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S MATRIX THEORY OF THE MASSIVE THIRRING MODEL

by

B. Berg
II. Institut für Theoretische Physik der Universität Hamburg

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B.Berg
(II.Institut für Theoretische Physik, Universität Hamburg)

Abstract:

The S matrix theory of the massive Thirring model, describing the exact quantum scattering of solitons and their boundstates, is reviewed. Treated are: Factorization equations and their solution, boundstates, generalized Jost functions and Levinson's theorem, scattering of boundstates, "virtual" and anomalous thresholds.

+) Invited lecture given at the "International colloquium of complex analysis, microlocal calculus and relativistic quantum field theory" Les Houches, Sept.3 - Sept.13, 1979, to be published by Springer (Berlin Heidelberg New York).

CONTENTS

I. Introduction

For a variety of two dimensional relativistic field theories physicists know now the exact quantum S matrix. I like to call these models "quantum field theories with soliton behaviour". Examples are the Sine-Gordon theory (alias massive Thirring model)/1,2/, the O(N) non-linear σ-model /3/, the Gross Neveu model /4/, supersymmetric extensions of the last two models /5/ and the chiral invariant SU(N) Thirring model /6,7/.

I will up to some side remarks concentrate on the massive Thirring model (MTM), because it is the prototype of a field theory with soliton behaviour and it's S matrix theory is with respect to a number of properties best understood. Someone who understands this model should have no difficulties with the original literature concerning the other models. I also have tried to be complementary to the review of Zamolodchikov and Zamolodchikov /8/, which relies on the results of ref./1-4/ and previous literature.

Following the original development, we use the method of factorization equations to derive the S matrix and the spectrum of the MTM. This is "old fashioned", because it is now possible to get the same results in the more fundamental Hamiltonian approach by means of the Bethe ansatz. This is treated in the lectures of Honerkamp. So far one has, however, not succeeded to solve the Bethe ansatz for all models where the exact S matrix is known.

Our lecture is concerned with details of the MTM S matrix. The S matrix is calculable, because in the MTM infinitely many higher conserved classical currents survive quantization /9/ and imply /1o/ factorization of the S matrix (section II). Although there is only elastic scattering, nevertheless the S matrix exhibits a suprisingly rich structure. After solving the factorization equations for soliton-antisoliton scattering (section III) one discovers the boundstate(breather) spectrum, which was previously known from the quasiclassical treatment/11/, to be described through simple poles in the physical sheet of the forward scattering amplitude. In the nonrelativistic limit this is consistent with the two dimensional version of Levinson's theorem. More generally it can be shown /12/, that for the relativistic S matrix Jost functions

exist in close analogy to potential scattering (section IV). The distinction between boundstate and CDD poles is also discussed in section IV.

The complete S matrix of the MTM /13/ includes the amplitudes for soliton-breather and breather-breather scattering (section V). In these channels of the complete S matrix one finds double poles, which populate the physical sheets of the forward scattering amplitudes. They have a natural interpretation as anomalous thresholds /14/ (section VI). Another interesting feature arises from the fact, that breather-breather boundstates have to leave the spectrum of physical particles at a value of the coupling, where their mass is two times the soliton mass and still less than the sum of the masses of their breather constituents. This gives rise to a "virtual threshold" (section VI).

In summary the S matrix theory of the MTM is well understood and provides a lot of material, which could be used for illustration in an introductory course about S matrix theory. An outlook is given in the final section VII.

II. The factorization equations

The MTM is described by the Lagrangian

$$\mathcal{L} = \bar{\Psi} i \gamma_\mu \partial^\mu \Psi - m \bar{\Psi}\Psi - \tfrac{1}{2} g j_\mu j^\mu \quad \text{with} \quad j_\mu = \bar{\Psi}\gamma_\mu \Psi \qquad (1.a)$$

of a selfinteracting fermi field in 1 + 1 dimensions. By Coleman's /15/ equivalence the MTM is known to be equivalent to the Sine Gordon (SG) theory described by the Langrangian

$$\mathcal{L} = \partial_\mu \varphi \partial^\mu \varphi + \frac{\alpha}{\beta^2}\left(\cos(\beta\varphi) - 1\right) \qquad (1.b)$$

of aselfinteracting bose field. The correspondence is such that the elementary fermions $f, \bar{f}$ of the MTM correspond to the solitons $s, \bar{s}$ of the SG theory. The elementary SG boson corresponds in the MTM to the lowest $f\bar{f}$ boson boundstate b_1.

The coupling constants have to be defined in terms of physical quantities. Using the parametrization of /15/ the coupling constants are related by

$$\gamma = \pi \left(\frac{4\pi}{\beta^2} - 1 \right) \qquad (2)$$

α has dimension $(\text{mass})^2$ and is related to the MTM fermion mass.

The SG theory is classically completely integrable by means of the inverse scattering method /16/. This implies the existence of an infinite set of higher conservation laws /16/ which govern the dynamics of the model. Their physical meaning is conservation of higher powers of in- and outgoing momenta, i.e.

$$(p_1^{in})^m + (p_2^{in})^m + \dots + (p_n^{in})^m = (p_1^{out})^m + (p_2^{out})^m + \dots + (p_{n'}^{out})^m \quad (m=1,3,5..)$$

for $n \to n'$ particle scattering, similar higher conserved currents exist in the classical (Grassmann) MTM /17/. In the quantized theory these conservation laws survive to all orders of perturbation theory /9/ and have the following severe consequences for particle scattering:

a) There is no production or annihilation.
b) The sets of incoming and outgoing momenta are identical
$$\{p_i^{in}, \dots, p_n^{in}\} = \{p_1^{out}, \dots, p_n^{out}\}.$$

Relying on axiomatic S matrix theory it has been proven rigorously by Iagolnitzer /1o/ that properties a) and b) imply factorization of the S matrix in terms of two particle scattering amplitudes.
Roughly:

$$S_n(p_1 \dots p_n) = \prod_{i<j} S_2(p_i, p_j) \qquad (3)$$

This factorization has first been recognized for the SG boson in SG perturbation theory ($1/\beta^2$ small) /18/ and then also been verified for the MTM fermion in MTM perturbation theory (g small) /19/.

As it stands equation (3) is not well defined. Let us consider in detail elastic fermion-fermion and fermion-antifermion 2- particle scattering. Energy momentum conservation $p_1+p_2 = p_1' + p_2'$ and the mass shell condition $p_i^2 = p_i'^2 = m^2$ $(i = 1,2)$ imply that only forward $(p_1=p_1'$, $p_2 = p_2')$ and (for distinguishable particles) backward $(p_1 = p_2'$, $p_2=p_1')$ scattering is possible. The S matrix acts on 2-particle states by

$$|p_1, p_2\rangle^{in} = S\,|p_1, p_2\rangle^{out} = u(\hat{s})\,|p_1, p_2\rangle^{out} \qquad (4.a)$$

and

$$|p_1, \bar{p}_2\rangle^{in} = S\,|p_1, \bar{p}_2\rangle^{out} = t(\hat{s})\,|p_1, \bar{p}_2\rangle^{out} + r(\hat{s})\,|\bar{p}_1, p_2\rangle^{out} \qquad (4.b)$$

Here $\hat{s} = (p_1 + p_2)^2$; particles with momentum p_i are denoted p_i and antiparticles with momentum p_i are denoted $\bar{p}_i$.

