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NOTE ON EXTREMES OF SU(n) HIGGS POTENTIALS

by

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Abstract

The problem of symmetry breaking of SU(n) is discussed more completely with Higgs scalar fields belonging to an adjoint representation or an adjoint together with a fundamental one.

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Recently, the symmetry breaking pattern for SU(n) has been discussed by Buccella et al [1] and Ruegg [2] for the most general form of the renormalizable Higgs potential with Higgs fields belonging to an adjoint representation or an adjoint together with a fundamental one. They first prove a lemma and then apply it to get the symmetry breaking patterns. But their work needs some supplements and also a few amendments. It is the purpose of this note to give these. The symbols used here are the same as those in Ref. 2, unless otherwise stated.

I. On the proof of the lemma

The statement of the lemma will not be repeated, only some supplements to the proof given in Ref. 2) are proposed.

For given φ and σ, a_1 and a_2 will vary within an ellipse

$$3(a_1^2+a_2^2)+2a_1a_2-2\sigma(a_1+a_2)+\sigma^2-2\varphi=0,$$

owing to the requirement that a_3 and a_4 must be real. Therefore when one looks for the extrema (including the absolute extremum), one must investigate not only over the interior region of the ellipse, but also over its boundary, which can be parametrized as

$$a_1=\frac{1}{2}A\cos\theta+\frac{1}{4}\sigma-\frac{1}{\sqrt{2}}A\sin\theta,$$
$$a_2=\frac{1}{2}A\cos\theta+\frac{1}{4}\sigma+\frac{1}{\sqrt{2}}A\cos\theta, \tag{1.1}$$

with

$$A=\sqrt{\varphi-\frac{1}{4}\sigma^2}.$$

The inequality

$$\varphi - \frac{1}{4}\sigma^2 \geq 0$$

holds so long as a_1, a_2, a_3 and a_4 are all real.

The value of F on the boundary turns out to be

$$F = -\frac{1}{2}\mu^2\varphi + \frac{1}{4}a\varphi^2 + \frac{1}{4}b\left(A^4 + \frac{3}{4}A^2\sigma^2 + \frac{1}{32}\sigma^4 + 3A^3\sigma\cos\theta + A^4\cos^2\theta - 3A^3\sigma\cos^3\theta - \frac{3}{2}A^4\cos^4\theta\right) + \frac{1}{2}d\left(\frac{3}{2}A^2\sigma + \frac{1}{8}\sigma^3 + 3A^3\cos\theta - 3A^3\cos^3\theta\right). \tag{1.2}$$

The solutions of the equation

$$\frac{dF}{d\theta} = 0 \tag{1.3}$$

are essentially of the following type /1/:

$$\begin{aligned}
&\mathrm{I.} \quad a_1 = a_2 = \frac{1}{2}A + \frac{1}{4}\sigma,\\
&\mathrm{II.} \quad a_1 = -\frac{1}{2\sqrt{3}}A + \frac{1}{4}\sigma, \quad a_2 = \frac{3}{2\sqrt{3}}A + \frac{1}{4}\sigma,\\
&\mathrm{III.} \quad a_1 = -\frac{3}{2\sqrt{3}}A + \frac{1}{4}\sigma, \quad a_2 = \frac{1}{2\sqrt{3}}A + \frac{1}{4}\sigma,\\
&\mathrm{IV.} \quad a_1 = -\frac{3}{2}\frac{d}{b} - \frac{1}{2}\sigma - A\sqrt{\frac{1}{2} - \frac{9}{2A^2}\left(\frac{d}{b} + \frac{1}{2}\sigma\right)^2}, \quad a_2 = -\frac{3}{2}\frac{d}{b} - \frac{1}{2}\sigma + A\sqrt{\frac{1}{2} - \frac{9}{2A^2}\left(\frac{d}{b} + \frac{1}{2}\sigma\right)^2}.
\end{aligned} \tag{1.4}$$

Among these, only solution I is substantially new, since the others all belong to the case for which the original proof can apply when viewed from a_3 and a_4 instead of a_1 and a_2. This is due to the fact that

$$3(a_3^2 + a_4^2) + 2a_3a_4 - 2\sigma(a_3 + a_4) + \sigma^2 - 2\varphi = -(a_1 - a_2)^2, \tag{1.5}$$

which means (a_3, a_4) lies inside its boundary ellipse unless $a_1 = a_2$. As for

/1/ The solution $a_1 = a_2 = -\frac{1}{2}A + \frac{1}{4}\sigma$ can be obtained from I by $a_1, a_2 \leftrightarrow a_3, a_4$.

the extra one (solution I), it is evident that it does not affect the validity of the lemma.

