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SOME TOPICS IN QUANTUM FIELD THEORY

by

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1. Introduction

Quantum field theory (QFT) I would here like to understand as four-dimensional (most-ly, Euclidean) continuum theory. Unfortunately, concerning this there are so far only negative rigorous results:

a) In ϕ^4_4, $0 \leq g_{ren} < \mathrm{const}$ [1].

b) An important recent result: In lattice approximation to ϕ^4_4, ϕ cannot have, in the massless case, an anomalous infrared (IR) dimension (i.e., $\eta = 0$), and in the conti- nuum limit, ϕ cannot have an anomalous ultraviolet (UV) dimension (which would also be given by η) [2]. This result explains the failure of all attempts so far to obtain a nonzero η for ϕ^4_4 by expansions, e.g. analytically: ε-expansion in $4-\varepsilon$ dimensions, 1/N-expansion in $(\vec{\phi}^2)^2_4$ theory, and numerically: strong-coupling expansion in ϕ^4_4 lattice theory [3]. - The strong-coupling-expansion result [4] that $g_{ren} \searrow 0$ as $\xi/a \nearrow \infty$ (ξ = correlation length, a = lattice constant) still remains to be proven rigorously.

Obviously, lattice regularization is the most natural starting point for constructing and analyzing continuum theories. Therefore, after a few general remarks on lattice theory (sect. 2), I shall describe the relation of lattice to continuum theory (sect. 3) on the basis of perturbation theory, and deduce herefrom the principles of con- structing "improved" lattice actions (sect. 4). Then I shall briefly describe some recent perturbative and nonperturbative results in continuum theory (sect. 5). - Finally, I shall point out a few recent approaches of more speculative nature that appear to merit particular attention (sect. 6). In the appendix, a few standard formulae from renormalization group analysis are collected for reference.

2. Remarks on lattice theory

For the continuum theory, lattice regularization is valuable mainly due to the follow- ing:

a) Strong-coupling expansions become possible. However, the needed extrapolation to infinite expansion parameter, as $a \searrow 0$, encounters difficulties (e.g., in gauge theory, due to the roughening transition [5] or other irregularities such as a peak in the specific heat in the SU(2) theory [6] caused, presumably, by a nearby critical point in a several-parameter phase diagram [7]).

b) Lattice approximation is the source of many concepts (see, e.g., ref. [8]) that may (eventually be proven to) be relevant also for qualitative features of the con- tinuum theory, e.g. vortices, condensation, defects, domain walls. An important new

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entry here is the topological charge Q_t recently constructed by M. Lüscher [9].

c) The lattice allows computer simulation by Monte-Carlo integration [10]. Hereby, phase diagrams can be explored and, in "asymptotically free" (AF, see appendix) theories, the weak-coupling limit, decisive for the continuum theory, can be extra- polated to. E.g., the topological susceptibility [9] should (except in particular cases, see ref. [11]) obey

$$\chi_t := \langle Q_t^2 \rangle / \text{4-volume} = \xi^{-4} \cdot \mathrm{const}, \tag{2.1}$$

with the constant here to be determined e.g. by computer simulation.

At this point we recall that continuum theories arise at (more generally, in the in- finitesimal neighbourhood of) phase transitions of the second kind of lattice systems, and they may have a direct Lagrangian description that is not obvious from the lattice Lagrangian. E.g., the continuum theory underlying the Ising_2 model at the critical point is the free Majorana theory [12]. The example of a free field theory at a scaling limit of $U(1)_3$ lattice gauge theory is reported by Mack [13]. On the basis of mean-field calculations, E. Brézin and J.M. Drouffe [14] argue for a critical point in the interior of the phase diagram of a Z(2) lattice gauge model.