The transmission amplitudes $u = u(s)$, $t = t(s)$ and the reflection amplitude $r = r(s)$ are supposed to be real meromorphic functions $\bar{f}(s)=f(\bar{s})$) ($f = u,t,s$) of the complex Mandelstam variables, which are defined on an (possibly) infinitely many sheeted Riemann surface with branching points at o and $4\ m^2$. By standard convention the cuts are choosen from $-\infty$ to o and from $4m^2$ to ∞. Because of the conservation laws no other singularities are required. The assumption is that only the minimal singularities are realized. By putting the considerations about Levinson's theorem (section IV) and anomalous thresholds (section VI) at the beginning one would have good arguments for this. The physical values $\hat{s}$ of s are real and above the elastic threshold: $\hat{s} = (p_1+p_2)^2 \geq 4m^2$.
Following the standard convention, the physical values of the amplitudes are obtained as limit

$$\lim_{\varepsilon \to 0^+} f(\hat{s} + i\varepsilon) \qquad (f=u,t,s)$$

on a certain Riemannian sheet, which is called the physical sheet.

The further discussion is much simplified by introducing the complex rapidity variable θ defined by

$$\operatorname{ch} \theta = \frac{s-2m^2}{2m^2} \qquad (5)$$

The variable θ plays the role of a global uniformizing variable of the amplitudes. The conformal mapping (5) is one-to-one between the infinitely many sheeted complex s-plane and the complex θ-plane. The physical sheet is required to be mapped into the strip $0 < \operatorname{Im}\theta < \pi$ of the θ -plane. All other sheets are mapped into similar strips above or below the physical strip (cf.Fig.1.a,b)). The amplitudes are defined as functions of θ by $f(\theta) = f(s(\theta))$. The Hermitian analyticity $\bar{f}(s)=f(\bar{s})$ carries over into $\bar{f}(\theta)=f(-\bar{\theta})$.

In the s-plane the crossing and unitarity relations of the amplitudes are

$$\lim_{\varepsilon \to 0^+} u(s+i\varepsilon) = \lim_{\varepsilon \to 0^+} t(s-4m^2-i\varepsilon)$$

$$\lim_{\varepsilon \to 0^+} r(s+i\varepsilon) = \lim_{\varepsilon \to 0^+} r(s-4m^2-i\varepsilon)$$

and

$$\lim_{\varepsilon \to 0^+} u(s+i\varepsilon)\cdot u(s-i\varepsilon) = 1$$

$$\lim_{\varepsilon \to 0^+} \left[t(s+i\varepsilon)\cdot t(s-i\varepsilon) + r(s+i\varepsilon)\cdot r(s-i\varepsilon)\right] = 1$$

$$\lim_{\varepsilon \to 0^+} \left[t(s+i\varepsilon)\cdot r(s-i\varepsilon) + r(s+i\varepsilon)\cdot t(s-i\varepsilon)\right] = 0$$

Using the Schwarz reflection principle we obtain from these relations, that the following equations hold for all complex θ.

Crossing:

$$u(\theta) = t(i\pi - \theta) \qquad (6.a)$$

$$r(\theta) = r(i\pi - \theta) \qquad (6.b)$$

Unitarity:

$$u(\theta)\cdot u(-\theta) = 1 \qquad (7.a)$$

$$t(\theta)\cdot t(-\theta) + r(\theta)\cdot r(-\theta) = 1 \qquad (7.b)$$

$$t(\theta)\cdot r(-\theta) + r(\theta)\cdot t(-\theta) = 0 \qquad (7.c)$$

We are now prepared for stating precisely the factorization equations. The 2-particle S matrix is defined to act on a multiparticle state by means of

$$S_2(p_i,p_j)\,|p_1 .. p_i .. p_j .. p_n\rangle^{out} = u(\hat{s}_{ij})\,|p_1 .. p_i .. p_j .. p_n\rangle^{out}$$

and

$$S_2(p_i,p_j)\,|p_1 .. p_i .. \bar{p}_j .. p_n\rangle^{out} = t(\hat{s}_{ij})\,|p_1 .. p_i .. \bar{p}_j .. p_n\rangle^{out} + r(\hat{s}_{ij})\,|p_1 .. \bar{p}_i .. p_j .. p_n\rangle^{out}$$

where $\hat{s}_{ij} = (p_i - p_j)^2$.

In lowest order MTM perturbation theory (cf.e.g./19/) one may check easily, that the reflection amplitude does not vanish and the 2-particle S matrix is really a matrix. The 2-particle S matrices of different arguments do not commute and therefore the factorization equation (3) makes only sense if we specify the order of the factors on the r.h.s. For this task it is useful to consider elastic scattering of well localized wave packets such that all the $\frac{1}{2}n(n-1)$ 2-particle interaction points are far apart from another. Let us assume $p_1^1 > p_2^1 > ... > p_n^1$, then an order which corresponds to a physically possible scattering process is

$$S_n(p_1 .. p_n) = \prod_{i=1}^{n-1} \left(\prod_{j=i+1}^{n} S_2(p_i,p_j) \right) \qquad (8)$$

The product factors have to be written down from left to right. For 3-particle scattering (cf.fig.2a) this reads

$$S_3(p_1,p_2,p_3) = S_2(p_1,p_2)\, S_2(p_1,p_3)\, S_2(p_2,p_3) \qquad (9)$$

By parallel shifting of particle lines ordering procedures equivalent to (8) can be obtained, which have to give the same result for the n-particle S matrix. Already the restriction obtained from 3-particle scattering enables factorization of n-particle scattering. By shifting one line the equivalent equation to (9)(cf.fig.2b)) implies

$$S_2(p_1,p_2)\, S_2(p_1,p_3)\, S_2(p_2,p_3) = S_2(p_2,p_3)\, S_2(p_1,p_3)\, S_2(p_1,p_2) \qquad (1o)$$

Equations of this type are known for a long time /2o/ as consistency equations for factorization of multiparticle amplitudes in non-relativistic δ-potential scattering. It has, however, only been recognized much later by Karowski et al /2/, that one may go the other way round and use these equations as input for the calculation of exact (even relativistic) S matrices.

Applying (1o) to $|p_1,p_2,\bar{p}_3\rangle^{in}$ and looking for the final state $|p_1,\bar{p}_2,p_3\rangle^{out}$ the following severe restriction on the scattering amplitudes is obtained:

$$h(\theta_{31}+\theta_{12}) = h(i\pi+\theta_{31})\cdot h(\theta_{12}) + h(\theta_{31})\cdot h(i\pi - \theta_{12})$$

where $h(\theta) = \frac{t(\theta)}{r(\theta)}$ and $\theta_{31} = -\theta_{13}$. For $i<j$ the rapidity variables θ_{ij} are defined by $ch\,\theta_{ij} = \frac{s_{ij} - 4m^2}{2\,m^2}$ (cf.(5)).

III. The exact S matrix

We are now able to derive the main result. Let us first note, that equation (11) has the unique solution

$$h(\theta) = \frac{\operatorname{sh}\lambda\theta}{\operatorname{sh} i\pi\lambda} \qquad (12)$$

here λ is an arbitrary real parameter, which will become related to the coupling constant. The proof of (12) is similar to that of the addition theorem for the exponential function although slightly more complicated. λ is real because $\bar{h}(\theta) = h(-\bar{\theta})$.

Let us assume the crossing and unitarity relations (6),(7) and equation (12). Under the following additional two assumptions the scattering amplitudes become uniquely determined:

(a) The phase $\delta(\theta)$ defined by $u(\theta) = |u(\theta)| e^{i2\delta(\theta)}$ is bounded in the physical strip.

