DEUTSCHES ELEKTRONEN-SYNCHROTRON DESY

DESY 81/036
July 1981

PROOF OF CONFINEMENT OF STATIC QUARKS IN 3-DIMENSIONAL U(1) LATTICE GAUGE THEORY FOR ALL VALUES OF THE COUPLING CONSTANT

by

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NOTKESTRASSE 85 · 2 HAMBURG 52

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Proof of confinement of static quarks in 3-dimensional U(1) lattice gauge theory for all values of the coupling constant.

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Abstract: We study the 3-dimensional pure U(1) lattice gauge theory with Villain action which is related to the 3-dimensional Z-ferromagnet by an exact duality transformation (and also to a Coulomb system). We show that its string tension α is nonzero for all values of the coupling constant g^2, and obeys a bound $\alpha \geq \text{const} \cdot m_D \beta^{-1}$ for small ag^2, with $\beta = 4\pi^2/g^2$ and $m_D^2 = (2\beta/a^3)e^{-\beta v c b(0)/2}$ (a = lattice spacing). A continuum limit $a \to 0$, m_D fixed, exists and represents a scalar free field theory of mass m_D. The string tension αm_D^{-2} in physical units tends to ∞ in this limit. Characteristic differences in the behavior of the model for large and small coupling constant ag^2 are found. Renormalization group aspects are discussed.

1. Introduction and discussion of results

In this paper we will study the Z-ferromagnet on a 3-dimensional cubic lattice $\Lambda \subseteq (a\mathbb{Z})^3$ of lattice spacing a. The spin variables n(x) of the model are attached to the sites x of the lattice. They take values which are integer multiples of 2π. The partition function is

$$Z_\Lambda = \sum_{n\in(2\pi\mathbb{Z})^\Lambda} \exp L(n) \quad , \quad \text{with} \quad L(n) = -\frac{1}{2\beta}\int_x [\nabla_\mu n(x)]^2 \tag{1.1}$$

We use the notations (e_μ = lattice vector of length a in μ-direction)

$$\int_x = a^3 \sum_{x\in\Lambda} \quad ; \quad \nabla_{\pm\mu} n(x) = a^{-1}[n(x\pm e_\mu) - n(x)] \tag{1.2}$$

β has dimension of a length, whereas n is dimensionless. Formula (1.1) must be supplemented by boundary conditions. We choose to immerse the system into an infinitely extended heat bath which is described by a massless free field theory, see eqs. (2.3) of section 2. [Formally, the partition function for the combined system is also given by eq. (1.1), but the variables n(x) are integrated over the reals outside Λ.] These boundary conditions break the global $\mathbb{Z}$-invariances of the model, because the symmetry $\phi(x) \to \phi(x)$ + const. in a massless free field theory is spontaneously broken. To determine the string tension we will need to consider a state of the system on an infinitely extended lattice $\Lambda = (a\mathbb{Z})^3$. This state will be defined in section 6.

The pioneering work on 3-dimensonal U(1) gauge theory was done by Polyakov [4]. It is now well known [5] that the Z-ferromagnet (1.1) is related by an exact duality transformation to 3-dimensional U(1) lattice gauge theory without matter fields with Villain action (in a heat bath that is described by noncompact electrodynamics, see Appendix A). The coupling constants are related by

$$\beta = 4\pi^2/g^2 \tag{1.3}$$

(β can be interpreted as an inverse temperature for the lattice gauge theory, and as a temperature for the Z-ferromagnet). Moreover, the string tension α in the U(1) gauge theory equals the surface tension in the Z-ferromagnet*. More precisely, the Wilson loop expection value is transcribed as follows.

Let C be a closed loop which is boundary of some surface Ξ on the dual lattice** Λ^* of Λ. Let U(C) be the parallel transporter around C in the U(1) lattice gauge theory, and $\chi_k(U) = U^k$ for $U\in U(1)$, k integer. Then***

$$\langle \chi_k(U(C))\rangle_{U(1)} = Z_\Lambda(k,\Xi)/Z_\Lambda \; . \tag{1.4}$$

$Z_\Lambda(k,\Xi)$ is partition function of the Z-ferromagnet with a modified action. It depends actually only on C and not on Ξ, and $Z_\Lambda = Z_\Lambda(0,\cdot)$.

$$Z_\Lambda(k,\Xi) = \sum_{n\in(2\pi\mathbb{Z})^\Lambda} \exp\left(-\frac{1}{2\beta}\int_x [\nabla_\mu n(x) - k j_\mu(x)]^2\right) \tag{1.5}$$

where

$$j_\mu(x) = \begin{cases} 2\pi a^{-1} & \text{if } (x, x+e_\mu)\in\Xi \\ -2\pi a^{-1} & \text{if } (x+e_\mu, x)\in\Xi \\ 0 & \text{otherwise} \end{cases} \tag{1.6}$$

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* The surface tension of a ferromagnet is given by the free energy per unit area of a domain wall between domains with different spontaneous magnetization. The Wilson loop definition of α corresponds to a mathematical definition of surface tension that is not quite the standard one [22], unless the loop C lies on the boundary of Λ.

** By definition, the sites, links, plaquettes and cubes of Λ^* are the cubes, plaquettes, links and sites of Λ. Therefore Ξ consists of links in Λ and C consists of plaquettes in Λ. We write ∂^* for the boundary operator on Λ^*. It is called the coboundary operator on Λ. For further explanation of the dual lattice the reader is referred to refs. [18, 27].

*** This formula can be generalized as follows. Let Ξ be a 2-chain with integer coefficients on Λ^*, $\Xi = \Sigma c(p)p$ (sum over plaquettes p in Λ^* = links in Λ) and $\partial^*\Xi = \Sigma n_i C_i$, where C_i are closed paths on Λ^*. Then

$$\langle \prod_i \chi_{kn_i}(U(C_i))\rangle = Z_\Lambda(k,\Xi)/Z_\Lambda \, ,$$

where $Z_\Lambda(k,\Xi)$ is given by eq. (1.5), with $j_\mu(x) = c(p)2\pi a^{-1}$ for $p = (x, x+e_\mu)$. Truncated expectation values of this form will be briefly considered at the end of section 7.

The string tension is obtained by considering a rectangular loop C in a lattice plane of side length L_1, L_2.

$$\alpha = -\lim_{L_2\to\infty} \lim_{L_1\to\infty} \frac{1}{L_2L_1} \ln \langle \chi_1(U(C))\rangle_{U(1)}$$
$$= -\lim \frac{1}{L_2L_1} \ln [Z_\Lambda(1,\Xi)/Z_\Lambda] , \qquad (1.7)$$

in the infinite volume limit.

Finally we define a mass m_D by*

$$a^2 m_D^2 = (2\beta/a)\, e^{-\beta v_{Cb}(0)/2} \qquad (1.8a)$$

$v_{Cb}(x) = (-\Delta)^{-1}(x0)$ is the Coulomb potential on the infinitely extended lattice $(a\mathbb{Z})^3$. Its numerical value at zero distance is known [11]

$$v_{Cb}(0) = 0.2527\, a^{-1} \qquad (1.8b)$$

Now we are ready to state our main result.

Theorem 1. The surface tension α of the Z-ferromagnet (as defined by eq. (1.7)) satisfies the following inequality with a constant $c > 0$ for β/a sufficiently large

$$\alpha \geq c\, m_D \beta^{-1} . \qquad (1.9)$$

We believe that the r.h.s. of (1.9) represents the true asymptotic behaviour of α, apart from the value of the constant c, but the estimates to prove it are lacking. A classical approximation based on the effective action (1.22) (without the correction terms ...) would give

$$\alpha_{\text{class.approx.}} = 8 m_D \beta^{-1}$$

in the limit $a^{-1} \gg M \gg m_D$.

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*The Z-ferromagnet and U(1) lattice gauge theory are also related to a Coulomb system by another exact transformation [5]. m_D^{-1} is equal to the prediction of a Debye Hückel approximation for the screening length of that Coulomb system.

Corollary 2. The string tension α in the U(1) lattice gauge theory with Villain action is bigger than zero for all values of the coupling constant $g^2 > 0$.

The corollary follows from theorem 1 and the remarks preceding it because αa^2 is a monotone decreasing function of β/a by the 3-dimensional version of Guth's inequality (lemma 4.1 of ref. 18 with $K = \Delta^{-1}$).

We will see from theorem 4 below that m_D is asymptotically equal to the physical mass (= mass gap , or inverse correlation length). It follows that the coupling constant g^2 and the string tension α in units of physical mass (squared) go to infinity in the continuum limit $a \to 0$,

$$\alpha/m_D^2 \geq c g^2 (4\pi^2 m_D)^{-1} = c\,(2\beta^3/a^3)^{\frac{1}{2}}\, e^{\beta v_{Cb}(0)/4} \longrightarrow \infty . \qquad (1.10)$$

as $\beta/a \to \infty$. To the best of our knowledge, this kind of behaviour of the string tension had not been anticipated.

The proof of theorem 1 is based on a rigorous block spin calculation. (The block spin method was invented by Kadanoff [14]. It forms the back bone of Wilson's renormalization group procedure [13,15]. The Glimm Jaffe Spencer expansion [9] of constructive field theory also employs it. Other rigorous applications have been made by Gallavotti et al. [17], and Gawedzki and Kupiainen [16].)

The first step in the proof is to integrate out the high frequency components of the field $\varphi(x) = \beta^{-1/2} n(x)$. This produces are effective action $L_{eff}(\phi)$ for a real field ϕ with Pauli Villars cutoff $M = \lambda^{-1} m_D$ of order m_D. $L_{eff}(\phi)$ is obtained in the form of a series expansion. It is a nontrivial problem to establish bounds on the correction terms in this expansion. Such bounds are needed in the second part of the proof. Our solution of this problem is based on splitting the Yukawa interaction $v(xy) = (-\Delta + M^2)^{-1}(xy)$ that is mediated by the high frequency parts of the field into pieces of decreasing strength and increasing range, and treating one after the other of these pieces by cluster expansion. This is our method of iterated Mayer expansions which was described in ref. 6. We will obtain bounds which show, in particular, that no interactions of range significantly larger than the cutoff length M^{-1} are generated by integrating out the high frequency parts of the field. (We come back to this point in remark C below.)

To state the bounds we need a measure $L(a_1, \ldots, a_n)$ of the spatial separation between subsets $a_1 \ldots a_n$ of Λ. Let T be a tree graph with vertices $i = 1 \ldots t \geq n$. It consists of t-1 ordered pairs $(i_1 i_2)$ such that all but one of the vertices $i = 1 \ldots t$ occur exactly once as i_1. Set

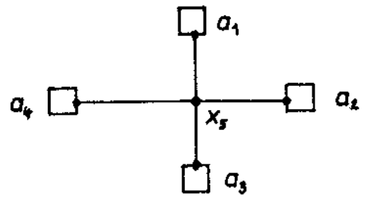

Fig. 1 An example to illustrate the construction of L_T and L

$$L_T(a_1,\ldots,a_n) = \min_{(x_k \in a_k)} \sum_{(ij)\in T} \|x_i - x_j\|, \quad (1.11a)$$

with $\|x\| = 3^{-1/2}\sum_\mu |x_\mu|$, and

$$L(a_1,\ldots,a_n) = \min_T L_T(a_1,\ldots,a_n) \quad (1.11b)$$

The first minimum is over all t-tuples $x_1, \ldots x_t$ with $x_1 \in a_1, \ldots, x_n \in a_n$, $x_s \in \Lambda$ for $n < s \leq t$. The second minimum is over all tree graphs with n or more vertices, $i = 1 \ldots t \geq n$. Fig. 1 illustrates the construction.

We use the notations (1.1) and the following definitions of f, J_μ.

$$f(x) = \int_y v_{cb}(x-y)\nabla_{-\mu} j_\mu(y) \quad (1.12)$$

$$J_\mu(x) = \varepsilon_{\mu\sigma\tau}\nabla_\sigma j_\tau(x) \quad (1.13)$$

$$(J_\mu, v_{cb}J_\mu) = \int_x \int_y J_\mu(x) v_{cb}(x-y) J_\mu(y) \ , \ \text{etc.} \quad (1.14)$$

J_μ is a current supported on the Wilson loop.

We use the abbreviations

$$\xi = (m,x) \quad (m = \pm 1, \pm 2, \ldots ; x \in \Lambda) \ , \ \int d\xi = \sum_{m=\pm 1,\pm 2,\ldots} \int_{x\in\Lambda} \quad (1.15)$$

$\Delta = -\nabla_{-\mu}\nabla_\mu$ is the lattice Laplacian on $(a\mathbb{Z})^3$, see eq. (1.2), and $Vol(a) \equiv \int_{x\in a} 1$.

Proposition 3. Fix λ sufficiently small, and suppose that β/a is sufficiently large (depending on λ). Set $\varepsilon(\xi) = -1 + \exp[im\beta^{-1/2}\phi(x)]$. Then

$$Z_\Lambda(k,\Xi) = e^{-\frac{1}{2}\beta^{-1}k^2(J_\mu, v_{cb}J_\mu)} \int d\mu_{u,k\beta^{-1/2}f}(\phi)\, e^{-V_{eff}(\phi)} \quad (1.16)$$

where $d\mu_{u,g}(\phi)$ is the Gaussian measure with mean g and covariance u,

$$u = (-\Delta)^{-1} - (-\Delta + M^2)^{-1} \quad , \quad M = \lambda^{-1}m_D \ ,$$

and V_{eff} is of the form

$$-V_{eff}(\phi) = \sum_{s\geq 0} \frac{1}{s!}\int d\xi_1 \ldots d\xi_s\, \varepsilon(\xi_1)\ldots\varepsilon(\xi_s)\, \rho_s(\xi_1,\ldots,\xi_s) \quad (1.17)$$

Given $\kappa < \infty$, $\mu < \infty$ arbitrarily large, and $C > 0$, $\delta > 0$ arbitrarily small and < 1, then the following bound is true for $s \geq 1$, uniformly in Λ

$$\int_{x_1\in a_1} d\xi_1 \ldots \int_{x_s \in a_s} d\xi_s\, |\rho_s(\xi_1,\ldots,\xi_s)|\, e^{2\kappa(-1+\Sigma|m_i|)}$$
$$\leq s!\, Vol(a_1)\, m_D^2 \beta^{-1} e^{-(1-\delta)ML(a_1,\ldots,a_s) - \mu(s-1)} (1-C)^{-s-10} \quad (1.18)$$

provided λ is sufficiently small (depending on κ, μ, C, δ) and β/a is sufficiently large (depending on κ, μ, C, δ and λ). Finally

$$2\beta m_D^{-1}\rho_1(\pm 1, x) \to 1 \quad \text{if} \quad \beta/a \to \infty \quad \text{and} \quad M\beta \to 0 \ .$$

The effective action L_{eff} (ϕ) is the sum of $-V_{eff}$ (ϕ) and a kinetic term that comes out of the Gaussian measure. V_{eff} (ϕ) does not depend on k, Ξ, and it still has the global $\mathbb{Z}$-invariance of the model under

$$\phi(x) \to \phi(x) + 2\pi\beta^{-1/2} I \quad , \quad (I \in \mathbb{Z}) \quad (1.19)$$

The heat bath, which breaks this symmetry, is incorporated into the Gaussian measure. Formally

$$d\mu_{u,g}(\phi) = \frac{1}{Z_0} \exp\left[-\tfrac{1}{2}(\phi - g, u^{-1}(\phi - g))\right] \prod_{x \in (a\mathbb{Z})^3} d\phi(x) \quad (1.20)$$

where

$$(f_1, f_2) = a^{-3} \sum_{x \in (a\mathbb{Z})^3} \bar{f}_1(x) f_2(x) \quad (1.21)$$

[The reader should imagine that the sum and product over x run over a subset Λ_1, $\Lambda \subset \Lambda_1 \subset (a\mathbb{Z})^3$ to begin with, with O-Dirichlet boundary conditions which put $\phi(x) = 0$ on the boundary $\partial\Lambda_1$; then the limit $\Lambda_1 \nearrow (a\mathbb{Z})^3$ is taken. This limit procedure is already incorporated in the construction of the Gaussian measure on $(a\mathbb{Z})^3$.]
The effective action is therefore of the form

$$L_{eff}(\phi) = -\tfrac{1}{2}\left(\phi, -\Delta\left(1 - \tfrac{\Delta}{M^2}\right)\phi\right) - \frac{m_D^2}{\beta^3} \int_{x \in \Lambda} \left[1 - \cos\beta^{1/2}\phi(x)\right] + \dots \quad (1.22)$$

The terms represented by dots come from terms with $s \geq 2$ in (1.17), and from the part of the $s = 1$ term with $|m_1| \geq 2$. In addition, the deviation of $\rho_1(\pm 1, x)$ from its asymptotic value leads to a small change of the coefficient m_D^2/β. The bound (1.18) shows that these correction terms can be suppressed by making β/a large (and m_D/M and βM small). In the further analysis of the theory with action L_{eff}, the part of the $s = 2$ term that is quadratic in $\phi - 2\pi\beta^{-1/2} \cdot$ integer is treated as a small correction to the first term in eq. (1.22).

The second step in the proof of theorem 1 consists in the analysis of a theory with the action specified by proposition 3. Such an analysis has already been carried out by Brydges and Federbush [2], and we can use their results. [They also derive this effective

action for a dilute Coulomb gas. The derivation of this part of their results requires a small value of the reduced fugacity $\bar{z}$ though ($\bar{z} = a^3 \cdot$ fugacity when the self-interaction of the charges is included in the potential) or, if $\bar{z}=1$ as in the Coulomb gas representation of our model, a cutoff length M^{-1} that is less than one lattice spacing a - see the discussion in the introduction of ref. 6.] Their analysis is based on the Glimm Jaffe Spencer expansion around mean field theory. The expansion and its convergence proof are readily adapted to cover the case of the Wilson loop expection value (the ration (1.4) of partition functions).

We will now give a brief and informal discussion of the main ideas that are involved in this analysis in order to explain how the string tension comes out. Technical aspects such as justification of approximations and control over correction terms are discussed in later sections.

Imagine that a lattice Λ' of lattice spacing L is superimposed on the dual lattice Λ^*. L is chosen of order M^{-1}, LM = sufficiently small constant. The presence of the Pauli-Villars cutoff M in the effective action ensures that the ϕ-integral, which gives the partition functions, is dominated by functions ϕ of x which are very nearly constant on block cells of side length L. Therefore

$$e^{-V_{eff}(\phi)} \approx \prod_{x' \in \Lambda'} e^{-\mathcal{V}_{eff}(\bar{\phi}(x'))} \quad (1.23a)$$

$$\mathcal{V}_{eff}(\bar{\phi}(x')) = -m_D^2 \beta^{-1} L^3 \left[1 - \cos\beta^{1/2}\bar{\phi}(x')\right] \quad (1.23b)$$

where $\bar{\phi}(x')$ = average of ϕ over the block cell specified by $x' \in \Lambda'$. We see that $\mathcal{V}_{eff}$ $(\bar{\phi}(x'))$ has isolated minima at $\bar{\phi}(x') = 2\pi\beta^{-1/2} I$, I integer, which are separated by maxima. These maxima are very high. We choose M, L^{-1} of order m_D, so that $m_D^2 L^3/\beta$ is of order $(m_D\beta)^{-1}$ which grows exponentially with β/a. Based on this observation, the field $\phi(x)$ is decomposed into a sum of an "integer" part $h(x)$, $\beta^{1/2}h(x)/2\pi$ = integer and constant on block cells, which selects a minimum of V_{eff}, and a spin wave part that describes fluctuations around it.

We see now that there are two kinds of excitations, spin waves and domains walls. The domain walls consist of those plaquettes on the block lattice Λ' where $h(x)$ jumps by $2\pi\beta^{-1/2}\cdot$integer.

In the Glimm Jaffe Spencer expansion [9], the domain walls are treated by a low temperature expansion (Peierls expansion), and the spin waves by a cluster expansion [10]. The presence of a domain wall costs very much energy per plaquette in Λ', due to the presence of the high maxima in $\mathcal{V}_{eff}(\bar{\phi}(x'))$. The configurational entropy of the domain walls on the block lattice cannot compete against this. As a result the density of domain walls on the block lattice is very low, the surface tension α (= free energy of domain walls per unit area) is not zero and the Peierls expansion converges.

The cluster expansion converges because the spin waves interact weakly. The large factor $m_D^2L^3/\beta$ in eq. (1.23b) implies that the Gibbs measure assigns sizable probability only to values of the average field $\bar{\phi}(x')$ which are close to a minimum of $\mathcal{V}_{eff}$. Since the dominant field configurations ϕ are nearly constant on block cells (see before eq. (1.23)), the same is then also true for $\phi(x)$. For such $\phi(x)$ the later terms in the Taylor expansion of the cosine in eq. (1.22) around this minimum are very small compared to the leading term. The leading term tells us that the spin waves have bare mass m_D.

The same method also enables us to derive a result about the existence of a continuum limit. For the purpose of deriving a lower bound on the surface tension, it was most efficient to use a ratio $M/m_D = \lambda^{-1}$ of cutoff M to mass m_D which is sufficiently large but independent of β/a. If we want to construct a continuum limit by letting $\beta/a \to \infty$ we must at the same time remove the Pauli Villars cutoff. We do this in such a way that the cutoff M goes to infinity in units of physical mass (slowly) and to zero in units of inverse lattice spacing a^{-1} (fast). We define a new field

$$\Psi(x) = \beta^{-1/2}\sin\beta^{1/2}\phi(x) \tag{1.24}$$

Theorem 4. Let the field $\Psi(x)$ be defined by eq. (1.24) with a choice of cutoff $M = (\beta/a)^{1/12}m_D$. Then the correlation functions $\langle\Psi(x_1)\ \dots\ \Psi(x_n)\rangle$ for fixed nonzero distances $m_D|x_i - x_j|$ (in units of m_D^{-1}) approach the correlation functions of the (Lorentz invariant) Euclidean scalar free field theory with mass m_D as $\beta/a \to \infty$. The string tension in units of physical mass squared goes to infinity in this limit.

We see that only the spin waves survive in this continuum limit. For their interpretation (as a "magnetic matter field") in gauge theory language see remark D below. An infinite string tension means that the Wilson loop operator $U(C)$ - renormalized by a perimeter law behaved factor to remove the self-energy of the static quarks - is zero.

Theorem 4 is concerned with a continuum limit in which the inverse physical mass (= mass gap) is taken as the standard of length. One could think of trying to construct another, massless, continuum limit by taking the string tension as the standard of $(\text{length})^{-2}$. Again one would have to let the cutoff $M \to \infty$ in physical units. It is not clear that this limit will exist because the sine-Gordon theory is nonrenormalizable in 3 dimensions (compare eq. (1.22)).

We conclude this section with some remarks about A) symmetry breaking aspects and physical interpretation, B) a possible roughening transition and characteristic differences between the behavior of the model for weak and strong coupling, C) renormalization group aspects, D) the interpretation of our effective action in gauge theory language, E) mass generation, F) unicity of the equilibrium state, and G) our choice of cutoff.

Remark A (Symmetry breaking) The spin waves are Goldstone modes of a spontanously broken symmetry. They need not be exactly massless because there exists explicit breaking from a continuous symmetry $\mathbb{R}$ to a discrete symmetry $\mathbb{Z}$. The approximate symmetry $\mathbb{R}$ can be seen by replacing $\beta^{-1/2}n(x)$ in eq. (1.1) by a field $\varphi(x)$ that is integrated over the reals, and replacing the Boltzmann factor by a serrated Gaussian which depends only on the integer part of $\beta^{1/2}\varphi(x)/2\pi$. If β/a is large then its saw teeth, which break the symmetry $\varphi(x) \rightarrow \varphi(x) + \text{const} \in \mathbb{R}$ are small. If the explicit symmetry breaking were switched off then the $\mathbb{R}$ invariance would remain spontaneously broken - the result is a massless free field theory which is known to have a spontaneously broken symmetry $\varphi \rightarrow \varphi + \text{const}$. Note that the global $\mathbb{Z}$-invariance is always spontaneously broken if $\langle n(x) \rangle$ exists at all, since the equation $\langle n(x) \rangle = \langle n(x) \rangle + 2\pi I$ has no solution for $I \neq 0$. So the explicit breaking from $\mathbb{R}$ to $\mathbb{Z}$ is not important to get spontaneous magnetization (symmetry breaking) but it is important to produce a surface tension. Ferromagnets with spontaneously broken continuous symmetry have no surface tension. (This is of course trivial for the massless free field theory. It can also be proven in general, for Heisenberg ferromagnets etc. [24] .)

When β/a is small then the $\mathbb{R}$-invariance is completely destroyed. In this case the only excitations of our $\mathbb{Z}$-ferromagnet are a dilute gas of small domain walls on the original lattice (see section 5), and there are no spin waves.

Remark B (roughening transition etc.) We have made no attempts to prove the existence of a roughening transition [23] , but we emphasize that nothing in our results speaks against its existence. In particular the convergence of the Peierls expansion does not imply that a domain wall with prescribed boundary C is not rough. [It is called rough when the amplitude of the fluctuations in its position in the transverse direction tend to infinity when the area enclosed by C becomes infinite.] Convergence of the Peierls expansion requires that energy and chemical potential of a domain wall are bounded below by positive constants times area. Such a bound alone is not even sufficient to guarantee that among all surfaces with prescribed boundary the one with least area is also the one which costs the least energy, and it certainly does not imply that all others cost so much more energy that they are assigned negligible probability by the Gibbs measure. An argument for rough domain walls which starts from an effective action like eq. (1.22) has recently been given by Ogilvie [25].

It is instructive to compare the behavior of the Z-ferromagnet for small and large coupling parameter ("temperature" - see after eq. (1.2)) β/a. At low β/a the theory can be analyzed by Peierls expansions on the original lattice (see section 5), these are related to the familiar high temperature expansions for the U(1) gauge theory by a duality transformation. The only excitations are domain walls (on the original lattice), and these are dilute. Domain walls with prescribed boundary are not rough when they are aligned with the lattice planes. The proof of this property makes essential use of the fact that there is only nearest neighbour interaction (single plaquette interaction in the U(1) gauge theory). At large β/a the low lying excitations are spin waves. In addition there are domain walls on the block lattice. They cost much energy and are very dilute. In principle, the spin waves can be integrated out, the result is an effective action $L^{Z}_{eff}(h)$ for a Z-ferromagnet on the block lattice with $2\pi\cdot$integer spins. However, the spin waves have a mass m_D less than one inverse lattice spacing L^{-1}, so they mediate interactions between domain walls over a range larger than one lattice spacing L (see remark C below). As a result, $L^{Z}_{eff}(h)$ does not have nearest neighbour interaction only and the argument against rough domain walls does not carry over to it.

