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On the Geometric Lattice Approximation to a Realistic Model of QCD

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Erratum:

The lattice versions of equations (3.4) and (3.4') should read:

$$(\overset{\vee}{\Delta}_A \varphi)_c(x, \#) = \sum_{\mu} \rho_{\mu, \#} (\mathcal{U}(x, \mu)_c \overset{c'}{\varphi}_c(x + e_{\mu}, \# - \mu) - \varphi_c(x, \# - \mu)),$$

$$(\overset{\vee}{\nabla}_A \varphi)_c(x, \#) = -\sum_{\mu} \rho'_{\mu, \#} (\varphi_c(x, \# + \mu) - \mathcal{U}^{-1}(x - e_{\mu}, \mu)_c \overset{c'}{\varphi}_c(x - e_{\mu}, \# + \mu)).$$

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1. Introduction

By the inclusion of quark fields in non-perturbative calculations QCD becomes an even more realistic theory of hadron dynamics. The first determinations of realistic hadron mass spectra [1] by a numerical evaluation of the lattice approximation illustrate this point. However, these are only first results on elementary problems for which the relativistic dynamics of quark pair creation does not play an essential rôle. The inclusion of dynamical pair creation is not always only a small correction to pure Yang-Mills theory but implies sometimes a qualitative change of physics. For example, the charge screening by pair creation modifies the linearly rising confinement potential into one which is constant at large distances. This is explicitly demonstrated in the Schwinger model [2].

The treatment of these problems in the lattice approximation of QCD meets the difficulty of a correct description of fermions on the lattice. There are three proposals to overcome the degeneracy problem of the naive lattice approximation of the Dirac equation due to K.G. Wilson [3], L. Susskind [4] and the SLAC group [5]. In spite of their unsatisfactory features they seem to lead to qualitatively similar results for the elementary problems. However, one would expect that an appropriate treatment of fermions is decisive for the advanced problems. For this reason we performed an analysis of the lattice fermion problem [6,7] based on a geometric description of Dirac particles by differential forms due to E. Kähler [8]. For this Dirac-Kähler equation there exists a straightforward correspondence between the continuum and the lattice description with no degeneracy problem as a lattice artifact.

Abstract

We suggest a model of QCD with four flavour degrees of freedom on the lattice. This model has a well-defined continuum limit and no spurious quark degrees of freedom. The formulation is realistic insofar as the different quarks may have different bare masses. For Monte-Carlo calculations on finite lattices our suggestion should be superior to comparable other models. The model is formulated within the Dirac-Kähler description of fermions which we repeat in a short glossary.

are independent; $d\mu^\# \equiv d\mu_1 \wedge \dots \wedge d\mu_n$. The exterior differentiation d can be written as

$$d\phi = d\mu^\# \wedge \partial_\mu \phi = \sum_{\mu} \partial_\mu \phi(x, \mu) d\mu^\#, \quad (2.3)$$

where $\partial_\mu \phi$ is the partial differentiation of the coefficients $\phi(x, \mu)$ of ϕ . The sign factor $\rho_{\mu, \#}$ is zero, if μ does not belong to H and is $(-1)^p$ if ν is the number of transpositions which are necessary in order to commute μ in $H = (\mu_1, \dots, \mu_n)$ to the left. $H - \mu$ is the ordered set H without μ . The adjoint δ of d is defined by a similar expression:

$$\delta \phi = -e^{\mu} \partial_\mu \phi = -\sum_{\mu} \rho_{\mu, \#} (\partial_\mu \phi)(x, \mu) d\mu^\#. \quad (2.4)$$

Here, e^{μ} denotes the differentiation with respect to the differential $d\mu^\mu$ defined by the formulae

$$e^{\mu} 1 = 0, \quad e^{\mu} d\mu^\# = \rho_{\mu, \#} d\mu^{\#-\mu},$$

especially: $e^{\mu} d\mu^\nu = \rho_{\mu\nu}$.

The sign factor $\rho_{\mu, \#}$ is zero, if μ belongs to the index set H and is equal to $\rho_{\mu, \#-\mu}$ otherwise. $H \cup \mu$ is the union of H with μ in natural order $\mu_1 < \dots < \mu_n$. The operations d and δ have the properties $d^2 = 0$, $\delta^2 = 0$, and therefore it follows:

$$(d-\delta)^2 = -(d\delta + \delta d) = 0, \quad (2.6)$$

where \square is the Laplacean. In this sense $d-\delta$ is a square root of the Laplacean $d-\delta \sim \square^{1/2}$, a property which it shares with the Dirac operator $\mu^\# \partial_\mu$.