(b) There exists a repulsive region in the parameter λ, where $t(\theta)$ has no poles and zeros in the physical strip and for arbitrary λ $t(\theta;\lambda)$ can be obtained by analytic continuation from the repulsive region.

The explicit results for the amplitudes are:

$$t(\theta) = \frac{F(\theta)}{F(2\pi i - \theta)} \qquad (13.a)$$

where

$$F(\theta) = \prod_{k=1}^{\infty}\prod_{l=0}^{\infty} \frac{\left(2l + \frac{k}{\lambda} + \frac{\theta}{i\pi}\right)\left(2l + \frac{k-1}{\lambda} + \frac{\theta}{i\pi}\right)}{\left(2l - 1 + \frac{k}{\lambda} + \frac{\theta}{i\pi}\right)\left(2l + 1 + \frac{k-1}{\lambda} + \frac{\theta}{i\pi}\right)} .$$

One could write $F(\theta)$ as infinite product of Γ-functions, but this is not really a simplification. The other amplitudes are

$$r(\theta) = \frac{t(\theta)}{h(\theta)} = \frac{\sin\pi\lambda}{\sin\pi\lambda\frac{\theta}{i\pi}}\, t(\theta) \qquad (13.b)$$

and

$$u(\theta) = t(i\pi - \theta) = \frac{\sin\pi\lambda\left(1 - \frac{\theta}{i\pi}\right)}{\sin\pi\lambda\frac{\theta}{i\pi}}\, t(\theta) \qquad (13.c)$$

Remark:

For the MTM the region with $g < o$ is known to be repulsive (cf.e.g./15/). The assumptions of a bounded phase and the absence of poles and zeros of $t(\theta)$ in the physical stripe for the repulsive region are required by Levinson's theorem (cf.next section).

Proof of (13.a):

Unitarity and crossing relations yield

$$\frac{h(\theta)}{h(i\pi - \theta)} = t(\theta)\, t(i\pi + \theta) \qquad (14)$$

Let C denote a contour, which is identical with the boundary of the physical strip , except that possible poles or zeros of $t(\theta)$ on the boundary of the physical stripe are circumvented by small half-circles (lying inside the physical strip) and let θ be an element of the physical strip . Under the done assumptions we have in the repulsive region the integral representation

$$\ln t(\theta) = \frac{1}{2\pi i}\int_C \frac{dz}{\operatorname{sh}(z-\theta)} \ln t(z)$$

$$= \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{dx}{\operatorname{sh}(x-\theta)} \ln\left(\frac{\operatorname{sh}\lambda x}{\operatorname{sh}\lambda(i\pi - x)}\right)$$

The last equality is obtained by using (12), (14).
After some calculation one obtains

$$t(\theta) = \exp i \int_0^{\infty} \frac{dx}{x} \frac{\operatorname{sh}\frac{x}{2}(1-\lambda)}{\operatorname{sh}\frac{x}{2}\operatorname{ch}\frac{x}{2}\lambda} \sin\frac{x\lambda}{\pi}(i\pi - \theta)$$

Using Malmstén's formula /21/, we finally arrive at (13.a).

Discussion of the result:

From (13a) we can read off the poles and zeros of $t(\theta)$ by looking for the zeros of the factors. One immediately recognizes, that all poles and zeros are lying on the imaginary θ-axis. Simple linear relations between $1/\lambda$ and Re φ with $\varphi = \frac{\theta}{i\pi}$ are obtained. They are drawn in fig.3. In the physical strip there are only simple poles given by

$$\theta_k = i\pi\left(1 - \frac{k}{\lambda}\right) \qquad (k=1,2,\ldots, [\lambda]) \qquad (15.a)$$

$[\lambda]$ is the largest integer smaller then λ . The poles describe $f\bar{f}$ boundstates (bosons). By translating the θ-dependence back in the s-plane we obtain a spectrum of quasiclassical type /11/

$$m_k = 2m\sin\left(\pi\frac{k}{2\lambda}\right) \qquad (k=1,2,\ldots, [\lambda]) \qquad (15.b)$$

With the relationship

$$\lambda = 1 + \frac{2g}{\pi} = \frac{8\pi}{\beta^2} - 1 \qquad (16)$$

between λ and the MTM coupling g (SG coupling β, cf.(2)) the quasiclassical spectrum becomes exact as has been conjectured in ref./11/. Before the exact S matrix was known, the spectrum (15) has also been derived by lattice methods /22/ and by assuming factorisation for the SG boson S matrix /23/.

In contrary to the spectrum the quasiclassical S matrix /24/ is only exact for integer λ. These are precisely the points, where the reflection amplitude (13.b) vanishes and a new particle (cf.(15.a)) enters the physical spectrum. Zamolodchikov /1/ first obtained the exact MTM S matrix by looking for the simplest analytic interpolation between these points, which has a nonvanishing reflection amplitude. It has been checked /25/ that $t(\theta)$ agrees up to third order in g with MTM perturbation theory.

IV Boundstates and Levinsons's theorem

In the previous section we have associated simple poles (we consider always poles and zeros in the physical strip now) of the amplitude $t(\theta)$ with $f\bar{f}$ boundstates. It is, however, known in potential scattering, that only those simple poles of the S matrix describe boundstates, which correspond to zeros in a related Jost function. In the nonrelativistic limit of the MTM $f\bar{f}$ scattering is known to be described by a smooth potential /26/.One may therefore conjecture, that results similar to potential scattering hold for the relativistic MTM S matrix. This question has been answered affirmative by explicit inspection of the S matrix /12/.

Let us first review potential scattering in one space dimension (e.g. /27/) for a well behaved symmetric potential V(x):

$$\int_{-\infty}^{+\infty} dx\ (1+|x|)\ |V(x)| < \infty \quad \text{and} \quad V(x) = V(-x).$$

The Jost solutions f(x,k) of the Schrödinger equation

$$-\frac{1}{2m}\frac{d^2}{dx^2}\Psi(x) + V(x)\,\Psi(x) = E\,\Psi(x), \qquad E=\frac{k^2}{2m}$$

are defined by the boundary condition

$$f(x,k) \approx e^{ikx} \qquad \text{as} \qquad x \to \infty$$

f(x,k) is analytic in the open and continuous on the closed upper half of the complex k-plane. The scattering solution of the Schrödinger equation corresponding to an incoming plane wave e^{-ikx} from the right is given by

$$\Psi_r(x,k) = f(x,-k) + r(k)\cdot f(x,k) = t(k)\cdot f(x,-k)$$

where t and r are the transmission and reflection coefficients. The parity even (+) and odd (-) channels of the S matrix are

$$s_\pm(k) = t(k) \pm r(k).$$

They obey the unitarity condition $|s_\pm(k)| = 1$ for k real. The Jost functions of the parity $\pm$ channels are defined to be ($f_x = \frac{df}{dx}$)

$$f_+(k) = \frac{f_x(0,k)}{ik} \qquad \text{and} \qquad f_-(k) = f(0,k).$$

It can be proved (e.g. /12,27/), that the Jost functions are related to the amplitudes by

$$s_\pm(k) = \frac{f_\pm(-k)}{f_\pm(k)} \tag{17}$$

and

$$t(k) = \frac{1}{f_+(k)\cdot f_-(k)} \tag{18}$$

Furthermore it can be proved, that the Jost functions $f_\pm(k)$ have in the upper half complex **k-plane no** poles and only a **finite** number of simple zeros. The zeros are located on the imaginary k-axis. They are different for f_+ and f_- and correspond to boundstates with even or odd wave functions respectively. If a zero is located at $k=i\varkappa$, $\varkappa > 0$ the energy of the corresponding boundstate is $E = -\varkappa^2$. We recognize that all poles in t(k) correspond to physical boundstates, whereas in the $s_\pm$ channels redundant poles are possible.