Moreover, comparison of the excitations suggests that there is also a change in bulk behavior when one goes from small to large β/a. The question naturally poses itself whether it is associated with a phase transition. The methods of the present paper are not powerful enough to decide the question whether the free energy is indeed nonanalytic in β at some value $\beta = \beta_c$. This is so because we do not have the region of intermediate β/a under control, except for some pieces of information that rely on correlation inequalities, like corollary 2 and the results of Fröhlich and Spencer [12].

Remark C (renormalization group aspects) All our results are perfectly consistent with the general principles of the renormalization group. This will become evident from the discussion at the end of section 2. In addition our results contain dynamical information that is specific to the model and could not be deduced from general principles alone.

All this is as it should be. It must be emphasized, however, that a good choice of the block spin (= the field that appears in the effective action) is absolutely crucial. What is a good choice and what is not depends on the low lying excitations of a model. If, by mischief, one integrates out degrees of freedom that are associated with low lying excitations in the course of an iterative renormalization group procedure then one will encounter disaster - the effective action becomes very nonlocal. If the disaster goes unnoticed by courtesy of uncontrolled approximations, then one may get qualitatively wrong results. We will now illustrate the danger at the example of our model.

For large β/a the low lying excitations are spin waves. This is not a priori obvious, especially not if one starts from the U(1) gauge theory. (Imagine that you had never heard of duality transformations ...)

Suppose that someone wanted to set up a renormalization goup procedure in which the effective action at each intermediate step does not depend on a real field ϕ, but on an integer valued field on a block lattice. He might be motivated by the hope of obtaining effective actions which are approximately of the form of the original action (1.1), except for the replacement of β by a running coupling constant. Such an action would be the dual transform of a U(1) gauge theory on a block lattice with one-plaquette action of the Villain form, with a running coupling constant.

Our proposition 3 is valid for arbitrarily large values of the ratio M/m_D of cutoff to mass. Therefore it also gives us information about what the effective action $L_{eff}(\phi)$ would be at some intermediate stage of a renormalization group procedure when M/m_D is still very large but $M\beta$ is already small. To get an effective action $L^Z_{eff}(h)$ for a theory with integer valued block spins one would need to integrate out the spin wave part of ϕ. But this leads to disaster. The spin waves always have mass $\approx m_D$ as soon as $M\beta \ll 1$, as a result they will lead to interactions of range m_D^{-1} rather than M^{-1} in the resulting effective action $L^Z_{eff}(h)$.

For the purpose of illustration we can use the approximation (1.23), and substitute a periodized Gaussian for the exponential of the cosine so that

$$e^{-V_{eff}(\phi)} = \sum_{h\in(2\pi/\beta^{1/2}\mathbb{Z})^{\Lambda}} \exp\left[-\tfrac{1}{2}m_D^2\int_x (\phi(x)-h(x))^2+\dots\right] \tag{1.25}$$

h is constant on cells of the block lattice Λ'. Its lattice spacing L is taken of order M^{-1}. If the correction terms ... are dropped the resulting Gaussian integration can be performed with the result that ($m_D' \approx m_D$, $M' \approx M$, $r \approx 1$ if $M \gg m_D$)

$$L^Z_{eff}(h) = -\tfrac{1}{2}m_D^2\,(\nabla_\mu h, G\nabla_\mu h) + \dots \tag{1.26}$$

$$G = r\left[(-\Delta+m_D'^2)^{-1} - (-\Delta+M'^2)^{-1}\right]\left(1-\frac{\Delta}{M^2}\right) \approx (-\Delta+m_D^2)^{-1}$$

We see that the spin waves have generated an interaction of range m_D^{-1} between domain walls. (The domain walls are where h jumps, i.e. $\nabla_\mu h \neq 0$.)

If one wants to determine the surface tension α one has to integrate out the spin waves eventually. The above considerations show that one should not do this in the course of an iterative renormalization group procedure which lowers the cutoff sequentially, but postpone this step until the cutoff M has been brought down to order m_D (physical mass). This is what we have done. One obtains an action for an effective Z-theory of the form (1.26). [Actually the periodezed Gaussian is not a very good approximation for the purpose of computing L^Z_{eff}, because it overestimates the height of the maxima of V_{eff} by a factor $\approx\pi^2/4$. Therefore the correction terms represented by dots in (1.26) are not small. But one proves that they can at most reduce the surface tension to a finite fraction > 0.] Of course, the resulting L^Z_{eff} still does not have nearest neighbour interaction only. But it is interesting to note that it can be replaced by the action of a simplified effective Z-theory* which has only nearest neighbour interaction for the purpose of finding a lower bound on the surface tension.** Since h is constant on block cells, we can replace G by $\bar{G}$ in eq. (1.26) with the kernel of $\bar{G}$ equal to the block average of the kernel of G. The assertion follows now from the fact that $\bar{G}$, when considered as an operator in the Hilbert space of functions on the block lattice, is bounded below by a multiple of the unit operator $\mathbb{1}$,

$$\bar{G} \geq c_1 L^{-1} \cdot \mathbb{1} . \tag{1.27}$$

For details see section 7.

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* In ref. 7, an effective $\mathbb{Z}(N)$ theory of quark confinement in pure SU(N) Yang Mills theory on a 4-dimensional lattice was presented (N = 2,3). It uses only the simplest if approximations. Nevertheless it is hoped that it will become possible one day to prove its validity mathematically, with rigorous bounds in place of approximate equalities. The effective $\mathbb{Z}(N)$ theory is a $\mathbb{Z}(N)$ gauge theory on a block lattice with positive coupling constant and the standard one-plaquette action. It can be subject to a duality transformation. The resulting $\mathbb{Z}(N)$ gauge theory can be regarded as an analog of the simplified effective Z-theory mentioned in the text. $\mathbb{Z}(N)$ gets replaced by $\mathbb{Z}$ because the dual of the center of the gauge group U(1) is $\mathbb{Z}$.

** This is not the first result of such a kind. The inequalities of ref. 29 are another example. They bound the string tension of an SU(N) theory by that of a Z(N) theory.

Remark D (Effective action in gauge theory language) To obtain the effective action in gauge theory language, we should apply an inverse duality transformation. We do not know how to do this for the action given by proposition 3. Instead we present here the solution of another simplified auxiliary problem. We consider the action $L'_{eff}(\phi)$ which is obtained from $L_{eff}(\phi)$ by substituting a lattice cutoff L^{-1} for the Pauli Villars cutoff M. Thus, ϕ is regarded as constant on cells of the block lattice Λ' of lattice spacing L, and $u = (-\Delta')^{-1}$ where $\Delta' = -\nabla'_{-\mu}\nabla'_{\mu}$ is the Laplacian on the block lattice, $\Delta' F(x') = L^{-2}\sum_{\pm,\mu}[F(x' \pm L e_\mu/a) - F(x')]$. The other formulae are the same, for k = 0.

We perform an inverse duality transformation on this modified action. Such a duality transformation is a Fourier transformation on the symmetry group $\mathbb{Z}$. To apply it one must decompose $\mathbb{R} \ni \phi(x')$ into $\mathbb{Z}$-orbits. This is achieved by writing

$$\phi(x') = h(x') + \theta(x')\beta^{-1/2} \quad , \quad -\pi \leq \theta(x') \leq \pi \ , \ h(x') \in 2\pi\beta^{-1/2}\mathbb{Z} . \tag{1.28}$$

The symmetry group acts only on h. Therefore the "spin wave" variables θ remain unaffected by the duality transformation. We note that $V_{eff}(\phi)$ depends only on θ or, equivalently, on the field χ to be introduced below, because

$$\varepsilon(\xi) = -1 + \exp[im\beta^{1/2}\phi(x)] = -1 + \exp[im\theta(x')] \quad \text{for } \xi = (m,x) . \tag{1.29}$$

$x' \in \Lambda'$ specifies the block cell in which is $x \in \Lambda$.

It is straight-forward to perform the duality transfomation. The result is a U(1) gauge theory on a block lattice with Villain action which is coupled nonminimally to a "magnetic" matter field $\chi(x')$. The action becomes

$$\begin{aligned} \hat{L}'_{eff}(F_{\mu\nu}, \chi) = -L^{-3}\sum_{x' \in \Lambda'^*} \Big\{ & \tfrac{1}{2} g^{-2} F_{\mu\nu}(x') F_{\mu\nu}(x') + \\ & + \tfrac{i}{4\pi}\varepsilon_{\mu\nu\rho} F_{\mu\nu}(x') \bar{\chi}(x') i \overleftrightarrow{\nabla}'_{\rho} \chi(x') \Big\} - V_{eff} . \end{aligned} \tag{1.30}$$

Λ'^* is the dual of the block lattice Λ'. If y' is the site of Λ'^* which corresponds to the cube of Λ' with corner points x', $x'+e'_\mu$, $x'+e'_\nu$, $x'+e'_\rho$, $\mu\nu\rho$ = 123 or cyclic, then

$$\chi(y') = \exp i\theta(x') \tag{1.31}$$

Under space reflections

$$\chi(x') \to \bar{\chi}(-x') \tag{1.32}$$

in accordance with its "magnetic" character. V_{eff} is a power series in χ and $\bar{\chi}$ because $\varepsilon(\xi)$ is a polynomial in χ or $\bar{\chi}$ by eq. (1.29). $F_{\mu\nu}$ depends on the auxilary Villain variables $\ell_{\mu\nu}(x') \in 2\pi L^{-2}\mathbb{Z}$,

$$F_{\mu\nu} = \nabla_\mu' A_\nu - \nabla_\nu' A_\mu - \ell_{\mu\nu} \quad ; \quad -\pi L^{-1} \le A_\mu \le \pi L^{-1} \tag{1.33}$$

and the partition function is obtained by summing over $\ell_{\mu\nu}$ and integrating over A_μ and χ.

Apart from the appearance of the matter field χ the most spectacular feature of expression (1.30) is that the last term in $\{\}$ is pure imaginary. It would be real in Minkowski space because the antisymmetric tensor $\varepsilon_{\mu\nu\rho}$ ensures that always one of the indices μ, ν, ρ is 3 (time).

Remark E (mass generation) The result (1.10) indicates that the mass is "too small" in the present model. Some years ago one of us suggested that mass generation in nonabelian gauge theories could be understood as a dynamical Higgs mechanism [26]. In an abelian pure gauge theory this mechanism cannot work because the gauge field carries no colour charge.

Remark F (Unicity of equilibrium state) Convergence of this infinite volume cluster expansions (of section 6) for the Z-ferromagnet at low temperatures makes it most unlikely that its equlibrium state is nonunique [in the sense that there is another one which assigns expectation values to Z-invariant observables that are different from those determined by solution of the infinite volume Kirkwood Salsburg equations of section 6. Z-invariant observables do not distinguish between Gibbs states that are obtained from each other by a symmetry transformations.] A proof of unicity would be desirable

Remark G (choice of cutoff) For a block spin calculation, it might seem appropriate to define a field $\phi(x')$ on a block lattice $(L\mathbb{Z})^3$ by $\phi(x') = L^{-3}\int_{x\in \text{block}} \beta^{-1/2} n(x)$. Since $n(x)/2\pi$ is integer, such a choice would require that $\phi(x')$ is an integer multiple of $2\pi(a/L)^3\beta^{-1/2}$. We avoid this by the use of a Pauli Villars cutoff. In this way we can work with a real field and a smooth action for it.

2. Yukawa gas representation

The most convenient starting point for a rigorous treatment of the 3-dimensional U(1) lattice gauge theory and its dual transform, the Z-ferromganet, is another well known representaion of this model which is intermediate between the Coulomb gas representation and the Z-ferromagnet. It uses the gas picture to deal with short distance problems (up to M^{-1}) and the ferromagnetic language for longer distances. This idea was used by Fröhlich [30] in his work on 2-dimensional Coulomb systems, and later also by Brydges[1](See also [31] In this section we review the derivation of this representation. Some informal remarks on renormalization group aspects are added at the end.

Since we are only interest in ratios (1.3) of partition functions, we will allow ourselves to add constants to the action, and redefine the partition functions by corresponding factors independent of k, Ξ, without introducing a new notation each time. We set the lattice spacing

$$a = 1 \text{ throughout sections } 2 - 5. \tag{2.0}$$

Accordingly, the scalar product (,) in the space of square summable functions on the infinitely extended lattice takes the form

$$(f,g) = \sum_{x\in \mathbf{Z}^3} \bar{f}(x)g(x) \tag{2.1}$$

Using eq. (1.13) and completing the square, the action in eq. (1.5) can be rewritten as

$$\frac{1}{2\beta}\int_x [\nabla_\mu n(x) - kj_\mu(x)]^2 = \frac{1}{2\beta}(n - kv_{cb}\nabla_{-\mu} j_\mu, -\Delta[n - kv_{cb}\nabla_{-\mu} j_\mu]) + \frac{k^2}{2\beta}(J_\mu, v_{cb}J_\mu). \tag{2.2}$$

We introduce a real field $\varphi(x)$ on the lattice $\mathbf{Z}^3$ which is going to replace $\beta^{-1/2}n(x)$. The integrality condition on $n(x)$ for $x\in\Lambda$ will be imposed through δ-functions. Let $d\mu_{v,f}(\varphi)$ be the Gaussian measure with mean f and covariance v. We can use it to write the partition functions for the Z-ferromagnet with boundary conditions as described in the introduction in the form

$$Z_\Lambda(k,\Xi) = e^{-k^2(J_\mu, v_{cb}J_\mu)/2\beta}\, \tilde{Z}_\Lambda(k,\Xi) \tag{2.3a}$$

$$\tilde{Z}_\Lambda(k,\Xi) = \int d\mu_{v_{cb},\, k\beta^{-1/2}f}(\varphi) \prod_{x\in\Lambda}\left\{2\pi\beta^{-1/2}\sum_{n_x\in 2\pi\mathbf{Z}} \delta(\varphi(x) - \beta^{-1/2}n_x)\right\} \tag{2.3b}$$

with $f = v_{cb}\nabla_{-\mu} j_\mu$ as in eq. (1.12). f depends on Ξ. The periodized δ-function can be expanded in a Fourier series. This gives

$$\prod_{x\in\Lambda}\left\{2\pi\beta^{-1/2}\sum_{n_x\in 2\pi\mathbf{Z}}\delta(\varphi(x) - \beta^{-1/2}n_x)\right\} = \sum_{m\in\mathbf{Z}^\Lambda} e^{i\beta^{1/2}(m,\varphi)} \tag{2.4}$$

One inserts this into eq. (2.3b) and performs the φ-integrations with the help of the well known formula for the characteristic function of a Gaussian measure

$$\int d\mu_{v,f}(\varphi)\, e^{i(g,\varphi)} = e^{-\frac{1}{2}(g,vg) + i(g,f)} \tag{2.5}$$

In this way one obtains the Coulomb gas representation of Banks, Myerson and Kogut [5]. The partition functions become

$$\tilde{Z}_\Lambda(k,\Xi) = \sum_{m\in\mathbf{Z}^\Lambda} e^{ik(m,f)}\, e^{-\beta(m, v_{cb}m)/2} \tag{2.6}$$

For k = 0 this is the partition function of a Coulomb gas with insulating boundary conditions [12]: $v_{Cb}(x-y)$ is the Coulomb potential on the infinitely extended lattice $\mathbf{Z}^3$, but the charged particles (monopoles*) are constrained to be in Λ; $m(x) = 0$ for x outside Λ.

Now one splits the Coulomb potential into a Yukawa potential v of range M^{-1} which dominates for distances much shorter than M^{-1}, and the rest

$$v_{cb} = v + u \tag{2.7}$$

with

$$v = (-\Delta + M^2)^{-1}, \tag{2.8}$$

$$u = (-\Delta)^{-1} - (-\Delta + M^2)^{-1}. \tag{2.9}$$

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*They are monopoles of the gauge theory, see. refs. 4,5. Monopoles in SO(3) lattice gauge fields are discussed in ref. 33.

Inserting the split (2.7) into eq. (2.6) one obtains

$$\tilde{Z}_\Lambda(k,\Xi) = \sum_{m\in \mathbb{Z}^\Lambda} e^{ik(m,f)-\frac{1}{2}\beta(m,um)} \; e^{-\frac{1}{2}\beta(m,vm)}$$

Now one uses the formula (2.5) for the characteristic function of a Gaussian measure again to reexpress the first factor

$$\tilde{Z}_\Lambda(k,\Xi) = \sum_{m\in \mathbb{Z}^\Lambda} \int d\mu_{u,k\beta^{-1/2}f}(\phi)\, e^{i\beta^{1/2}(m,\phi)} \; e^{-\frac{1}{2}\beta(m,vm)} \qquad (2.10)$$

ϕ is a real field on $\mathbb{Z}^3$. m-summations and ϕ-integrations can be interchanged (by the dominated convergence theorem). Using eq. (2.3a) we can therefore write

$$Z_\Lambda(k,\Xi) = e^{-k^2(J_\mu, v_{cb}J_\mu)/2\beta} \int d\mu_{u,k\beta^{-1/2}f}(\phi)\, Z_\Lambda(\phi) \qquad (2.11)$$

with

$$Z_\Lambda(\phi) = \sum_{m\in \mathbb{Z}^\Lambda} e^{i\beta^{1/2}(m,\phi)} \; e^{-\beta(m,vm)/2} \qquad (2.12)$$

$Z_\Lambda(\phi)$ is partition function of a Yukawa gas with complex space dependent fugacity. It turns out to be positive [12] so that one can define $V_{eff}(\phi) = -\ln Z_\Lambda(\phi)$. The problem is to show that V_{eff} admits an expansion with the properties stated in proposition 3. The natural thing to try is a Mayer expansion, as in the work of Brydges, and Brydges and Federbush. Unfortunately the known methods for proving convergence of such a Mayer expansion fail for the values of β and M in which we are interested. Therefore a new piece of technology had to be developed. This is discussed in the next section.

We add some informal remarks. Inserting eq. (2.4) into eq. (2.3b) with k = 0 one can write

$$Z_\Lambda = \int d\mu_{v_{cb},0}(\varphi) \prod_{x\in\Lambda} \left\{ 1 + 2\cos\beta^{1/2}\phi(x) + \textit{terms with } |m(x)|>1 \right\} \qquad (2.13)$$

For temporary use only we introduce normal ordering :: with respect to the Gaussian measure $d\mu_{v_{cb}} \equiv d\mu_{v_{cb},0}$. It is defined by

$$:e^{i(f,\varphi)}: \;\equiv e^{i(f,\varphi)} \Big/ \int d\mu_{v_{cb}}(\varphi)\, e^{i(f,\varphi)} = e^{\frac{1}{2}(f,v_{cb}f)}\, e^{i(f,\varphi)} \qquad (2.14)$$

Eq. (2.13) becomes

$$Z_\Lambda = \int d\mu_{v_{cb}}(\varphi) \prod_{x\in\Lambda} \left\{ 1 + 2z :\cos\beta^{1/2}\varphi(x): + \textit{terms with } |m(x)|>1 \right\} (2.15)$$

with

$$z = e^{-\beta v_{cb}(0)/2} = m_D^2/2\beta \qquad (2.16)$$

If we could neglect the monopoles with $|m|>1$ and also the hard core of the others, we could approximate this expression by

$$Z_\Lambda = \int d\mu_{v_{cb}}(\varphi) \exp\left[m_D^2 \beta^{-1} \int_x :\cos\beta^{1/2}\varphi(x): \right] + \dots \qquad (2.17)$$

This formula was first derived in Appendix A of ref. 5 . (Normal ordering is presumably tacitly understood there.) The normal ordering is crucial here, it is responsible for the exponentially small factor $2z = m_D^2/\beta$, compare eqs. (2.15) with (2.13). It absorbs the self-interaction of the charged particles.

The integrand in (2.14), which is a smooth function of φ, looks like a very drastic approximation to the exact integrand in eq. (2.3b) with k = 0 , which involves δ-functions. But it is instructive to compare eq. (2.17) with eq. (1.22) for the effective action. This reveals that the main effect of integrating out the high frequency components of the field $\varphi(x)$ is that $\varphi(x)$ in (2.17) gets replaced by a cutoff field $\phi(x)$, and the correction terms ... in (2.17) get suppressed. [One can rewrite eq. (1.22) in normal ordered form, but this makes little difference there because of the presence of the low cutoff mass M.] There are other correction terms appearing, including some that could be absorbed in part by mass and wave function renormalization, but they are all small so long as the ratio cutoff/phys. mass $\gg 1$ and β is large enough - see proposition 3. All this is perfectly consistent with the general principles of the renormalization group [13,14,15]. The presence of the ultraviolet cutoff M will make it possible to write down convergent expansions in domain walls on a block lattice with lattice spacing L of order M^{-1}. A similar expansion on the original lattice would not converges if β/a is large, because a jump of the spin n(x) by 2π across a link costs very little energy.

3. Iterated Mayer expansions.

We wish to examine expression (2.12) for $Z_\Lambda(\phi)$. We interpret it as a partition function of a Yukawa gas on Λ with a complex space dependent fugacity. It consists of particles with a hard core which prevents two of them occupying the same lattice site. They can exist in infinitely many states labelled by charge $m = \pm 1, \pm 2, \pm 3$ ..

We introduce abbreviations

$$\xi_i = (m_i, x_i) \quad , \quad \int d\xi_i (\cdots) = \sum_{m_i = \pm 1, \pm 2, \ldots} \int_{x \in \Lambda} (\cdots) \tag{3.1}$$

$$v(\xi_i, \xi_j) = \begin{cases} +\infty & \text{if } x_i = x_j \ , \ i \neq j \\ m_i m_j v(x_i - x_j) & \text{otherwise} \end{cases} \tag{3.2}$$

With this notation we can rewrite $Z_\Lambda(\phi)$ in grand canonical form

$$Z_\Lambda(\phi) = \sum_{N=0,1,2,\ldots} Z_\Lambda^N(\phi)$$
$$= \sum_{N=0,1,2,\ldots} \int d\xi_1 \ldots d\xi_N \, \Xi^N(\xi_1, \ldots, \xi_N) \prod_{j=1}^N [1 + \varepsilon(\xi_j)] \ , \tag{3.3}$$

with

$$\varepsilon(\xi) = -1 + \exp[im\beta^{1/2}\phi(x)] \tag{3.4}$$

as in proposition 3, and

$$\Xi^N(\xi_1, \ldots, \xi_N) = \frac{1}{N!} \exp\left[-\frac{\beta}{2} \sum_{i,j=1}^N v(\xi_i, \xi_j)\right] \ . \tag{3.5}$$

We split the potential v into R pieces of increasing range and decreasing strength.

$$v(\xi_i, \xi_j) = v^0(\xi_i, \xi_j) + \sum_{r=1}^{R-1} v^r(\xi_i, \xi_j) \ . \tag{3.6}$$

v^0 incorporates the hard core .

$$v^0(\xi_i, \xi_j) = \begin{cases} +\infty & \text{if } x_i = x_j \ , \ i \neq j \\ 0 & \text{otherwise} \ . \end{cases} \tag{3.7}$$

The other pieces of the interaction shall assume finite values only. They will be specified below in eqs. (3.19) ... (3.22).

We write down an iterated cluster expansion [6]. We consider clusters of clusters of ... of clusters of particles = constituents. They are called ℓ-vertices, $\ell = 0, 1, \ldots R$. A 0-vertex is a single particle i which is its own constituent. Assiciated with it is a variable $\xi_i = (m_i, x_i)$ which specifies its state and position, and a vertex function

$$\sigma^0(\xi_i) = 1 \tag{3.8}$$

Higher vertices are defined inductively. An ℓ-vertex α' is a non-empty finite collection $\{\alpha\}$ of $(\ell-1)$-vertices such that no two of them share a constituent. There is an associated variable

$$\xi_{\alpha'} = (\{\xi_\alpha\}_{\alpha \in \alpha'}) \quad ; \quad d\xi_{\alpha'} = \prod_{\alpha \in \alpha'} d\xi_\alpha \tag{3.9}$$

A particle i is constituent of α' if it is constituent of one of the $(\ell-1)$-vertices $\alpha \epsilon \alpha'$. We write $i \underline{\epsilon} \alpha'$ in this case, and $C(\alpha')$ for the set of all constituents of α'. One also defines the type $[\alpha']$ of an ℓ-vertex. $[\alpha']$ is an equivalence class of ℓ-vertices. All 0-vertices are equivalent. Two ℓ-vertices belong to the same equivalence class if they contain the same number of $(\ell-1)$-vertices of each type $[\alpha]$. We write T_ℓ for the set of all types of ℓ-vertices. To every ℓ-vertex α there is an associated vertex function $\sigma_\alpha^\ell(\xi_\alpha)$. It is defined inductively as follows. Write

$$v^\ell(\alpha\gamma) \equiv \sum_{i \underline{\epsilon} \alpha} \sum_{j \underline{\epsilon} \gamma} v^\ell(\xi_i, \xi_j) \tag{3.10}$$

and

$$\exp[-\beta v^\ell(\alpha\gamma)] = 1 + f_{\alpha\gamma}^\ell \tag{3.11}$$

Then one defines, for $\ell \geq 0$,

$$\sigma_{\alpha'}^{\ell+1} = \left\{ \prod_{[\beta] \in T_\ell} \frac{1}{N_{[\beta]}^{\alpha'}!} \right\} \sum_{\mathcal{G}_{\alpha'}} \left\{ \left(\prod_{\alpha \in \alpha'} \sigma_\alpha^\ell e^{-\beta v^\ell(\alpha\alpha)/2} \right) \left(\prod_{(\alpha\gamma) \in \mathcal{G}_{\alpha'}} f_{\alpha\gamma} \right) \right\} \tag{3.12}$$

Summation is over all connected graphs $\mathcal{G}_{\alpha'}$ on vertices $\alpha\epsilon\alpha'$. Such a graph is specified by a set of links = unordered pairs (α,β), with $\alpha \neq \beta$, and $\alpha,\beta\epsilon\alpha'$. $N^{\alpha'}_{[\beta]}$ is the number of ℓ-vertices of type $[\beta]$ in the $(\ell+1)$ vertex α'.

The idea of the iterated cluster expansion is to treat one after the other of the pieces v^r of the potential by a cluster expansion, short range interactions first. In each step one uses (3.11) and expands in products of f's. The result is a sum of contributions which can be represented by graphs, and each such contribution factorizes according to the decomposition of this graph into connected pieces. The final result of this analysis is a formula for the Boltzmann factor $Z^N(\xi_1,\dots,\xi_N)$, proposition 2 of ref. 6.

$$Z^N(\xi_1,\dots,\xi_N) = \sum_{\substack{\{\alpha\} \\ \Sigma C(\alpha)=\{1\dots N\}}}' \Big(\prod_{[\beta]\epsilon T_R} N^R_{[\beta]}!\Big)^{-1} S^{(c)} \prod_{\alpha\epsilon\{\alpha\}} \sigma^R_\alpha(\xi_\alpha) \tag{3.13}$$

Summation if over partitions of the set 1 ... N of N constituents into R-vertices. (We write Σ for the union of disjoint sets; R = numbers of pieces into which v has been split.) The prime $'$ indicates that only one representative $\{\alpha\}$ out of every collection $\{[\alpha]\}$ of types of R-vertices is to be included in the sum. (This restriction could be dropped, but the combinatorial factor would then be different.) The symmetrizer $S^{(c)}$ averages over all N! permutations of the N constituents 1 ... N. $N^R_{[\beta]}$ is the number of R-vertices of type $[\beta]$ in $\{\alpha\}$.