In this letter, we give a short glossary of our results and we apply them in a suggestion of an economic formulation of QCD with four quarks with arbitrary bare masses. This model uses Susskind's idea [9] to identify the quark degeneracy with their flavours. However, in spite that there is an equivalence between the free Dirac-Kähler equation on the lattice and Susskind's formulation the interacting theories are different. For our formulation there is a well-defined formal continuum limit which leads to the conventional QCD Lagrangian with four fermions. This procedure minimizes the quark degrees of freedom on the lattice and thus promises advantages for numerical calculations. Apart from these practical aspects we believe that the geometrical approach opens a more fundamental point of view on the problem of quantum chromodynamics with fermions.

2. Glossary of the Dirac-Kähler Formalism

In the Dirac-Kähler equation (DKE)

$$(d - \delta + m) \phi = 0 \quad (2.1)$$

the Dirac field is described by a general real or complex differential form

$$\phi = \int_{\mathbb{R}^{3+1}} \left(\int_{\mu} \phi(x) d\mu^\mu + \frac{1}{2!} \int_{\mu\nu} \phi(x) d\mu^\mu \wedge d\mu^\nu + \frac{1}{3!} \int_{\mu\nu\rho} \phi(x) d\mu^\mu \wedge d\mu^\nu \wedge d\mu^\rho + \dots \right) d\mu^\# \quad (2.2)$$

Since the Grassmann product of the differentials is antisymmetric

$d\mu^\mu \wedge d\mu^\mu = 0$, only $2^d = 16$ coefficients $\rho_{\mu_1, \dots, \mu_n}$ ($\mu_i \in \{1, 2, 3, 4\}$) are needed depending on the ordered set of indices $\# = (\mu_1, \dots, \mu_n)$, $\mu_1 < \dots < \mu_n$.

There is a natural correspondence between quantities and operations defined on the lattice and differential forms and their manipulations. This correspondence expresses the fact that the lattice is considered as an approximation of the continuum. It allows a straightforward transcription of the DKE to the lattice:

$$(\overset{\vee}{\Delta} - \overset{\vee}{\nabla} + m) \phi = 0. \tag{2.7}$$

In this equation, ϕ denotes a linear functional defined on the elements of a (cubic) lattice: points (x) , links (x, μ) , plaquettes (x, μ, ν) , cubes (x, μ, ν, λ) , ..., in general: (x, R) , see Fig. 1:

$$\phi = \sum_{x, \#} \varphi(x, \#) d^{x, \#} \text{ where } d^{x, \#}((y, \lambda)) = \delta_y^x \delta_\lambda^\#. \tag{2.8}$$

$\overset{\vee}{\Delta}$ and $\overset{\vee}{\nabla}$ are the dual boundary and dual coboundary operator, respectively. Besides the well-known geometric meaning of $\overset{\vee}{\Delta} \phi((x, \#)) = \varepsilon^{-1} \phi(\Delta(x, \#))$ and $\overset{\vee}{\nabla} \phi((x, \#)) = \varepsilon^{-1} \phi(\nabla(x, \#))$ explained in Fig. 1, there exist representations of these operators similar to (2.3) and (2.4):

$$\overset{\vee}{\Delta} \phi = \sum_{x, \#} \left(\sum_{\mu, \#} \rho_{\mu, \#} (\Delta_{\mu}^+ \varphi)(x, \# - \mu) \right) d^{x, \#}, \tag{2.9}$$

$$\overset{\vee}{\nabla} \phi = - \sum_{x, \#} \left(\sum_{\mu, \#} \rho'_{\mu, \#} (\Delta_{\mu}^- \varphi)(x, \# + \mu) \right) d^{x, \#}. \tag{2.9'}$$

Δ_{μ}^+ (Δ_{μ}^-) is the forward (backward) derivative of the coefficients:

$$\begin{aligned} (\Delta_{\mu}^+ \varphi)(x, \#) &= \frac{1}{\varepsilon} (\varphi(x + \varepsilon e_{\mu}, \#) - \varphi(x, \#)), \\ (\Delta_{\mu}^- \varphi)(x, \#) &= \frac{1}{\varepsilon} (\varphi(x, \#) - \varphi(x - \varepsilon e_{\mu}, \#)), \end{aligned} \tag{2.10}$$