Rigorous results, however, or even reliable approximative ones are very scarce, and thus at present physicists set their hopes on Monte-Carlo treatment of lattice approximations to presumed realistic theories (e.g., QCD = $SU(3)_4$ gauge theory with fermions). So far, results have been consistent with and, with good will, even indi- cative of AF (by comparing computer results with theoretical expectations, cp. appendix). While it would be of utmost impact if then AF should not ultimately be borne out where it should apply (cp. Brézin's report [15] on critical phenomena, where analogous renormalization group (RG) deductions always abode so far), the sur- prise is that AF should manifest itself on so small lattices as are computerizable so far. This calls for examining the shortcomings of lattice approximations:

a) Lack of rotation (and continuous translation) invariance. E.g., for the Gaussian model in D dimensions with mass m, the correlation length in direction $\vec{n}$, $|\vec{n}| = 1$, is

$$\xi_{\vec{n}} = m^{-1}\Big[1 + \frac{1}{24}(am)^2 \sum_{\mu=1}^{D} n_\mu^4 + O((am)^4)\Big]. \tag{2.2}$$

The corrections here are consequences of "irrelevant" terms in the corresponding local effective action (LEL) to be described later.

b) Deviations from AF formulae (see appendix). E.g., in the O(N) vector_2 model with action

$$L = \beta \sum_j \sum_{\mu=1}^{2} \vec{\sigma}_j \cdot \vec{\sigma}_{j+\hat{\mu}} \quad , \quad \beta^{-1} =: g_B \tag{2.3}$$

($j \in \mathbb{Z}^2$, $\vec{\sigma} \in S^{N-1}$, $\hat{\mu}$ = unit vector in the positive μ-direction) the deviations can be given in 1/N expansion at fixed $g_B N$. Simplest is the spherical model, i.e. the $N \nearrow \infty$ limit. Then [16] (cp. (2.2))

$$2\pi (g_B N)^{-1} = [1+\tfrac{1}{4}a^2m^2]^{-1} K([1+\tfrac{1}{4}a^2m^2]^{-1})$$

which yields (cp. (A.5) and remarks there)

$$m = 4\sqrt{2}a^{-1} \exp[-2\pi(g_B N)^{-1}]\{1+[-8\pi(g_B N)^{-1}+2]\exp[-4\pi(g_B N)^{-1}] + O(\exp[-8\pi(g_B N)^{-1}])\} \tag{2.4}$$

such that the "β-function for the correlation length" m^{-1}

$$\begin{aligned}\bar\beta(g_B N) &:= -m\, \partial(g_B N)/\partial m\,|_a = \\ &= -\tfrac{1}{2\pi}(g_B N)^2 + [8(g_B N) + \tfrac{4}{\pi}(g_B N)^2]\exp[-4\pi(g_B N)^{-1}] + O(\exp[-8\pi(g_B N)^{-1}])\end{aligned} \tag{2.5}$$

also has "exponentially small" correction terms. In AF gauge theory the formula for the "string tension" (of mass dimension two) should have (by perturbation theory alone, not computable) corrections (cp. (A.5)) similarly as in (2.4), with different corrections than for e.g. the "glueball" mass, i.e. the lightest particle mass. (In this gauge theory case, one expects also corrections from entirely nonperturbative effects, like instantons and vortex formation.)

c) Finite-size effects on computerizable lattices. These have recently received attention in gauge-theory computations (e.g., ref. [17]), and should do more so in the future. The finite size introduces a new scale, which persists in the continuum limit but is unrelated to the microscopic renormalization procedure. (For the related theory of "finite-size scaling", see ref. [18].)

Points a) and b) here can be mitigated by computing with an improved lattice action. To explain this, I must give some details on the lattice-versus-continuum relation.

3. Local effective Lagrangian for lattice theory

For simplicity, consider [19] the ϕ^4_4 theory (sufficient for the present formal explications) with lattice action (for uses below, written in D dimensions)

$$L = a^D \sum_{j\in\mathbb{Z}^D} \Big[-\tfrac{1}{2}\sum_{\mu=1}^{D} \partial_\mu\phi_j\, \partial_\mu\phi_j - \tfrac{1}{2}\Delta m_B^2 \phi_j^2 - \tfrac{1}{4!} g_B \phi_j^4 - \tfrac{1}{2} m_{B0}^2 \phi_j^2\Big] \tag{3.1}$$

where $\partial_\mu \phi_j = a^{-1}(\phi_{j+\hat\mu} - \phi_j)$.