From eq. (3.13) it follows that

$$Z^N(\xi_1,\dots,\xi_N)\prod_{j=1}^{N}[1+\varepsilon(\xi_j)] = \sum_{\substack{\{\alpha\} \\ \Sigma C(\alpha)=\{1\dots N\}}}' \Big(\prod_{[\beta]\epsilon T_R} N^R_{[\beta]}!\Big)^{-1} S^{(c)} \prod_{\alpha\epsilon\{\alpha\}} \Big\{\sigma^R_\alpha(\xi_\alpha)\prod_{j\epsilon\alpha}[1+\varepsilon(\xi_j)]\Big\}$$

This can be inserted into eq. (3.3). $1+\varepsilon(\xi)$ is a complex number of modulus 1. Proposition 3 of ref. 6 generalized therefore to assert that

$$Z_\Lambda(\phi) = \exp\sum_{[\alpha]\epsilon T_R}\int d\xi_\alpha\, \sigma^R_\alpha(\xi_\alpha)\prod_{j\epsilon\alpha}[1+\varepsilon(\xi_j)] \tag{3.14}$$

provided $\sum_{[\alpha]}\int d\xi_\alpha |\sigma^R_\alpha(\xi_\alpha)| < \infty$. We write $|C(\alpha)|$ for the number of constituents in α, and we assume in the rest of this section that the following stronger bound is satisfied

$$\sum_{[\alpha]\epsilon T_R} 3^{|C(\alpha)|}\int d\xi_\alpha |\sigma^R_\alpha(\xi_\alpha)| < \infty . \tag{3.15}$$

Its validity will be proven in the following sections. The final step is to expand the product

$$\prod_{j=1}^{t}[1+\varepsilon(\xi_j)] = \sum_{s=0}^{t}\binom{t}{s} S^{(c)}\varepsilon(\xi_1)\dots\varepsilon(\xi_s) \tag{3.16}$$

$S^{(c)}$ symmetrizes in the arguments $\xi_1\dots\xi_t$. This is inserted into expansion (3.14). Since $\sum_s\binom{t}{s}2^s = 3^t$, the bound (3.15) assures that the resulting series is absolutely convergent and may therefore be reordered. Define

$$\tilde\rho_t(\xi_1\dots\xi_t) = \sum_{\substack{[\alpha] \\ C(\alpha)=\{1\dots t\}}} \sigma^R_\alpha(\xi_\alpha) , \tag{3.17a}$$

and

$$\rho_s(\xi_1,\dots,\xi_s) = s!\sum_{t\geq s}\binom{t}{s}\int d\xi_{s+1}\dots d\xi_t\,\tilde\rho_t(\xi_1\dots\xi_t) . \tag{3.17b}$$

Then it follows from eq. (3.14) that

$$Z_\Lambda(\phi) = \exp\Big\{\rho_0 + \sum_{s=1,2,\dots}\frac{1}{s!}\int d\xi_1\dots d\xi_s\,\varepsilon(\xi_1)\dots\varepsilon(\xi_s)\rho_s(\xi_1\dots\xi_s)\Big\} . \tag{3.18}$$

The series in the exponent is absolutely convergent if the inequality (3.15) is fulfilled.

If we insert eq. (3.18) into (2.11), we obtain formulae (1.16), (1.17) of proposition 3. To complete the proof of proposition 3 we must show that inequality (3.15) holds and that the bounds (1.18) are satisfied, for a suitable split (3.6) of the Yukawa potential. In addition we should determine the asymptotic behavior of $\rho_1(\pm 1, x)$ for $\beta/a \to \infty$. This will be done in the next section.

We will now specify the split (3.6) of the potential. There are three intrinsic length scales in our problem: the lattice spacing a, the Landau length β, and the Debye length m_D^{-1} (m_D was defined in eq. (1.8)). If β/a is large then we have $a \ll \beta \lll m_D^{-1}$. We split the potential v into R = 3 pieces. v^0 incorporates the hard core - i.e. an interaction of range a - and is defined by eq. (3.7). v^1 will have a range of order β. The rest of v is called v^2, it has range $M^{-1} = \lambda m_D^{-1}$. Explicitly

$$v^1(\xi_i,\xi_j) = \begin{cases} 0 & \text{if } x_i = x_j\ ,\ \text{sign}\, m_i = -\text{sign}\, m_j \\ m_i m_j v^1(x_i - x_j) & \text{otherwise}\ , \end{cases} \tag{3.19}$$

$$v^1(x_i-x_j) = (-\Delta + M_1^2)^{-1}(x_i,x_j)\ ,\ \text{with } M_1 = -4\pi\beta^{-1}\ln(1-C)\ , \tag{3.20}$$

and

$$v^2(\xi_i,\xi_j) = m_i m_j v^2(x_i - x_j) \tag{3.21}$$

$$v^2(x_i-x_j) = (-\Delta + M^2)^{-1}(x_i,x_j) - (-\Delta + M_1^2)^{-1}(x_i,x_j) \tag{3.22}$$

C is a constant; the same constant appears in the bounds of proposition 3. In eq. (3.19) use is made of the presence of the hard core to adjust the value of the potential at zero distance conveniently.

To study fall-off properties of vertex functions we introduce the distance

$$\| x \| = 3^{-1/2} \sum_\mu |x_\mu| \tag{3.23}$$

and norms depending on a constant $A \geq 0$,

$$\| v^r \|_A = \int_{x\in(a\mathbb{Z})^3} |v^r(x)|\, e^{A\|x\|} \tag{3.24}$$

We will need information on the energies

$$E^r(m) = \tfrac{1}{2}(m, v^r m) = \tfrac{1}{2}\sum_{i,j=1}^{N} v^r(\xi_i,\xi_j)$$

of a charge configuration m,

$$m(x) = \sum_{i=1}^{N} m_i\, \delta_{x_i,x}\ .$$

It is provided by the following lemma. It tells us in particular that among all possible configurations of $N \geq 1$ particle, a single particle with charge $m_1 = \pm 1$ has the lowest possible energy $E^1 = \frac{1}{2}v^1(0)$. In particular, neutral dipoles have higher energy E^1. We state the lemma for an arbitrary choice of lattice spacing.

<u>Lemma 5</u> If aM_1 is sufficiently small and $M \equiv M_2 \leq M_1$ then there exists a constant $\varepsilon_1 > 0$ such that the following inequality holds for $\ell = 1,2$.

$$\tfrac{1}{2}\sum_{1\leq i,j\leq N} v^\ell(\xi_i,\xi_j) \geq \gamma_\ell + \varepsilon_\ell a^{-1}\Big(-1 + \sum_{i=1}^{N} m_i^2\Big) \quad \text{for } N \geq 1\ , \tag{3.25}$$

with $\gamma_2 = \varepsilon_2 = 0$, $\gamma_1 = \frac{1}{2}v^1(0)$.

Moreover, $v^\ell(x) \geq 0$ and

$$\| v^1 \|_A \leq 2(M_1 - A)^{-2} \quad ; \quad \| v^2 \|_A \leq 2(M - A)^{-2} \tag{3.26}$$

for $A < (1-\delta)M$, $\delta > 0$, and β/a sufficiently large, depending on δ.

The proof of the first part of this lemma is given in ref. 6 (proposition 8). The proof of the inequalities (3.26) is based on explicit computation in momentum space. Details are given in Appendix B.

4. Estimates for vertex functions

Equations (3.17) express the coefficient functions $\rho_t(\xi_1 \dots \xi_t)$ in the series expansion (1.17) of V_{eff} in terms of vertex functions σ^R_α. In this section we will derive recursive bounds for vertex functions σ^ℓ_α. Validity of inequality (3.15) (which was assumed in deriving (3.17)) and of the bounds (1.18) in proposition 3 will be deduced from them.

We start with the 1-vertex functions. They incorporate only the hard core and are known explicitly, see eq. (3.2) of [6]. There is only one type of 1-vertex, and

$$\sigma^1(\xi_1,\dots,\xi_n) = (-1)^{n-1}\, n^{-1}\, \delta_{x_2 x_1} \dots \delta_{x_n x_1} \tag{4.1}$$

for the choice (3.8) of 0-vertex function.

To obtain estimates for higher vertex functions, it is convenient to start from the tree formula, eq. (4.3) below. The tree formalism was used extensively in constructive field theory. Proposition 4 of [6] asserts that the tree formula (4.3) for the $(\ell+1)$-vertex function ($\ell \geq 1$) is equivalent to the defining equation (3.12).

Consider the set of all functions η which assign to every integer $a = 1 \dots t-1$ a positive integer $\eta(a)$ satisfying $\eta(a) \leq a$. Every such function specifies a tree graph T with t vertices. Its links are the pairs $(a+1, \eta(a))$; $a = 1 \dots t-1$. One also introduces real variables $s_1 \dots s_{t-1}$ which take values $0 \dots 1$.

Consider now an $(\ell+1)$-vertex α' which consists of t ℓ-vertices. We shall label them in some arbitrary way $\alpha_1 \dots \alpha_t$. The sympol S will stand for symmetrization on labels $\alpha_1 \dots \alpha_t$. It acts on symbolic expressions F carrying such labels. The defining expression (3.12) for the vertex function $\sigma^{\ell+1}_{\alpha'}$ is symmetric in the labels $\alpha_1 \dots \alpha_t$.

Given the potential $v^\ell(\alpha_a\alpha_b)$ of eq. (3.10) one defines a partially decoupled interaction W^ℓ. It depends on $\underline{s} = (s_1, \dots, s_{t-1})$

$$W^\ell(\underline{s}|\alpha') = \frac{1}{2}\sum_{a=1}^{t} v^\ell(\alpha_a\alpha_a) + \sum_{1\leq a<b\leq t} s_a s_{a+1} \cdots s_{b-1}\, v^\ell(\alpha_a\alpha_b) \tag{4.2}$$

The tree formula reads for $\ell \geq 1$

$$\sigma^{\ell+1}_{\alpha'} \equiv \frac{t!}{\prod_{[\beta]\in T_\ell} N^{\alpha'}_{[\beta]}!}\, \hat\sigma^{\ell+1}_{\alpha'} , \tag{4.3a}$$

$$\hat\sigma^{\ell+1}_{\alpha'}(\xi_{\alpha'}) = \frac{(-\beta)^{t-1}}{t}\, S \int ds_1 \dots ds_{t-1} \sum_\eta \Big\{ f(\eta,\underline{s}) \Big[\prod_{a=1}^{t-1} v^\ell(\alpha_{a+1},\alpha_{\eta(a)})\Big] e^{-\beta W^\ell(\underline{s}|\alpha')} \Big\} \cdot \prod_{b=1}^{t} \sigma^\ell_{\alpha_b}(\xi_{\alpha_b}) . \tag{4.3b}$$

Summation is over trees η as described above, and

$$f(\eta,\underline{s}) = \prod_{a=1}^{t-1} [s_{a-1} s_{a-2} \cdots s_{\eta(a)}] . \tag{4.4}$$

Empty products which arise when $\eta(a) = a$ or $t = 1$ are read as 1.

As a consequence of their definition and of inequality (3.23) in lemma 5, the partially decoupled interaction $W^\ell(\underline{s}|\alpha')$ also satisfy the inequalities (see eq. (4.3) of [6])

$$W^\ell(\underline{s}|\alpha') \geq \delta_\ell + \varepsilon_\ell \sum_{j\in\alpha'} m_j^2 \tag{4.5}$$

where

$$\delta_\ell = \gamma_\ell - \varepsilon_\ell \tag{4.6}$$

Inserting the bound (4.5) for $W^\ell(\underline{s}|\alpha')$ and the definition (3.10), (3.17), (3.19) of $v^\ell(\alpha_a,\alpha_b)$ into eq. (4.3b) one obtains the inequality

$$|\hat\sigma^{\ell+1}_{\alpha'}(\xi_{\alpha'})| \leq \frac{\beta^{t-1}}{t}\, e^{-\beta(\delta_\ell + \varepsilon_\ell \Sigma_{j\in\alpha'} m_j^2)} \Big\{ \prod_{b=1}^{t} |\sigma^\ell_{\alpha_b}(\xi_{\alpha_b})| \Big\} \cdot S \int_0^1 ds_1 \dots ds_{t-1} \sum_\eta \prod_{a=1}^{t-1} \Big\{ \sum_{j\in\alpha_{a+1}} \sum_{k\in\alpha_{\eta(a)}} s_{a-1} s_{a-2} \cdots s_{\eta(a)} |m_j m_k v^\ell(x_j - x_k)| \Big\} \tag{4.7}$$

In eq. (1.11) we introduced quantities $L_T(a_1, \dots, a_n)$ and $L(a_1, \dots, a_n)$ which measure the spatial separation between subsets $a_1, \dots a_n$ of Λ. If a_i consist of single sites x_i, we write $L_T(x_1, \dots, x_n)$ in place of $L_T(a_1, \dots, a_n)$, etc. It is convenient to introduce also

$$\tilde{L}(x_1,\dots,x_n) = \min_T L_T(x_1,\dots,x_n) \equiv \min_T \sum_{(ij)\in T} \|x_i - x_j\|$$

minimum over tree graphs T with exactly n vertices 1 ... n. (4.8)

It follows that

$$L(a_1,\dots,a_n) = \min_{t\geq n} \min_{\substack{(x_i)\\ x_1\in a_1 \dots x_n \in a_n}} \tilde{L}(x_1,\dots,x_t) \tag{4.9}$$

Let α be some ℓ-vertex with n constitutents $i_1 \dots i_n$. The variable ξ_α specifies the position x_i of every constituent $i \in \alpha$. We set

$$\tilde{L}(x_\alpha) \equiv \tilde{L}(x_{i_1},\dots,x_{i_n}) \tag{4.10}$$

<u>Lemma 6</u> Let α' be a $(\ell+1)$-vertex which consists of t ℓ-vertices $\alpha_1 \dots \alpha_t$, and let η be a map as described above which specifies a tree graph with t vertices 1 ... t. Suppose that constituents $j(a) \in \alpha_{a+1}$ and $k(a) \in \alpha_{\eta(a)}$ are chosen in some arbitrary way for every a = 1 ... t-1. Then

$$\tilde{L}(x_{\alpha'}) \leq \sum_{a=1}^{t} \tilde{L}(x_{\alpha_a}) + \sum_{a=1}^{t-1} \|x_{k(a)} - x_{j(a)}\| \tag{4.11}$$

<u>Proof</u>: By definition (4.8) of $\tilde{L}$, there exist tree graphs T_a whose vertices are the constitutents of α_a such that

$$\tilde{L}_T(x_\alpha) = L_{T_a}((x_i)_{i\in\alpha_a})$$

We construct a tree graph T whose vertices are the constituents of α', and such that

$$L_T(x_{\alpha'}) = \sum_{a=1}^{t} L_{T_a}(x_{\alpha_a}) + \sum_{a=1}^{t-1} \|x_{k(a)} - x_{j(a)}\|$$

Inequality (4.11) follows from this by definition (4.8) of $\tilde{L}$. The tree graph T consists of all the links of all the trees T_a, plus the links (k(a), j(a)), a = 1 ... t-1 □

Now we return to our vertex functions. We define norms (n = no. of constituents of α)

$$\|\sigma_\alpha^\ell\|_{A;\varepsilon,\kappa} \equiv \int_{x_\alpha \in \mathbb{Z}^{3n}} d\xi_\alpha |\sigma_\alpha^\ell(\xi_\alpha)| \exp\Big[A\tilde{L}(x_\alpha) + \sum_{j\in\alpha}\big(2\kappa|m_j| - \varepsilon m_j^2\big)\Big]\, \delta_{x_i,x} \tag{4.12}$$

and

$$\|\sigma^\ell\|_{A;\varepsilon,\kappa} \equiv \sum_{[\alpha]\in T_\ell} \|\sigma_\alpha^\ell\|_{A;\varepsilon,\kappa} \tag{4.13}$$

In formula (4.12) the x-summations run over the infinitely extended lattice $\mathbb{Z}^3$, and not only over Λ as would be implied by the notational convention (3.1). i is an arbitrary constituent of α. The result does not depend on x or i because of translation invariance of the potentials. Summation in (4.13) is over all types of ℓ-vertices.

We will derive recursive bounds for these norms from inequality (4.7). First we incorporate factors exp $[A\tilde{L}(\cdot)]$ into inequality (4.7), using lemma 6. This gives

$$|\hat{\sigma}_{\alpha'}^{\ell+1}(\xi_{\alpha'})| e^{A\tilde{L}(x_{\alpha'})} \leq \frac{\beta^{t-1}}{t} e^{-\beta(\delta_\ell + \varepsilon_\ell \Sigma_{j\in\alpha'} m_j^2)} \Big\{\prod_{b=1}^{t} |\sigma_{\alpha_b}^\ell(\xi_{\alpha_b})| e^{A\tilde{L}(x_{\alpha_b})}\Big\}$$
$$\cdot \mathcal{S}\int_0^1 ds_1 \dots ds_{t-1} \sum_\eta \prod_{a=1}^{t-1} \Big\{\sum_{j\in\alpha_{a+1}} \sum_{k\in\alpha_{\eta(a)}} s_{a-1}s_{a-2}\dots s_{\eta(a)} |m_j m_k v_A^\ell(x_j - x_k)|\Big\} \tag{4.14}$$

where

$$v_A^\ell(x-y) = v^\ell(x-y)e^{A\|x-y\|} \tag{4.15}$$

From here on the procedure is literally the same as in section 4 of ref. 6. One carries out the x-summations, the s-integrations and η-summations (using the tree estimate, lemma 5 of [6]), and the m-summations. This produces a recursive bound for $\|\sigma_{\alpha'}^\ell\|_{A;\varepsilon,\kappa}$. Then one carries out the α-summations in two steps. First one sums over $(\ell+1)$-vertices which consist of a fixed number $t \geq 1$ of ℓ-vertices, and finally one sums over t. As a result one obtains the following generalizations of proposition 6 and its corollary 7 of ref. 6.

<u>Proposition 7</u> The following inequalities hold for any ℓ = 1 ... R-1 and for arbitrary choice of $\kappa_\ell > 0$, provided the argument of the logarithm is positive

$$\|\sigma^{\ell+1}\|_{A;\varepsilon,\kappa} \leq -\left[\beta\|v^\ell\|_A/\kappa_\ell^2\right]^{-1} e^{-\beta\delta_\ell} \ln\left(1-\beta\|v^\ell\|_A \kappa_\ell^{-2}\|\sigma^\ell\|_{A;\varepsilon+\beta\varepsilon_\ell,\kappa+\kappa_\ell}\right) \quad (4.16)$$

ε_ℓ and $\delta_\ell = \gamma_\ell - \varepsilon_\ell$ are the constants in the bounds (3.23) for the potentials v^ℓ, and $\|v^\ell\|_A$ is defined in eq. (3.22).

<u>Corollary 8</u> Set $K_\ell = \Sigma_{k=\ell}^{R-1}\kappa_k$, $E_\ell = \beta\Sigma_{k=\ell}^{R-1}\varepsilon_k$, and $\Delta_\ell = \Sigma_{k=1}^{\ell}\delta_k$. Suppose that the following inequalities are fulfilled for some $A_1 > 0$, $0 < C < 1$, $\kappa' \geq 0$.

$$\|\sigma^1\|_{A;E_1,K_1+\kappa'} \leq A_1(1-C)^{-1} \quad (4.17a)$$

$$\beta\|v^\ell\|_A \kappa_\ell^{-2} \leq C(1-C)^\ell A_1^{-1} e^{\beta\Delta_{\ell-1}} \quad \text{for } \ell = 1 \ldots R-1 \quad (4.17b)$$

Then

$$\|\sigma^R\|_{A;0,\kappa'} \leq A_1(1-C)^{-R} e^{-\beta\Delta_{R-1}} . \quad (4.18)$$

The next step of our analysis will be to show that the hypotheses of Corollary 8 can be fulfilled by making β/a and $\lambda^{-1} = M/m_D$ sufficiently large. We choose a constant C in the interval $0 < C < 1$, as small as desired. This constant C will appear in the bounds (1.18) in the end. We set

$$\kappa_1 = \kappa_2 = -\ln(1-C) \quad (4.19)$$

The intermediate cutoff M_1 is chosen as in eq. (3.18), with the same C.

We begin by estimating the 1-vertex function. Consider a 1-vertex α made of constituents 1 ... n. Since $\sigma_\alpha^1(\xi_\alpha) \equiv \sigma^1(\xi_1,\ldots,\xi_n)$ is nonzero only for coinciding positions x_i, according to eq. (4.1), we have

$$\sigma_\alpha^1(\xi_\alpha)\, e^{AL(x_\alpha)} = \sigma_\alpha^1(\xi_\alpha) . \quad (4.20)$$

There is only one type of 1-vertex, summation over [α] amounts therefore to summing over the number of constituents n. Thus, by eq. (4.11)

$$\|\sigma^1\|_{A;\beta\varepsilon_1,\kappa''} = \|\sigma^1\|_{0;\beta\varepsilon_1,\kappa''} = \sum_{n\geq 1}\frac{1}{n}\Big(\sum_{q=\pm1,\pm2,\ldots} e^{-\beta\varepsilon_1 q^2 + 2\kappa''|q|}\Big)^n$$

Setting $\kappa'' = \kappa' + \kappa_1 + \kappa_2 = \kappa' + K_1$ and noting that $E_1 = \beta\varepsilon_1$ by definition, we find that

$$\|\sigma^1\|_{A;E_1,K_1+\kappa'} \leq A_1(1-C)^{-1} \quad (4.21)$$

with

$$A_1 = 2e^{-\beta\varepsilon_1+2\kappa''} = 2e^{-\beta\varepsilon_1+2\kappa'}(1-C)^{-4} , \quad (4.22)$$

provided β is sufficiently large (depending on C, κ'). This establishes hypothesis (4.17a) of corollary 8, for arbitrary A.

Next we turn to hypothesis (4.17b) with $\ell = 1$. Since $\Delta_0 = 0$ we must show that

$$\beta\|v^1\|_A \leq C(1-C)^5\left[\ln^2(1-C)\right]\tfrac{1}{2}\, e^{\beta\varepsilon_1 - 2\kappa'} \quad (4.23)$$

Lemma 5 asserts that this is fulfilled for $A < (1-\delta)M_1$ and β sufficiently large, depending on C, κ' and δ. Similarly, for $\ell = 2$ we have $\Delta_1 = \delta_1$, and we should show that

$$\beta\|v^2\|_A \leq C(1-C)^6\left[\ln^2(1-C)\right]\tfrac{1}{2}\, e^{\beta\gamma_1 - 2\kappa'} \quad (4.24)$$

We have (see eq. (5.6) of [6])

$$\gamma_1 = \tfrac{1}{2}v^1(0) = \tfrac{1}{2}v_{Cb}(0) - (4\pi)^{-1}M_1 + O(M_1^2)$$

Therefore, with the choice (3.18) of M_1

$$\beta\gamma_1 \geq \tfrac{1}{2}\beta v_{Cb}(0) + 2\ln(1-C) \quad (4.25)$$

if β is large enough. We assume that β is large enough so that $M \leq M_1$. Lemma 5 tells us that inequality (4.24) is fulfilled for $A \leq M(1-\delta)$ if

$$1 \leq \tfrac{1}{4}\delta^2 C^3(1-C)^8 e^{-2\kappa'} M^2\beta^{-1} e^{\beta v_{Cb}(0)/2}$$
$$= \tfrac{1}{4}\delta^2 C^3(1-C)^8 e^{-2\kappa'} M^2/m_D^2$$

by definition (1.8) of m_D (a = 1 in this section).

This is true if $\lambda = m_D/M$ is small enough, depending on C, κ' and δ.

Having verified validity of its hypothesis, we can now apply corollary 8. By definition, $\beta(\varepsilon_1 + \Delta_2) = \beta\gamma_1$. Using inequality (4.25) again and setting $\kappa' = \kappa + \frac{1}{2}\mu$ we obtain (R = 3 always)

Proposition 9 Under the hypotheses of proposition 3

$$\|\sigma^R\|_{A;0,\kappa+\frac{1}{2}\mu} \leq (m_D^2/\beta)\, e^{2\kappa+\mu}\, (1-C)^{-9} \tag{4.26}$$

for $A \leq (1-\delta)M$.

Now we are ready to proceed to the proof of proposition 3. First we note that $m_j^2 \geq 1$ implies

$$\Sigma_{j\in\alpha}\, m_j^2 \geq |C(\alpha)| \tag{4.27}$$

Therefore

$$\|\sigma^R\|_{0;0,\frac{1}{2}\mu} \geq \sum_{[\alpha]\in T_R} e^{\mu|C(\alpha)|} \int_{(x_\alpha\in\Lambda^n)} d\xi_\alpha\, |\sigma_\alpha^R(\xi_\alpha)| \tag{4.28}$$

Setting $\mu = \ln 3$ we see that proposition 9 ensures validity of the bound (3.15), under the hypotheses of proposition 3. Therefore, $\ln Z_\Lambda(\phi)$ can be represented by an absolutely convergent series (3.18). We set

$$-V_{eff}(\phi) = \ln Z_\Lambda(\phi) \tag{4.29}$$

Then eqs. (1.16) and (1.18) of proposition 3 follows from eqs. (2.11) and (3.18), and eqs. (3.27) give the following formula for the coefficient functions ρ_s

$$\rho_s(\xi_1\ldots\xi_s) = s!\sum_{t\geq s}\binom{t}{s}\sum_{\substack{[\alpha]\\ C(\alpha)=\{1\ldots t\}}}\int d\xi_{s+1}\ldots d\xi_t\;\sigma_\alpha^R(\xi_\alpha) \tag{4.30}$$

If $s \geq 1$ it follows from expression (4.9), (4.10) for $L(a_1,\ldots,a_s)$ that

$$\int_{x_2\in a_2}\ldots\int_{x_s\in a_s} d\xi_2\ldots d\xi_s \sum_{m_1} |\rho_s(\xi_1\ldots\xi_s)|\, e^{2\kappa\Sigma|m_i|}$$
$$\leq s!\, e^{-\mu(s-1)}\Big[\sum_{t\geq s}\binom{t}{s}e^{-\mu(t-s)}\Big]e^{-AL(a_1\ldots a_s)}\;.$$
$$\cdot \max_{t\geq s}\sum_{\substack{[\alpha]\\ C(\alpha)=\{1\ldots t\}}}\int d\xi'_\alpha\; e^{A\tilde{L}(x'_\alpha)+\mu(t-1)+2\kappa\Sigma_{i=1}^t|m_i|}\,|\sigma_\alpha^R(\xi_{\alpha'})|\,\delta_{x'_1x_1}$$
$$\leq s!\, e^{-\mu(s-1)}e^{-AL(a_1\ldots a_s)}\Big[\sum_{t\geq s}\binom{t}{s}e^{-\mu(t-s)}\Big]\,\|\sigma^R\|_{A;0,\kappa+\frac{1}{2}\mu}\, e^{-\mu}$$
$$\leq s!\, e^{-\mu(s-1)}e^{-AL(a_1\ldots a_s)}\,(m_D^2/\beta)\,e^{2\kappa}\,(1-C)^{-9}\Big[\sum_{t\geq s}\binom{t}{s}e^{-\mu(t-s)}\Big]$$

by proposition 9. Without loss of generality we may assume that μ is large enough so that

$$\sum_{t\geq s}\binom{t}{s}e^{-\mu(t-s)} = \left[1-e^{-\mu}\right]^{-s-1} \leq (1-C)^{-s-1} \tag{4.32}$$

The bound (1.18) of proposition 3 follows from inequalities (4.31), (4.32) by summing over $x_1 \in a_1$.