We now state Levinson's theorem. For real k the phase shifts defined modulo π by

$$s_\pm(k) = e^{2i\delta_\pm(k)}$$

are real functions. For the Jost functions $\bar{f}_\pm(k) = f_\pm(-\bar{k})$ holds and the phase shifts are determined modulo 2π (cf.(17)) from

$$f_\pm(k) = |f_\pm(k)|\ e^{-i\delta_\pm(k)}.$$

The functions $f_\pm(k)$ are analytic in the upper half plane and we can apply the argument principle to get the numbers of parity $\pm$ boundstates

$$n_{\pm} = \frac{1}{2\pi i} \int_C dk \frac{f_{\pm}'(k)}{f_{\pm}(k)} = \frac{1}{\pi} \left(\delta_{\pm}(o) - \delta_{\pm}(\infty) \right) - \frac{1}{2}\alpha_{\pm} \qquad (19.a)$$

Here the contour C encloses the upper half plane, avoiding k=o on a small half circle inside the upper half plane, where the behaviour $f(k) \approx k^{\alpha_{\pm}}$ for $k \to o$ is assumed. The integers $\alpha_{\pm}$ can be obtained (e.g./12/) from the threshold behaviour of $s_{\pm}$ via

$$\alpha_{\pm} = \pm \frac{1}{2} \left(s_{\pm}(o) - 1 \right) \qquad (19.b)$$

Equations (19) are Levinson's theorem for one dimensional potential scattering.

In the nonrelativistic limit of the MTM the physical strip of the complex θ-plane becomes the upper half complex k-plane. Let us now turn to the relativistic case.

Because of CPT and T invariance parity $\pm$ eigenstates in the MTM are identical with $C=\pm 1$ eigenstates

$$|\pm\rangle = \frac{1}{2} \left(|p_1, \bar{p}_2\rangle \pm |\bar{p}_1, p_2\rangle \right) \qquad (21)$$

On these states the two-particle S matrix becomes diagonal

$$S_2 = \begin{pmatrix} s_+ & \\ & s_- \end{pmatrix} \qquad \text{with} \qquad s_{\pm} = t \pm r .$$

In the $s_{\pm}$ channels the unitarity relations (7.b,c) read

$$s_+(\theta) \cdot s_+(-\theta) = s_-(\theta) \cdot s_-(-\theta) = 1 .$$

Explicit formulas for $s_{\pm}$ are

$$s_+(\theta) = - \frac{\mathrm{sh}\frac{\lambda}{2}\ (\theta + i\pi)}{\mathrm{sh}\frac{\lambda}{2}\ (\theta - i\pi)}\ t(i\pi - \theta) \qquad (22.a)$$

and

$$s_-(\theta) = - \frac{\mathrm{ch}\frac{\lambda}{2}\ (\theta + i\pi)}{\mathrm{ch}\frac{\lambda}{2}\ (\theta - i\pi)}\ t(i\pi - \theta) \qquad (22.b)$$

We note from these equations that the $s_{\pm}$ amplitudes have in the physical strip only poles for certain imaginary θ and no zeros. Let us exclude

λ =integer for the moment, then the poles are given graphically as function of λ in fig.4a,b). The boson boundstates (cf.(15)) energe as simple poles in s_- for k=odd and in s_+ for k=even. Therefore the boundstates have parity $P=(-1)^k$. This is consistent with the opinion that the lowest (=lightest) boundstate b_1 is a pseudoscalar, which corresponds to the elementary SG field.

It is not difficult to compute the residua of the poles with the result

$$(+i) \cdot \operatorname*{Res}_{\theta=\theta_k} s_+ = R_k > o \qquad \text{for} \quad P=+1 \qquad \text{(i.e. k even)} \qquad (23.a)$$

$$(+i) \cdot \operatorname*{Res}_{\theta=\theta_k} s_- = R_k < o \qquad \text{for} \quad P=-1 \qquad \text{(i.e. k odd)} \qquad (23.b)$$

Here $\operatorname*{Res}_{\theta=\theta_k} f(\theta) = +\lim_{\theta \to \theta_k} (\theta - \theta_k) \cdot f(\theta)$. This is in agreement with the result+):

a) Symmetric wave functions correspond to negative and antisymmetric wave functions correspond to positive residua. This is required by general S matrix theory (e.g./28/); poles with the wrong residuum give rise to ghosts with negative norm.

b) Fermion-antifermion boundstates of parity P=-1 have symmetric wave functions and of parity P=+1 have antisymmetric wave functions (e.g./29/). Putting a) and b) together we precisely arrive at (23).

Besides the physical boundstate poles we notice in $s_{\pm}$(fig.4) further poles which are redundant. Following the suggestions of potential scattering (17),(18) one can introduce "generalized Jost functions" by the ansatz

$$s_{\pm}(\theta) = \frac{g_{\pm}(-\theta)}{g_{\pm}(\theta)} \qquad (24)$$

and

$$t(\theta) = \frac{1}{g_+(\theta) \cdot g_-(\theta)} \qquad (25)$$

+) Formulated in the s-plane. Residua in s- and θ-plane are related by $(i\alpha) \operatorname*{Res}_{\theta_k} = \operatorname*{Res}_{s_k}$ with $\alpha > o$.

and finds the solution

$$g_+(\theta) = \prod_{l=0}^{\infty}\prod_{k=1}^{\infty} \frac{(2l-1+\frac{2k}{\lambda}+\frac{\theta}{i\pi})\cdot(2l+1+\frac{2k-1}{\lambda}+\frac{\theta}{i\pi})}{(2l+\frac{2k}{\lambda}+\frac{\theta}{i\pi})\cdot(2l+\frac{2k-1}{\lambda}+\frac{\theta}{i\pi})}$$

$$\times \frac{(2l+2+\frac{2k+1}{\lambda}-\frac{\theta}{i\pi})\cdot(2l+2+\frac{2k-2}{\lambda}-\frac{\theta}{i\pi})}{(2l+1+\frac{2k+1}{\lambda}-\frac{\theta}{i\pi})\cdot(2l+3+\frac{2k-2}{\lambda}-\frac{\theta}{i\pi})} \qquad (26)$$

and $g_-(\theta)=g_+(\theta-\frac{i\pi}{\lambda})$. The generalised Jost functions have no poles and for imaginary θ a finite number of simple zeros in the physical strip. Parity $\pm$ boundstates correspond presisely to these zeros of $g_\pm$.