ε is the lattice constant. There are the well-known relations $\overset{\vee}{\Delta}^2 = 0$, $\overset{\vee}{\nabla}^2 = 0$, and $(\overset{\vee}{\Delta} - \overset{\vee}{\nabla})^2 = -(\overset{\vee}{\Delta} \overset{\vee}{\nabla} + \overset{\vee}{\nabla} \overset{\vee}{\Delta}) = \square$ represents the correct

Lattice approximation to the Laplacean \square :

$$(\square \varphi)(x, \#) = \sum_{\mu} (\Delta_{\mu}^+ \Delta_{\mu}^- \varphi)(x, \#) = - \sum_{\mu} \left(\frac{\varepsilon}{2} \sin \frac{2\pi \mu \cdot \varepsilon}{2} \right)^2 \varphi(x, \#), \tag{2.11}$$

where the last equality sign holds for a plane wave $\varphi(x, \#) = e^{i(x, \#)} = e^{i(p, \#)}$. $\varepsilon \varphi(p, \#) / \varepsilon \approx \varphi(p, \#)$. For this reason the lattice approximation to the Dirac-Kähler equation has no spectrum problem. However, the continuum formulation of the DKE has already 2^d components compared to the $2^{d/2}$ components of the Dirac equation.

The relation between the DKE and the Dirac equation is clarified by the existence of an associative Clifford product $\phi \vee \psi$ for differential forms. It is defined for the basic elements $d\mu^{\mu}$ by

$$d\mu^{\mu} \vee d\mu^{\nu} = d\mu^{\mu, \nu} = d\mu^{\nu} \vee d\mu^{\mu} + g^{\mu\nu} \varepsilon \tag{2.12}$$

$$d\mu^{\mu} \vee \phi = d\mu^{\mu, \#} = d\mu^{\#} \vee \phi + \varepsilon^{\mu, \#} \phi, \tag{2.12'}$$

$$\phi \vee d\mu^{\mu} = \phi \wedge d\mu^{\mu} - \varepsilon^{\mu, \#} \phi \phi. \tag{2.12''}$$

$\phi \phi = (-1)^k \phi$, if ϕ is an h-form, and $\phi \phi$ is linear. From (2.3,4) and (2.12') it follows for the DKE

$$(d - \delta + m) \phi = (d\mu^{\mu} \vee \partial_{\mu} + m) \phi = 0. \tag{2.13}$$

According to (2.12), the Clifford product of the $d\mu^{\mu}$ satisfies the defining relation of the μ^{μ} matrices: $\mu^{\mu} \mu^{\nu} + \mu^{\nu} \mu^{\mu} = 2g^{\mu\nu}$. However, the representation $\mu^{\mu} \mapsto d\mu^{\mu} \vee$ of the Clifford algebra in the 16 dimensional space of differential forms is reducible. On irreducible subspaces the DKE implies the Dirac equation according to (2.13). This means that in a basis

$Z_a^{(b)}$, which satisfies

$$d\psi^{(b)} \wedge (Z_a^{(b)}) = (\sum_c \psi^{(c)T})_{ac} Z_c^{(b)} = \psi^{(b)T} Z, \quad Z \wedge d\psi^{(b)} = Z \psi^{(b)T}, \quad (2.14)$$

the components $\psi_a^{(b)}$ of the differential form $\psi = \sum_{a,b} \psi_a^{(b)} \psi_a^{(b)}$ satisfy the Dirac equation $(\not{D} \psi + m) \psi = 0$ iff ψ satisfies the DKE. An explicit calculation [8] gives the following relations between Dirac components and cartesian components of a differential form:

$$\begin{aligned} \psi(x, \mathcal{H}) &= \sum_{a,b} \psi_a^{(b)}(x) \psi_a^{(b)}, \quad \psi_a^{(b)}(x) = \frac{1}{4} \sum_{\#} \psi(x, \mathcal{H}) \psi^{\#} a_b, \\ \bar{\psi}(x, \mathcal{H}) &= \sum_{a,b} \bar{\psi}_a^{(b)}(x) \psi_a^{(b)T} b_a, \quad \bar{\psi}_a^{(b)}(x) = \frac{1}{4} \sum_{\#} \bar{\psi}(x, \mathcal{H}) \psi^{\#*} a_b, \\ \psi^{\#} &= \psi^{\mu_1 \mu_2 \dots \mu_k}, \quad \mu_1 < \dots < \mu_k \in \mathcal{H}. \end{aligned} \quad (2.15)$$