It remains to determine the symptotic behavior of $\rho_1(\xi)$ when $\beta/a \to \infty$. We single out the term with t = s = 1 in eq. (4.30). There is only a single type of ℓ-vertex α with a single constitutent i = 1, and recursion relation (3.12) specializes in this case to

$$\sigma_\alpha^{\ell+1}(\xi) = \sigma_\alpha^\ell(\xi)\exp\left[-\beta m^2 v^\ell(0)/2\right] \qquad \text{for } \xi = (m,x)$$

With eq. (3.8) is follows that

$$\sigma_\alpha^R(\xi) = \exp\left[-\beta m^2 v(0)/2\right] \qquad \text{if } C(\alpha) = \{1\}\,. \tag{4.33}$$

Thus, eq. (4.30) gives

$$\rho_1(\xi_1) = \exp\left[-\beta m_1^2 v(0)/2\right] + \sum_{t\geq 2}\binom{t}{1}\sum_{\substack{[\alpha]\\ C(\alpha)=\{1\ldots t\}}}\int d\xi_2\ldots d\xi_t\;\sigma_\alpha^R(\xi_\alpha) \tag{4.34}$$

The second term is bounded by

$$e^{-2\mu}\,\|\sigma^R\|_{0,0,\frac{1}{2}\mu}\Big[\sum_{t\geq 2}\binom{t}{1}e^{-\mu(t-1)}\Big] \leq e^{-\mu}(1-e^{-\mu})^{-2}\, m_D^2\beta^{-1}\,(1-C)^{-9}$$

by proposition 3. C can be chosen arbitrarily small and μ arbitrarily large, if β is large enough. Since $v(0) = v_{Cb}(0) - O(M)$, the first term in (4.34) is asymptotically equal to $\exp[-\beta m_1^2 v_{Cb}(0)/2] = (m_D^2/2\beta)^{m_1^2}$ if $\beta\to\infty$ and $\beta M \to 0$. The last assertion of proposition 3 follows from this. □

5. Low temperature behavior of the Z-ferromagnet

When β/a is small then the U(1) lattice gauge theory can be treated by high temperature expansions and the area law for the Wilson loop expectation value (1.3) follows. This was proven by Osterwalder and Seiler [19] . Under the exact duality transformation, the U(1) lattice gauge theory at high temperature β^{-1} goes into the Z-ferromagnet at low temperatures β, (see the remark after eq. (1.2)), and high temperature expansions for the gauge theory become low temperature expansions for the Z-ferromagnet. They are expansions in domain walls = Peierls contours on Λ. In this section we will write down these low temperature expansions. We indulge in this pedagogical exercise because it will be instructive to compare the result with corresponding formulae that will be valid for large β/a.

We start with expression (1.5) for the partition functions. Setting $a = 1$ it becomes

$$Z_\Lambda(k,\Xi) = \sum_{n\in(2\pi\mathbb{Z})^\Lambda} \exp\left[-\frac{1}{2\beta}\sum_{x\in\Lambda}\sum_\mu \left(n(x+e_\mu)-n(x)-kj_\mu(x)\right)^2\right] \tag{5.1}$$

For simplicity we consider Dirichlet boundary conditions, $n(x) = 0$ on $\partial\Lambda$, instead of the heat bath described in the introduction.

The domain walls are positioned at those links $b = (x,x+e_\mu)$ in Λ where $2\pi N(b) \equiv n(x+e_\mu) - n(x) - kj_\mu(x) \neq 0$. These links may be considered as plaquettes of the dual lattice Λ^* so that the domain walls become surfaces. To specify them completely, one must also specify the magnitude of the jumps across. In order to maintain a simple geometrical picture , it is most convenient to do this by counting every link b in the domain wall with the (positive or negative integer) multiplicity $N(b)$. We introduce the 1-chain T with coefficients in the abelian group $2\pi\beta^{-1/2}\mathbb{Z}$ by

$$T = 2\pi\beta^{-1/2}\sum_b N(b)\,b \tag{5.2}$$

$$N(b) = (2\pi)^{-1}\left\{n(x+e_\mu)-n(x)-kj_\mu(x)\right\} \quad \text{for } b=(x,x+e_\mu)$$

In the following we will also call this T the domain wall for short. The reader may picture it as a surface in Λ^* which passes $N(b)$ times through the plaquette b of Λ^*. (The sign of $N(b)$ gives the orientation of the surface.) Application of the coboundary operator ∂^* amounts to forming the boundary of T on the dual lattice Λ^*. The result is a 2-chain on Λ (= set of plaquettes counted with multiplicity). For a single link b, $\partial^* b$ is the sum of the four oriented plaquettes shown in Fig. 2, and $\partial^* T = 2\pi\beta^{-1/2}\Sigma_b N(b)\,\partial^* b$. An example is shown in Fig. 3.

It follows from its definition (5.2), (5.3) and expression (1.6) for j_μ that

$$\partial^* T = 2\pi\beta^{-1/2} k C \tag{5.4}$$

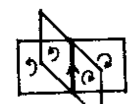

Fig. 2 A link b and its coboundary $\partial^* b = p_1+p_2+p_3+p_4$. p_i are the four plaquettes with orientation as shown. The plaquette with reversed orientation is called $-p_i$.

C is the Wilson loop in Λ^*, it is a sum of plaquettes in Λ.

Conversely, every 1-chain T with the prescribed coboundary (5.4) specifies a unique spin configuration with Dirichlet boundary conditions $n(x) = 0$ on $\partial\Lambda$. Using the notation

$$\|T\|^2 = 4\pi^2\beta^{-1}\sum_b N(b)^2 \tag{5.5}$$

the partition functions (5.1) can therefore be rewritten as

$$Z_\Lambda(k,\Xi) = \sum_{\substack{T \\ \partial^* T = 2\pi\beta^{-1/2}kC}} \exp\left[-\tfrac{1}{2}\|T\|^2\right]. \tag{5.6}$$

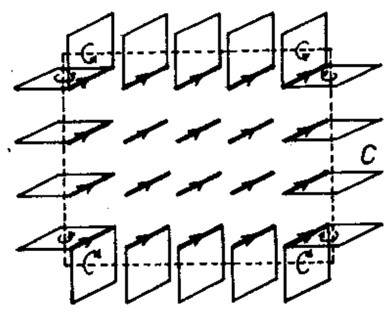

Fig. 3 A 1-chain $T = \Sigma b$(sum over the dark links) and its coboundary $\partial^* T = C$ (formal sum of the plaquettes shown with the orientation inherited from the links b). As an object on the dual lattice Λ^*, C is given by the dashed line.

Summation is over all 1-chains (5.2) on Λ with the prescribed coboundary. As a special case we have for $k = 0$

$$Z_\Lambda \equiv Z_\Lambda(0,\cdot) = \sum_{\substack{T \\ \partial^* T = 0}} \exp\left[-\tfrac{1}{2}\|T\|^2\right] \tag{5.7}$$

Next we decompose the domain walls into connected components. The notion of connectedness used here is the obvious one on the dual lattice Λ^* where the domain walls are surfaces. Two such surfaces are disjoint if they do not touch along a link of Λ^*, and a domain wall is connected if it cannot be decomposed into two that are positioned on disjoint surfaces. In the language of chains on Λ this can be formulated as follows. We say that

$$T_1 \wedge T_2 = 0$$

for two 1-chains $T_i = 2\pi\beta^{-1/2}\Sigma N_i(b)b$ if there is no plaquette p on Λ with the property that there are (not necessarily distinct) links b_1, b_2 in the boundary of p such that $N_1(b_1) \neq 0$ and $N_2(b) \neq 0$. The 1-chain T represents a connected domain wall if it is impossible to find 1-chains T_1, T_2 such that $T = T_1 + T_2$ and $T_1 \wedge T_2 = 0$.

If C is a simple loop on Λ^* we may decompose any domain wall T with coboundary $2\pi\beta^{-1/2}kC$, $k \neq 0$, into a connected piece T_1 with coboundary $2\pi\beta^{-1/2}kC$, and a rest T_2 which is disjoint from T_1 and has no coboundary:

$$T = T_1 + T_2 \tag{5.8}$$

such that

$$\partial^* T_1 = 2\pi\beta^{-1/2}kC \quad , \qquad T_1 \text{ is connected} \tag{5.8a}$$

and

$$\partial^* T_2 = 0 \quad , \quad T_1 \wedge T_2 = 0 \ . \tag{5.8b}$$

It can happen that $T_2 = 0$.

Correspondingly the partition functions become, for $k \neq 0$

$$Z_\Lambda(k,\Xi) = \sum_{\substack{T_1\ \mathrm{conn.} \\ \partial^* T_1 = 2\pi\beta^{-1/2}kC}} e^{-\frac{1}{2}\|T_1\|^2} \sum_{\substack{T_2 \\ \partial^* T_2 = 0,\, T_1 \wedge T_2 = 0}} e^{-\frac{1}{2}\|T_2\|^2} \tag{5.9}$$

Comparing with eq. (5.7) we see that the second factor is partition function of a system on a smaller lattice $\Lambda - X(T_1)$ which is, roughly speaking, the complement of a neighbourhood of 1 lattice spacing of the domain wall T_1. More precisely, let $T_1 = 2\pi\beta^{-1/2}\Sigma N_1(b)b$. We define $X(T_1)$ as the union of all cubes c of Λ such that there exists an edge b of c where $N_1(b) \neq 0$.

Any domain wall T_2 which satisfies the conditions (5.8b) lies completely inside the smaller lattice $\Lambda - X(T_1)$. Conversely any domain wall T_2 in $\Lambda - X(T_1)$ with $\partial^* T_2 = 0$ fulfills (5.8b).

Therefore we can write

$$\frac{Z_\Lambda(k,\Xi)}{Z_\Lambda} = \sum_{\substack{T_1\ \mathrm{conn.} \\ \partial^* T_1 = 2\pi\beta^{-1/2}kC}} \frac{Z_{\Lambda - X(T_1)}}{Z_\Lambda} K(T_1) \tag{5.10}$$

with

$$K(T_1) = \exp\left[-\tfrac{1}{2}\|T_1\|^2\right] . \tag{5.11}$$

Summation is over connected domain walls with coboundary $2\pi\beta^{-1/2}kC$. $Z_{\Lambda - X(T_1)}$ is the partition function of a system on $\Lambda - X(T_1)$ with zero Dirichlet boundary conditions on the part of $\partial\Lambda$ that is in its boundary. It has no dependence on the Wilson loop.

Eqs. (5.10), (5.11) are the desired low temperature expansions for the ratio $Z_\Lambda(k,\Xi) / Z_\Lambda$ which determines the surface tension α by eq. (1.7).

It follows from formula (5.7) that

$$Z_{\Lambda - X(T_1)} / Z_\Lambda \leq 1 , \tag{5.12}$$

because every term in the sum over (coclosed) domain walls T on Λ includes in particular all (coclosed) domain walls T on $\Lambda - X(T_1)$.

A lower bound on the surface tension α can be obtained from eqs. (5.10) ... (5.12). We formulate the result for general lattice spacing a.

<u>Theorem 10</u> There is a constant $c > 0$ such that for β/a sufficiently small

$$\alpha \geq c\,\beta^{-1} a^{-1} \tag{5.13}$$

<u>proof</u>. We set $k = 1$ and observe first that the leading term in the expression (5.10) comes from $T_1 = 2\pi\beta^{-1/2}\Xi$ (Ξ is the minimal surface on Λ^* with boundary C). With the definition (1.7) of α we obtain for the leading term

$$2\pi^2\beta^{-1} \tag{5.14}$$

in units where $a = 1$.

Now we come to the proof of inequality (5.13). Inserting inequality (5.12) into (5.10) with $k = 1$ we obtain

$$\langle \chi_1(u(C))\rangle_{u(1)} \leq \sum_{\substack{T_1\ \mathrm{conn.} \\ \partial^* T_1 = 2\pi\beta^{-1/2} C}} \exp\left[-\tfrac{1}{2}\|T_1\|^2\right] \tag{5.15}$$

Any T_1 which appears in (5.15) obeys

$$\|T_1\|^2 \geq 4\pi\beta^{-1}|\Xi| \tag{5.16}$$

because Ξ is the minimal surface in Λ^* with boundary C. We choose some $\varepsilon \in (0,1)$, arbitrarily close to 1, and define $\kappa(N)$ as the number of terms in the sum over T_1 in (5.15) which satisfy $\|T_1\|^2 = 4\pi^2\beta^{-1}N$. We conclude that

$$\langle \chi_1(u(C))\rangle \leq e^{-2\pi^2\beta^{-1}\varepsilon|\Xi|} \sum_{N\geq 1} \kappa(N)\, e^{-2\pi^2\beta^{-1}(1-\varepsilon)N} \tag{5.17}$$

Standard combinatorial arguments [10,28] (Euler's solution of the Königsberg bridge problem) assert that

$$\kappa(N) \leq e^{c_1 N} \tag{5.18}$$

with some constant c_1. Therefore the sum over N converges for sufficiently small β (depending on c_1 and ε). As a result, the surface tension α defined in eq. (1.7) satisfies

$$\alpha \geq 2\pi^2\beta^{-1}\varepsilon \tag{5.19}$$

This proves theorem 10 (with $c = 2\pi^2\varepsilon$). For general lattice spacing a the factor a^{-1} appears on dimensional grounds . □

6. The Glimm Jaffe Spencer expansion

Now we turn to the analysis of the theory with Pauli Villars cutoff M and an effective action given by proposition 3. We assume that λ has been chosen sufficiently small and β/a is sufficiently large so that the hypotheses of proposition 3 are fulfilled. We adapt the analysis of Brydges and Federbush [2]. It is based on the Glimm Jaffe Spencer expansion around mean field theory [9].

The idea is to find a substitute for expansion (5.10) of section 5 which is valid for large β/a. It will take the form

$$\frac{Z_\Lambda(k,\Xi)}{Z_\Lambda} = \sum_X \sum_{\substack{T_1 \\ \partial^* T_1 = 2\pi\beta^{-1/2}kC}} \frac{Z_{\Lambda-X}}{Z_\Lambda} K(X,T_1) \tag{6.0}$$

It differs from expansion (5.10) in the following respects. T_1 are domain walls on a block lattice of lattice spacing L. There is a summation over neighbourhoods X of the support of the domain wall T_1. This is due to the effect of the spin waves; they can mediate interactions between the domain wall T_1 and its surroundings. They can also mediate interactions between different connected pieces of the domain wall; therefore T_1 need not be connected now. Finally, $K(X,T_1)$ is much more complicated now; in particular it will involve an integral over dynamical variables associated with the spin waves. Eq. (6.35) below should be thought of as an expansion of the form (6.0) in which the summation over domain walls T_1 inside X has been carried out, viz. $\mathcal{K}(X) \equiv \Sigma_{T_1} K(X,T_1)$.

We obtain a bound on the surface tension α one needs bounds on the factors on the right hand side of (6.0). First one needs a bound on $K(X,T_1)$ for fixed T_1 that ensures convergence of the X summations when the volume Λ is infinite. It should suppress contributions in which X (minus a standard neighbourhood of the domain wall) is large. In addition one needs bounds that replace those of section 5 - a bound on $K(X,T_1)$ that suppresses large domain walls (by a factor $e^{-\text{const.}\mathcal{F}_1}$ with $\mathcal{F}_1 \propto \|T_1\|^2$) and allows to pull out an area law behaved factor as in eq. (5.17), and a bound on the ratio $Z_{\Lambda-X}/Z_\Lambda$ that replaces inequality (5.12). The bounds on K are combined in lemma 15,

which in turn depends on the bound of lemma 12 for F_1. The bound on the ratio $Z_{\Lambda-X}/Z_\Lambda$ of partition functions is supplied by lemma 17.

After these preliminaries we return to proposition 3. To be in agreement with the notations of ref. 2 we rewrite the result in terms of a Gaussian measure with mean zero, using the identy $d\mu_{u,g}(\phi) = d\mu_u(\phi - g)$.

$$Z_\Lambda(k,\Xi) = e^{-k^2(J_\mu, v_{cb} J_\mu)/2\beta} \int d\mu_u(\phi)\, e^{-V_{eff}(\phi|k\Xi)} \tag{6.1}$$

Because of the shift in the field, V_{eff} now depends on k,Ξ. In terms of the vertex functions it is given by

$$-V_{eff}(\phi|k\Xi) = \sum_{[\alpha]\in T_R} \int d\xi_\alpha\, \sigma_\alpha^R(\xi_\alpha)\, e^{i(m,\phi)\beta^{1/2}} e^{ik(m,f)} , \tag{6.2}$$

see section 3. It is convenient to introduce

$$\varepsilon_k(\xi) = e^{im(\beta^{1/2}\phi(x)+kf(x))} - 1 \tag{6.3}$$

As a result of the shift one obtains the following replacement for eq. (1.17)

$$-V_{eff}(\phi|k\Xi) = \rho_0 + \sum_{s=1,2,\dots} \frac{1}{s!} \int d\xi_1 \dots d\xi_s\, \rho_s(\xi_1 \dots \xi_s)\, \varepsilon_k(\xi_1) \dots \varepsilon_k(\xi_s) \tag{6.4}$$

We superimpose on the dual lattice Λ^* another lattice Λ' with lattice spacing L (an integer multiple of a). We call it the "block lattice". Its cubes are typically denoted* by Ω_α. We consider Wilson loops which are positioned in such a way that Ξ is a union of plaquettes of the block lattice Λ'.

In addition we superimpose on Λ^* a lattice of lattice spacing $\tilde{m}_D^{-1}$ with $\tilde{m}_D$ given by eq. (6.7) below. It is called "unit lattice" and its cubes are denoted by Δ_α. It will suffice to consider values of β such that $\tilde{m}_D^{-1}$ is an integer multiple of a. We choose L such that $\tilde{m}_D L$ is small and independent of β/a.

The Glimm Jaffe Spencer expansion consists of three basic steps: The Peierls expansion, the translation of ϕ, and the cluster expansion.

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*Indices α will label cubes on some lattice from now on. We will write $\partial\Lambda^*$ for the boundary of Λ^*, etc. ~~(instead of $\partial^*\Lambda^*$)~~.

(A) The Peierls expansion:

We split the effective potential

$$-V_{eff}(\phi|k\Xi) = \int d\xi\, \rho_1(\xi)\left[e^{im(\beta^{1/2}\phi(x)+kf(x))}-1\right] + E' \tag{6.5}$$

E' includes the terms with $s \geq 2$ in the expansion (6.4). For simplicity the ϕ-indepent constant ρ_o is dropped (it cancels after taking expectation values). We introduce functions h(x) that are constant on cubes Ω_α. On each of them h(x) equals an integral multiple of $2\pi\beta^{-1/2}$; h = 0 outside Λ^*. ($2\pi\beta^{-1/2}$ is the period of $\varepsilon_k(\xi)$, considered as a function of $\phi(x)$). The leading term of $-V_{eff}(\phi|k\Xi)$, i.e. the first term on the r.h.s. of (6.5), is approximated by a periodized Gaussian via the following identity:

$$e^{\int d\xi\, \rho_1(\xi)\varepsilon_k(\xi)} = \sum_h e^{-\frac{1}{2}\tilde{m}_D^2\int_\Lambda(\phi(x)+k\beta^{-1/2}f(x)-h(x))^2}\, e^G \tag{6.6}$$

with

$$\tilde{m}_D^2 \equiv \sum_m \rho_1(m,x=0)m^2\beta \tag{6.7}$$

By proposition 3, $m_D^{-1}\tilde{m}_D \to 1$ if $\beta/a \to \infty$ and $M\beta^{-1} \to 0$.
e^G is a correction factor (close to one for the most important fields ϕ, at least away from the domain walls). It differs from that of ref. 2 due to the presence of f in (6.6) and due to the slight dependence on x of $\rho_1(m_1,x)$ for a finite lattice Λ. Therefore we give an explicit expression:

$$G = G_1 + G_2 \quad , \quad e^{G_1} = \prod_\alpha r_\alpha(A_\alpha) \tag{6.8}$$

$$\begin{aligned} e^{G_2} = \exp\Big\{ &\int d\xi\, \rho_1(m,0)\left[e^{im\tilde{\delta}(x)\beta^{1/2}}-1-im\tilde{\delta}(x)\beta^{1/2}+\tfrac{1}{2}m^2\tilde{\delta}(x)^2\beta\right] + \\ &+\int d\xi\, \rho_1(m,0)\left[e^{im(\beta^{1/2}A(x)+k\bar{f}(x))}-1\right]\left[e^{im\tilde{\delta}(x)\beta^{1/2}}-1-im\tilde{\delta}(x)\beta^{1/2}\right]+ \\ &+\int d\xi\, (\rho_1(m,x)-\rho_1(m,0))\left[e^{im(\beta^{1/2}\phi(x)+kf(x))}-1\right]\Big\} \end{aligned} \tag{6.9}$$

where $\tilde{\delta}(x) = \delta(x) + k\beta^{-1/2}\delta f(x)$.

$$r_\alpha(A_\alpha) = \frac{\exp\left\{\sum_m \rho_1(m,0)L^3\left[e^{im(\beta^{1/2}A_\alpha+k\bar{f}_\alpha)}-1\right]\right\}}{\sum_{n\in\mathbb{Z}}\exp\left\{-\frac{1}{2}\tilde{m}_D^2L^3(A_\alpha+k\beta^{-1/2}\bar{f}_\alpha-2\pi n\beta^{-1/2})^2\right\}} \tag{6.10}$$

A(x) ($\bar{f}(x)$) is the average of $\phi(x)$ (f(x)) over the cubes Ω_α, and $\delta(x)$ ($\delta f(x)$) is its fluctuating part:

$$A(x) = L^{-3}\int_{x\in\Omega_\alpha}\phi(x) = A_\alpha \quad , \quad \delta(x) = \phi(x) - A_\alpha \tag{6.11}$$

$$\bar{f}(x) = L^{-3}\int_{x\in\Omega_\alpha} f(x) = \bar{f}_\alpha \quad ; \quad \delta f(x) = f(x) - \bar{f}(x)$$

for $x \in \Omega_\alpha$.

We use charge symmetry which implies that $\sum_m \rho_1(m,0)m = 0$.

(B) Translation of ϕ:

The combination of (6.5) and (6.6) yields ($\tilde{Z}_\Lambda$ is related to Z_Λ by eq. (2.3a))

$$\tilde{Z}_\Lambda(k,\Xi) = \sum_h \int d\mu_u(\phi)\, e^{-\frac{1}{2}\tilde{m}_D^2\int_\Lambda(\phi-k\beta^{-1/2}f-h)^2}\, e^{E'}\, e^G \tag{6.12}$$

For any h we define a function $g(x) = g_h(x)$. It differs from that of [2] and its precise form will be given below.
We write

$$\phi(x) = \psi(x) + g(x) \tag{6.13}$$

and include in the Gaussian measure the quadratic terms in ψ, namely $-\frac{1}{2}\tilde{m}_D^2\int_\Lambda\psi^2$ and $-\frac{1}{2}\int\psi\nu\psi$. The second expression is the part of E' which is quadratic in ψ and comes from the term with s = 2 in the expansion (6.4). We obtain the explicit from of ν if we write $\varepsilon_k(\xi) = \tilde{\varepsilon}_k(\xi) + im\beta^{1/2}\psi(x)$ with

$$\tilde{\varepsilon}_k(\xi) = e^{im(\beta^{1/2}\phi(x)+kf(x))} - 1 - im\beta^{1/2}\psi(x) \tag{6.14a}$$

This gives

$$-\tfrac{1}{2}\int\psi\nu\psi = -\tfrac{1}{2}\int d\xi_1 d\xi_2\, \rho_2(\xi_1,\xi_2)m_1m_2\beta\psi(x_1)\psi(x_2) \tag{6.14b}$$

We define the quantity E by the split

$$E' = E - \tfrac{1}{2}\int\psi\nu\psi \tag{6.14c}$$

We introduce new covariances (χ_Λ is the characteristic function for the region Λ)

$$C_o^{-1} = u^{-1} + \tilde{m}_D^2\chi_\Lambda \quad , \quad C^{-1} = C_o^{-1} + \nu \tag{6.15}$$

and the normalized Gaussian measure $d\mu(\psi)$ with covariance C, i.e.

$$N\,d\mu(\psi) = d\mu_u(\psi)\, e^{-\frac{1}{2}\tilde{m}_D^2 \int_\Lambda \psi^2 - \frac{1}{2}\int \psi v \psi} \tag{6.16}$$

(N is a normalization factor.)