$m_{Bo}^2 = a^{-2} f(g_B a^{-D+4})$ is the bare-mass-squared of the massless (i.e., critical) theory. The full-propagator-amputated one-particle-irreducible connected parts of the correlation functions are called vertex functions Γ_B (subscript for "bare"). Their Fourier transforms (on the infinite lattice, momenta in the first Brillouin zone), with the lattice-translation delta-function taken out, have in perturbation theory the small-a expansion (for D = 4)

$$\begin{aligned}\Gamma_B(p_1 \cdots p_{2n}; a, g_B, \Delta m_B^2) &= \\ &= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a^{2j}(\ln a)^k \bar f_{jk}(p_1 \cdots p_{2n}; g_B, \Delta m_B^2).\end{aligned} \tag{3.2}$$

Renormalization absorbs the j = 0, k ≥ 1 terms herein: With $g_B(g, a\mu)$ from (A.4), there are series $Z_3(g, a\mu)$ and $Z_2(g, a\mu)$ such that

$$\begin{aligned}Z_3(g,a\mu)^n\, \Gamma_B(p_1 \cdots p_{2n}; a, g_B(g,a\mu), Z_2^{-1}(g,a\mu)m^2) &= \\ = \Gamma(p_1 \cdots p_{2n}; \mu, g, m^2) &+ \\ + \sum_{j=1}^{\infty}\sum_{k=0}^{\infty} a^{2j}(\ln a)^k \tilde f_{jk}(p_1 \cdots p_{2n}; g_B(g,a\mu), Z_2^{-1}(g,a\mu)m^2)\end{aligned} \tag{3.3}$$

and the aim of action improvement is to remove the $O(a^2 \ln a^{\cdot\cdot})$, $O(a^4 \ln a^{\cdot\cdot})$ etc. terms on the r.h.s.. This is possible due to their remarkable structure: They can be generated from a local effective Lagrangian (LEL) [19]. Writing down such Lagrangian requires to adopt an interpretation rule for such Lagrangians. By far the most convenient rule is dimensional regularization, i.e. to consider the lattice and continuum theory in $4+\varepsilon$ dimensions, $\mathrm{Re}\,\varepsilon > 0$ generic. ($4+\varepsilon$-dimensional lattice theory poses in perturbation theory no problems, i.e. none more than the corresponding continuum theory which is standard nowadays [20].) Then the LEL density for the $4+\varepsilon$-dimensional version of (3.2) (where $a^{2j-\varepsilon k}$ appears in place of $a^{2j}(\ln a)^k$) is

$$\mathcal{L}_{eff} = -\frac{1}{2}\bar{Z}_3 \sum_\mu \partial_\mu\phi\,\partial_\mu\phi - \frac{1}{2}\bar{Z}_2\,\Delta m_B^2\,\phi^2$$
$$-\frac{1}{4!}\bar{Z}_1\, g_B\,\phi^4 + \frac{1}{24}a^2\bar{Z}_4 \sum_\mu \phi\,\partial_\mu^4\phi +$$
$$+\frac{1}{2}a^2\bar{Z}_5\Big(\sum_\mu\partial_\mu^2\phi\Big)^2 + \frac{1}{3!}a^2\bar{Z}_6\, g_B\,\phi^3\sum_\mu\partial_\mu^2\phi +$$
$$+\frac{1}{6!}a^2\bar{Z}_7\, g_B^2\,\phi^6 - \frac{1}{2}a^2\Delta m_B^2\bar{Z}_8\sum_\mu\partial_\mu\phi\,\partial_\mu\phi -$$
$$-\frac{1}{4!}a^2\Delta m_B^2\bar{Z}_9\, g_B\,\phi^4 - \frac{1}{2}a^2\Delta m_B^4\bar{Z}_{10}\,\phi^2 +$$
$$+\ a^4\text{-terms} + a^6\text{-terms} + \cdots \tag{3.4}$$

Here

$$\left.\begin{matrix}\bar{Z}_{1,2,3,4}-1\\ \bar{Z}_{5,6,7,8,9,10}\end{matrix}\right\}(g_B a^{-\varepsilon},\varepsilon) = \sum_{\mathcal{L}=1}^{\infty} f_{1,\dots 10}(\varepsilon)(g_B a^{-\varepsilon})^{\mathcal{L}} \tag{3.5}$$

where $\mathcal{L}$ is the number of lattice-graph loops entering the computation. (The $\bar{Z}_i$ can be expressed [19] simply in terms of vertex functions of the $\Delta m_B^2 = 0$ lattice theory at zero momenta and derivatives there.)