In E. Kähler's formulation the spin 1/2 Dirac field is represented as a coherent superposition of tensor fields:

The complex ψ represents in Kähler's formulation four Dirac fields. Right multiplication $\psi \vee c$ with constant differential forms c induces linear transformations of these Dirac fields according to (2.14):

$$\psi' = \psi \vee c \rightsquigarrow \psi'_a{}^{(b)}(x) = \sum_{\bar{c}} \psi_a^{(b)}(x) c_{\bar{c}}^{(b)} \quad (2.16)$$

where $\frac{1}{4} c_{\bar{c}}^{(b)}$ are the Dirac components of c . The group of unitary transformations $(c_a^{(b)}) \in \mathcal{U}(\mathcal{H})$ is called the global "flavour" group, following a suggestion of L. Susskind [9] for the equivalent lattice case. Its possible physical significance will be discussed in the next section. The DKE is invariant with respect to right \vee -multiplication $\psi \mapsto \psi \vee c$, i.e. flavour transformations. This follows immediately from the associativity of the Clifford product and (2.13). The decomposition of the DKE into Dirac equations can be de-

scribed by this symmetry. For example, the transformation properties of ψ with respect to the "reduction" group $(\mathcal{R}, \nu) = \{1, \tau = i d\psi^{\mu} \wedge d\psi^{\mu}\}$, $\varepsilon = d\psi^1 \wedge \dots \wedge d\psi^k$, $\varepsilon \vee \tau$ characterize the non-vanishing subspaces $\{\psi^{(b)}\}$ spanned by the Dirac components $\psi_a^{(b)}$ with b fixed. If

$$\psi^{(b)} \vee \tau = \pm \psi^{(b)}, \quad \psi^{(b)} \vee \varepsilon = \pm \psi^{(b)}, \quad (2.17)$$

the sign combinations $(\tau, \varepsilon) = (-, -), (+, -), (-, +), (+, +)$ correspond to $b = 1, 2, 3, 4$. This follows from (2.14) for ψ -matrices in the Weyl representation where $\not{1}_2$ and $\not{1}_3 \not{1}_2$ are diagonal.

In order to describe the decomposition of the DKE in Dirac equations on the lattice in the same way, the natural lattice operations corresponding to $d\psi^{\mu} \wedge \psi$, $e^{\mu} \psi$ and $d\psi^{\mu} \vee \psi$ must be considered. This has been done in ref. [7]. The essential difference to the continuum analysis is a weak non-locality of the Grabmann and Clifford product. The lattice definitions involve nearest neighbours. As a consequence, the reduction group on the lattice is intertwined with translations. The reduction of the DKE to four Dirac equations can be done in momentum space, but not in coordinate space. In momentum space the decomposition into Dirac components is given by

$$\psi(x, \mathcal{H}) = \exp(-i\varepsilon \frac{\not{p} \cdot \not{e}_\mu}{2}) \cdot \text{trace}(\not{1}_2^{\dagger} \psi(p)) e^{ipx}, \quad e_{\mu} = \sum_{\alpha \in \mathcal{H}} e_{\alpha}, \quad (2.18)$$

similar to (2.15). The Dirac components $\psi_a^{(b)}(p)$ of a solution of the DKE satisfy the Dirac equation $(-2i\varepsilon^{-1} \sin(p_{\mu} \varepsilon / 2) \not{1}_2^{\dagger} \not{1}_2 + m) \psi^{(b)}(p) = 0$ with the correct energy momentum spectrum.

Finally, we would like to mention that the free Dirac-Kähler equation on the lattice is equivalent to the Susskind reduction of the free naive lattice Dirac equation [10]. For interacting fermions, however, this is no longer true.