Putting everything together we obtain finally:

$$\tilde{Z}_\Lambda(k,\Xi) = \sum_h N \int d\mu(\psi)\, e^{E} e^{G} e^{-F_1} e^{-F_2} \tag{6.17}$$

$$F_1 = \frac{1}{2}\tilde{m}_D^2 \int_\Lambda (g - h + k\beta^{-1/2} f)^2 + \frac{1}{2}\int g u^{-1} g \tag{6.18}$$

$$F_2 = \int \psi C_o^{-1}(g - g_c) \tag{6.19}$$

with

$$g_c = \tilde{m}_D^2 C_o \chi_\Lambda [h - k\beta^{-1/2} f] \tag{6.20}$$

If we set g equal to g_c then F_2 is zero. For technical reasons we define g to be slightly different from g_c. For any h we define a 2-chain T on the block lattice with coefficients in the group $2\pi\beta^{-1/2}\mathbb{Z}$

$$T \equiv T(h) \equiv 2\pi\beta^{-1/2} \sum N(p)p = \sum_p \delta h(p) p - 2\pi k \beta^{-1/2} \Xi \tag{6.21}$$

In this formula, Ξ is to be read as "sum of oriented block plaquettes in the surface Ξ ". Summation is over all plaquettes p of the block lattice Λ'. $\delta h(p)$ denotes the discontinuity in h across p. For later use we introduce the abbreviation (compare with (5.5))

$$\| T \|^2 = 4\pi^2 \beta^{-1} \sum_p N(p)^2 \tag{6.22}$$

The domain wall Σ is a union of block plaquettes p where $N(p) \neq 0$. In contrast with the situation in ref. 2, Σ has a boundary which is given by the Wilson loop C. (More precisely, $\partial' T = 2\pi\beta^{-1/2} kC$, ∂' = boundary operator on the block lattice Λ' , $C \equiv$ sum of block links in C.) We let $\Sigma^\wedge$ be the union of unit lattice cubes in Λ^* whose distance from Σ is less than L'. L' is a new parameter. It will be chosen later on, such that $\tilde{m}_D L'$ is large and independent of β/a. $\Sigma^\wedge$ is a neighborhood of the domain wall Σ.

Now we will define g. We introduce the propagator

$$C_{oo} = (u^{-1} + \tilde{m}_D^2)^{-1}$$

We consider first the case when no domain wall comes near the boundary of Λ^*, i.e. $\partial\Lambda^* \cap \Sigma^\wedge = \emptyset$. In this case we set $\mathcal{S} = \Sigma^\wedge$. We let $\{R_\alpha\}_{\alpha\in I}$ be the set of connected components of the complement of $\mathcal{S}$ in the infinitely extended lattice $(a\mathbb{Z})^3$, and $\{S_\beta\}_{\beta\in J}$ the set of the connected components of $\Sigma^\wedge$. The union of unit cubes inside S_β having non-empty intersection with ∂S_β is denoted by BS_β. We choose a smooth function χ_β for every $\beta \in J$ such that

$$0 \le \chi_\beta \le 1 \quad , \quad \chi_\beta = 0 \text{ outside } S_\beta \ , \quad \chi_\beta = 1 \text{ on } S_\beta - BS_\beta \tag{6.23}$$

and all finite difference derivatives of χ_β are uniformely bounded (by powers of $\tilde{m}_D$ times constants that are independent of $\Sigma^\wedge$). For every set S_β we define h^e_β to be equal to h inside S_β, and constant on connected components of its complement in $(a\mathbb{Z})^3$. We define g by

$$g + k\beta^{-1/2} f = \left\{ \begin{array}{ll} h & \text{in } R_\alpha \\ \tilde{m}_D^2 C_{oo} h^e_\beta & \text{in } S_\beta - BS_\beta \\ \chi_\beta \tilde{m}_D^2 C_{oo} h^e_\beta + (1-\chi_\beta) h & \text{in } BS_\beta \end{array} \right\} \text{ if } \Xi \cap S_\beta = \emptyset \tag{6.24}$$

If $\Xi \cap S_\beta \neq \emptyset$ the following definition is used instead

$$g = \left\{ \begin{array}{ll} \tilde{m}_D^2 C_{oo} (h^e_\beta - k\beta^{-1/2} f) & \text{in } S_\beta - BS_\beta \\ \chi_\beta \tilde{m}_D^2 C_{oo} (h^e_\beta - k\beta^{-1/2} f) + (1-\chi_\beta)(h - k\beta^{-1/2} f) & \text{in } BS_\beta \end{array} \right\} \text{ if } \Xi \cap S_\beta \neq \emptyset \tag{6.25}$$

If $\partial\Lambda^* \cap \Sigma^\wedge \neq 0$, $\mathcal{S}$ is chosen to consist of $\Sigma^\wedge$, the complement of Λ^*, and the union of unit lattice cubes in Λ^* that have a distance less than L' from the boundary $\partial\Lambda^*$ of Λ^*. The other definitions are the same, except that C_{oo} is replaced by C_o on the set S_β which contains $\partial\Lambda^*$.

The usefulness of this definition comes from the properties of the quantity

$$F_2' = C_o^{-1}(g - g_c) \tag{6.26}$$

We note that C_o^{-1} and C_{oo}^{-1} agree as differential operators in the interior of Λ. We distinguish five possibilities for the arguments of $F_2'(x)$:

(i) $x \in R_\alpha$

$$F_2' = u^{-1}(h - k\beta^{-1/2} f) = M^{-2}(-\Delta + M^2)\nabla_\mu(\nabla_\mu h - k\beta^{-1/2} j_\mu) = 0 \, , \tag{6.27}$$

because $\nabla_\mu h - k\beta^{-1/2} j_\mu$ has its support in $\Sigma^\wedge$ (in deriving (6.27) we made use of the decomposition (C.13) of Appendix C).

(ii) $x \in S_\beta - BS_\beta \, , \, \Xi \cap S_\beta = \emptyset$

$$F_2' = -u^{-1} k\beta^{-1/2} f = 0 \tag{6.28}$$

because $u^{-1}f$ has its support in a neighborhood of two lattice spacings a around Ξ. This is a pleasant feature of the Pauli-Villars cutoff.

(iii) $x \in S_\beta - BS_\beta \, , \, \Xi \cap S_\beta \neq \emptyset$

$$F_2' = 0 \tag{6.29}$$

which follows by inspection of (6.25).

(iv) $x \in BS_\beta \, , \, \Xi \cap S_\beta = \emptyset$

$$F_2' = -C_o^{-1} \chi_\beta C_{oo} u^{-1} h_\beta^e + u^{-1}(h - k\beta^{-1/2} f) = -C_{oo}^{-1} \chi_\beta C_{oo} u^{-1} h_\beta^e \tag{6.30}$$

by the same arguments as in case (i)

(v) $x \in BS_\beta \, , \, \Xi \cap S_\beta \neq \emptyset$

$$F_2' = -C_{oo}^{-1} \chi_\beta C_{oo} u^{-1}(h_\beta^e - k\beta^{-1/2} f) \tag{6.31}$$

We see that F_2' is zero except on BS_β. This matches will the situation in [2]. It will be used later on to prove smallness of F_2.

(C) <u>The cluster expansion</u>:

The variables h and ψ in (6.17) describe the two kinds of excitations, the domain walls and the spin waves, respectively. The spin waves are treated via a cluster expansion [10], first introduced by Glimm, Jaffe, and Spencer. The setup of the cluster expansion and the proof of its validity is rather involved. A detailed proof is given in Appendix E together with an explanation of all notations. [The statement of the s-dependence of $E(X,s)$ in the text of ref. 2 contains a misprint. The correct definition is found in our Appendix E before lemma E.1.]. The cluster expansion for (6.17) reads

$$\tilde{Z}_\Lambda(k,\Xi) = \sum_h \sum_X K(X,h)\, N \int d\mu(\psi)\, e^{E(X^c)} e^{G(X^c)} e^{-F_1(X^c)} e^{-F_2(X^c)} \tag{6.32}$$

$$K(X,h) = \sum_{\bar{Y}} \int ds \int d\mu_s(\psi)\, e^{E(X,s)} \kappa(\bar{Y},s)\, e^{G(X)} e^{-F_1(X)} e^{-F_2(X)} \tag{6.33}$$

The validity of (6.32), (6.33) is proved in Appendix E (proposition E.2).

These formulae are the same as (8.3), (8.4) in [2], except that in our case $\mathcal{A} = 1$. The set X_1 is chosen to be the set of unit lattice cubes touching the Wilson surface Ξ. Another modification is due to the f-dependence of E, G, F_1, F_2. h specifies a domain wall Σ. X is summed over subsets of Λ that are union of unit cubes and contain X_1 and the connected component of $\Sigma^\wedge$ in which the Wilson loop C is located.

The sum over h in (6.32), (6.33) factorizes in a natural way

$$h = h_X + h_{X^c} \quad , \quad X^c = \Lambda - X = \text{complement of X in } \Lambda \tag{6.34}$$

where h_X and h_{X^c} are restricted to a form compatible with X and X^c respectively (see [1]). For instance h_X is constant in certain "collar" neighborhoods of width L' of connected components of X^c.

In this way we arrive at the final form of the expansion:

$$\tilde{Z}_\Lambda(k,\Xi) = \sum_X \mathcal{K}(X)\, Z'(\Lambda,X) \tag{6.35}$$

where

$$\mathcal{K}(X) = \sum_{\bar{y}} \sum_h \int ds \int d\mu_s(\psi)\, e^{E(X,s)} \varkappa(\bar{y},s)\, e^{G(X)} e^{-F_1(X)} e^{-F_2(X)} \tag{6.36a}$$

and

$$Z'(\Lambda,X) = \sum_h N \int d\mu(\psi)\, e^{E(X^c)} e^{G(X^c)} e^{-F_1(X^c)} e^{-F_2(X^c)} \tag{6.36b}$$

We notice that $Z_\Lambda = \tilde{Z}_\Lambda(k = 0,\Xi)$ for arbitrary Ξ, and so the expansion for Z_Λ is given by

$$Z_\Lambda = \sum_X \tilde{\mathcal{K}}(X)\, Z'(\Lambda,X) \tag{6.37}$$

where $\tilde{\mathcal{K}}(X) = \mathcal{K}(X)\restriction_{k=0}$ and $Z'(\Lambda,X)$ is again given by eq. (6.36b), X_1 can now be chosen as an arbitrary unit cube $\Delta_1 \subset \Lambda$.

We have to show that $Z'(\Lambda,X)$ as defined by eq. (6.36b) does not depend on kf. Only in this case the quantities $Z'(\Lambda,X)$ in eq. (6.35), (6.37) are identical [except for the change in the definition of X_1]. $Z'(\Lambda,X)$ can be considered as a partition function of a system in Λ-X. (This system interacts with the heat bath if $X^c \cap \partial\Lambda \neq \emptyset$. In addition one has boundary conditions h = 0 on a collar neighborhood of every connected component of X that meets $\partial\Lambda$). To verify our claim for $E(X^c)$ and $G(X^c)$ we must calculate

$$\Delta(g,h) \equiv g - h + k\beta^{-1/2} f \tag{6.38}$$

and show its independence of f in the region X^c.

In X^c we have always $\Xi \cap S_\beta = \emptyset$ and so definition (6.24) of g applies:

$$\Delta(g,h) = -\chi_\beta\, C_0\, u^{-1} h_\beta^c \tag{6.39}$$

This is obviously independent of f.

From eq. (6.26), (6.27), (6.28), and (6.30) we see that $F_2(X^c)$ is also independent of f. It remains to show it for $F_1(X^c)$. We return

to the definition (6.18) use (6.38) and (6.39) and remember that $u^{-1}f$ has its support in $X_1 \subset X$. This proves our claim.

The combined expansion (6.35) ... (6.37) should be compared with the simple expansion stated in (5.10), (5.11) which is valid for small β/a. The domain walls are living on the block lattice with spacing L (L is exponentially increasing in β/a) and their action $-F_1$ is more complicated than the simple expression (5.5), (5.6) for the low temperature expansion. In the next section (lemma 12) we state an lower bound on F_1 which gives an upper bound on e^{-F_1} of the same form as $K(T_1)$ in eq. (5.11).

The expansion (6.35) ... (6.37) are formulated for a finite lattice Λ. To compute the string tension we need to consider an infinite lattice $\Lambda = (a\mathbb{Z})^3$. If we assume that the thermodynamic limit $\Lambda \longrightarrow (a\mathbb{Z})^3$ exists we can define the quantity

$$\rho(X) = \lim_{\Lambda \to (a\mathbb{Z})^3} Z'(\Lambda,X)/Z_\Lambda \tag{6.40}$$

From the definitions (6.15), (6.15') we see that in the infinite volume limit

$$C_0 = (u^{-1} + \tilde{m}_D^2)^{-1} = C_{00}$$

If $X \cap \partial\Lambda = \emptyset$, the only Λ-dependence of $\mathcal{K}(X)$ and $\tilde{\mathcal{K}}(X)$ comes from the covariance $C = (C_0^{-1} + \nu)^{-1}$ which appears in $\varkappa(\bar{y},s)$ and in the measure $d\mu_s(\psi)$. Therefore the limit $\Lambda \longrightarrow (a\mathbb{Z})^3$ for $\mathcal{K}(X)$, $\tilde{\mathcal{K}}(X)$ causes no problems: we only have to replace everywhere C by $(u^{-1} + \tilde{m}_D^2 + \nu)^{-1}$, i.e. to drop the characteristic function χ_Λ in eq. (6.15). The Λ dependence of $\rho_s(\xi_1 \dots \xi_s)$ in G and E is controlled by proposition 3. We use the same notations $\mathcal{K}(X)$, $\tilde{\mathcal{K}}(X)$ for the corresponding quantities in the limit $\Lambda \longrightarrow (a\mathbb{Z})^3$. In the infinite volume limit we have the following system of equations to determine $\rho(X)$ and $\langle \chi_k(U(C)) \rangle_{U(1)}$:

$$\langle \chi_k(U(C)) \rangle_{U(1)} = e^{-k^2 (J_\mu, v_{cb} J_\mu)/2\beta} \cdot \sum_X \mathcal{K}(X)\, \rho(X) \tag{6.41a}$$

$$1 = \sum_X \tilde{\mathcal{K}}(X)\, \rho(X) \tag{6.41b}$$

The summation over X is over all finite unions of unit cubes of the infinitely extended unit lattice.

Similar equations hold for expectation values of other $\mathbb{Z}$-invariant observables* (compare ref. 2). From now on we will study the state of the infinite volume $\mathbb{Z}$-ferromagnet at low temperature that is defined by solution of these equations. As we shall see in the next section, the Glimm Jaffe Spencer expansion (6.41a,b) converges for the range of the expansion parameters in which we are interested (lemma 15, 16), and the Kirkwood Salsburg equations (6.41b) are sufficient to determine $\rho(X)$ uniquely, by iteration (compare with Appendix 4 of ref. 2). Of course, once we know $\rho(X)$ we can calculate $\langle \chi_k(U(C)) \rangle_{U(1)}$ with the help of eq. (6.41a).

For future reference, the infinite volume propagators are given by

$$C = (C_0^{-1} + \nu)^{-1} \quad , \quad C_0 = (u^{-1} + \tilde{m}_D^2)^{-1} \tag{6.42}$$

*We are only interested in $\mathbb{Z}$-invariant observables because others have no interpretation in gauge theory language. Accordingly a state is defined as a positive linear functional on an algebra of $\mathbb{Z}$-invariant observables.

7. Convergence of the Glimm Jaffe Spencer expansion

The proof of convergence of the expansion (6.41a,b) requires an estimate of the quantity ($c_1 \geq 0$, $|X|$ is the volume of X measured in units of $\tilde{m}_D^{-1}$)

$$\sum_{X,\, X \supset X_1} |\mathcal{K}(X)|\, e^{c_1 |X|} \tag{7.1}$$

The methods used by Brydges and Federbush [2] to obtain this upper bound, namely the method of combinatoric factors (section 9.1), the Hölder inequality (section 9.2), the vacuum energy estimates (sections 9.6 and 9.7), the bounds on derivatives of r(A) (section 9.5), and the bounds on functional derivatives (section 9.8 with $\mathcal{A} = 1$) can be adapted to our situation word by word, except for our use of lattice Gaussian measures [see the remark at the beginning of Appendix D].

We state only the results and give the basic definitions. For proofs the reader is referred to [2]. More details can be found in ref. [32].

Lemma 11

We assume that the iterated Mayer expansion of proposition 3 is valid, i.e. β/a is sufficiently large and $\tilde{\lambda} \equiv M^{-1}\tilde{m}_D$ is sufficiently small.

For any δ_1, δ_2, c_1, $c_1' > 0$ there are c_2, r such that for any subset W of Λ

$$\sum_{\substack{X \\ X \supset X_1,\, X \cap W \neq \emptyset}} |\mathcal{K}(X)|\, e^{c_1|X|} \leq \mathcal{P} \cdot \sup_{(\cdot)} e^{c_1' F_1 + c_2|X| + (1-2\delta_1+\delta_2) d} \cdot e^{-(1-2\delta_1)\, \mathrm{dist}(X_1, W)} \cdot \int d\mu_s \left| e^{E(X,s)} \chi' e^{\mathcal{E}(X)} e^{-F_1(X)} e^{-F_2(X)} \right|_0 , \tag{7.2}$$

where $\mathcal{P}$ is a perimeter law behaved factor [see eq. (7.8) and (7.7a) below]. The distance $\mathrm{dist}(X_1, W)$ [with respect to the norm (3.23)] is measured in units of $\tilde{m}_D^{-1}$.

This lemma replaces lemma 9.4 of [2].

We explain the notations in (7.2).

$\mathcal{X}'$ is the same as $\mathcal{X}$ except that any term in $\mathcal{X}$ which contains a quantity like $\mathcal{E}(a_1, \ldots, a_t)$ is multiplied by $e^{\tau\mathcal{O}_o} e^{\bar{\delta}L_o}$ $(\bar{\delta} = 1-2\delta_1+\delta_2)$.

The formal operators L_o and $\mathcal{O}_o$ are defined by

$$e^{\bar{\delta}L_o} : \mathcal{E}(a_1,\ldots,a_t) \longrightarrow e^{ML(a_1,\ldots,a_t)\cdot\bar{\delta}}\, \mathcal{E}(a_1,\ldots,a_t) \tag{7.3}$$

and

$$e^{\tau\mathcal{O}_o} : \mathcal{E}(a_1,\ldots,a_t) \longrightarrow e^{\tau t}\, \mathcal{E}(a_1,\ldots,a_t) \tag{7.4}$$

The symbol $|\cdot|_o$ means that one performs first all derivatives in $\mathcal{X}'$, this produces a finite sum. The absolute value is taken inside this sum. The supremum in (7.2) is over all parameters that occur in the combined Glimm Jaffe Spencer and iterated Mayer expansion. These are: n, (m_i), (y_{ij}), h, $\int ds$, T, types of terms in $\mathcal{X}'$, t,

$$(\Delta_i', \Delta_i''), \quad (a_1,\ldots,a_t) \tag{7.5}$$

Their precise meaning is given in section 9.1. of [2].

The definition of d is

$$d = \sum_{i=1}^{n-1} dist(\Delta_i', \Delta_i'') \tag{7.6}$$

where $dist(\Delta_i', \Delta_i'')$ is the distance between unit cubes Δ_i', Δ_i''. It is calculated with the norm (3.23) and is measured in units of $\tilde{m}_D^{-1}$.

The proof of lemma 11 is literally the same as the corresponding one in [2]. The only modification is due to the replacement of lemma 9.2. of [2] by the following

Lemma 12

There is a constant $c_F > 0$ such that

$$F_1(X,h) \geq c_F\, \tilde{m}_D^2 L^3\, \| T(h) \|^2 - P_F \tag{7.7}$$

The meaning of $T(h)$ and $\| T(h) \|$ was defined in (6.21) and (6.22).

The quantity P_F is independent of all parameters listed in (7.5) and it is "perimeter behaved" in the sense that P_F obeys an inequality of the form

$$0 \leq P_F \leq \hat{c}\, |C|\, \ln(|C|/a) \tag{7.7a}$$

with some constant $\hat{c}$. $|C|$ is the length of the Wilson loop.

The proof of lemma 12 is given in the Appendix C. An explicit expression for P_F can be found there also.
The factor P in (7.2) is related to P_F by

$$P = e^{c_1' P_F} \tag{7.8}$$

The lemma 11 is a purely combinatorial one. It remains to be shown that the r.h.s. of (7.2) is finite for a suitable choice of parameters δ_1, δ_2, c_1, c_1'.

The first step in the estimate of the Gaussian integrals in (7.2) is the use of a bound on the functional derivatives in $\mathcal{X}'$ which act on $e^G e^{-F_1} e^{-F_2}$. It is given by eq. (9.823), (9.22), (9.23) of [2].

In a schematic notation one obtains an upper bound

$$\int d\mu_s \,| e^E \mathcal{X}' e^G e^{-F_1} e^{-F_2} |_o \leq \int_{x_i \in X} k((x_i)) \int d\mu_s\, e^E e^{G_2} e^{-F_1} e^{-F_2} \,.$$
$$\cdot \prod_i |\psi(x_i)| \cdot \exp\left[\tfrac{1}{2}\gamma \int (\psi + g - h + k\beta^{-1/2} f)^2\right] \tag{7.9}$$

The integrals in (7.9) are restricted to the region X. $k((x_i)) \geq 0$ are given by a complicated expression, see eq. (9.823) of [2]. The constant γ comes from a bound on derivatives of $r(A)$ which is defined in eq. (6.10). This bound is stated in lemma 9.7. of [2]. γ has the form

$$\gamma = \eta\, \tilde{m}_D^2 \,, \quad 1-4\pi^{-2} < \eta < 1 \tag{7.10}$$

where η can be made arbitrarily close to $1-4\pi^{-2}$ (compare the discussion after eq. (1.26) in the introduction).

(7.9) is further estimated with the help of Hölders inequality. We choose four numbers $p_1 \ldots p_4$ with $\sum_{i=1}^{4} p_i^{-1} = 1$, p_1 = even integer, $p_i > 1$, $p_3\gamma < \tilde{m}_D^2$ and obtain the equivalent of inequality (9.25) of ref. [2].

$$|(7.9)| \leq \int k((x_i)) \left[\int d\mu_s \prod_i |\psi(x_i)|^{p_1} \right]^{p_1^{-1}} \cdot \left[\int d\mu_s\, e^{-p_2 F_2} \right]^{p_2^{-1}} \cdot e^{-F_1} \cdot$$
$$\cdot \left[\int d\mu_s\, e^{\frac{1}{2} p_3 \gamma \int (\psi + g - h + k\beta^{-1/2} f)^2} e^{2 p_3 \tilde{m}_D^2 \int \tilde{\delta}^2} \right]^{p_3^{-1}} \cdot$$
$$\cdot \left[\int d\mu_s\, e^{p_4 E} e^{p_4 G_2} e^{-2 p_4 \tilde{m}_D^2 \int \tilde{\delta}^2} \right]^{p_4^{-1}} \tag{7.11}$$

The fluctuation part $\tilde{\delta}$ of ψ is defined by eqs. (6.9), (6.11), (6.13).

<u>Lemma 13</u>

There are constants c, c' > 0, c' < 1 such that the product of the last four factors on the r.h.s. of (7.11) is bounded by

$$P' e^{c|X|} e^{-c' F_1} \tag{7.12}$$

where P' is a perimeter behaved factor [see eq. (7.14) below] that does not depend on the parameters listed under (7.5). The validity of (7.12) requires the $\tilde{\lambda}$ is sufficiently small, $\tilde{m}_D L$ is sufficiently small, and $\tilde{m}_D L'$ is sufficiently large.

<u>proof</u>:

The corresponding proof in [2] applies in our case, except for one modification. It leads to the factor P' in front of (7.12) and involves the following estimate of $\int F_2'^2$ which replaces the estimates of section 9.4. in ref. 2.

<u>Lemma 14</u>

$$\int F_2'^2 \leq c(L') \left[F_1 + P_F \right] \tag{7.13}$$

where c(L') becomes arbitrarily small (exponentially) as $\tilde{m}_D L'$ goes to infinity. P_F is the same quantity as in lemma 12.

This lemma is proved in Appendix D.

The factor on the r.h.s. of (7.11) involving F_2 is estimated by the same methods as in section 9.4. of [2] together with our lemma 14. The last factor but one is bounded by lemma 9.8. of [2] (together with our lemma 14) and finally the last factor can be estimated by use of lemma 9.9. of [2].

For details of the rather lenthy proofs the reader is referred to [2].

Lemma 14 shows that P' is of the form (with constant $\tilde{c} > 0$):

$$P' = e^{\tilde{c} P_F} \tag{7.14}$$

P' can be absorbed in the factor P of lemma 11.
Proof of lemma 13 completed. □

The first factor on the r.h.s. of (7.11) is estimated by using Wick's theorem. As shown in [2,10] it produces the bound [in units where $\tilde{m}_D = 1$]

$$\left[\int d\mu_s \prod_i |\psi(x_i)|^{p_1} \right]^{p_1^{-1}} \leq c^{\sum_j n_j} \prod_j n_j! \tag{7.15}$$

where n_j is the number if x_i's in the unit cube Δ_j.

Now we are ready to finish the proof of convergence of the Glimm Jaffe Spencer expansion. We use (7.9), (7.11), (7.15), and lemma 13. We end up with an object like that of eq. (9.824) of [2]. At this point the bound (1.18) of proposition 3 comes in. We see that the combinatoric factors $e^{\bar{\delta} L_0}$, $e^{\gamma \theta_0}$ can be controlled if we choose $\bar{\delta} = 1-\delta$ in $e^{\bar{\delta} L_0}$ sufficiently small and μ in (1.18) sufficiently large. The factor $e^{\bar{\delta} d}$ is controlled by the exponential decay of the covariances $C(s)$. This exponential decay is assured by lemma B.4 (Appendix B). Although (B.27) is not a pointwise estimate on $C(xy)$ it suffices in our situation. This can be seen by inspection of (9.824) of ref. [2].[The tree structure of the cluster expansion is essential here.]

We use our estimates above to bound the r.h.s. of (7.2). For details the reader is again referred to ref. 2.

In this way we arrive at the main lemma which is the complete transcription of lemma 9.12. of [2].

Lemma 15

Assume that all parameters in our theory like $\tilde{\lambda} = M^{-1}\tilde{m}_D$, L, L', δ_1 are fixed as above and let c_1 be an arbitrary constant. Then there are constants c_2 (independent of β) and P (which depends on β and is perimeter behaved) such that for β/a sufficiently large:

$$\sum_{\substack{X \\ X\supset X_1,\, X\cap W\neq\emptyset}} |\mathcal{K}(X)|\, e^{c_1|X|} \leq P\, e^{c_2|X_1|}\, e^{-(1-2\delta_1)\,dist(X_1,W)} \tag{7.16}$$

Lemma 15 is sufficient to derive the lower bound on the surface tension α which is stated in theorem 1. This is shown in the next section.

The proof of our theorem 4 (on the continuum limit) requires a generalization of lemma 15. This generalization deals with expectation values of observables like

$$\mathcal{A}_1 = \prod_{j=1}^{\omega} e^{i\beta^{1/2} a_j \phi(x_j)} \quad (a_j \in \mathbb{Z}), \quad \text{or} \quad \mathcal{A}_2 = \prod_{j=1}^{\omega} \beta^{-1/2} \sin \beta^{1/2} \phi(x_j) \tag{7.17}$$

instead of the Wilson loop. In this case the connection with the work of Brydges and Federbush [2] is even closer. The problems caused by the f-dependence of $V_{eff}(\phi|\, k,\Xi)$ in (6.2) are absent here.