Remarks: 1) Since the $\varepsilon \to 0$ vertex functions exist due to $a > 0$, the first three terms in (3.4) have the 't Hooft form [21] whereby $g_B a^{-\varepsilon} \equiv \bar{g}$ plays the role of renormalized coupling constant (though the renormalization convention implied here differs from 't Hooft's) and we may set $a^{-1} = \mu$, the normalization mass (cp. appendix). The beta-function of the unrenormalized theory ($\bar{\beta}(g_B)$ in (A.3-5)) is

$$\bar{\beta}(g_B) = \Big\{\varepsilon\big[\partial/\partial(g_B a^{-\varepsilon})\ \ln\big(g_B a^{-\varepsilon}\bar{Z}_1(g_B a^{-\varepsilon};\varepsilon)\cdot$$
$$\cdot\,\bar{Z}_3(g_B a^{-\varepsilon};\varepsilon)^{-2}\big)\big]^{-1}\Big\}\Big|_{\varepsilon\to 0}$$

2) In $\mathcal{L}_{eff}$, there are as many terms with factor a^{2j} as there are linearly independent local monomials (modulo total derivatives) of dimension $4+2j$. The $j \geq 1$ terms generate, treated as insertions in ordinary ($j = 0$) graphs with dimensional integration rules [20], upon letting $\varepsilon \to 0$ the additional terms on the r.h.s. of (3.3).

3) Δm_B^2 appears in the LEL, like a coupling constant, only in positive integer powers. (However, Δm_B^2 resp. m^2 need be used, of course, in the denominators of the propagators.)

4) The first three terms in (3.4), i.e. the "marginal" (dim. 4) and "relevant" (dim. 2) ones alone, determine the symmetries of the continuum theory, which leads to symmetry enhancement of that theory: it is rotation invariant as a consequence of the lattice theory being invariant under the discrete rotations alone. - Note, however, that nontrivially scale-(and thus also conformally) invariant theories, as they underlie universality [15], do not have a Lagrangian description but must be described indirectly (e.g., ref. [22]). This allows intricate cases of internal symmetry enhancement [23].

5) An LEL exists also for lattice spacing unequal in different directions and, in particular, for (Euclidean) time continuous. (However, the LEL then is not directly related to an "effective local Hamiltonian". Cp. also sect. 5.)

6) The existence of an LEL to describe large-cutoff behaviour is not trivial. In ref. [24] it was proven for Pauli-Villars regularization (the local effective light-particle Lagrangians incorporating virtual heavy-particle effects [25] are related hereto), and the method used there is extendible [26] to the lattice case. An LEL does not exist for sharp cutoff, in view of

$$\int_{\substack{|k|<\Lambda\\ |k+p|<\Lambda}} d^4k\ |k|^{-2}|k+p|^{-2} = 2\pi^2\ln(\Lambda|p|^{-1}) + \pi^2 - \frac{4\pi}{3}|p|\Lambda^{-1} - \frac{1}{90\pi}|p|^3\Lambda^{-3} + O(|p|^5\Lambda^{-5}) \tag{3.6}$$

and ($c_{1,2}$ are nonnamed constants)

$$\int_{|k|<\Lambda} d^4k \int_{\substack{|l|<\Lambda\\ |k+p+l|<\Lambda}} d^4l\ |k|^{-2}|l|^{-2}|k+l+p|^{-2} = c_1\Lambda^2 +$$
$$+\big(c_2 + 2\pi^4\ln(\Lambda|p|^{-1})\big)p^2 + \frac{16\pi^3}{15}|p|^3\Lambda^{-1} + O\big(|p|^{\frac{7}{4}}\Lambda^{-\frac{3}{4}}\big).$$

The difference to the lattice cutoff is that the latter figures in the denominator of the (momentum space) lattice propagator; the limitation to the Brillouin zone is not harmful due to the periodicity in momentum space of all lattice functions and effects only "finite renormalizations" (cp. ref. [19]).