II. On the symmetry breaking due to the adjoint representation

As indicated in Ref. 2), I is a quadratic function with respect to χ :

$$I = \frac{b\rho^4}{n}\left[\frac{1}{2}(\chi^2 + 1) - \frac{d}{b\rho}\sqrt{n}\,\chi\right], \tag{2.1}$$

where χ is a monotonic increasing function of n_1, since

$$\frac{d\chi}{dn_1} = \frac{2}{\sqrt{n_1(n-n_1)}} + \frac{1}{2}\frac{(n-2n_1)^2}{[n_1(n-n_1)]^{3/2}} > 0 .$$

For the case $b < 0$, the minimum of I occurs at the maximum value of $|\chi|$, which corresponds to $n_1 = n-1$ or $n_1 = 1$, so that the symmetry breaking pattern can only be of the form [2)]

$$SU(n) \rightarrow SU(n-1) \times U(1).$$

For $b > 0$, it is convenient to regard χ as a continuous variable, and draw conclusions from the results so obtained. Doing this, one gets that I reaches its minimum at [2)]

$$\chi = \frac{d}{b\rho}\sqrt{n} . \tag{2.2}$$

But the relation of χ to the coefficients is not only through the factor d/b, it is also through the factor ρ, since ρ must take the value which minimizes the Higgs potential.

Substituting (2.2) into V, one gets

$$V=\frac{1}{2}\left(\frac{a}{\lambda}+\frac{b}{n}\right)\rho^4-\frac{1}{2}\left(\mu^2+\frac{d^2}{b}\right)\rho^2 . \tag{2.3}$$

The condition for V to develop a minimum at a non-zero value of ρ is

$$\frac{a}{\lambda}+\frac{b}{n}>0 , \qquad \mu^2+\frac{d^2}{b}>0 . \tag{2.4}$$

And the value of ρ^2 for the minimum will be

$$\rho^2=\frac{n(b\mu^2+d^2)}{b(na+2b)} , \tag{2.5}$$

therefore

$$\chi=\frac{d}{b}\sqrt{\frac{na+2b}{\mu^2+\frac{d^2}{b}}} . \tag{2.6}$$

It is this formula which relates χ , and hence the symmetry breaking pattern, to the coefficients. One easily sees that χ can take all the value from $-\frac{n-2}{\sqrt{n-1}}$ to $\frac{n-2}{\sqrt{n-1}}$ when the coefficients run through their permissible range constrained by (2.4), so that the remaining symmetry group can be any one among the series $SU(n-1) \times U(1)$, $SU(n-2) \times SU(2) \times U(1)$, ..., $SU(\frac{n}{2}) \times SU(\frac{n}{2}) \times U(1)$ or $SU(\frac{n+1}{2}) \times SU(\frac{n-1}{2}) \times U(1)$.

III. On the symmetry breaking due to adjoint and fundamental representations

The problem is to minimize

$$F=\frac{b}{2}(n_1a_1^4+n_2a_2^4+a_n^4)+d(n_1a_1^3+n_2a_2^3+a_n^3) + H^\dagger H\, a_n(\beta a_n+\gamma) , \qquad n_1+n_2=n-1. \tag{3.1}$$

under the condition

$$n_1a_1+n_2a_2+a_n=0, \quad n_1a_1^2+n_2a_2^2+a_n^2=\rho^2 , \tag{3.2}$$

and also under the constraints

$$a_n(\beta a_n+\gamma)\le a_1(\beta a_1+\gamma), \qquad a_n(\beta a_n+\gamma)\le a_2(\beta a_2+\gamma). \tag{3.3}$$

Solving (3.2) for a_1 and a_2, taking a_1 to be the larger one and replacing the independent variables n_1 and a_n with

$$x=\frac{n_1-n_2}{\sqrt{n_1n_2}} , \qquad y=\sqrt{\frac{n}{n-1}}\,\frac{a_n}{\rho} , \tag{3.4}$$

one gets for F

$$F=\frac{b\rho^4}{2n(n-1)}\Big[(n^2-3n+3)y^4+6y^2(1-y^2)+n(1+x^2)(1-y^2)^2 + 2\sqrt{n}\,x(1-y^2)^{3/2}\Big(2y-\frac{d}{b\rho}\sqrt{n(n-1)}\Big)+\frac{2d}{b\rho}\sqrt{n(n-1)}\big((n+1)y^3-3y\big)\Big] + H^\dagger H\sqrt{\frac{n-1}{n}}\rho y\Big(\beta\sqrt{\frac{n-1}{n}}\rho y+\gamma\Big). \tag{3.5}$$