All the above arguments (and those of [2]) can be applied and combine to prove the following

Lemma 16

Let the parameters ($\tilde{\lambda}$, L, L', δ_1) be fixed as in lemma 15, let $\mathcal{A}$ be an observable of the form (7.17), and consider the quantity:

$$\mathcal{K}_{\mathcal{A}}(X) = \sum_{\bar{Y}} \sum_{h} \int ds \int d\mu_s\, e^{E(X,s)}\, \chi(\bar{Y},s)\, e^{\epsilon(X)}\, e^{-\mathcal{F}_1(X)}\, e^{-\mathcal{F}_2(X)}\, \mathcal{A} \tag{7.18}$$

For arbitrary $c_1 > 0$ there are constants c_2, c_3 (independent of β) such that for β/a sufficiently large:

$$\sum_{\substack{X \\ X\supset X_1,\, X\cap W\neq\emptyset}} |\mathcal{K}_{\mathcal{A}}(X)|\, e^{c_1|X|} \leq c_3^{\omega}\, e^{c_2|X_1|} \cdot e^{-(1-2\delta_1)\,dist(X_1,W)} \tag{7.19}$$

X_1 is a union of unit cubes that contains the support of $\mathcal{A}$. If we require that X strictly contains X_1, i.e., $X - X_1 \neq \emptyset$, we may replace c_3 by $c \cdot c_3$ where $c \to 0$ as β/a .

By a standard "doubling the measure" argument [1,2] and our lemma 15 one can also investigate truncated correlation functions such a

$$\langle U(C_1)U(C_2)\rangle_{U(1)} - \langle U(C_1)\rangle_{U(1)} \langle U(C_2)\rangle_{U(1)} \tag{7.20}$$

where C_1, C_2 are closed loops separated by a large distance.

The exponential decay of (7.20) with respect to the distance of C_1, C_2 determines the glue-ball mass. It turns out that this glue-ball mass is asymptotically bounded from below by m_D. The details are left to the reader*. We believe that this upper bound on expression (7.20) represents its true asymptotic behavior so that the glue-ball mass is asymptotically equal to m_D for large β/a, but the estimates to prove it are lacking.

*We thank G. Münster for a private communication concerning the glue-ball mass in the U(1) lattice gauge theory.

8. Proof of the area law

We will now apply our results to prove theorem 1. For this purpose we need an upper bound for the "ratio of partition functions" $\rho(X)$, which is the solution of (6.41b).

Lemma 17

Under the same conditions on parameters an in lemma 16 there exists a unique solution of the Kirkwood Salsburg equations (6.41b) and there is a constant c such that

$$|\rho(X)| \leq e^{c|X|} \tag{8.1}$$

proof:

The proof is the same as that for lemma A 4.1. of [2] to which the reader is referred. □

proof of theorem 1:

With lemma 17, and eq. (1.4), (6.1), and (6.41a) we obtain the following bound

$$\langle \chi_k(U(C))\rangle_{U(1)} \leq e^{-k^2(J_\mu, v_{cb} J_\mu)/2\beta} \cdot \sum_{X, X \supset X_1} |K(X)|\, e^{c|X|} \quad . \tag{8.2}$$

where c is the same constant as in lemma 17.

We return to inequality (7.11) and extract from e^{-F_1} a small fraction $e^{-\varepsilon F_1}$ where ε is so small that the proof of lemma 15 remains unaffected if e^{-F_1} in the r.h.s. of (7.11) is replaced by $e^{-(1-\varepsilon)F_1}$.

$e^{-\varepsilon F_1}$ is then estimated from above with the help of lemma 12 and the inequality (C.22) of Appendix C.

Finally we apply lemma 15 with $W = X_1$.

In this way we arrive at

$$\langle \chi_k(U(C))\rangle_{U(1)} \leq e^{-k^2(J_\mu, v_{cb} J_\mu)/2\beta} \cdot P \cdot e^{c_2|X_1|} \cdot e^{-\varepsilon c_F 4\pi^2 k^2 \tilde{m}_D^2 L \beta^{-1}|\Xi|} \cdot e^{\varepsilon P_F} \tag{8.3}$$

The upper bound on $\langle \chi_k(U(C))\rangle_{U(1)}$ in (8.3) implies a lower bound on α as seen from the definition (1.7) of α.

The perimeter law behaved factors in (8.3), namely $P' = e^{-k^2(J_\mu, v_{cb} J_\mu)}$, P, $e^{\varepsilon P_F}$, do not contribute to the string tension because

$$- \lim_{|\Xi| \to \infty} \frac{1}{|\Xi|} \ln P' = 0 \tag{8.4}$$

By construction we have

$$|X_1| = 2\tilde{m}_D^2 |\Xi| + 2\tilde{m}_D |\partial\Xi| + 8 \tag{8.5}$$

For β/a sufficiently large the factor $e^{2c_2 \tilde{m}_D^2 |\Xi|}$ which comes from the insertion of (8.5) in (8.3) is controlled by the factor $\exp[-\varepsilon c_F 4\pi^2 k^2 \tilde{m}_D^2 L \beta^{-1} |\Xi|]$ at the cost of lowering ε to $\varepsilon' < \varepsilon$.

In conclusion the surface tension obeys the inequality (k = 1)

$$\alpha \geq \varepsilon' c_F\; 4\pi^2 \tilde{m}_D L \cdot \tilde{m}_D \beta^{-1} \tag{8.6}$$

(8.6) is just the inequality (1.9) of theorem 1 if we remember the relation between m_D and $\tilde{m}_D$ mentioned below eq. (6.7) [$\tilde{m}_D L$ is a small but β independent quantity]. □

9. Continuum limit

In this section we prove theorem 4. We recall the dependence of the various length scales on the lattice spacing a. In the continuum limit which we consider here, a shrinks to zero and m_D is held fixed.

Therefore

$$\beta = \beta(a) \quad \text{is the solution of} \quad a^2 m_D^2 = z(\beta/a)\, e^{-\frac{1}{2} c_0 \beta/a} \tag{9.1}$$

where $c_0 = a\, v_{cb}(0) \approx .2527$ by eq. (1.8b).

The Pauli-Villars cutoff is chosen as follows

$$M = M(a) = (\beta(a)/a)^{1/12}\, m_D \tag{9.2}$$

As a goes to zero, $\beta(a)/a$ tends to infinity.

The quantities $\tilde{m}_D L$ and $\tilde{m}_D L'$ are independent of β and a.

After these replacements everything depends on a (and m_D) and we denote the expectation value by the symbol $\langle \cdot \rangle_a$.

We consider the observable

$$\mathcal{A}_a = \prod_{i=1}^{w} \beta(a)^{-1/2} \sin \beta(a)^{1/2} \phi(x_i) \tag{9.3}$$

$\mathcal{A}_a$ is a periodic function of ϕ and therefore the expansion given in the previous sections can be applied.

In the limit $a \to 0$ we have $\beta(a) \to 0$ (in units of physical length m_D^{-1}) and therefore, formally

$$\lim_{a \to 0} \mathcal{A}_a = \prod_{i=1}^{w} \phi(x_i) \equiv \mathcal{A}_0 \tag{9.4}$$

If $v_0 \equiv (-\Delta + m_D^2)^{-1}$ is the propagator of the free field theory in the continuum with mass m_D and $d\mu_{v_0}(\phi)$ is the corresponding Gaussian measure, we have to show that the following equality is true:

$$\lim_{a \to 0} \langle \mathcal{A}_a \rangle_a = \int d\mu_{v_0}(\phi)\, \mathcal{A}_0(\phi) \tag{9.5}$$

To begin with we choose X_1 to be the smallest union of unit lattice cubes which contains the points $x_1, \ldots, x_w$. We decouple v_0 between X_1 and $\sim X_1$:

$$\tilde{v}_0(xy) = \begin{cases} v_0(xy) & \text{if either } x,y \in X_1 \text{ or } x,y \in \sim X_1 \\ 0 & \text{otherwise} \end{cases} \tag{9.6}$$

From supp $\mathcal{A}_0 \subset X_1$ and well-known properties of Gaussian measures one concludes that

$$\int d\mu_{\tilde{v}_0}(\phi)\, \mathcal{A}_0(\phi) = \int d\mu_{v_0}(\phi)\, \mathcal{A}_0(\phi) \tag{9.7}$$

Now we start to prove (9.5) with v_0 replaced by $\tilde{v}_0$.

We consider the expansion for $\langle \mathcal{A}_a \rangle_a$ presented in the section 6 and 7:

$$\langle \mathcal{A}_a \rangle_a = \sum_{X,\, X \supset X_1} \mathcal{K}(X)\, \varrho(X) \tag{9.8}$$

with

$$\mathcal{K}(X) = \sum_{\bar{\gamma}} \sum_{h} \int ds \int d\mu_s(\psi)\, e^{E(X,s)}\, \chi(\bar{\gamma},s)\, e^{G(X)} e^{-F_1(X)} e^{-F_2(X)} \mathcal{A}_a \tag{9.9}$$

We split the sum over X in $X = X_1$ and those X which strictly contain X_1.

From the last statement of lemma 16 we see that

$$\lim_{a \to 0} \sum_{\substack{X,\, X \supset X_1 \\ \text{strictly}}} \mathcal{K}(X)\, \varrho(X) = 0 \tag{9.10}$$

This implies (9.11)

$$\lim_{a \to 0} \langle \mathcal{A}_a \rangle_a = \lim_{a \to 0} \mathcal{K}(X_1)\, \varrho(X_1)$$

So the continuum limit is governed by the leading term of the expansion (9.8).

The expansion for $\rho(X)$ reads

$$1 = \tilde{\mathcal{K}}(X_1)\,\rho(X_1) + \sum_{\substack{X,\, X \supset X_1 \\ \text{strictly}}} \tilde{\mathcal{K}}(X)\,\rho(X) \tag{9.12}$$

$\tilde{\mathcal{K}}(X)$ is given by the same expression as $\mathcal{K}(X)$, eq. (9.9), except that $\mathcal{A}_a = 1$. By the same arguments as above it follows that the second term on the r.h.s. of (9.12) tends to zero when $a \to 0$.

Therefore

$$1 = \lim_{a \to 0} \tilde{\mathcal{K}}(X_1)\,\rho(X_1) \tag{9.13}$$

We combine (9.11) and (9.13) to obtain

$$\lim_{a \to 0} \langle \mathcal{A}_a \rangle_a = \lim_{a \to 0} \frac{\mathcal{K}(X_1)}{\tilde{\mathcal{K}}(X_1)} \tag{9.14}$$

The definition (9.9) of $\mathcal{K}(X)$ and $\tilde{\mathcal{K}}(X)$ reduces for $X = X_1$ to the formulae

$$\mathcal{K}(X_1) = \sum_h \int d\mu(\psi)\, e^{E(X_1)}\, e^{G(X_1)}\, e^{-F_1(X_1)} e^{-F_2(X_1)}\, \mathcal{A}_a \tag{9.15a}$$

and

$$\tilde{\mathcal{K}}(X_1) = \sum_h \int d\mu(\psi)\, e^{E(X_1)}\, e^{G(X_1)}\, e^{-F_1(X_1)}\, e^{-F_2(X_1)} \tag{9.15b}$$

We want to show that the contributions from $h \neq 0$ vanish as $a \to 0$. To this end we use the Schwartz inequality:

$$\sum_{h \neq 0} \int d\mu(\psi)\, e^{E} e^{G} e^{-F_1} e^{-F_2} \mathcal{A}_a \leq \sum_{h \neq 0} \left[\int d\mu(\psi)\, e^{2E} e^{2G} |\mathcal{A}_a|^2 \right]^{1/2} \cdot e^{-F_1} \cdot \left[\int d\mu(\psi)\, e^{-2F_2} \right]^{1/2} \tag{9.16}$$

By lemma 9.5. of [2] we have $\left[\int d\mu(\psi)\, e^{-2F_2} \right]^{1/2} \leq e^{c(L')F_1}$ (9.17)

with $c(L') \to 0$ as $L' \to \infty$ [see also Appendix D].
From lemma 12 (modified for the situation that there is no Wilson loop, i.e. $k = 0$) we get

$$F_1 \geq c_F\, m_D^2\, L^3 \sum_p |\delta h(p)|^2 \tag{9.18}$$

In our situation the quantity $m_D^2 L^3 \beta^{-1}$ goes to infinity as $a \to 0$ (β^{-1} comes from the fact that h is an integer multiple of $2\pi\beta^{-1/2}$). Because $h \neq 0$ the sum over plaquettes p (of the block lattice) is non-zero and therefore the product of the last two factors in the r.h.s. of (9.16) vanishes in the limit $a \to 0$. The first factor is bounded uniformly in a by $c_1 e^{c_2 F_1}$ with an arbitrarily small constant $c_2 > 0$ and some constant c_1.

We use the bounds (9.22) and (9.23) below on E and G and the easy estimate

$$|\mathcal{A}_a|^2 \leq \prod_{i=1}^{w} (\phi(x_i) - h(x_i))^2 \tag{9.19}$$

Finally we end up with the equation

$$\lim_{a \to 0} \langle \mathcal{A}_a \rangle_a = \lim_{a \to 0} \frac{\int d\mu(\phi)\, e^{E(X_1)} e^{G(X_1)} \mathcal{A}_a}{\int d\mu(\phi)\, e^{E(X_1)} e^{G(X_1)}} \tag{9.20}$$

provided the dominator has a non-zero limit.

<u>Lemma 18</u>

$$\lim_{a \to 0} e^{E(X_1)} e^{G(X_1)} \mathcal{A}_a = \mathcal{A}_0 \quad \text{and} \quad \lim_{a \to 0} e^{E(X_1)} e^{G(X_1)} = 1 \tag{9.21}$$

The limits in (9.21) are understood as pointwise in ϕ.

To prove this lemma we need the following

<u>Lemma 19</u>

(i) $e^{E(X_1)} \leq c(a)$ uniformly in ϕ, $c(a) \to 1$ as $a \to 0$ (9.22)

(ii) $e^{G(X_1)} \leq \exp \frac{1}{2} \eta\, m_D^2 \int \phi^2$, with $\eta \in (1 - 4\pi^{-2}, 1)$ (9.23)

(iii) $\lim_{a \to 0} e^{G(X_1)} = 1$, $\lim_{a \to 0} e^{E(X_1)} = 1$ (9.24)

The η in (9.23) is the same as that of (7.10).

<u>proof of lemma 19</u>:

(i)

$$|E(X_1)| \le \sum_{t=2}^{\infty} \frac{2^t}{t!} \int_{X_1} d\xi_1 \cdots \int_{X_1} d\xi_t \, | \rho_t(\xi_1, \ldots, \xi_t) | \tag{9.25}$$

The r.h.s. of (9.25) goes to zero as $a \to 0$, $\beta/a \to \infty$, $M = (\beta/a)^{1/12} m_D \to \infty$. This follows from proposition 3 and the bound (1.18) given there. (9.22) is then established.

(ii)

$$e^{G} = \prod_{\alpha} \frac{\exp \int_{x \in \Omega_\alpha} \sum_m \rho_1(m,0) \left[\cos m\beta^{1/2}\phi(x) - 1 \right]}{\sum_{n \in \mathbb{Z}} \exp\left[-\frac{1}{2} \tilde{m}_D^2 \int_{x \in \Omega_\alpha} (\phi(x) - 2\pi n \beta^{-1/2})^2 \right]} \tag{9.26}$$

By periodicity of G we may assume that $\beta^{1/2}\phi(x) \in [-\pi, +\pi]$.

denominator $\ge \exp\left[-\frac{1}{2}\tilde{m}_D^2 \int_{x\in\Omega_\alpha} \phi(x)^2\right] = \exp\left[-\frac{1}{2}\sum_m \rho_1(m,0)\, m^2\beta \int_{x\in\Omega_\alpha} \phi(x)^2\right]$

So we get:

$$e^{G} \le \prod_{\alpha} \exp \int_{x\in\Omega_\alpha} \sum_m \rho_1(m,0) \left[\cos m\beta^{1/2}\phi(x) - 1 + \tfrac{1}{2} m^2\beta\,\phi(x)^2 \right] \tag{9.27}$$

By the same techniques used in bounding derivatives of r(A) [lemma 9.7. of ref. 2] one can show that

$$\sum_m \rho_1(m,0) \left[\cos m\beta^{1/2}\phi(x) - 1 + \tfrac{1}{2} m^2\beta\,\phi(x)^2 \right] \le \eta \tfrac{1}{2} \tilde{m}_D^2 \, \phi(x)^2 \quad , \quad \eta \in (1 - 4\pi^{-2}, 1) \tag{9.28}$$

(9.27) and (9.28) imply (9.23) if we remember the relation between m_D and $\tilde{m}_D$ stated after eq. (6.7).

(iii) The second part of (9.24) follows from the bound (9.25). For fixed ϕ the denominator in (9.26) approaches (as $a \to 0$)

$$\exp\left[-\tfrac{1}{2}\tilde{m}_D^2 \int_{x\in\Omega_\alpha} \phi(x)^2\right] = \exp\left[-\int_{x\in\Omega_\alpha} \phi(x)^2 \tfrac{1}{2} \sum_m \rho_1(m,0)\, m^2\beta\right]$$

and we would have succeeded if we could show that

$$\exp \int_{x\in X_1} \sum_m \rho_1(m,0) \left[\cos m\beta^{1/2}\phi(x) - 1 + \tfrac{1}{2} m^2\beta\,\phi(x)^2 \right] \longrightarrow 1 \tag{9.29}$$

We expand the cosine in (9.29) in a Taylor series up to fourth order with remainder

$$\int_{x\in X_1} \sum_m \rho_1(m,0) \left[\cos m\beta^{1/2}\phi(x) - 1 + \tfrac{1}{2} m^2\beta\,\phi(x)^2 \right] =$$

$$= \int_{x\in X_1} \sum_m \rho_1(m,0) \frac{1}{4!} \beta^2 m^4 \phi(x)^4 \cdot \cos(m\beta^{1/2}\phi(x)\,\theta_{xm}) \tag{9.30}$$

$$\le \int_{x\in X_1} \phi(x)^4 \sum_m |\rho_1(m,0)| \frac{1}{4!} \beta^2 m^4 \qquad [\text{ with } \theta_{xm} \in (0,1)]$$

Now we can use proposition 3 to show

$$\sum_m |\rho_1(m,0)|\, m^4\beta \le c\, m_D^2, \text{ with some constant } c > 1 \tag{9.31}$$

The remaining factor of β in (9.30) ensures that (9.29) is indeed true as $a \to 0$. This implies that $\lim_{a\to 0} e^{E(X_1)} = 1$. The convergence is only pointwise in ϕ.
End of the proof of lemma 19. □

Lemma 18 follows from lemma 19 and (9.4). □

With these preparations we are able to carry out the limit in (9.20) and to prove (9.5). We use the dominated convergence theorem. Its applicability is proved by the lemmata 18, 19 together with the bound:

$$|\mathcal{A}_\alpha| \le \prod_{i=1}^{\omega} |\phi(x_i)| \tag{9.32}$$

In addition we remember that $M \to \infty$ (in units of physical length m_D^{-1}) as $a \to 0$.

It is well known that the lattice Gaussian measure (with covariance C in our case) converges to the continuum Gaussian measure with the corresponding covariance [see for instance ref. 20,21]. In our situation this covariance is just $\tilde{v}_0$, eq. (9.6) [as $M \to \infty$].

Therefore eq. (9.5) is now proven and theorem 4 is established. □

Appendix A. Relation between the U(1) lattice gauge theory and the Z-ferromagnet

We give a short derivation of the duality transformation mentioned in the introduction with special emphasis on the boundary conditions [the reader should compare with ref. 5]. Let Λ, Λ_1 be finite cubic lattices such that

$$\Lambda \subset \Lambda_1 \subset (a\mathbb{Z})^3 \tag{A.1}$$

Λ, Λ_1 should be closed cell complexes under the boundary operator ∂. The center of Λ is located at the origin $0 \in (a\mathbb{Z})^3$. The dual lattices Λ^*, Λ_1^* are then closed under the coboundary operator ∂^*. We define $\tilde{\Lambda}_1^*$, to be the minimal lattice which is closed under ∂ and contains Λ_1^*: $\Lambda_1^* \subset \tilde{\Lambda}_1^*$.

The 3-dimensional U(1) lattice gauge theory embedded in a heat bath of noncompact electrodynamics is given by the measure*

$$d\mu_{\Lambda^*}^{\Lambda_1^*}(\theta) = (Z_{\Lambda^*}^{\Lambda_1^*})^{-1}\, d\nu_{\Lambda^*}^{\Lambda_1^*}(\theta) \quad , \quad Z_{\Lambda^*}^{\Lambda_1^*} = \int d\nu_{\Lambda^*}^{\Lambda_1^*}(\theta)$$

$$d\nu_{\Lambda^*}^{\Lambda_1^*}(\theta) = \prod_{b\in\tilde{\Lambda}_1^*} (2\pi)^{-1} d\theta(b) \prod_{p\in\Lambda_1^*} e^{\mathcal{L}_p(\theta(\dot{p}))} \tag{A.2}$$

The product over b, p is over all links, plaquettes in $\tilde{\Lambda}_1^*$, Λ_1^*. $\theta(b)$ is integrated over the range $[-\pi, +\pi]$ for $b \in \Lambda^*$ and over $\mathbb{R}$ for $b \in \tilde{\Lambda}_1^* - \Lambda^*$. $\theta(-b) = -\theta(b)$ where $-b$ denotes the link b with opposite orientation. $\theta(\dot{p})$ is the oriented sum over all $\theta(b)$ with $b \in \partial p$, considered modulo 2π if p lies inside Λ^*.

The function $\mathcal{L}_p(\theta)$ outside Λ^* is that of noncompact electrodynamics:

$$\mathcal{L}_p(\theta) = -\tfrac{1}{2}\beta_1\theta^2 \,, \quad \text{for} \quad p \in \Lambda_1^* - \Lambda^* \tag{A.3a}$$

[The parameter β_1 is related to β by eq. (A.10) below]

* The reason for the unusual boundary conditions is that they can be dually transformed without producing nonlocal constraints. In the Coulomb gas representation (see section 2) they imply that the monopoles are only in Λ ("insulating boundary conditions").

Inside of Λ^*, $\mathcal{L}_p(\theta)$ has to be a periodic function with period 2π. The standard Wilson action for lattice gauge theories would be

$$\mathcal{L}_p^W(\theta) = \beta_1(\cos\theta - 1) \tag{A.4}$$

Instead of this action we choose the Villain action

$$\mathcal{L}_p(\theta) = \mathcal{L}_p^V(\theta) = \ln \sum_{n\in\mathbb{Z}} e^{-\frac{1}{2}\beta_1(\theta - 2\pi n)^2} \,, \quad \text{for} \quad p \in \Lambda^* \tag{A.3b}$$

We are interested in the limit $\Lambda_1^* \to (a\mathbb{Z})^3$ while Λ^* is held fixed. This limit causes no problems because outside of Λ^* the action is purely Gaussian (quadratic in $\theta(b)$, see (A.3a)).

The duality transformation is obtained if we expand the Boltzmann factors $e^{\mathcal{L}_p}$ (and the obervable) in a Fourier series or Fourier integral respectively and then integrate over the θ-variables. The Fourier decomposition of $e^{\mathcal{L}_p}$ reads

$$e^{\mathcal{L}_p(\theta(\dot{p}))} = \int_{\mathbb{R}} d\ell(p)\, (2\pi\beta_1)^{-1/2}\, e^{i\ell(p)\theta(\dot{p})}\, e^{-\ell(p)^2/2\beta_1} \,, \quad \text{for} \quad p \in \Lambda_1^* - \Lambda^* \tag{A.5a}$$

and

$$e^{\mathcal{L}_p(\theta(\dot{p}))} = \sum_{\ell(p)\in\mathbb{Z}} (2\pi\beta_1)^{-1/2}\, e^{i\ell(p)\theta(\dot{p})}\, e^{-\ell(p)^2/2\beta_1} \,, \quad \text{for} \quad p \in \Lambda^* \tag{A.5b}$$

The integration over the θ-variables produces δ-functions $\delta(\ell(\partial^* b))$ for $b \in \tilde{\Lambda}_1^* - \Lambda^*$ and Kronecker δ's $\delta(\ell(\partial^* b))$ for $b \in \Lambda^*$ [Here it is crucial that Λ^* is closed under ∂^*, i.e., if $b \in \Lambda^*$ then also $\partial^* b \subset \Lambda^*$.]. $\ell(\partial^* b)$ is the oriented sum over all $\ell(p)$ with $p \in \partial^* b$.

In particular we have

$$\ell(p) = 0 \,, \quad \text{for} \quad p \in \partial\Lambda_1^* \tag{A.6}$$

The constraint

$$\ell(\partial^* b) = 0 \,, \quad \text{for} \quad b \in \tilde{\Lambda}_1^* - \Lambda^* \tag{A.7}$$

can be solved in the standard way [5] :

there is a function n living on cubes of Λ_1^* (equivalenty on sites of Λ_1) which is integer valued in Λ^* and real valued in $\Lambda_1^* - \Lambda^*$ and satisfies

$$\ell(p) = n(\partial^* p) \quad \text{and} \quad n(c) = 0 \quad \text{for} \quad c \in \partial\Lambda_1^* \tag{A.8}$$

The final expression for the dually transformed measure is then given by

$$d\hat{\mu}_\Lambda^{\Lambda_1}(n) = (\hat{Z}_\Lambda^{\Lambda_1})^{-1} d\hat{\nu}_\Lambda^{\Lambda_1}(n) \quad , \quad \hat{Z}_\Lambda^{\Lambda_1} = \int d\hat{\nu}_\Lambda^{\Lambda_1}(n)$$

$$d\hat{\nu}_\Lambda^{\Lambda_1}(n) = \prod_{x\in\Lambda_1} dn(x) \prod_{b\in\Lambda_1,\, b=(x,y)} \exp\left[-\tfrac{1}{2\beta_1}(n(x)-n(y))^2\right] \tag{A.9}$$

where the product over x,b is over all sites, links in Λ_1.

There is a constraint on n, namely $n(x) = 0$ for $x \in \Lambda_1$.

$n(x)$ is integrated over $\mathbb{R}$ for $x \in \Lambda_1 - \Lambda$ (with the ususal Lebesgue measure) and summed over the integers $\mathbb{Z}$ for $x \in \Lambda$.

Therefore (A.9) is a $\mathbb{Z}$-ferromagnet embedded in a heat bath described by a massless free field theory.