4. Improved lattice action [26]

The $O(a^2)$ terms in (3.4) and, consequently, the $O(a^2\ln a^{\cdot\cdot})$ terms in (3.3) can be removed by adding (four if massless, three additional ones if massive) "irrelevant" terms to the original nearest-neighbour-coupling lattice action (3.1) with suitable, at $\varepsilon = 0$ finite coefficients. These terms can be chosen identical in form with the $O(a^2)$ irrelevant continuum terms, with ∂_μ replaced by the lattice difference quotient and Δm_B^2 again appearing only polynomially. Corresponding statements hold for $O(a^{2j})$ terms, $j \geq 2$.

Remarks: 1) The relation between the coefficients of irrelevant terms in the LEL and on the lattice is nonlinear: The insertion of two $a^2 \dim 6$ vertices in lattice graphs generates, upon summing over the lattice, via a short-distance expansion [27] with lattice cutoff new such operators, an effect absent for continuum graphs under dimensional integration rule.

2) Determination of the correct coefficients of irrelevant terms on the lattice is straightforward in perturbation theory. Outside of perturbation theory it can be effected e.g. by "trial and error" (which also ensures that, with the correct coefficients, the improved lattice action is bounded above as is the unimproved one). This inelegant procedure can be replaced by other ones [26] exploiting the IR properties of the massless theory in four dimensions [19]. - The lowest-order terms (zeroth order in g_B) in (3.4) leading to (2.2),

$$\frac{1}{4!} a^2 \sum_\mu \partial_\mu^2 \phi \, \partial_\mu^2 \phi - \frac{1}{6!} a^4 \sum_\mu \partial_\mu^3 \phi \, \partial_\mu^3 \phi + - \cdots$$

are removed by using in the lattice action the "SLAC Laplacian" [28], yielding the series of negative terms

$$a^{4+\varepsilon} \sum_j \sum_\mu \left(-\frac{1}{24} a^2 \partial_\mu^2 \phi_j \, \partial_\mu^2 \phi_j - \frac{1}{180} a^4 \partial_\mu^3 \phi_j \, \partial_\mu^3 \phi_j - \cdots \right)$$

3) Irrelevant terms on the lattice lead in general to violation of Osterwalder-Schrader (OS) positivity [29] due to presence of non-nearest-neighbor interactions. Such a violation should not be more harmful than the related one in continuum theory by Pauli-Villars regularization, which is an unobjectionable regularization in Euclidean perturbation theory and probably beyond [30].

4) The present improvement prescription is related to Wilson's [31] but is more systematic since it eliminates $O(a^2 \ln a^{\cdot\cdot})$ corrections in all quantities. (This means, presumably, removal of the second term on the r.h.s. of (A.5) as far as it stems from irrelevant LEL terms, as it does in (2.4)). The price to pay is the sacrifice of OS positivity.

5) The LEL allows one version of "integrating out degrees of freedom": One changes the LEL parametrization in (3.4) from lattice constant a to e.g. 2a, which implies g_B change according to (A.4) and ϕ-normalization change. Then one determines those irrelevant terms on the 2a-lattice that would altogether yield an unchanged LEL. Since hereby, due to the definition of the LEL, the Fourier transforms of the correlation functions remain unchanged in the 2a-Brillouin zone, the relation between the new

field variables (i.e., on the 2a-lattice) and the old ones (i.e., on the a-lattice) is linear but nonlocal.

The procedures described here for $\phi^4_{4+\varepsilon}$ theory are adaptable without difficulty to the O(N) vector$_{2+\varepsilon}$ model [26] where David's result [32] on IR convergence of O(N) invariant quantities renders the coefficients in the improved lattice action IR finite also at $\varepsilon = 0$. Comparison with the 1/N expansion is here an interesting check transgressing perturbation theory. - More important is the application of these methods to lattice gauge theory. With original action e.g. the Wilson one, the improved lattice action will also contain traces of products of links bordering several adjacent plaquettes as in Wilson's improved action [31]. Again, the lowest-order contributions to all coefficients of such traces are merely kinematical.