For the original variables (n_1, a_n), the region of their independent variation can be taken as

$$-\sqrt{\frac{n-1}{n}}\,\rho\le a_n\le\sqrt{\frac{n-1}{n}} , \qquad 1\le n_1\le n-2 . \tag{3.6}$$

Here we exclude the value $n_1 = n-1$ /1/, with the understanding that when a_n reaches its extreme values $\pm\sqrt{\frac{n-1}{n}}\,\rho$, the symmetry breaking will be to SU(n-1) not to $SU(n_1) \times SU(n-n_1-1) \times U(1)$. This is due to the fact that once $a_n^{\,2}$ reaches its extreme value $\frac{n-1}{n}\rho^2$, a_1 will equal to a_2.

As for the new variables x,y, the region of variation is accordingly

$$- 1 \leq y \leq 1,$$
$$-\frac{n-3}{\sqrt{n-2}} \leq x \leq \frac{n-3}{\sqrt{n-2}} \qquad (3.7)$$

also with the understanding that once the y^2 reaches its extreme value 1, the symmetry breaking will be to SU(n-1).

We shall mainly consider the simpler case $d = \gamma = 0$, and discuss for $b < o$ and $b > o$ separately.

1) $b < o$

The minimum of F is obtained for the largest value of $|x|$:

$$x = \frac{n-3}{\sqrt{n-2}}\ , \quad \text{if } y > o \qquad (3.8)$$

$$x = -\frac{n-3}{\sqrt{n-2}}\ , \quad \text{if } y < o$$

Since now $d = \gamma = o$ and therefore F is invariant under the transformation $y \rightarrow -y$,

/1/ Because in this case, a_n must be $\sqrt{\frac{n-1}{n}}\,\rho$ or $-\sqrt{\frac{n-1}{n}}\,\rho$, so that n_1 and a_n are not independent. Furthermore F becomes independent of n_1 when $a_n^{\,2} = \frac{n-1}{n}\rho^2$.

one can always limit oneself to the region $y \geqslant o$, so that

$$F = \frac{b}{2n(n-1)}\rho^4\Big[(n^2-3n+3)y^4 + 6y^2(1-y^2) + \frac{n(n^2-5n+7)}{n-2}(1-y^2)^2 + 4\sqrt{\frac{n}{n-2}}\,(n-3)\,y\,(1-y^2)^{3/2}\Big] + H^{+}H\beta\,\frac{n-1}{n}\rho^2 y^2, \qquad 0 \leq y \leq 1. \qquad (3.9)$$

In case the second term in F is absent, the minima will be degenerate, with the location

$$y_1 = 1, \quad y_2 = \frac{1}{n-1}\,. \qquad (3.10)$$

The presence of the second term removes the degeneracy and also alters the value of y_2.

For $\beta < 0$, the second term makes the absolute minimum to occur at

$$y = 1$$

so that the symmetry will break to SU(n-1); and for $\beta > 0$, the second term makes the absolute minimum to occur at a value smaller than $\frac{1}{n-1}$, namely

$$y = \frac{1}{n-1} - \Delta\ , \quad \Delta > 0 \qquad (3.11)$$

so that the symmetry will break to SU(n-2)xU(1). It remains to show that the criterion (3.3) holds for these solutions. In the present case, it turns to be

$$a_n^{\,2} \geqslant a_1^{\,2}\ , \quad a_n^{\,2} \geqslant a_2^{\,2}\ , \quad \text{when } \beta < 0 \qquad (3.12)$$

$$a_n^{\,2} \leqslant a_1^{\,2}\ , \quad a_n^{\,2} \leqslant a_2^{\,2}\ , \quad \text{when } \beta > 0 \qquad (3.13)$$

(3.12) is evident. For the case $\beta > 0$, from (3.11) one gets

$$a_n^2 < \frac{1}{n(n-1)}\,\rho^2 \tag{3.14}$$

Making use of (3.14) and the expression for a_1

$$a_1 = \frac{1}{n-1}\left[-a_n + \sqrt{\frac{(n-1)\rho^2 - n a_n^2}{n-2}}\,\right],$$

one easily confirms

$$a_n < a_1.$$

Likewise one can show

$$a_2 < -a_n.$$

These prove the validity of (3.13).