The functions $g_\pm(\theta)$ can be used to obtain the generalized version of Levinson's theorem. The numbers of parity $\pm$ boundstates are

$$n_\pm = \frac{1}{2\pi i}\int_C d\theta \frac{g_\pm'(\theta)}{g_\pm(\theta)} = \qquad (27.a)$$

$$\frac{1}{\pi}(\delta_\pm(o) - \delta_\pm(\infty)) + \frac{1}{\pi}(\delta(o) - \delta(\infty)) - \frac{1}{2}\alpha_\pm$$

Here the contour C is identical with the boundary of the physical strip, except that $\theta=o$ is avoided on a small half circle inside the physical strip, where we have the behaviour $g(\theta)\approx\theta^{\alpha_\pm}$ for $\theta\to o$. The integers $\alpha_\pm$ can be obtained from the threshold behaviour of $s_\pm$ via

$$\alpha_\pm = \mp\frac{1}{2}(s_\pm(o) - 1) \qquad (27.b)$$

The sign difference compared with (19.b) relies on the different threshold behaviour of bosons (potential scattering) and fermions (present case), cf./12/. Equations (27) constitute the generalized Levinson theorem. By explicit calculation of the phase shifts we may obtain the already known results

$$n_+ = [\lambda/2], \qquad n_- = [(\lambda+1)/2], \qquad n_+ + n_- = [\lambda]$$

The term $\frac{1}{\pi}(\delta(o)-\delta(\infty))$, which reflects crossing, is defined by

$u(\theta) = |u(\theta)|\cdot e^{2i\delta(\theta)}$ and gives the contribution $\frac{1}{4}\cdot(\lambda-1)$.

In the semiclassical limit all results agree with those computed previously /3o/. This shows again, that the semiclassical method is reliable in the SG theory.

Let us close this section with some remarks on the so far excluded case λ =integer. For this case the reflection amplitude vanishes and we have

$$s_+(\theta) = s_-(\theta) = t(\theta)$$

Therefore poles with the wrong residuum (ghosts) appear in the $s_\pm$ channels. Following Karowski /31/ a consistent particle interpretation is guaranteed if there are no transitions between genuine particles and ghosts. Let us denote the ghost with mass m_n by b_n'. Anticipating method and notation of the next section the f $b_n\to$f b_n' transition amplitude is

$$_{out}\langle b_n' f | b_n f\rangle_{in} = u_{23}\cdot t_{13} - t_{23}\cdot u_{13}$$

This vanishes for $u=\pm t$, as is indeed the case for λ integer (13.c). Similarly all other transitions between genuine particles and ghosts vanish.

For S matrices with higher symmetries (e.g. the chiral SU(n) Thirring model /6/) analog considerations are quite restrictive and imply, that only a finite number of boundstate poles can be introduced consistently. Further there are poles in the physical strip possible, which are not interpreted as boundstates or ghosts. They are called CDD /32/ poles. If such poles are allowed, i.e.assumption b) of section III is given up, the solution of the factorization equations for the transmission amplitude becomes

$$T(\theta) = \prod_{i=1}^{L} \frac{sh\frac{1}{2}(\theta+\theta_i)\cdot ch\frac{1}{2}(\theta-\theta_i)}{sh\frac{1}{2}(\theta-\theta_i)\cdot ch\frac{1}{2}(\theta+\theta_i)}\cdot t(\theta)$$

where $t(\theta)$ is the previous (minimal) solution and the θ_i are imaginary. If the relativistic model (like the SG theory) is described by a well behaved potential in the nonrelativistic limit, CDD poles are excluded /33/, because they would be in contradiction to the existence of Jost functions.

V The complete S matrix

The complete S matrix includes scattering of boundstates. Relying on ref./13/, this section is devoted to the derivation of the boundstate scattering amplitudes.

The $f\bar{f}$ boson boundstate of mass m_k as given by (15.b) is denoted b_k. There is the possibility of boson-boson scattering $b_k b_l$ and of boson-fermion scattering $b_k f$. For the description of scattering of particles with different masses m_k, m_l we use the rapidity variable θ_{kl}, defined analog to (5) by

$$s_{kl} = m_k^2 + m_l^2 + 2m_k m_l \operatorname{ch}(\theta_{kl}). \quad (28)$$

s_{kl} is the complex Mandelstam variable, which for physical values reduces to $\hat{s}_{kl}=(P_k+P_l)^2$. The l.h. cut in the s_{kl}-plane starts at $s_{kl}=(m_k-m_l)^2$ and the r.h. cut at $s_{kl}=(m_k+m_l)^2$. We further use the variable

$$\varphi_{kl} = \frac{\theta_{kl}}{i\pi}. \quad (29)$$

If no misunderstanding is possible the indices k,l are dropped.

The fermion-fermion amplitude $u(\theta)$ (13.c) is continuous at the boundstate poles θ_k as may be seen from fig.3. Therefore and from the discussion of the $s_\pm$ amplitudes (cf. (22) ff.) we conclude, that in the fermion sectors the residua of the S matrix at $s_{12}= (P_1+P_2)^2=m_k^2$ (P_1 and P_2 are at unphysical values) act as projection operators. More presisely, the operators

$$P_{12}^k \overset{def}{=} \frac{1}{R_k} \operatorname*{Res}_{s_{12}=m_k^2} S(p_1,p_2) \quad (30)$$

project for $\lambda \neq$ integer onto chargeless states

$$\frac{1}{\sqrt{2}}\left(|p_1,\bar{p}_2\rangle + (-1)^k |\bar{p}_1,p_2\rangle \right)$$

with momentum $p=p_1+p_2$, mass m_k and parity $(-1)^k$. They are identified with the boundstates b_k.

Let us first consider the S matrix for boundstate-fermion scattering. It is defined by

$$S(p_1+p_2,p_3)\Big|_{\substack{(p_1+p_2)^2=m_k^2 \\ p_3^2=m^2}} = \frac{1}{R_k} \operatorname*{Res}_{s_{12}=m_k^2} S_{12}S_{13}S_{23} \quad (31.a)$$

Using the factorization equation (10) and the definition of the projector we find

$$\frac{1}{R_k} \operatorname*{Res}_{s_{12}=m_k^2} S_{12}S_{13}S_{23} = P_{12}^k S_{13}S_{23} = S_{23}S_{13}P_{12}^k = P_{12}^k S_{23}S_{13}P_{12}^k \quad (31.b)$$

Formulas (31) have a graphical interpretation as given in fig.5a). One can associate with b_k a fermion and antifermion with "parallel" momenta p_1 and p_2. Since for real and parallel momenta the mass shell condition $p_i^2=m^2$ implies $(p_1+p_2)^2=4m^2$, the momenta have to be complex and only the real parts are parallel. $p=p_1+p_2$ is real, because it is the physical momentum of b_k. A convenient choice is $p_1=(p^0, \mathrm{Re}\,p^1+i\,\mathrm{Im}\,p^1)$ and $p_2=(p^0, \mathrm{Re}\,p^1-i\,\mathrm{Im}\,p^1)$ with p^0 real. $(p_1+p_2)^2$ can take all values between 0 and $4m^2$ for convenient choices of $\mathrm{Im}\,p^1$.

We now calculate the amplitudes explicitly. There is no reflection, since the projector P_{12}^k appears on both sides of equation (31b). It is convenient to use amplitudes defined as functions of φ (cf.(29)) by $f(\varphi) \equiv f(\varphi(\theta))$. The boson-fermion scattering amplitude $t_{b_k f}(\varphi)$ becomes now

$$t_{b_k f}(\varphi) = {}^{out}\langle b_n f | S_{23}S_{13} | b_n f\rangle^{out}$$

$$= \frac{1}{2}\left(t_{23}u_{13} + (-1)^n r_{23}r_{13} + u_{23}t_{13}\right)$$

where $t_{23} = t(\varphi_{23})$ etc.

and $\varphi_{13} = \varphi + \frac{1}{2}(1-\frac{n}{\lambda})$, $\varphi_{23} = \varphi - \frac{1}{2}(1-\frac{n}{\lambda})$.