3. A Minimal Realistic Model of QCD

The geometric lattice approximation of Dirac fields with help of the DKE leads to a complete clarification of the problems with free lattice fermions. The DKE is equivalent to the Susskind reduction of the naive lattice approximation of the Dirac equation. However, it is superior to that method, as it has a well-defined continuum limit, and the spectrum degeneracy is not a lattice artifact but is already described by the continuum theory. In the following we want to use these advantages for a realistic lattice model of QCD which is no longer equivalent to the Susskind formulation. For this, we give the multiplicity of coloured Dirac fields contained in the coloured Dirac-Kähler field a physical interpretation. It is allowed to do so, because this multiplicity is given by the continuum theory. Spurious multiplicities appearing in approximations with different lattice shapes [11] are therefore controlled [12]. We consider a model of QCD with u,d,s and c quarks described by the euclidean continuum Lagrangian density

$$\mathcal{L} = \sum_{G=u,d,s,c} \bar{\psi}^{(G)}(x) (\gamma^\mu D_\mu + m^{(G)}) \psi^{(G)}(x) - \frac{1}{4} G_{\mu\nu} G^{\mu\nu} \quad (3.1)$$

With help of the relations (2.15) between the cartesian and Dirac components of the DK field this Lagrangian can be written in the Dirac-Kähler form

$$\mathcal{L} = (\bar{\psi}, (\mathcal{D}_4 - \mathcal{D}_1) \psi) + \sum_G m^{(G)} (\bar{\psi}, \psi \nu P^{(G)}) - \frac{1}{4} G_{\mu\nu} G^{\mu\nu} \quad (3.2)$$

with the notations given in ref. [7]. We see in this form not necessarily a profound explanation of the appearance of flavours [13] but more a convenient and straightforward starting point for realistic lattice calculations. For practical use we repeat the explicit definitions of the different terms in the action and their natural lattice correspondences. The inner product $(\bar{\psi}, \mathcal{Z})$ is expressed by the cartesian components in the continuum, equation (2.2), or so

the lattice, equation (2.8) as

$$(\bar{\psi}, \mathcal{Z}) = \sum_{c=\nu, \mathcal{D}_4} \sum_{\#} \bar{\psi}_c(x, \#) \mathcal{Z}_c(x, \#), \quad (3.3)$$

c ≡ colour index. The covariant exterior differentiations $\mathcal{D}_A, \mathcal{D}_A$ and their lattice analogues are given by

$$(\mathcal{D}_4 \psi)_c(x, \#) = \sum_{\mu} \rho_{\mu, \#} (\delta_c^{\nu} \mathcal{D}_\nu - ig A_\mu^a(x) \frac{\partial}{\partial x}) \psi_c(x, \# - \mu), \quad (3.4)$$

$$(\mathcal{D}_A \psi)_c(x, \#) = \sum_{\mu} \rho_{\mu, \#} (\mathcal{D}(x, \mu)_c^{\nu} \mathcal{D}_\nu(x, \# - \mu) - \mathcal{D}_\nu(x, \# - \mu)),$$

$$(\mathcal{D}'_4 \psi)_c(x, \#) = - \sum_{\mu} \rho'_{\mu, \#} (\delta_c^{\nu} \mathcal{D}'_\nu - ig A_\mu^a(x) \frac{\partial}{\partial x}) \psi_c(x, \# + \mu), \quad (3.4')$$

$$(\mathcal{D}'_A \psi)_c(x, \#) = - \sum_{\mu} \rho'_{\mu, \#} (\mathcal{D}'(x, \mu)_c^{\nu} \mathcal{D}'_\nu(x, \# + \mu) - \mathcal{D}'_\nu(x, \# + \mu)).$$

These are the gauge invariant generalizations of (2.3,9) and (2.4,9') respectively. The $U(x, \mu)$'s are the familiar link variables representing the gauge field on the lattice. The mass operator described by $m^{(b)}$ and the projection operators $P^{(b)}$ can be derived from the characterization of the Flavours by the reduction group \mathcal{R} according to (2.17). In the continuum it has the form

$$\sum_G m^{(G)} (\bar{\psi}, \psi \nu P^{(G)}) = \sum_C \sum_{\#} \bar{\psi}_C(x, \#) M_{\#k} \psi_C(x, k). \quad (3.5)$$

The 16×16 matrix $M_{H,K}$ can be calculated with help of the equations (2.15) as

$$M_{\#k} = \frac{1}{16} \text{trace} (\gamma^k M(\psi, \#)^{\dagger}) \quad (3.6)$$

with

$$M = \frac{1}{4} \{ 1 (m^{(1)} + m^{(2)} + m^{(3)} + m^{(4)}) - 2 \gamma^1 \gamma^2 (m^{(1)} - m^{(2)} + m^{(3)} - m^{(4)}) - 2 \gamma^1 \gamma^3 (m^{(1)} - m^{(2)} + m^{(3)} - m^{(4)}) - 2 \gamma^1 \gamma^4 (m^{(1)} - m^{(2)} + m^{(3)} - m^{(4)}) + \gamma^2 \gamma^3 \gamma^4 (m^{(1)} - m^{(2)} + m^{(3)} - m^{(4)}) \} \quad (3.7)$$