2) $b > 0$

Treating χ as a continuous variable [1)-2)], the minimum for F is obtained at

$$\chi = -\frac{2y}{\sqrt{n(1-y^2)}}\,, \tag{3.15}$$

Substituting it into F, one gets

$$F = \frac{b}{2n(n-1)}\,\rho^4\left[(n-1)^2 y^4 - 2(n-1)y^2 + n\right] + \frac{n-1}{n}\,H^\dagger H\beta\,\rho^2 y^2. \tag{3.16}$$

The value of y^2 which minimizes F is

$$y^2 = \frac{1}{n-1} - \frac{H^\dagger H\beta}{b\rho^2}\,, \quad \text{if} \quad 0 \le \frac{1}{n-1} - \frac{H^\dagger H\beta}{b\rho^2} \le 1\,, \tag{3.17}$$

otherwise y^2 will take its extreme value 0 or 1.

When $\beta > 0$,

$$y^2 < \frac{1}{n-1}\,, \tag{3.18}$$

from (3.15) and (3.18) one can deduce

$$\chi^2 < \frac{4}{n(n-2)} \tag{3.19}$$

(3.19) means [1)]

i) The condition (3.13) is satisfied,

ii) The symmetry breaking pattern will be

$$\begin{aligned} &SU(\tfrac{n}{2}) \times SU(\tfrac{n}{2}-1) \times U(1), \text{ for n even.} \\ &SU(\tfrac{n-1}{2}) \times SU(\tfrac{n-1}{2}) \times U(1), \text{ for n odd.} \end{aligned} \tag{3.20}$$

When $\beta < 0$, (3.17) reads

$$y^2 = \frac{1}{n-1} - \frac{H^\dagger H\beta}{b\rho^2}\,, \quad \text{if} \quad \frac{H^\dagger H\beta}{b\rho^2} > -\frac{n-2}{n-1}\,, \tag{3.21}$$

$$y^2 = 1 \qquad \text{if} \quad \frac{H^\dagger H\beta}{b\rho^2} < -\frac{n-2}{n-1}\,. \tag{3.22}$$

For the case (3.22), evidently (3.12) is satisfied. For the case (3.21),

$$y^2 > \frac{1}{n-1}\,, \tag{3.23}$$

therefore by (3.15)

$$\chi^2 > \frac{4}{n(n-2)}\,, \tag{3.24}$$

which also implies that (3.12) holds.

To get the symmetry breaking pattern, we substitute (3.17) into (3.16) to get

$$F = \frac{b}{2n}\left[\rho^4 - (n-1)\,\frac{(H^\dagger H\beta)^2}{b^2} + 2\,\frac{H^\dagger H\beta}{b}\,\rho^2\right], \tag{3.25}$$

so that

$$\begin{aligned} V = {} & \left(\tfrac{1}{4}a + \tfrac{b}{2n}\right)\rho^4 - \tfrac{1}{2}\mu^2\rho^2 - \tfrac{1}{2}\nu^2(H^\dagger H) \\ & + \left(\tfrac{1}{4}\lambda - \tfrac{n-1}{2n}\tfrac{\beta^2}{b}\right)(H^\dagger H)^2 + \left(\alpha + \tfrac{\beta}{n}\right)\rho^2(H^\dagger H) \end{aligned} \tag{3.26}$$

The condition for V developing a minimum is

$$\begin{aligned} & \tfrac{1}{2}(a+\lambda) + \tfrac{b}{n}\left[1-(n-1)\tfrac{\beta^2}{b^2}\right] > 0, \\ & \left(\tfrac{a}{2}+\tfrac{b}{n}\right)\left(\tfrac{\lambda}{2} - \tfrac{n-1}{n}\tfrac{\beta^2}{b}\right) - \left(\alpha+\tfrac{\beta}{n}\right)^2 > 0, \end{aligned} \tag{3.27}$$