Using equations (13) for t,r and u one finds after some calculation

$$t_{b_k f}(\varphi) = (-1)^k \frac{\sin\frac{\pi}{2}(\varphi + \frac{k}{2\lambda} + \frac{1}{2})\ \sin\frac{\pi}{2}(\varphi - \frac{k}{2\lambda} + \frac{1}{2})}{\sin\frac{\pi}{2}(\varphi - \frac{k}{2\lambda} - \frac{1}{2})\ \sin\frac{\pi}{2}(\varphi + \frac{k}{2\lambda} - \frac{1}{2})} \times \left(\prod_{j=1}^{k-1} \frac{\sin\frac{\pi}{2}(\varphi + \frac{k-2j}{2\lambda} + \frac{1}{2})}{\sin\frac{\pi}{2}(\varphi - \frac{k-2j}{2\lambda} - \frac{1}{2})} \right)^2 \tag{32}$$

Similarly boson-boson scattering can be treated. The S matrix for scattering $b_n b_m$ is defined by

$$S(p_1+p_2, p_3+p_4) \Big|_{\substack{(p_1+p_2)^2=m_k^2 \\ (p_3+p_4)^2=m_1^2}} =$$

$$\frac{1}{R_k} \operatorname*{Res}_{s_{12}=m_k^2} \frac{1}{R_1} \operatorname*{Res}_{s_{34}=m_1^2} S_{12}S_{13}S_{14}S_{23}S_{24}S_{34}$$

cf.fig.6b).Again there is only forward scattering. After some calculation one finds for $k \leq 1$:

$$t_{b_k b_1}(\varphi) = \frac{tg\frac{\pi}{2}(\varphi + \frac{k+1}{2\lambda})\ tg\frac{\pi}{2}(\varphi + \frac{1-k}{2\lambda})}{tg\frac{\pi}{2}(\varphi - \frac{k+1}{2\lambda})\ tg\frac{\pi}{2}(\varphi - \frac{1-k}{2\lambda})} \times \left(\prod_{j=1}^{k-1} \frac{tg\frac{\pi}{2}(\varphi + \frac{k+1-2j}{2\lambda})}{tg\frac{\pi}{2}(\varphi - \frac{k+1-2j}{2\lambda})} \right)^2 \tag{33}$$

A consistency check is to build up the $t_{b_k b_1}$ amplitude from $b_k f\bar{f}$ scattering. The result is the same. This and analog checks were carried out by Karowski and Thun /13/. All amplitudes are in agreement with previous calculations /24/ in the semiclassical limit.

Particularly simple is the scattering amplitude for two elementary SG bosons b_1:

$$t_{b_1 b_1}(\varphi) = \frac{\sin\pi\varphi + \sin\frac{\pi}{\lambda}}{\sin\pi\varphi - \sin\frac{\pi}{\lambda}}$$

At $\lambda=3$ or equivalently $\beta^2=2\pi$, where $m_1=m$ and $m_2=m\sqrt{3}$, one reccgnizes /34/

$$-u(\theta) = -t(\theta) = t_{b_1 f}(\theta) = t_{b_1 \bar{f}}(\theta) = t_{b_1 b_1}(\theta)$$

and

$$t_{b_2 f}(\theta) = t_{b_2 \bar{f}}(\theta) = t_{b_1 b_2}(\theta) .$$

This can be interpreted as a hidden SU(2) symmetry. The particles $f, b_1, \bar{f}$ are in an isotriplet and b_2 is in an isosinglet. Using different methods this symmetry was first observed by Coleman /35/ in his investigation of the massive Schwinger model.

Finally let us consider the poles (there are no zeros) of $t_{b_k f}$ and $t_{b_k b_1}$ in the physical strip . In the boson fermion amplitude $t_{b_k f}$ there are two simple poles at

$$\varphi = \frac{\pm k}{2\lambda} + \frac{1}{2} \tag{34.a}$$

and double poles at

$$\varphi = \frac{\pm j}{2\lambda} + \frac{1}{2} \tag{34.b}$$

with j=o,2,.. ,k-2 if k is even and j=1,3.., k-2 if k is odd.

Let us choose $k \leq 1$ for the boson-boson amplitude $t_{b_k b_1}$. Then we find simple poles at

$$\varphi = \frac{k+1}{2\lambda}, \qquad \varphi = \frac{1-k}{2\lambda} \tag{35.a}$$

and double poles at

$$\varphi = \frac{k+1-2j}{2\lambda} \qquad (j=1,\dots,k-1) \tag{35.b}$$

Further we encounter the poles required by crossing symmetrie at $1-\varphi$.

For the special cases $t_{b_4 f}$ and $t_{b_2 b_4}$ the poles are drawn in fig. 6a) and b). At the first look it is surprising, that besides simple poles double poles are found. This and related questions are considered in the next section.

The simple pole at $\varphi = (k+1)/2\lambda$ in the boson-boson amplitude $t_{b_k b_1}$ has the interpretation as boson boundstate b_{k+1}. This is seen by calculating the associated mass and checking the residuum.

Also the simple pole in the boson-fermion amplitude $t_{b_k f}$ has a natural interpretation. It's associated mass is precisely m. The meaning is, that the fermion f can be interpreted as $b_k f$ boundstate. This is consistent with the picture /36/ of the SG soliton being a coherent boundstate of SG bosons. The soliton may "eat" a boson and remains a soliton. Again the check of the residuum is consistent.
The sign of the residuum R_b of a boundstate b is given by the compact formula (b=boundstate)

$$R_b \eta_b / (\eta_\alpha \eta_\beta) < 0 \qquad (36)$$

where $\eta_{b_k} = (-1)^k$, $\eta_f = \eta_{\bar{f}} = i$ and α, β are the boundstate constituents.

More recently /37/ the interpretation of the fermion as boson-fermion boundstate has been used to reconstruct the MTM fermion S matrix from the boundstate S matrices. This method can be used /38/, to obtain the kink S matrix of the Gross Neveu model, where the boundstate S matrix is known /4/.

VI Anomalous and virtual thresholds

The double poles (34.b), (35.b) of the previous section turn out /14/ to

be anomalous threshold. We follow closely ref./14/. Let us first recall what an anomalous threshold is.

For physical external momenta normal Landau singularities of the S matrix occur for those values of the momenta, for which one can draw a space-time graph of a process involving classical particles, which are all on mass shell, all moving forward in time, and interacting only through energy-momentum conserving interactions localized at a space-time point. The discontinuity of the singularity is obtained by the Cutkosky rules /39/. One evaluates the space-time graph as if it were a Feynman graph, with two exceptions:

a) The point interactions are replaced by actual S matrix elements.
b) Feynman propagators are replaced by mass-shell δ-functions, $\theta(p^o)\,\delta(p^2-m^2)$.

Singularities below threshold are called anomalous singularities. In this case one may choose the time components of the external momenta to be real and the space components to be imaginary. Such a two-vector is called pseudomomentum. The rules for finding an anomalous singularity are /4o/:

(1) The graph must be a geometrical figure in Euclidean space.
(2) With every internal line there must be associated a pseudomomentum of lenght m, the mass of the particle associated with this line.
(3) The pseudomomentum must be parallel to the associated line.
(4) All pseudomomenta at a given vertex must sum to zero.

Thus, locating an anomalous singularity becomes a problem in Euclidean plane geometry. As an example let us consider the graph of fig.7).