5. Other recent results

To all orders in perturbation theory, the Schrödinger representation exists in renormalizable theories [33]. Hereby, the field operator that is diagonalizable on a smooth three-dimensional hypersurface differs from the renormalized (and from the unrenormalized) one by a factor that diverges logarithmically if the cutoff is removed. As cutoff serves the (inverse) distance from the surface, and there is a close relationship to the short-distance analysis of operator products [27]. The need for this cutoff requires a limiting procedure to be employed if matrix elements of renormalized field operators are to be computed in the Schrödinger representation.

The diagonalization of Fermion fields leads to Grassmann variables. Their treatment in Volterra expansions leads to antisymmetric functions with properties otherwise analogous to the ones of the expansion functions in the Bose case. The noteworthy point is that (homogeneous and inhomogeneous) Dirichlet boundary conditions do not, in renormalizable theories, require a new renormalization parameter to be introduced. - In AF theories, the "precise" relation between the renormalized and the diagonalized field operator could be determined.

There is also a Schrödinger equation that involves point splitting and a limit as already needed in the free-field case. The new feature with interaction is that in the removal of the point splitting, again logarithmically divergent factors must be applied where otherwise none are needed.

To the half-space problem for renormalizable theories are closely related the properties of semi-infinite ferromagnets at the critical point [34].

A new nonperturbative result of, however, "conventional" nature is the rigorous construction of Pauli-Villars regularized Euclidean Yang-Mills theory in a space-"time" box by M. Asorey and P.K. Mitter [35], who proved the existence of the relevant gauge-covariant diffusion process. The regularization is hereby so chosen that OS positivity will not be impeded if the "time"-infinity limit is taken. Cutoff removal in the formulation, rather than in the final Green's functions, would be equivalent to setting up the Schrödinger representation and -equation as just discussed.

A recent less conventional topic that seems worth mentioning is G. Münster's treatment [36] of the dense instanton gas. Here the IR divergence due to the dilation zero-mode has been a long-standing problem. Münster estimates the (convergent) collective-coordinate integrations in a finite Euclidean box and postulates, after summing over all winding numbers, the existence of the thermodynamic limit. A consequence of this procedure is a correction factor to the one-instanton action as it appears in the approximation to the free energy, leading to formulae that differ significantly from those of the "dilute-gas approximation" that has been used almost exclusively so far. In particular, a previous paradox in the CP_2^{N-1} model is hereby explained. However, so far only the pure instanton gas is taken into account, and more detailed calculations to support the estimates employed are desirable.

6. Unconventional approaches

Finally, I would like to draw attention to recent, in part still speculative, work of A.M. Polyakov [37] and of A.A. Migdal [38]. Guided by the (widespread) conviction that local QFT (e.g., QCD), even if "correct", is too far removed from the objects (hadrons) observed to be useful for the description of their "IR properties" (relative to the UV ones where AF should apply) both authors look for better suited mathematical models. These are "string" models that become surface models in Euclidean four-space. Polyakov [37] defines an integration over (so far, closed) two-surfaces and is hereby led to the quantum Liouville equation in two-space, possibly a giant step relative to "old" string theory [39]. Migdal [38] proposes a solution (in terms of a Fermion field on a two-surface) for a functional differential equation for the vacuum expectation value of a Wilson loop in SU(∞) gauge theory proposed earlier by him and Make'enko [40]. While this approach is highly suggestive, the mathematics as well as the connection with local gauge theory are not clear. In particular, R.A. Brandt, F. Neri, and M. Sato [41] have recently analyzed, to all perturbation orders, the renormalization of Wilson loops with corners and self crossings, and have been unable to relate their results, which should be a constraint also on nonperturbative approaches, to the Migdal-Make'enko equation.

Appendix: Some renormalization group formulae

For details to the following, see e.g. ref. [42]. The case of several coupling constants is dealt with in ref. [43].