β_1 is related to the parameter β of section 1 by

$$\beta_1 a = (4\pi^2)^{-1}\beta = g^{-2} \tag{A.10}$$

Appendix B. Decay properties of the lattice Yukawa and Coulomb potential

We consider the Yukawa potential of mass $m > 0$ on an infinitely extended lattice $(a\mathbb{Z})^3$ of lattice spacing a

$$v_m(x-y) = (-\Delta + m^2)^{-1}(x,y) \tag{B.1}$$

Its Fourier representation is

$$v_m(x-y) = (2\pi)^{-3} \cdot \int_{-\frac{\pi}{a} \le k_\mu < \frac{\pi}{a}} d^3k \quad e^{ik\cdot x} \quad \tilde{v}_m(k) \tag{B.2}$$

$$\tilde{v}_m(k) = \left[m^2 + 2a^{-2} \sum_{\mu=1}^{3} (1 - \cos k_\mu a) \right]^{-1} \tag{B.3}$$

We wish to estimate the quantity

$$\| v_m \|_A = a^3 \sum_{x\in(a\mathbb{Z})^3} v_m(x)\, e^{A\|x\|} = a^3 \sum_{x\in(a\mathbb{Z})^3} v_m(x)\, e^{\alpha\|x\|_1} \tag{B.4}$$

where $\alpha = 3^{-\frac{1}{2}} A$, and $\|x\|_1 = \sum_\mu |x_\mu|$

For the purpose of this paper it suffices to have estimates which are valid when $a \cdot m$ is sufficiently small (for fixed m,α). Such estimates can be obtained by verifying that the sum in eq. (B.4) converges if

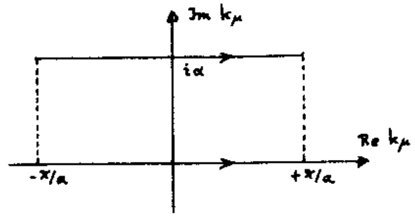
$$0 \le \alpha < 2a^{-1} \cdot \mathrm{Arsh}\left[a \cdot m / 2\sqrt{3}\right] \tag{B.5}$$

and converges to the corresponding expression in the continuum when $a \cdot m \to 0$.

Let us consider $v_m(x)$ for arguments x with $x_\mu \ge 0$ (all μ). $\tilde{v}_m((k_\mu + i\alpha))$ is free from singularities in the intervall (B.5).

Therefore the path of the k_μ-integrations in eq. (B.2) can be shifted as shown in figure 4. The contributions from the dashed pieces of the closed path shown in figure 4 cancel by periodicity.

Fig. 4. Shift of the contour of the k_μ-integrations



Therefore

$$v_m(x) = (2\pi)^{-3} e^{-\alpha\|x\|_1} \int_{-\frac{\pi}{a} \le k_\mu < \frac{\pi}{a}} d^3k \; e^{ik\cdot x} \left[m^2 + 2a^{-2}\sum_{\mu=1}^{3}(1-\cos(k_\mu + i\alpha)a)\right]^{-1} \tag{B.6}$$

for $x_\mu \geq 0$ ($\mu = 1,2,3$) and α in the interval (B.5). The expression $[\cdots]$ is bounded below by

$$|[\cdots]| \geq m^2 - 6a^{-2}(\operatorname{ch}\alpha a - 1) \tag{B.7}$$

in the interval (B.5).

This produces the bound of the following lemma, with $c_\alpha = a^{-3}[m^2 - 6a^{-2}(\operatorname{ch}\alpha a - 1)]^{-1}$
By symmetry the bound generalizes to all x_μ.

Lemma B.1

For α in the interval (B.5)

$$|v_m(x)| \leq c_\alpha e^{-\alpha\|x\|_1} \tag{B.8}$$

The bound (B.8) is not yet good enough for our purpose because c_α can blow up when $a\cdot m \to 0$. But it assures convergence of the x-summations in eq. (B.4).

$v_m(x)$ is positive. From eq. (B.4) it follows that

$$\|v_m\|_A \leq 8a^3 \sum_{x_\mu \geq 0} v_m(x)\, e^{\alpha\|x\|_1} \tag{B.9}$$

Because of lemma (B.1) we may insert eq. (B.6) [with α replaced by some $\alpha' > \alpha$, α' in the interval (B.5)] and interchange the x-summations with the k-integrations.

The x-summations can be done since they are geometric series.

As a result

$$\|v_m\|_A \leq 8(2\pi)^{-3}a^3 \int_{-\frac{\pi}{a} \le k_\mu < \frac{\pi}{a}} d^3k \; \left[m^2 + 2a^{-2}\sum_\mu(1-\cos(k_\mu + i\alpha')a)\right]^{-1} \cdot \prod_\mu \left(1 - e^{-(\alpha'-\alpha-ik_\mu)a}\right)^{-1}$$

The limit $a\cdot m \to 0$ of this integral exists and gives

$$\limsup_{a\cdot m \to 0} \|v_m\|_A \leq 8(2\pi)^{-3} \int_{-\infty < k_\mu < +\infty} d^3k \; \left[m^2 + \sum_\mu (k_\mu + i\alpha')^2\right]^{-1} \cdot \prod_\mu (\alpha' - \alpha - ik_\mu)^{-1}$$

$$= \int d^3x \; e^{\alpha\|x\|_1} v_m^\infty(x) \tag{B.10}$$

v_m^∞ is the Yukawa potential in the continuum, and the second equality follows by repeating the above calculations in the continuum.

Finally we can use the explicit expression for v_m^∞

$$v_m^\infty(x) = \frac{1}{4\pi|x|} e^{-m|x|} \quad \text{with} \quad |x| = \left(\sum_\mu |x_\mu|^2\right)^{1/2} \tag{B.11}$$

Since $\|x\|_1 \leq \sqrt{3}\,|x|$ it follows that

$$\lim_{a\cdot m \to 0} \|v_m\|_A \leq \int_0^\infty dr \; r\, e^{-(m-\sqrt{3}\alpha)r} = (m - \sqrt{3}\alpha)^{-2} \tag{B.12}$$

provided that $\sqrt{3}\cdot\alpha < m$. This is true in the interval (B.5).

Finally we choose some $c > 1$. If $0 \leq A < m$ then the condition (B.5) is fulfilled for sufficiently small $a\cdot m$.

Therefore

$$\|v_m\|_A \leq c\cdot(m - A)^{-2} \tag{B.13}$$

for $0 \leq A < m$, $c > 1$, and $a\cdot m$ sufficiently small, depending on c and $\frac{A}{m}$.

The assertions of lemma 5 for $\|v^\ell\|_A$ ($\ell = 1,2$) follow from this because $v^1(x) = v_{M_1}(x) \geq 0$ and $0 \leq v^2(x) \leq v_M(x)$, with M, M_1 as in the hypotheses of lemma 5.

In section 7 and Appendix D we need estimates for the exponential decay of the covariances C, C_o and the finite difference derivatives of C_o.

We start with the following

Lemma B.2

(i) For arbitrary constants $c_1 > 0$, $\eta \in (0,1)$, $\delta > 0$ and any integer $j \geq 0$ there is a constant c_2 depending on c_1, η, j, δ such that

$$|\nabla_{\mu_1} \cdots \nabla_{\mu_j} v_m(x)| \leq c_2 \cdot m^{j+1} \cdot e^{-\alpha \|x\|_\infty}, \quad \text{for } \|x\|_\infty \geq c_1 m^{-1} \text{ and } a \cdot m \leq \delta \tag{B.14a}$$

where

$$\alpha = \eta m \frac{\mathrm{Arsh}\, am/2}{a \cdot m/2} \tag{B.14b}$$

and

c_2 is independent of a, m.

(ii) For any integer $j \geq 0$ there is a constant c depending on j such that

$$|\nabla_{\mu_1} \cdots \nabla_{\mu_j} v_{cb}(x)| \leq c \cdot \|x\|_\infty^{-(j+1)}, \quad \text{for } x \neq 0 \tag{B.15}$$

c is independent of the lattice spacing a.

proof:

Because of the lattice symmetry we may assume that $\|x\|_\infty = |x_1| = x_1 > 0$.

(i) We rescale the integration variables $k_\mu = mp_\mu$

$$v_m(x) = (2\pi)^{-3} m \cdot \int\limits_{-\frac{\pi}{am} \leq p_\mu < \frac{\pi}{am}} d^3p \; e^{ip \times m} \left[1 + 4a^{-2}m^{-2} \sum_{\mu=1}^{3} \sin^2 \tfrac{1}{2} p_\mu am\right]^{-1} \tag{B.16}$$

The finite difference derivatives ∇_μ act on (B.16) in the following way

$$\nabla_\mu e^{ip \cdot xm} = m\, e^{ip \cdot xm} \cdot a^{-1}m^{-1} \left(e^{ip_\mu am} - 1\right) \tag{B.17}$$

We have $\nabla_{\mu_1} \cdots \nabla_{\mu_j} = \nabla_1^{n_1} \nabla_2^{n_2} \nabla_3^{n_3}$, with $n_1 + n_2 + n_3 = j$.

The p_1-integration can be done with the help of the Cauchy formula:

$$\int\limits_{-\pi/am}^{+\pi/am} dp_1 \; e^{ip_1 m x_1} \left[M(p)^2 + 4a^{-2}m^{-2} \sin^2 \tfrac{1}{2} p_1 am\right]^{-1} \cdot a^{-n_1} m^{-n_1} \left(e^{ip_1 am} - 1\right)^{n_1} = \tag{B.18}$$

$$= \pi M(p)^{-1} \left(1 + \tfrac{1}{4} a^2 m^2 M(p)^2\right)^{-\frac{1}{2}} \exp\left[-mx_1 2a^{-1}m^{-1} \mathrm{Arsh}\, \tfrac{1}{2} am M(p)\right] \cdot a^{-n_1} m^{-n_1} \left(e^{-2\,\mathrm{Arsh}\,\frac{1}{2} am M(p)} - 1\right)^{n_1}$$

where

$$M(p) = \left[1 + 4a^{-2}m^{-2} \sum_{\mu=2}^{3} \sin^2 \tfrac{1}{2} p_\mu am\right]^{1/2} \tag{B.19}$$

Eq. (B.16), (B.17), (B.18) produce an integral representation of $\nabla_{\mu_1} \cdots \nabla_{\mu_j} v_m(x)$.

We estimate it using the following inequalities:

$$|e^{ip_\mu x_\mu m}| \leq 1 \tag{B.20a}$$

$$a^{-1}m^{-1} |e^{ip_\mu am} - 1| \leq |p_\mu|, \quad \text{for } \mu = 2,3 \tag{B.20b}$$

and

$$a^{-1}m^{-1} |e^{-2\,\mathrm{Arsh}\,\frac{1}{2} am M(p)} - 1| \leq 2a^{-1}m^{-1} \mathrm{Arsh}\, \tfrac{1}{2} am M(p) \leq M(p) \tag{B.20c}$$

In addition we use $M(p) \geq 1$ and $mx_1 \geq c_1$ and obtain finally

$$|\nabla_{\mu_1} \cdots \nabla_{\mu_j} v_m(x)| \leq (2\pi)^{-3} m^{j+1} \pi \int\limits_{-\frac{\pi}{am} \leq p_\mu < \frac{\pi}{am}} dp_2\, dp_3 \; e^{-\eta m x_1 2a^{-1}m^{-1} \mathrm{Arsh}\, \frac{1}{2} am} \cdot$$
$$\cdot e^{-c_1(1-\eta) 2a^{-1}m^{-1} \mathrm{Arsh}\, \frac{1}{2} am M(p)} \cdot M(p)^{-1} \left(1 + \tfrac{1}{4} a^2 m^2 M(p)^2\right)^{-\frac{1}{2}} \cdot |p_2|^{n_2} |p_3|^{n_3} \cdot M(p)^{n_1} \tag{B.21}$$

We get an estimate on M(p) if we remember that

$$\frac{4}{\pi^2} s^2 \leq \sin^2 s \leq s^2, \quad s \in [-\pi/2, +\pi/2] \tag{B.22}$$

This implies

$$M_u(p)^2 \equiv 1 + \tfrac{4}{\pi^2}(p_2^2 + p_3^2) \leq M(p)^2 \leq 1 + p_2^2 + p_3^2 \equiv M_o(p)^2 \tag{B.23}$$

for $-\frac{\pi}{am} \leq p_\mu < \frac{\pi}{am} \quad (\mu = 2,3)$

In the same region we have

$$2\hat{a}^{-1}m^{-1}\left[1+\tfrac{4}{\pi^2}(p_2^2+p_3^2)\right]^{-\frac{1}{2}}\cdot \mathrm{Arsh}\,\tfrac{1}{2}am\left[1+\tfrac{4}{\pi^2}(p_2^2+p_3^2)\right]^{\frac{1}{2}} \geq$$
$$\geq \left(2+\tfrac{1}{4}\delta^2\right)^{-\frac{1}{2}}\cdot \mathrm{Arsh}\left(2+\tfrac{1}{4}\delta^2\right)^{\frac{1}{2}} \equiv \hat{c} \qquad \text{(B.24)}$$

provided $a\cdot m \leq \delta$.

We insert (B.23) and (B.24) in (B.21). In this way we arrive at

$$|\nabla_{\mu_1}\cdots\nabla_{\mu_j} v_m(x)| \leq m^{j+1} e^{-\alpha\|x\|_\infty}\cdot(8\pi^2)^{-1}\int_{\mathbb{R}^2} dp_2\,dp_3\, e^{-c_1(1-\eta)\hat{c}M_u(p)}\cdot M_u(p)^{-1}\cdot M_o(p)^j \qquad \text{(B.25)}$$

where α is given by the expression (B.14b), viz.

$$\alpha = \eta m\cdot 2\hat{a}^{-1}m^{-1}\,\mathrm{Arsh}\,\tfrac{1}{2}am \qquad \text{(B.26)}$$

(B.25) is true for $\|x\|_\infty \geq c_1 m^{-1}$ and $a\cdot m \leq \delta$.

We compare with (B.14a) and see that part (i) of our lemma is proved.

(ii) Now we rescale the integration variables $k_\mu = \frac{1}{x_1}p_\mu$. Then we can apply the same techniques as above. The details are left to the reader.

End of the proof of lemma B.2. □

The propagator C_o on the infinitely extended lattice $\Lambda = (a\mathbb{Z})^3$ can be written in the form

$$C_o = r\left\{(-\Delta + m_D'^2)^{-1} - (-\Delta + M'^2)^{-1}\right\} \qquad \text{(B.27)}$$

with $r \to 1$, $m_D' \to \tilde{m}_D$ and $M' \to M$ in the limit $\tilde{m}_D^{-1}M \to \infty$.

It follows that

$$0 \leq C_o(xy) \leq r\cdot(-\Delta + m_D'^2)^{-1}(xy) \qquad \text{(B.28)}$$

and

$$|\nabla_{\mu_1}\cdots\nabla_{\mu_j} C_o(xy)| \leq r\,|\nabla_{\mu_1}\cdots\nabla_{\mu_j} v_{m_D'}(xy)| + r\,|\nabla_{\mu_1}\cdots\nabla_{\mu_j} v_{M'}(xy)| \qquad \text{(B.29)}$$

We apply lemma B.2 to the r.h.s. of (B.28) and (B.29).

This proves our

Lemma B.3

(i) For arbitrary constants $c_1 > 0$, $\eta \in (0,1)$, $\delta > 0$ and any integer $j \geq 0$ there is a constant c_2 depending on c_1, η, j, δ [but independent of a, m_D', M'] such that

$$|\nabla_{\mu_1}\cdots\nabla_{\mu_j} C_o(xy)| \leq r c_2 M'^{j+1} e^{-\alpha\|x-y\|_\infty}, \quad \text{for } \|x-y\|_\infty \geq c_1 m_D'^{-1} \text{ and } a\cdot M' \leq \delta \qquad \text{(B.30)}$$

where $\alpha = \eta m_D'\cdot 2\hat{a}^{-1}m_D'^{-1}\,\mathrm{Arsh}\,\tfrac{1}{2}am_D'$

(ii) For arbitrary constants $c_1 > 0$, $\eta \in (0,1)$, $\delta > 0$ there is a constant c_2 depending in c_1, η, δ [but independent of a, m_D'] such that

$$|C_o(xy)| \leq r c_2 m_D' e^{-\alpha\|x-y\|_\infty}, \quad \text{for } \|x-y\|_\infty \geq c_1 m_D'^{-1} \text{ and } a\cdot m_D' \leq \delta \qquad \text{(B.31)}$$

where $\alpha = \eta m_D'\cdot 2\hat{a}^{-1}m_D'^{-1}\,\mathrm{Arsh}\,\tfrac{1}{2}am_D'$

An estimate similar to (B.31) is needed in section 7 for the propagator $C(xy)$ defined by eq. (6.15).

$$C = \left[-\Delta\left(1-\frac{\Delta}{M^2}\right) + \tilde{m}_D^2 + v\right]^{-1} = \left[C_o^{-1} + v\right]^{-1} \qquad \text{(B.32)}$$

$$v(x_1x_2) = \sum_{m_1}\sum_{m_2} \rho_2(m_1x_1, m_2x_2)\, m_1 m_2 \beta \qquad \text{(B.33)}$$

To obtain such an estimate, one writes down a convergent Neumann series in which v is treated as the perturbation

$$C(xy) = C_o(xy) - \int_{x_1}\int_{x_2} C_o(xx_1)\, v(x_1x_2)\, C_o(x_2y) + - \cdots \qquad \text{(B.34)}$$

Absolute convergence of this series will follow from the bounds below.

We introduce for an arbitrary subset α_1 of the lattice $(a\mathbb{Z})^3$

$$L(x,\alpha_1) = \min_{y\in\alpha_1} \|x-y\| \qquad \text{(B.35)}$$

This is in agreement with the definition (1.11) in the main text.

Now we can estimate

$$\int_{y\in a_1} |C(xy)| \le e^{-AL(x,a_1)} \cdot \int_y |C(xy)|\, e^{A\|x-y\|} \le$$

$$\le e^{-AL(x,a_1)} \cdot \Big\{ \int_y C_0(xy)\, e^{A\|x-y\|} + \int_{x_1}\int_{x_2}\int_y C_0(xx_1)\, e^{A\|x-x_1\|} \cdot |\nu(x_1x_2)|\, e^{A\|x_1-x_2\|} \cdot C_0(x_2y)\, e^{A\|x_2-y\|} + \dots \Big\} \tag{B.36}$$

The summations can be performed in the order y, x_{2n}, ..., x_2, x_1 (n = no. of factors ν).

Both C_0 and ν are translation invariant, and the sums are extended over the infinite lattice $(a\mathbb{Z})^3$. The sums of C_0 are estimated with the help of (B.28) and (B.13). The sums of ν are estimated with the help of inequality (1.18) of proposition 3.

As a result one obtains

Lemma B.4

Under the hypotheses of proposition 3

$$\int_{y\in a_1} |C(xy)| \le c_A \cdot e^{-AL(x,a_1)} \tag{B.37}$$

for $A < m_D'$ and sufficiently small lattice spacing a.

The constant $c_A < \infty$ is independent of a (in units where $\tilde{m}_D = 1$), and $m_D' \longrightarrow m_D$ as $\lambda = M^{-1} m_D \to 0$.

Appendix C. Lower bound on F_1 (lemma 12)

Lemma 12 asserts a lower bound on F_1 by an expression which involves only nearest neighbor interactions in the h variables.

Due to the presence of f in the definition of F_1, the proof of lemma 12 is slightly different from that in [1,2].

We start with the obvious equality

$$\sum_p |\delta(h - k\beta^{-1/2}\bar{f})(p)|^2 = L^{-6} \sum_{\langle\Omega,\Omega'\rangle} \int_{x\in\Omega} \int_{y\in\Omega'} |h(x) - h(y) - k\beta^{-1/2}\bar{f}(x) + k\beta^{-1/2}\bar{f}(y)|^2 \tag{C.1}$$

Summation over p is over all plaquettes of the block lattice.

$\delta(h - k\beta^{-\frac{1}{2}}\bar{f})(p)$ is the discontinuity in $h - k\beta^{-\frac{1}{2}}\bar{f}$ aross p.

Ω, Ω' are cubes of the block lattice and the sum over $\langle\Omega, \Omega'\rangle$ is over all nearest neighbors.

$$|h(x)-h(y)-k\beta^{-\frac{1}{2}}\bar{f}(x)+k\beta^{-\frac{1}{2}}\bar{f}(y)|^2 \le 5\cdot|h(x)-k\beta^{-\frac{1}{2}}f(x)-g(x)|^2 + 5\cdot|h(y)-k\beta^{-\frac{1}{2}}f(y)-g(y)|^2 + 5\cdot|g(x)-g(y)|^2 + 5k^2\beta^{-1}|\delta f(x)|^2 + 5\cdot k^2\beta^{-1}|\delta f(y)|^2 \tag{C.2}$$

For any $x\in\Omega$, $y\in\Omega'$ we consider the path from x to y which consists of segments parallel to the coordinate axes. It has a maximal length of 4L. So we get by use of the Schwartz inequality

$$|g(x)-g(y)|^2 = |\int_x^y \nabla g\cdot ds|^2 \le 4L\cdot\int_x^y |\nabla g|^2 ds \tag{C.3}$$

Furthermore we have:

$$\int_{x\in\Omega}\int_{y\in\Omega'}\int_x^y |\nabla g|^2 ds \le 3L^4 \int_{\Omega\cup\Omega'} |\nabla g|^2 \tag{C.4}$$

We collect our estimates:

$$\text{r.h.s. of (C.1)} \le L^{-6}\, 5L^3 2 \cdot \sum_{\langle\Omega,\Omega'\rangle} \int_\Omega |h - k\beta^{-\frac{1}{2}}f - g|^2 + L^{-6}\, 5\cdot 3L^4\cdot 4L \sum_{\langle\Omega,\Omega'\rangle} \int_{\Omega\cup\Omega'} |\nabla g|^2 + L^{-6}\, 5L^3\cdot 2 \sum_{\langle\Omega,\Omega'\rangle} \int_\Omega k^2\beta^{-1}|\delta f|^2 \tag{C.5}$$

The r.h.s. of (C.5) is less than

$$30L^{-3}\int|h-k\beta^{-\frac{1}{2}}f-g|^2 + 360L^{-1}\int|\nabla g|^2 + 30L^{-3}k^2\beta^{-1}\int|\delta f|^2 \tag{C.6}$$

We use the fact that

$$u^{-1} \geqslant -\Delta \tag{C.7}$$

and compare with the definition (6.18) of F_1 to get

$$(C.6) \leqslant L^{-3}\tilde{m}_D^{-2}\,\tilde{c}\,\left(F_1 + \tfrac{1}{2}k^2\beta^{-1}\tilde{m}_D^2\int|\delta f|^2\right) \tag{C.8}$$

where $\tilde{c} = \max\{60,\ 720\,\tilde{m}_D^2L^2\}$.
Finally we conclude from (C.1), (C.8):

$$F_1 \geqslant c\,\tilde{m}_D^2L^3\cdot\sum_p|\delta(h-k\beta^{-\frac{1}{2}}\bar{f})(p)|^2 - \tfrac{1}{2}k^2\beta^{-1}\tilde{m}_D^2\int|\delta f|^2 \tag{C.9}$$

with

$$c = \min\left\{\frac{1}{60},\ \frac{1}{720}\tilde{m}_D^{-2}L^{-2}\right\}.$$

The expansion is such that $\tilde{m}_D L \ll 1$. So the actual value of c should be $\frac{1}{60}$.

The above arguments remain true if the integrals and the sum over $\langle\Omega,\Omega'\rangle$ and block plaquettes p are restricted to X, a union of unit lattice cubes.

We are going to bound the first term on the r.h.s. of (C.9) from below. To this end we rewrite it in the following form:

$$\sum_p|\delta(h-k\beta^{-\frac{1}{2}}\bar{f})(p)|^2 = \sum_{\langle\Omega,\Omega'\rangle}|(h-k\beta^{-\frac{1}{2}}\bar{f})(\Omega)-(h-k\beta^{-\frac{1}{2}}\bar{f})(\Omega')|^2 = $$
$$= a^2L^{-2}\sum_{\langle xy\rangle}|(h-k\beta^{-\frac{1}{2}}\bar{f})(x)-(h-k\beta^{-\frac{1}{2}}\bar{f})(y)|^2 \tag{C.10}$$

Summation over $\langle\Omega,\Omega'\rangle$ ($\langle xy\rangle$) is over all nearest neighbors of cubes of the block lattice (original lattice). The factor a^2L^{-2} compensates for multiple counting.

The r.h.s. of (C.10) is nothing else than

$$a\,L^{-2}\int|\nabla(h-k\beta^{-\frac{1}{2}}\bar{f})|^2 \tag{C.11}$$

For further estimation of (C.11) it is useful to consider the quantity

$$\int|\nabla(h-k\beta^{-\frac{1}{2}}f)|^2 \tag{C.12}$$

We decompose

$$\nabla_\mu f = j_\mu + R_\mu \tag{C.13}$$

A straightforward calculation using (1.12) and (1.13) verifies that R_μ is given by the expression

$$R_\mu = v_{cb}\ \varepsilon_{\mu\sigma\tau}\ \nabla_{-\sigma}\ J_\tau \tag{C.14}$$

The term R_μ is associated with the perimeter and it satisfies

$$\nabla_{-\mu}R_\mu = 0 \tag{C.15}$$

This follows trivially from (C.14).

With help of (C.15) we see that

$$\int|\nabla(h-k\beta^{-\frac{1}{2}}f)|^2 = \int|\nabla_\mu h - k\beta^{-\frac{1}{2}}j_\mu|^2 + k^2\beta^{-1}\cdot\int R_\mu j_\mu \tag{C.16}$$

The calculation of $\int R_\mu j_\mu$ is easy

$$\int R_\mu j_\mu = -(J_\mu, v_{cb}J_\mu) \tag{C.17}$$

Now we consider the quantity (C.11) and relate it to (C.12) by $\bar{f} = f - \delta f$

$$\int|\nabla(h-k\beta^{-\frac{1}{2}}\bar{f})|^2 = \int|\nabla(h-k\beta^{-\frac{1}{2}}f)|^2 + k^2\beta^{-1}\int|\nabla\delta f|^2 + $$
$$+2k\beta^{-\frac{1}{2}}\cdot\int\nabla_\mu\delta f\cdot(\nabla_\mu h - k\beta^{-\frac{1}{2}}j_\mu) \tag{C.18}$$

We use (C.16), (C.17) and arrive at:

$$\int|\nabla(h-k\beta^{-\frac{1}{2}}\tilde{f})|^2 = \int|\nabla_\mu h-k\beta^{-\frac{1}{2}}j_\mu|^2 + 2\cdot k\beta^{-\frac{1}{2}}\int\nabla_\mu\delta f\cdot(\nabla_\mu h-k\beta^{-\frac{1}{2}}j_\mu) + \tag{C.19}$$

$$+k^2\beta^{-1}\cdot\int|\nabla\delta f|^2 - k^2\beta^{-1}(J_\mu, v_{cb}J_\mu)$$

A short computation shows that (for $\varepsilon\in(0,1)$)

$$\varepsilon\cdot\int|\nabla_\mu h-k\beta^{-\frac{1}{2}}j_\mu|^2 + 2k\beta^{-\frac{1}{2}}\int\nabla_\mu\delta f\cdot(\nabla_\mu h-k\beta^{-\frac{1}{2}}j_\mu) \geqslant -\varepsilon^{-1}k^2\beta^{-1}\int|\nabla\delta f|^2 \tag{C.20}$$

uniformly in h.

