>Phys. Rev</u>. D19:3682 (1979).

6. G. Mack, Quark and colour confinement through dynamical Higgs mechanism, DESY-report DESY 77/58, Hamburg (Aug. 1977).
7. J. Finger, D. Horn and J.E. Mandula, Quark condensation in QCD, preprint, Cambridge/Princeton (1979).
8. G. Mack, Phys. Letters B78:263 (1978).
9. G.F. De Angelis, D. De Falco and F. Guerra, Acta Physica Austriaca Suppl. XIX:205 (1978), and references in ref. 5 above.
10. D. Ruelle, "Statistical mechanics: rigorous results", Benjamin, New York (1969).
11. Ch. Gruber and A. Kunz, Commun. Math. Phys.22:133 (1971).
12. G. Mack and V.B. Petkova, Comparison of lattice gauge theories with gauge groups Z_2 and SU(2), DESY-report DESY 78/68 Hamburg, (Nov. 1978), to appear in Ann. Phys. (N.Y.)
13. J. Fröhlich, R. Israel, E.H. Lieb and B. Simon, Commun. Math. Phys.62:1 (1978).
B. Simon, New rigorous existence theorems for phase transitions in model systems, preprint, Princeton (1977).
14. E. Wegner, J. Math. Phys. 12: 2259 (1971).
15. B. Balian, J.M. Drouffe and C. Itzykson, Phys. Rev. D10:3376 (1974); D11:2098 (1975).
16. G. 't Hooft, Nucl. Phys. B138:1 (1978).
17. J.M. Drouffe, to appear.
18. J. Fröhlich, Phys. Letters 83B:195 (1979).
19. C.P. Korthals Altes, Inequalities for order and disorder parameters in SU(N) gauge theories, preprint, Utrecht (May 1979).
20. G. Mack, Commun. Math. Phys. 65:91 (1979).
21. R.B. Griffiths, J. Math. Phys. 8:978 (1967).
D.G. Kelly and S. Sherman, J. Math. Phys.9:466 (1968).
J. Ginibre, Commun. Math. Phys.16: 310 (1970).
22. A. Messager, S. Miracle Solé and C. Pfister, Commun. Math. Phys. 58:19 (1978).
23. G. 't Hooft, lectures presented at this school.
24. R. Peierls, Proc. Camb. phil. Soc. 32:477 (1936).
D. Ruelle, reference 10.
25. R. Marra and S. Miracle Solé, Commun. Math. Phys.67:233 (1979).
26. M. Göpfert, Peierls arguments for Z(2) lattice gauge theory, Diplomarbeit and DESY-report, Hamburg (in preparation).
27. M. Fisher, J. Appl. Phys. 38:981 (1967).
28. R.L. Dobrushin and S.B. Shlosman, Commun. Math. Phys.42:31 (1975).
29. G. Mack and V.B. Petkova, Sufficient condition for confinement of static quarks by a vortex condensation mechanism, DESY-report DESY 78/69 Hamburg (Nov. 1978), to appear in Ann. Phys. (N.Y.).
30. G. 't Hooft, Nucl. Phys. B153:141 (1979).
31. H.C. Tze and Z.F. Ezawa, Phys. Rev. D14:1006 (1976).
F. Englert, Cargèse lectures 1977, in: Hadron structure and lepton-hadron interactions, M. Lévy et al., eds., Plenum Press, New York (1979).
F. Englert and P. Windey, Les Houches lectures 1978.
Z.F. Ezawa, Quantum field theory of vortices and instantons, talk presented at DESY (June 1977); Phys. Rev. D18:2091 (1978).

32. G. Mack, Physical principles, geometrical aspects and locality properties in gauge field theories, Max Planck Institut preprint München (1979).
33. G. Münster, On the characterization of the Higgs phase in lattice gauge theories, DESY-report DESY 79/72 Hamburg (Nov. 1979).
34. G. Gallavotti, F. Guerra and S. Miracle Solé, in: Lecture Notes in Physics 80, Springer, Heidelberg (1978).
35. G. Münster, Ph.D. thesis and DESY-report, Hamburg (in preparation).
36. L.G. Yaffe, Confinement in SU(N) lattice gauge theories, preprint, Princeton University (Nov. 1979).
37. T. Yoneya, Nucl. Phys. B144:195 (1978).
38. K. Wilson, lectures presented at this school.
M. Creutz, Phys. Rev. Letters 43:553 (1979).
K. Binder, Monte Carlo investigations of phase transitions and critical phenomena, in: Phase transitions and critical phenomena, C. Domb and M. Green, eds., Academic Press, New York (1976), vol. 5B.
39. J. Glimm and A. Jaffe, Commun. Math. Phys. 56:195 (1977); Phys. Letters 66B:67 (1977)
A. Jaffe, Lattice instantons: What they are and why they are important, in: Lecture Notes in Physics 80, Springer, Heidelberg (1978).
40. H. B. Nielsen and P. Olesen, Nucl. Phys. B61:45 (1973)
41. W. Feller, "An introduction to probability and its applications" Wiley, New York (1961), vols. I,II.
B. Simon, "Functional integration and quantum physics", Academic Press, New York (1979).
M.C. Reed, Functional analysis and probability theory, in: Lecture Notes in Physics 25, Springer, Heidelberg (1973).
42. A. Ukawa, P. Windey and A.H. Guth, Dual variables for lattice gauge theories and the phase structure of Z(N) systems, preprint Princeton/Cornell (1979).
43. G. Mack and V.B. Petkova, Z_2 monopoles in the standard SU(2) lattice gauge theory model, DESY-report DESY 79/22 Hamburg (Apr. 1979), unpublished.
44. G.S. Rushbrooke, G.A. Baker and P.J. Woods, in: Phase transitions and critical phenomena, C. Domb and M. Green, eds., Academic Press, New York (1974), vol. 3.
C. Domb, ibid., p. 77; G.S. Rushbrooke, J. Math. Phys. 5:1106 (1964)
45. F. Vinciarelli, Phys. Letters B78:485 (1978).
46. J.R. Fontaine and Ch. Gruber, Commun. Math. Phys. 70:243 (1979), and references given there.
47. C. Callan and D. Gross, Phys. Rev. D16:2526 (1977).
48. J. Goldstone and R. Jackiw, Phys. Letters B74:81 (1978).