and the minimum occurs at

$$\begin{aligned} \rho^2 &= \frac{1}{2}\,\frac{\mu^2\left(\frac{1}{2}\lambda - \frac{n-1}{n}\frac{\beta^2}{b}\right) - \nu^2\left(\alpha+\frac{\beta}{n}\right)}{\left(\frac{1}{2}a+\frac{1}{n}b\right)\left(\frac{1}{2}\lambda - \frac{n-1}{n}\frac{\beta^2}{b}\right) - \left(\alpha+\frac{\beta}{n}\right)^2}, \\ H^\dagger H &= \frac{1}{2}\,\frac{\nu^2\left(\frac{1}{2}a+\frac{1}{n}b\right) - \mu^2\left(\alpha+\frac{\beta}{n}\right)}{\left(\frac{1}{2}a+\frac{1}{n}b\right)\left(\frac{1}{2}\lambda - \frac{n-1}{n}\frac{\beta^2}{b}\right) - \left(\alpha+\frac{\beta}{n}\right)^2}, \end{aligned} \tag{3.28}$$

if both of them are positive.

Substituting (3.28) into (3.21) one gets /1/

$$y^2 = \frac{1}{n-1} + \frac{\beta}{b}\,\frac{\nu^2\left(\frac{1}{2}a+\frac{1}{n}b\right) - \mu^2\left(\alpha+\frac{1}{n}\beta\right)}{\nu^2\left(\alpha+\frac{1}{n}\beta\right) - \mu^2\left(\frac{1}{2}\lambda - \frac{n-1}{n^2}\beta^2\right)}. \tag{3.29}$$

/1/ If the right hand side of (3.29) becomes larger than 1, y^2 will remain at 1.

By appropriate choice of the coefficients, y can go through from $\frac{1}{n-1}$ to 1, so that χ^2 runs from $\frac{4}{n(n-1)}$ to ∞, which means all possible values of n_1 and n_2 (except $n_1 = n_2$) can occur. But the symmetry breaking pattern is not solely dependent on $\frac{\beta}{b}$, it involves all coefficients in a somewhat complicated way through (3.29) and (3.15).

Lastly, we say a few words about the general case, $d, \gamma \neq 0$. For $b < 0$, the minimum also occur at the largest possible value of $|\chi|$, hence

$$\begin{aligned} F = {} & \frac{b}{2n(n-1)}\,\rho^4\Big[(n^2-3n+3)y^4 + 6y^2(1-y^2) + \frac{2d}{b\rho}\sqrt{n(n-1)}\,\big((n+1)y^3-3y\big) \\ & + \frac{n(n^2-5n+7)}{n-2}(1-y^2)^2 + 4(n-3)\sqrt{\frac{n}{n-2}}\,\left|y - \frac{d}{2b\rho}\sqrt{n(n-1)}\right|(1-y^2)^{3/2}\Big] \\ & + H^\dagger H\rho y\left(\beta\,\frac{n-1}{n}\rho y + \gamma\sqrt{\frac{n-1}{n}}\right). \end{aligned} \tag{3.30}$$

In case the second term is absent, the minimum is similarly degenerate and occurs at

$$\begin{aligned} & y = 1 \text{ and } y = -\frac{1}{n-1}, \quad \text{for } d < 0, \\ & y = -1 \text{ and } y = \frac{1}{n-1}, \quad \text{for } d > 0. \end{aligned} \tag{3.31}$$

The presense of the second term will also remove the degeneracy. But now we cannot make sure that for $\beta < 0$ the absolute minimum will occur at $|y| = 1$ so that the symmetry breaks to SU(n-1), since the location of the minimum also depend on γ and other coefficients. For example, when $-\frac{\gamma}{\beta} > \frac{n-2}{\sqrt{n(n-1)}}\rho_{min}$ holds in the case of $d > 0$, the value of F at $y = -\frac{1}{n-1}$ is surely smaller than that at y = 1. The same conclusion applies equally to the case $\beta > 0$.

Hence we shall only say that for $b < 0$ the symmetry breaking is to $SU(n-1)$ or $SU(n-2) \times U(1)$, without making further classification with reference to β.

As to $b > 0$, the conclusion remains as that in Ref. 1) and 2), namely the symmetry can break into anyone among the series $SU(n-1)$, $SU(n-2) \times U(1)$, $SU(n-3) \times SU(2) \times U(1)$, ..., $SU(\frac{n-1}{2}) \times SU(\frac{n-1}{2}) \times U(1)$ or $SU(\frac{n}{2}) \times SU(\frac{n}{2} - 1) \times U(1)$.

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