We collect all our estimates and get the result:

$$F_1 \geqslant c(1-\varepsilon)\,\tilde{m}_D^2 La\cdot\int|\nabla_\mu h-k\beta^{-\frac{1}{2}}j_\mu|^2 - c\tilde{m}_D^2 La\,k^2\beta^{-1}(J_\mu, v_{cb}J_\mu) -$$

$$-\tfrac{1}{2}k^2\beta^{-1}\tilde{m}_D^2\int|\delta f|^2 - (\varepsilon^{-1}-1)\,c\,\tilde{m}_D^2 La\,k^2\beta^{-1}\cdot\int|\nabla\delta f|^2 \tag{C.21}$$

In the first term in (C.21) the $\int$ can be converted back into a sum of discontinuities across plaquettes on the block lattice. Using the definition (1.6) of j_μ one finds that this term is equal to $c(1-\varepsilon)\tilde{m}_D^2L^3\|T(h)\|^2$, $\|T(h)\|$ defined in eq. (6.21) of section 6. $(J_\mu, v_{cb}J_\mu)$ is bounded by const.$|C|\cdot\ln(|C|/a)$ as is well known. (This quantity determines the expectation value of the Wilson loop operator in noncompact free electrodynamics).

To establish lemma 12 is only remains to verify that $\int|\nabla\delta f|^2$ and $\int|\delta f|^2$ is perimeter behaved. By definition

$$\delta f(x) = L^{-3}\cdot\int_{y\in\Omega\ni x}[f(x)-f(y)] = L^{-3}\cdot\int_{y\in\Omega\ni x}\int_{y'=y}^{x}\nabla f(y') \tag{C.21a}$$

where the inner $\int$ is over a path in the interior of Ω from y to x.

We insert eq. (C.13). j_μ has its support on faces of cubes Ω and contributes therefore nothing to the path integral. In conclusion

$$\delta f(x) = L^{-3}\cdot\int_{y\in\Omega\ni x}\int_{y'=y}^{x} R_t(y') \tag{C.21b}$$

R_t is the component of R_μ tangential to the path from y to x. Expression (C.14) for R_μ can now be inserted. From the decay properties of derivatives of the Coulomb potential (see Appendix B) and the fact that J_t is supported on the Wilson loop C it follows now that $\int|\nabla\delta f|^2$ is perimeter behaved. Applying the same methods as in the derivation of (C.3), (C.4) one concludes from (C.21b)

$$|\delta f(x)|^2 \leqslant 9L^{-1}\int_{y\in\Omega\ni x} R_\mu^2(y) \tag{C.21c}$$

Therefore

$$\int|\delta f|^2 \leqslant 9L^2\int R_\mu^2 = 9L^2(J_\mu, v_{cb}J_\mu) \tag{C.21d}$$

where the last equality follows from the expression (C.14) for R_μ. This completes the proof of lemma 12. □

Given that Ξ is the minimal surface whose boundary is the Wilson loop C it is obvious from the definition (6.21) of T(h) that

$$\|T(h)\|^2 \geqslant 4\pi^2k^2\beta^{-1}L^{-2}|\Xi| \tag{C.22}$$

$|\Xi|$ = area of the surface Ξ.

Appendix D. F_2 is small (lemma 14)

This appendix replaces section 9.4. of [2].

There are two modifications as against the situation in [2]. First our shift g is different and second we work on the (infinitely extended) lattice with spacing a whereas Brydges and Federbush deal with a continuum theory. Therefore we are not allowed to use the Leibniz rule if we want to evaluate the finite difference derivative of a product. Instead we have the following rule

$$\nabla_\mu(\varphi_1\varphi_2) = (\nabla_\mu\varphi_1)\varphi_2 + \varphi_1(\nabla_\mu\varphi_2) + (a\nabla_\mu\varphi_1)(\nabla_\mu\varphi_2) \tag{D.1}$$

The last term on the r.h.s. of (D.1) can be estimated if we recognize that

$$\| a\nabla_\mu \|_2 \le 2 \qquad \text{or} \qquad \| a\nabla_\mu \|_\infty \le 2 \tag{D.2}$$

In section 6 we have shown that $F_2'(x)$ vanishes except for $x \in BS_\beta$. We start to estimate $\int_{BS_\beta} F_2'^2$ and finally we sum over all connected components S_β of $\Sigma^\wedge$.

With the notations of section 6 we define

$$\tilde h^e_{\beta,\mu} = \begin{cases} \nabla_\mu h^e_\beta & \text{if } \Xi \cap S_\beta = \emptyset \\ \nabla_\mu h^e_\beta - k\beta^{-\frac12} j_\mu & \text{if } \Xi \cap S_\beta \neq \emptyset \end{cases} \tag{D.3}$$

This definition is such that

$$\nabla_\mu \tilde h^e_{\beta,\mu} = \begin{cases} -\Delta h^e_\beta & \text{if } \Xi \cap S_\beta = \emptyset \\ -\Delta(h^e_\beta - k\beta^{-\frac12} f) & \text{if } \Xi \cap S_\beta \neq \emptyset \end{cases} \tag{D.4}$$

(where we have used the decomposition (C.13), (C.14)) and therefore the cases (iv) and (v), eq. (6.30), (6.31), can be treated on an equal footing

$$F_2' = -C_0^{-1}\chi_\beta M^{-2}(-\Delta + M^2)\nabla_\mu C_0 \tilde h^e_{\beta,\mu} \tag{D.5}$$

C_0^{-1} is a lattice differential operator:

$$C_0^{-1} = M^{-2}\cdot(-\Delta)^2 + (-\Delta) + \tilde m_D^2 \tag{D.6}$$

Let $\eta_\beta(x)$ be the characteristic function for the support of $\tilde h^e_{\beta,\mu}(x)$ ($\mu = 1,2,3$) and define an operator K by its kernel

$$k_{\mu\nu}(xz) = \int_{y\in BS_\beta} \eta_\beta(x)\eta_\beta(z)\left[C_0^{-1}\chi_\beta(1-\tfrac{\Delta}{M^2})\nabla_\mu C_0\right](yx)\cdot \left[C_0^{-1}\chi_\beta(1-\tfrac{\Delta}{M^2})\nabla_\nu C_0\right](yz) \tag{D.7}$$

Then we have

$$\int_{BS_\beta} F_2'^2 = \sum_\mu \int_x \sum_\nu \int_z k_{\mu\nu}(xz)\, \tilde h^e_{\beta,\mu}(x)\, \tilde h^e_{\beta,\nu}(z) \tag{D.8}$$

We estimate the norm of the operator K

$$\| K \|_2 \le \sup_{x,\mu} \sum_\nu \int_z | k_{\mu\nu}(xz)| \le \tag{D.9}$$

$$\le \sup_{x,\mu} \sum_\nu \int_z \int_{y\in BS_\beta} \eta_\beta(x)\eta_\beta(z) \left|(C_0^{-1}\chi_\beta(1-\tfrac{\Delta}{M^2})\nabla_\mu C_0)(yx)\right|\cdot\left|(C_0^{-1}\chi_\beta(1-\tfrac{\Delta}{M^2})\nabla_\nu C_0)(yz)\right|$$

We notice that on the r.h.s. of (D.9) $\|y-x\| \ge \frac12 L'$ and $\|y-z\| \ge \frac12 L'$. This implies

$$\left|(C_0^{-1}\chi_\beta(1-\tfrac{\Delta}{M^2})\nabla_\mu C_0)(yx)\right| \le c_2 M^4 e^{-\frac13 \tilde m_D \|y-x\|} \tag{D.10}$$

provided that $\tilde m_D^{-1} M$ is sufficiently large. c_2 is a constant independent of a, $\tilde m_D$, M and β.

To derive (D.10) we make repeated use of (D.1), (D.2), remember the defining properties of χ_β, and then apply lemma B.3 of Appendix B with $j \le 5$. In addition we need the bound $\|y-x\|_\infty \ge 3^{-1/2}\|y-x\|$.

We evaluate the r.h.s. of (D.9) in two steps [using (D.10)]

(i)

$$\sum_\nu \int_z \eta_\beta(z) \left|(C_0^{-1}\chi_\beta(1-\tfrac{\Delta}{M^2})\nabla_\nu C_0)(yz)\right| \le c_3\, a\, \tilde m_D^2\, e^{-\frac13 \tilde m_D L' \frac13} \tag{D.11}$$

with a constant c_3. The factor a is due to the fact that the support of $\tilde{h}^{e}_{\beta,\mu}$ is a two dimensional surface (with thickness one lattice spacing a)

(ii)

$$\sup_{x,\mu} \int_{Y\in BS_\beta} \eta_\beta(x) \left| (C_0^{-1}\chi_\beta(1-\tfrac{\Delta}{M^2})\nabla_\mu C_0)(yx)\right| \le c_4\, \tilde{m}_D\, e^{-\frac{1}{3}\tilde{m}_D L'\frac{1}{3}} \tag{D.12}$$

with a constant c_4 .

Combining (D.11) and (D.12) we obtain

$$\| K \|_2 \le c_3 c_4\, a\, \tilde{m}_D^3 \cdot e^{-\frac{1}{3}\tilde{m}_D L'} \tag{D.13}$$

Now we are able to estimate (D.8)

$$\int_{BS_\beta} F_2'^2 \le \| K \|_2 \cdot \sum_\mu \int_x |\tilde{h}^{e}_{\beta,\mu}(x)|^2 \le c_3 c_4\, a\, \tilde{m}_D^3\, e^{-\frac{3}{3}\tilde{m}_D L'} \cdot \sum_\mu \int_x |\tilde{h}^{e}_{\beta,\mu}(x)|^2 \tag{D.14}$$

From lemma 12 and (C.21) of Appendix C we conclude

$$\sum_{S_\beta} \sum_\mu \int_x |\tilde{h}^{e}_{\beta,\mu}(x)|^2 \le c_F^{-1}\, \tilde{m}_D^{-2}\, L^{-1} a^{-1} \cdot (F_1 + P_F) \tag{D.15}$$

Summation over S_β is over all connected components of $\Sigma^\wedge$.

Finally we have

$$\int F_2'^2 \le c_3 c_4 c_F^{-1}\, \tilde{m}_D\, L^{-1} \cdot e^{-\frac{1}{3}\tilde{m}_D L'} \cdot (F_1 + P_F) \tag{D.16}$$

(D.16) is the assertion of lemma 14. ◻

Appendix E. The setup of the cluster expansion

The cluster expansion to be discussed here is a well-known technique used in constructive quantum field theory [1,2,10].

We give the basic definitions and a proof of its validity. For simplicity we restrict ourselves to the situation we are actually interested in, namely the expansion for the quantity

$$\int d\mu(\psi)\, e^{E}\, e^{G}\, e^{-F_1}\, e^{-F_2} \tag{E.1}$$

for fixed value of h. The domain wall Σ is fixed by h.

To begin with, we define a covering $\tilde{Y}$ of Σ. Its elements are the sets $S_\beta \cap \Lambda$ which contain the connected pieces of $\Sigma^\wedge$, and the unit cubes $\Delta \subset \Lambda - \cup S_\beta$. For any sequence $\tilde{Y} = (Y_1 \ldots Y_n)$ with mutually disjoint Y_i [each Y_i is a union of elements out of $\tilde{Y}$] and n-1 real variables $s_1 \ldots s_{n-1} \in [0,1]$ we define a covariance C(s) by its kernel

$$C(xy|s) = p(xy|s)\, C(xy) \tag{E.2}$$

with

$$p(xy|s) = \sum_{1\le i<j\le n+1} s_i s_{i+1} \cdots s_{j-1} \left[\chi_i(x)\chi_j(y) + \chi_i(y)\chi_j(x)\right] + \sum_{1\le i\le n+1} \chi_i(x)\chi_i(y) \tag{E.3}$$

where χ_i is the characteristic function of Y_i $(i = 1, \ldots, n)$, χ_{n+1} is the characteristic function of $\sim \bigcup_{i=1}^{n} Y_i$ and s_n is set to zero.

In (E.2) C(xy) is the kernel of the covariance C of the Gaussian measure $d\mu(\psi)$. $C(s)$ is a convex combination of covariances of the form $\chi C \chi$ where χ is a characteristic function [1,2,3]. The Gaussian measure with covariance $C(s)$ is denoted by $d\mu_s(\psi)$. If all variables $s_1 \ldots s_{n-1}$ vanish the interactions between different regions Y_i are switched off.

We set $Q = e^G e^{-F_1}\, e^{-F_2}$ and notice that Q factors across unit cubes. For any union of unit cubes $X \subset \Lambda$ this factorization leads to a natural splitting

$$Q = Q(X) \cdot Q(X^c)\ , \qquad X^c = \Lambda - X \tag{E.4}$$

Given any sequence $\overline{Y} = (Y_1 \dots Y_n)$ and variables $s_1 \dots s_{n-1}$ we have to define $E(X)$ and $E(X,s)$ $[\ X = \bigcup_{i=1}^{n} Y_i\ ]$.

For this purpose we write E as a sum of terms of the following four forms:

$$\mathcal{E}(a_1,\dots,a_t) = \frac{1}{t!}\int_{a_1} d\xi_1 \cdots \int_{a_t} d\xi_t\ \rho_t(\xi_1\cdots\xi_t)\ \varepsilon_k(\xi_1)\cdots\varepsilon_k(\xi_t), \quad \text{for } t \geq 3 \tag{E.5a}$$

$$\frac{1}{2}\int_{a_1} d\xi_1 \int_{a_2} d\xi_2\ \rho_2(\xi_1,\xi_2)\ \tilde{\varepsilon}_k(\xi_1)\ \tilde{\varepsilon}_k(\xi_2) \tag{E.5b}$$

$$\frac{1}{2}\int_{a_1} d\xi_1 \int_{a_2} d\xi_2\ \rho_2(\xi_1,\xi_2)\ \tilde{\varepsilon}_k(\xi_1)\ i m_2 \beta^{1/2}\,\psi(x_2) \tag{E.5c}$$

$$\frac{1}{2}\int_{a_1} d\xi_1 \int_{a_2} d\xi_2\ \rho_2(\xi_1,\xi_2)\ i m_1 \beta^{1/2}\,\psi(x_1)\ \tilde{\varepsilon}_k(\xi_2) \tag{E.5d}$$

Each a_i is a unit lattice cube.

$E(X)$ is the sum of all terms (E.5a,b,c,d) where $a_i \subset X$ for all i.

$E(X,s)$ is given by the same sum as $E(X)$ except that each term (E.5a,b,c,d) is multiplied by $\prod_{i \in I} s_i$. Here $i \in I$ if $1 \leq i \leq n-1$ and for some α, β, $1 \leq \alpha,\beta \leq t$, $a_\alpha \subset Y_j$ (for some $j > i$) and $a_\beta \subset \bigcup_{\ell=1}^{i} Y_\ell$.

Now we are ready to state the basic

<u>Lemma E.1</u>.

$$\int d\mu(\psi)\, e^{E} Q = \sum_X \left[\ \mathcal{K}(X) \int d\mu(\psi)\ e^{E(X^c)} Q(X^c) \ +\ R_n(X)\right] \tag{E.6}$$

where

$$\mathcal{K}(X) = \sum_{\overline{Y}} \int_{[0,1]^{n-1}} ds\ \int d\mu_s(\psi)\ e^{E(X,s)}\ \varkappa(\overline{Y},s)\ Q(X) \tag{E.7}$$

and

$$R_n(X) = \int_0^1 ds_n\, \partial_{s_n} \sum_{\overline{Y}} \int_{[0,1]^{n-1}} ds\ \int d\mu'_{ss_n}(\psi)\ e^{E(\Lambda, ss_n)}\ \varkappa(\overline{Y},s)\ Q \tag{E.8}$$

The notations in (E.6) ... (E.8) are as follows.

(i) $d\mu'_{ss_n}(\psi)$ is the Gaussian measure with the covariance given by (E.2), (E.3) except that $s_n \in [0,1]$ instead of being set to zero.

(ii) $ds = ds_1 \dots ds_{n-1}$

(iii) The summation over X is over all sets of the form $\bigcup_{i=1}^{m} Y_i$, $m \leq n$, with mutually disjoint Y_i's . Each Y_i $(i \geq 2)$ is a union of elements in $\tilde{Y}$ and $Y_1 = X_1$. $X_1 \subset \Lambda$ is a fixed union of unit cubes which is chosen before one starts with the cluster expansion.

(iv) In (E.7) $\overline{Y}$ is summed over all sequences $(Y_1 \dots Y_n)$ such that $X = \bigcup_{i=1}^{n} Y_i$. Here n is arbitrary and not necessarily the same as in $R_n(X)$. Summation over $\overline{Y}$ in (E.8) is over all sequences $(Y_1 \dots Y_n)$ such that $X = \bigcup_{i=1}^{n} Y_i$. If there is no such sequence then $R_n(X) = 0$.

(v) $E(\Lambda, ss_n)$ is defined as above with sequence $(Y_1,\dots,Y_n,X^c)$ and s-variables $s_1,\dots,s_{n-1}, s_n$.

(vi) It remains to define $\varkappa(\overline{Y},s)$:

$$\varkappa(\overline{Y},s) = \prod_{i=1}^{n-1} \varkappa(i) \tag{E.9}$$

where

$$\varkappa(i) = \partial_{s_i} E^{(i)}(X,s) + \int_{x \in Y_{i+1}} \int_{y \in \cup_{j \leq i} Y_j} \partial_{s_i} C(xy|s) \left[\frac{\delta}{\delta\psi(x)} + \frac{\delta E(X,s)}{\delta\psi(x)}\right]^{(i)} \cdot \left[\frac{\delta}{\delta\psi(y)} + \frac{\delta E(X,s)}{\delta\psi(y)}\right]^{(i)} \tag{E.10}$$

The superscripts (i) in (E.10) have the following meaning.

We expand the product of the two squares brackets and obtain a sum of five terms where the first one does not include a functional derivative or a propagator $C(s)$. We insert the decomposition of $E(X,s)$ given by (E.5a,b,c,d) and below. Then we drop all terms in this decomposition (with support in $\mathcal{O} \subset \Lambda$, $\mathcal{O}$ is a union of a_i's) which violate the following condition:

$$\sigma - \bigcup_{j \le i} Y_j \subset Y_{i+1} \text{ and } Y_{i+1} \text{ is the smallest union of sets} \quad \text{(E.11)}$$

from $\tilde{Y}$ that contains $\sigma - \bigcup_{j \le i} Y_j$

In particular for the term without $E(X,s)$ in (E.10) this means that Y_{i+1} has to be an element of $\tilde{Y}$: $Y_{i+1} \in \tilde{Y}$.

By construction $\varkappa(i)$ depends only on $s_1 \dots s_i$ and is independent of $s_{i+1} \dots s_n$.

<u>proof:</u>

Lemma E.1 is proved by induction in n [compare with [1]].

(i) <u>n=1</u>: Eq. (E.6) reads

$$\int d\mu(\psi)\, e^{E} Q = \int d\mu(\psi)\, e^{E(X_1)} Q(X_1) \cdot \int d\mu(\psi)\, e^{E(X_1^c)} Q(X_1^c) + \\ + \int_0^1 ds_1\, \partial_{s_1} \int d\mu'_{s_1}(\psi)\, e^{E(\Lambda,s_1)} Q \quad \text{(E.12)}$$

This equation is indeed true because $d\mu'_{s_1=1} = d\mu(\psi)$, $d\mu'_{s_1=0}$ factors between X_1 and X_1^c and $E(\Lambda, s_1=1) = E$, $E(\Lambda, s_1=0) = E(X_1) + E(X_1^c)$

(ii) <u>n → n+1</u>: Our strategy is to perform the derivative ∂_{s_n} in $R_n(X)$, eq. (E.8). $\varkappa(\bar{Y},s)$ does not depend on s_n and so the Leibniz rule gives

$$\partial_{s_n} \int d\mu'_{ss_n}\, e^{E(\Lambda,ss_n)} (\cdot) = \int d\mu'_{ss_n}\, e^{E(\Lambda,ss_n)} \partial_{s_n} E(\Lambda,ss_n)\, (\cdot) + \\ + \left(\partial s_n \int d\mu'_{ss_n}\right) e^{E(\Lambda,ss_n)} (\cdot) \quad \text{(E.13)}$$

By partial integration we see that

$$\left(\partial_{s_n} \int d\mu'_{ss_n}\right)(\cdot) = \int d\mu'_{ss_n} \tfrac{1}{2} \int_x \int_y \partial_{s_n} C(xy|ss_n) \frac{\delta}{\delta\psi(x)} \frac{\delta}{\delta\psi(y)} \quad (\cdot) \quad \text{(E.14)}$$

We insert (E.14) in (E.13) and obtain

$$\partial_{s_n} \int d\mu'_{ss_n}\, e^{E(\Lambda,ss_n)} (\cdot) = \int d\mu'_{ss_n}\, e^{E(\Lambda,ss_n)}\, \tilde{\varkappa}(n)\; (\cdot) \quad \text{(E.15)}$$

where

$$\tilde{\varkappa}(n) = \partial_{s_n} E(\Lambda,ss_n) + \tfrac{1}{2} \int_x \int_y \partial_{s_n} C(xy|ss_n) \left[\frac{\delta}{\delta\psi(x)} + \frac{\delta E(\Lambda,ss_n)}{\delta\psi(x)}\right] \cdot \\ \cdot \left[\frac{\delta}{\delta\psi(y)} + \frac{\delta E(\Lambda,ss_n)}{\delta\psi(y)}\right] \quad \text{(E.16)}$$

Next we use the fundamental theorem of calculus

$$d\mu'_{ss_n}\, e^{E(\Lambda,ss_n)} = d\mu'_{ss_n s_{n+1}}\, e^{E(\Lambda,ss_n s_{n+1})}\Big|_{s_{n+1}=0} + \\ + \int_0^1 ds_{n+1}\, \partial_{s_{n+1}} \left[d\mu'_{ss_n s_{n+1}}\, e^{E(\Lambda,ss_n s_{n+1})} \right] \quad \text{(E.17)}$$

to introduce a new s-variable s_{n+1}.

The set $Y_{n+1} \subset \Lambda - \bigcup_{i=1}^{n} Y_i$ (appearing in the definition of $E(\Lambda, ss_n s_{n+1}$ and $d\mu'_{ss_n s_{n+1}}$) has to be chosen conveniently. The choice depends on the term we get after expanding the product of the two brackets in (E.16).

This expansion produces a representation of $\tilde{\varkappa}(n)$ as a sum of five terms. For the term without any functional derivative we let Y_{n+1} be an arbitrary union of elements in $\tilde{Y}$ outside of $\bigcup_{i=1}^{n} Y_i$.

We collect all terms in $E(\Lambda, ss_n)$ under the decomposition (E.5) which satisfy the condition (E.11) for i = n. Obviously

$$\partial_{s_n} E(\Lambda,ss_n) = \sum_{Y_{n+1}} \partial_{s_n} E^{(n)}\left(\bigcup_{i=1}^{n+1} Y_i\, ,\, ss_n\right) \quad \text{(E.18)}$$

For the other four terms the Y_{n+1} are generated by expanding the integrals over x and y in a sum of integrals over finite regions. The factor $\frac{1}{2}$ in (E.16) cancels if we exploit the symmetry of the integrand in x and y. The region of integration outside Λ gives no contribution because the functional derivatives act on functions which are supported in Λ.

We illustrate the mechanism for the second term

$$\frac{1}{2}\int_X\int_Y \partial_{s_n} C(xy|ss_n)\,\frac{\delta}{\delta\psi(x)}\,\frac{\delta}{\delta\psi(y)}\,(\cdot) = \sum_{Y_{n+1}} \int_{x\in Y_{n+1}} \int_{y\in \bigcup_{i\le n} Y_i} \partial_{s_n} C(xy|ss_n)\,\frac{\delta}{\delta\psi(x)}\,\frac{\delta}{\delta\psi(y)}\,(\cdot) \tag{E.19}$$

where $Y_{n+1}\in\tilde{Y}$, $Y_{n+1}\cap \bigcup_{i\le n} Y_i = \emptyset$

In conclusion we have shown that

$$\tilde{\mathcal{K}}(n) = \sum_{Y_{n+1}} \varkappa(n) \tag{E.20}$$

where $\varkappa(n)$ is given by (E.10) for i = n.

By construction we have

(1) $$d\mu'_{ss_n s_{n+1}}\Big\lceil_{s_{n+1}=0} \text{ factors between } \bigcup_{i=1}^{n+1} Y_i \text{ and } \sim\bigcup_{i=1}^{n+1} Y_i \tag{E.21a}$$

(2) $$E(\Lambda, ss_n s_{n+1})\Big\lceil_{s_{n+1}=0} = E\Big(\bigcup_{i=1}^{n+1} Y_i, ss_n\Big) + E\Big(\Lambda-\bigcup_{i=1}^{n+1} Y_i\Big) \tag{E.21b}$$

(3) $$Q = Q\Big(\bigcup_{i=1}^{n+1} Y_i\Big)\cdot Q\Big(\Lambda-\bigcup_{i=1}^{n+1} Y_i\Big) \tag{E.21c}$$

We insert (E.20) and (E.17) in eq. (E.15) which is used to rewrite formula (E.8) for $R_n(X)$. The second term of (E.17) becomes $R_{n+1}(X)$ and the first term is just

$$\mathcal{K}(X)\int d\mu(\psi)\, e^{E(X^c)}\, Q(X^c) \quad \text{for} \quad X=\bigcup_{i=1}^{n+1} Y_i \tag{E.22}$$

The completes the induction step and the proof of lemma E.1. □

We notice that $R_n(X) = 0$ if n is sufficiently large (depending on Λ). This proves the final form of the cluster expansion given in

<u>Proposition E.2</u>.

$$\int d\mu(\psi)\, e^{E}\, Q = \sum_X \mathcal{K}(X)\int d\mu(\psi)\, e^{E(X^c)}\, Q(X^c) \tag{E.23}$$

with

$$\mathcal{K}(X) = \sum_{\bar{Y}} \int_{[0,1]^{n-1}} ds \int d\mu_s(\psi)\, e^{E(X,s)}\, \varkappa(\bar{Y},s)\, Q(X) \tag{E.24}$$

and $\varkappa(\bar{y},s)$ as given in (E.9), (E.10).

[$Q = e^{G} e^{-F_1} e^{-F_2}$]

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