The Cutkosky rules show that in four dimensions the singularity is a branch point. We have an eight-dimensional integral over six δ-functions; thus we obtain a finite discontinuity over the cut.
In two dimensions we have only a four-dimensional integral. Thus we

have two δ-functions left after integration. The associated singularity is a double pole. This is best shown by considering the singularity of the Feynman diagram associated with fig.7. By momentum conservation, the two virtual particles carry equal momenta, which we denote p_c. Thus the Feynman integral contains two factors $1/(p_c^2-m_c^2)$ and can be written as derivative with respect to m_c of an integral with only one such factor. The discontinuity of the new integral is $(p_c^2-m_c^2)$. Thus the singularity of the new integral is proportional to $1/(p_c^2-m_c^2)$ and the singularity of its derivative is a double pole.

It remains to show, that for each double pole (34.b), (35.b) there exists an associated diagram and vice versa. For the decay of a particle A the angle $\varphi(a,c;A)$ defined in fig.8 is related to the rapidity of a,c:

$$m_A^2 = m_a^2 + m_c^2 + 2m_a m_c \cos\{\pi\varphi(a,c;A)\} .$$

For fig.9 to be a possible graph, the lines a and b must intersect, thus

$$\varphi(c,a;A) + \varphi(b,c;B) < 1 \qquad (37)$$

The angle φ of fig.9 is the location of the singularity in the t_{AB} amplitude. By elementary geometry it is determined to be

$$\varphi = 2 - \varphi(A,c;a) - \varphi(B,c;b) \qquad (38)$$

These equations are all we need to locate the double poles for any two dimensional forward scattering process corresponding to fig.7. All we have to do is to plug in the particle masses.

For the SG spectrum (15.b) the relevant angles of the three particle vertices are readily computed to be

$$\varphi(f,\bar{f};b_k) = 1 - \frac{k}{\lambda} \qquad (39.a)$$

$$\varphi(b_k,f;\bar{f}) = \frac{1}{2}(1+\frac{k}{\lambda}) \qquad (39.b)$$

$$\varphi(b_k,b_1;b_{k+1}) = \frac{k+1}{2\lambda} \qquad (39.c)$$

$$\varphi(b_1,b_{k+1};b_k) = 1 - \frac{k}{2\lambda} \qquad (39.d)$$

Using these rules all double poles can be explained. For example fermion-boson scattering: Let A be f and B be b . While the initial fermion can only decay into a fermion and a boson, the initial boson may decay into either a fermion-antifermion pair or two bosons. We will for illustration only consider the case, where the initial boson decays into a fermion-antifermion pair. Therefore $a=b_1$, $b=\bar{f}$ and $c=f$ in fig.7. By equation (37) and (38) we obtain $2k-1>\lambda$. And by equation (38) the locations of the double poles are given by

$$\varphi = \frac{1}{2} + \frac{j}{2\lambda}$$

with $j=2l-k$, $l=1,\ldots [2k-\lambda]$ this is seen to describe a part of the range given in (34.b). The other part of the range is obtained by considering the decay of the initial boson into bosons.

The discussion can be completed /14/ and explains all double poles in $t_{b_k f}$ and $t_{b_k b_1}$. Especially there are no anomalous thresholds in the ff and $f\bar{f}$ amplitudes, because (37) cannot be fulfilled for this case. The simple pole at $\varphi=\frac{1-k}{2\lambda}$ (35.a) is explained by the tree graph of fig.9.

Let us finally consider fig.6. The pole given by $\varphi=\frac{6}{2\lambda}$ corresponds for $\lambda>6$ to the mass m_6 of the $b_2 b_4$ boundstate b_6. At $\lambda=6$ the boundstate b_6 ceases to be present in the MTM spectrum of $f\bar{f}$ boundstates (15). Precisely at this point the pole at $\varphi=6/2\lambda$ and the crossing symmetric pole give a double pole. For $4<\lambda<6$ we have again two simple poles in the physical strip . The poles have, however, changed the sign of their residuum, such that an interpretation as boundstate is no longer natural. In a theory with production, there would be a branch point corresponding to the anomalous threshold $b_2 b_4 \rightarrow f\bar{f}$, the boundstate pole would move into the second sheet and become a resonance. This is impossible here, because of the conservation laws. The remnant is a virtual threshold in form of a double pole. Analog considerations apply to the b_{1+k} boundstate in the $t_{b_k b_1}$ amplitude.

VI Summary and outlook

The S matrix theory of the MTM is a well understood and fairly closed topic. A challenging open problem is the complete solution of the MTM and other models with soliton behaviour. Complete solution means construction of the Green's functions. Different approaches to this problem are treated at this conference in the lectures of a) Honerkamp, b) Jimbo, Miwa, Sato and c) Karowski. The bootstrap program outlined in the lecture of Karowski tries to construct Green's functions from known exact S matrices and is therefore closely related to the present lecture. So far the bootstrap program has lead to a simple construction /41/ of the Green's functions of the Ising model in the scaling limit.

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Figures and figure captions

Figure 1: a) s-plane b) θ-plane .

Figure 2a) and 2b): Equivalent possibilities for factorization of 3-particle scattering.

Figure 3: Poles (———) and zeros (------) of $t(\theta)$ for arbitrary coupling.

Figure 4: a) Poles and zeros of the amplitude $s_+=t+r$.
b) Poles and zeros of the amplitude $s_-=t-r$.

Figure 5: a) Composition of a fermion-antifermion pair to a boson b_k.
b) Composition of two $f\bar{f}$ pairs to bosons b_k, b_l.

Figure 6: a) Poles (———) and double poles (═══) of $t_{b_4 f}$.
b) Poles and double poles of $t_{b_4 b_2}$.

Figure 7: The graph that produces double poles as anomalous thresholds in two dimensions.

Figure 8: Graphical representation of the angle $\varphi(a,c;A)$.

Figure 9: Graph explaining a simple pole.