In a renormalizable massless theory (i.e. one where the formal continuum limit Lagrangian has only dimensionless parameters; we use, for simplicity, the notation appropriate for massless ϕ_4^4 theory) the (Fourier transforms of the) renormalized Green's functions obey

$$\text{(A.1)}\quad \{\mu[\partial/\partial\mu] + \beta(g)[\partial/\partial g] + n\gamma(g)\}\cdot G(p_1\dots p_n;\mu,g) = 0$$

(g = renormalized coupling constant, μ = normalization (e.g., subtraction) momentum). $\beta(g)$ and $\gamma(g)$ depend on the renormalization convention. In the expansion

$$\text{(A.2)}\quad \beta(g) = b_0 g^2 + b_1 g^3 + \dots$$

b_0 and b_1 are, however, convention independent and so is for $\gamma(g)$ the coefficient in its first term. It is $c_0 g^2$ for ϕ_4^4. For nonabelian gauge theory (NAGT), e.g. QCD with massless quarks, it is $c_0 g$ where $g = g_G^2$, g_G the usual gauge coupling constant, and is gauge dependent.

The unrenormalized Green's functions that depend on the cutoff Λ (e.g., $= a^{-1}$, a the lattice constant) and the "bare" coupling constant g_B obey

$$\text{(A.3)}\quad \{\Lambda[\partial/\partial\Lambda] + \bar\beta(g_B)[\partial/\partial g_B] + n\bar\gamma(g_B)\}\cdot G_B(p_1\dots p_n;\Lambda,g_B) = O((\ln\Lambda)^{-\infty})$$

The r.h.s. is actually $O(\Lambda^{-2}\ln\Lambda^{\cdot\cdot})$ for lattice regularization (see sect. 4). $\bar\beta(g_B)$ and $\bar\gamma(g_B)$ depend (as do the G_B) on the manner of regularization. The first two coefficients of $\bar\beta(g_B)$ and the first coefficient of $\bar\gamma(g_B)$ are the same as for $\beta(g)$ (see (A.2)) resp. $\gamma(g)$. g_B obeys

$$\text{(A.4)}\quad \int^{g_B} dg_B'\,\bar\beta(g_B')^{-1} - \ln\Lambda = \int^{g} dg'\,\beta(g')^{-1} - \ln\mu$$

where, for definiteness, we here understand $\bar\beta(g_B)$ as defined by its weak-coupling series (see end of this appendix).

A "physical" quantity $P(\Lambda,g_B)$ (see, e.g., ref. [44]) has, if computed with $g_B(\mu\Lambda^{-1},g)$ from (A.4) and cutoff Λ, at fixed g,μ a $\Lambda\nearrow\infty$ limit and thus obeys (A.3) with $n = 0$. If then the r.h.s. is $O(\Lambda^{-2}\ln\Lambda^{\cdot\cdot})$ as for lattice regularization,

$$\text{(A.5)}\quad P(\Lambda,g_B) = \left[\Lambda\exp\left\{-\int_{c_P}^{g_B} dg_B'\,\bar\beta(g_B')^{-1}\right\}\right]^{\dim P} + \Lambda^{-2}F_P(g_B)\left[\Lambda\exp\left\{-\int_{c_P}^{g_B} dg_B'\,\bar\beta(g_B')^{-1}\right\}\right]^{\dim P+2} + \dots$$

follows, where the constant c_P and $F_P(g_B)$, meant to be at most mildly singular, are not computable in perturbation theory. A physical quantity of dimension -1 is the correlation length (cp. (2.1)); an example of (A.5) is equ. (2.4). In the interesting case of "asymptotic freedom", i.e. $b_o < 0$ (as in NAGT with not too many fermions [45], from (A.4) as $\Lambda \nearrow \infty$

$$g_B = [-b_0 \ln \Lambda]^{-1} + O((\ln \Lambda)^{-2} \ln \ln \Lambda) \searrow 0$$

such that the correction terms in (A.5) are "exponentially small" compared to the leading term. c_p and these correction terms are different for different physical quantities such that there is no universal β-function outside the weak-coupling expansion.

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