On the lattice, we use the same expression as a mass operator [14]. Thus, the action of the lattice approximation to the four quark QCD in the Dirac-Kähler formulation follows from these natural correspondences:

$$S = \sum_{\vec{x}} \left\{ (\bar{\psi}, (\Delta_4 - \nabla_4) \psi) + \sum_{\vec{y}} \kappa(\vec{y}) (\bar{\psi}, \phi \nu P(\vec{y})) \right\} + S_G(U_Q), \tag{3.8}$$

$S_G(U_Q)$ = Wilson action of pure gluon dynamics. We consider this action as the appropriate formulation of QCD with four flavours on the lattice:

- It does not contain any unphysical quark degrees of freedom, it is minimal in this sense.
- It has a formal continuum limit which is equivalent to the standard QCD action with four flavours.
- The model describes a quark mass splitting which is gauge invariant.

In comparison with other formulations of four quark QCD models we see the following advantages: In contrast to the Wilson formulation there are no unphysical fermion degrees which disappear only in the continuum limit. If we compare with the Susskind formulation we find that in the free theories the DK components $\psi(\vec{x}, \#)$ correspond to Susskind components $\chi(\vec{x} + \vec{e}_\mu/2)$. Therefore these are defined on a lattice with half the lattice spacing and 16 times the number of links. This means that the number of degrees of freedom of the gauge field is reduced by a factor 16 in the Dirac-Kähler formulation. Because of these facts we expect that numerical calculations with finite lattices based on the action (3.8) are more effective.

The existence of the formal continuum limit simplifies the formulation of currents and composite fields describing hadrons. For example, the isovector current formed by the up and down quark fields $j^k(x) = j^k_{\mu}(x) \alpha^k_{ij}$ with

$$j^k_{\mu}(x) = \bar{\psi}(x) \frac{\Sigma^k}{2} \gamma_{\mu} \psi(x) \tag{3.9}$$

can be expressed in the Dirac-Kähler formulation as

$$j^k_{\mu} = \frac{1}{4} \star g_{\mu} - (d\psi^{\mu} \nu \bar{\psi}, \phi \nu C^k) \tag{3.10}$$

with

$$C^k = \frac{(-1)^{k+1}}{4i} (d\psi^k + d\psi^4 + d\psi^1 d\psi^2 + d\psi^1 d\psi^3)$$

and (k, l, m) a cyclic permutation of (1,2,3). The lattice correspondence extended to the definition of the Clifford product ν and the duality operator \star gives an expression for the current

$$j^k_{\mu} = \sum_{\nu, \rho} j^k_{\mu\nu\rho}(x) d^{\nu, \rho}$$

completely equivalent to (3.10) [7]. More explicitly, this expression is of the form

$$j^k_{\mu}(x) = \frac{1}{4} \int \bar{\psi}(x + \vec{e}_{\mu}, \#) \xi^k(x, \nu, \rho) + \sum_{\nu, \rho} \rho'_{\mu, \nu \rho} \bar{\psi}(x + \vec{e}_{\mu}, \nu) \xi^k(x, \nu, \rho) + \sum_{\nu, \rho} \rho'_{\mu, \nu \rho} \bar{\psi}(x + \vec{e}_{\mu}, \nu) \xi^k(x, \nu, \rho) + \sum_{\nu, \rho} \rho'_{\mu, \nu \rho} \bar{\psi}(x + \vec{e}_{\mu}, \nu) \xi^k(x, \nu, \rho) \tag{3.11}$$