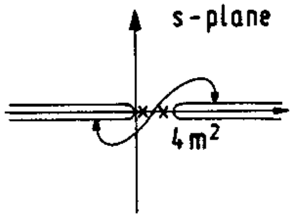

Fig. 1a

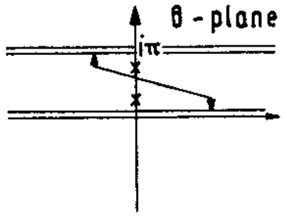

Fig. 1b

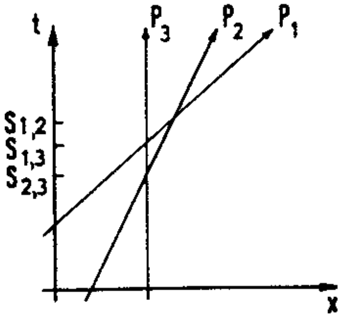

Fig. 2a

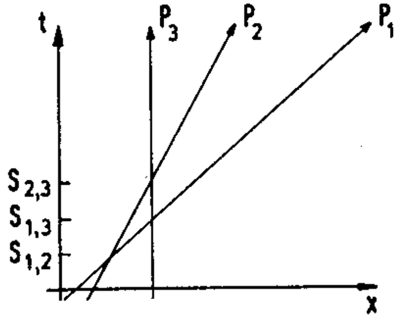

Fig. 2b

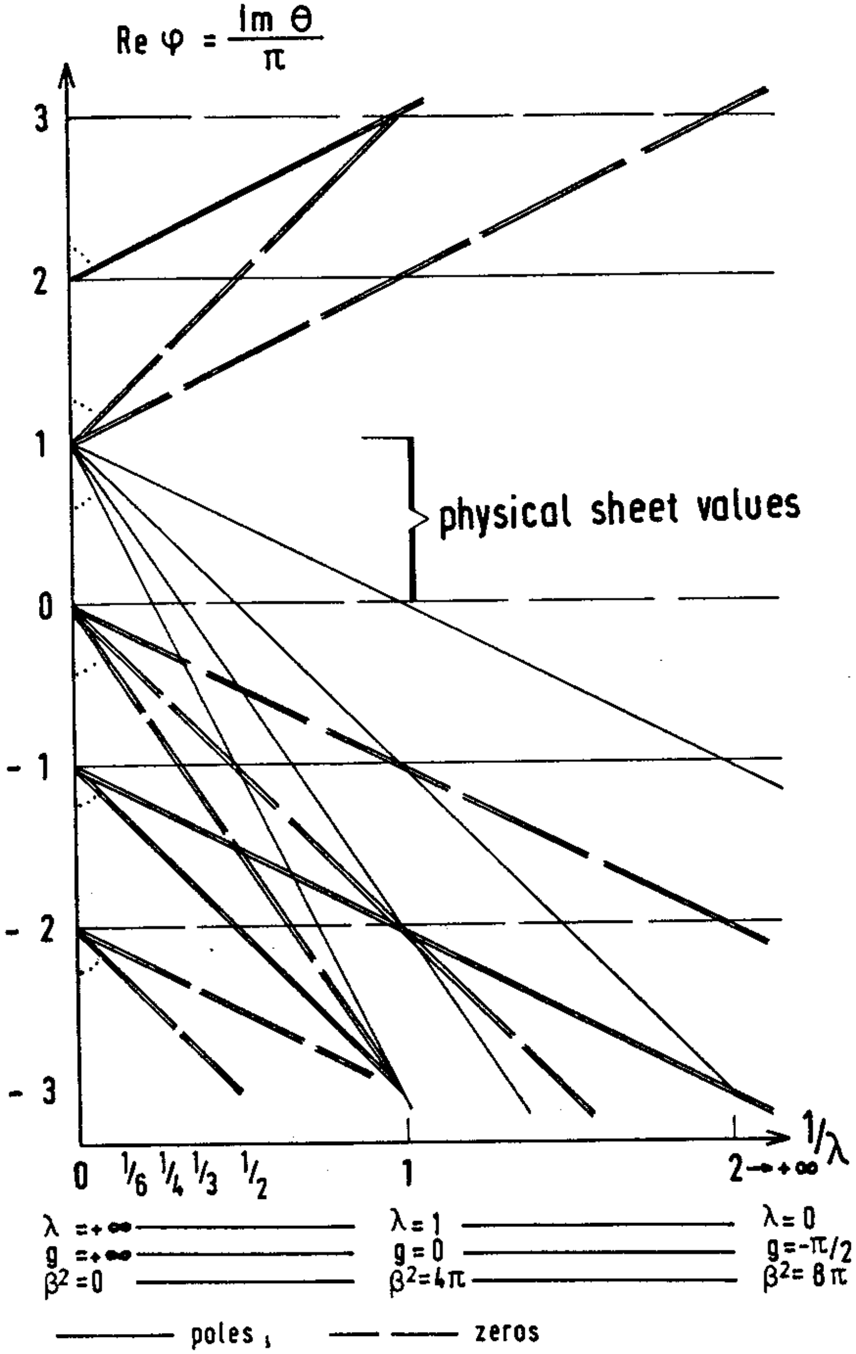

Fig. 3

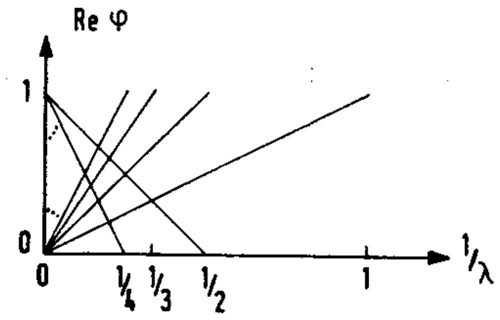

Fig. 4a

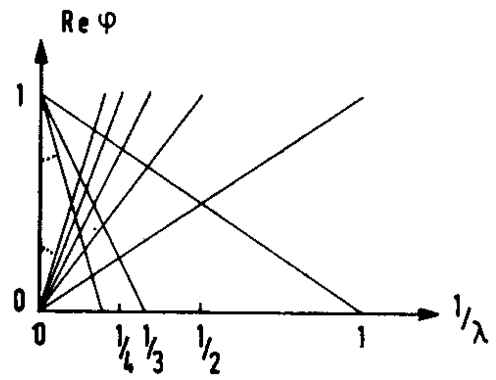

Fig. 4b

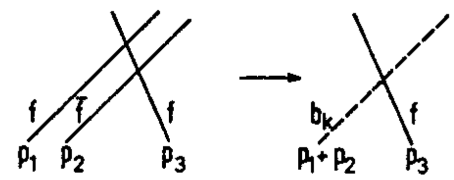

Fig. 5a

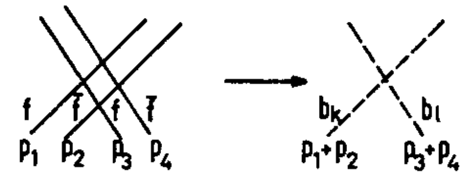

Fig. 5b

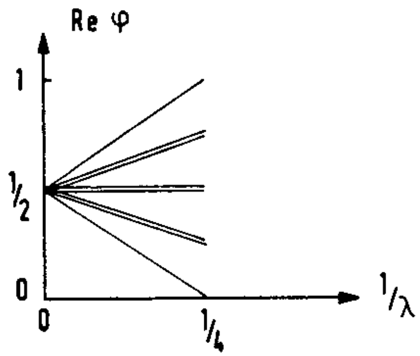

Fig. 6 a

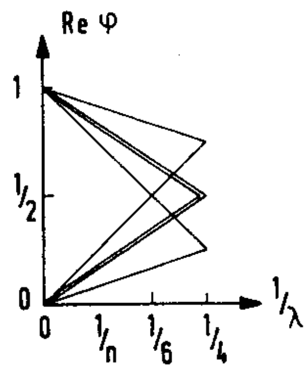

Fig. 6 b

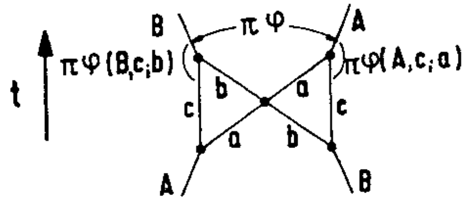

Fig. 7

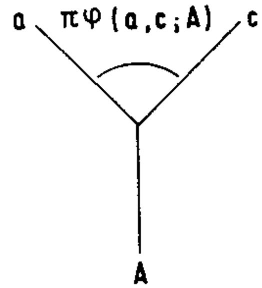

Fig. 8

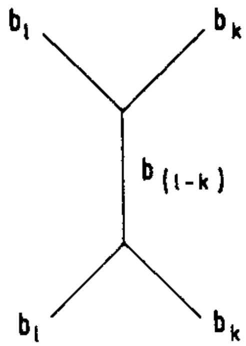

Fig. 9