Here, $\bar{\psi}(x, \#)$ are the cartesian components of the field $\bar{\psi}$ and ρ' is the sign function introduced in (2.4). $\xi^k(x, \#)$ are the coefficients of the field $\phi \nu C^k$ on the lattice. They are given by $\xi^k(x, \#) = \xi^k(x, \#) / \epsilon_i$,

$$\begin{aligned} \xi^1(x, \#) &= -\psi(x - \vec{e}_2, \#) - \psi(x - \vec{e}_3, \#) - \psi(x, \#) = \psi(x, \#) + \psi(x, \#) + \psi(x, \#), \\ \xi^2(x, \#) &= -\psi(x - \vec{e}_1, \#) - \psi(x - \vec{e}_3, \#) - \psi(x, \#) = \psi(x, \#) + \psi(x, \#) + \psi(x, \#), \\ \xi^3(x, \#) &= -\psi(x - \vec{e}_1, \#) - \psi(x - \vec{e}_2, \#) - \psi(x, \#) = \psi(x, \#) + \psi(x, \#) + \psi(x, \#). \end{aligned} \tag{3.12}$$

and similar expressions for the other coefficients, which can be obtained by appropriate permutations of the indices. The current (3.11) is conserved in the free case. The point splitting on the lattice introduced by the definition

of the Clifford product makes this current gauge non-invariant. Its gauge invariant version is a good candidate for the lattice analogue of (3.9). This current is not conserved on the lattice. However, one would expect that the renormalized continuum limit of this expression would describe the correct quantum mechanical current with possible anomaly [15]. This is at the moment a speculation. Systematic studies of this point are only at its beginning [16]. A similar procedure is possible for the description of general composite fields like $\psi(x)\psi(x)\psi(x)$.

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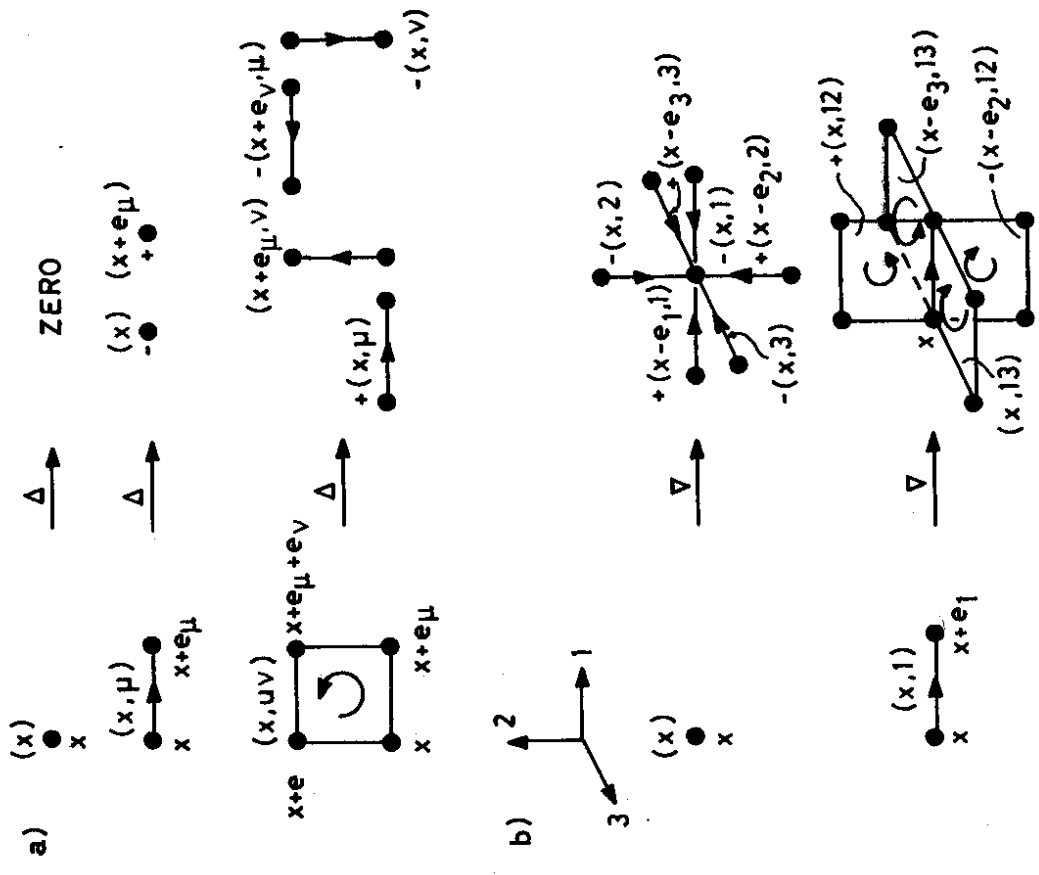


Figure Caption

Fig. 1: a) The boundary Δ of points, links and plaquettes.
 b) The coboundary ∇ of points and links in three dimensions.

Fig.1