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QUANTUM CHROMODYNAMICS ON THE LATTICE

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I. Introduction

II. Hamiltonian formulation

III. Strong coupling expansion

IV. Monte Carlo method

V. The spectrum of QCD

VI. Current amplitudes

VII. Finite temperatures

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I. Introduction

The most important feature of the development in theoretical high energy physics in the seventies is undoubtedly the recognition of the fundamental rôle of non-Abelian quantum gauge theories for the description of particle interactions. The proof of renormalizability by G. 't Hooft [1] opened up the way for the application of the powerful methods of perturbation theory. The obviously non-perturbative phenomena like confinement and spontaneous symmetry breaking, however, require a new approach: lattice gauge theory. The fundaments of this were laid down by K. Wilson [2]. The essence of lattice gauge theory is the discretization of space-time and the heavy use of the achievements of modern computer technology. The rapid development of the field began, however, only in the last 2-3 years and presently it is not possible to foresee its final range of capabilities. Besides the obvious difficulties there are also a lot of impressive and promising progresses.

In the present lecture notes some basic facts of lattice quantum chromodynamics are collected. It is, however, incomplete in many respects and the selection is subjective. For further information please study the recent reviews e.g. in [3-6].

LAGRANGEAN. The non-Abelian gauge theory for strong interactions is quantum chromodynamics (QCD). It is formulated in terms of quark- and gluon fields. The classical Lagrange-function is built up from the quark-field $\psi(x) = \{\psi^{\alpha c f}(x)\}$ (α is a Dirac-index for spin, $c = 1,2,3$ is an SU(3)-triplet colour index, $f = u,d,c,s,t?, b, \ldots$ is the flavour index) and from the gluon-field $A_a(x)_\mu$ ($a = 1,2,\ldots,8$ is the SU(3)-octet colour index, $\mu = 1,2,3,4$ is a four-vector index for spin). The colour field-strength tensor (analogous to $F(x)_{\mu\nu}$ in QED) is

$$F_a(x)_{\mu\nu} = \frac{\partial A_a(x)_\mu}{\partial x^\nu} - \frac{\partial A_a(x)_\nu}{\partial x^\mu} + g f_{abc} A_b(x)_\mu A_c(x)_\nu \quad , \qquad (1.1)$$

where g is the bare coupling constant of the gauge interaction and the

structure constants of SU(3) Lie-algebra f_{abc} are defined by the commutation relation of the Gell-Mann λ-matrices:

$$[\lambda_a, \lambda_b] = 2i f_{abc} \lambda_c \quad . \tag{1.2}$$

In terms of these fields the QCD Lagrange-function is

$$\mathcal{L} = -\frac{1}{4} F_a(x)_{\mu\nu} F_a(x)^{\mu\nu} + \tilde{\psi}(x)\left[\frac{i}{2}\left(\overleftarrow{\partial}\cdot\gamma - \overrightarrow{\partial}\cdot\gamma\right) - m\right]\psi(x) -$$

$$- g A_a(x)_\mu \tilde{\psi}(x) \gamma^\mu \frac{\lambda_a}{2} \psi(x) \quad . \tag{1.3}$$

This is a prototype of a large class of non-Abelian gauge field theory Lagrangeans which have important common general properties (the freedom is to choose the gauge group - here SU(3) for colour, and the group representation for ψ - here a sum of triplets).

Let us first collect some basic phenomenological facts and general theoretical expectations about QCD:

A. CONFINEMENT: one of the striking features of hadron physics is the absence of free quark and gluon states. This is in contrast with QED, where the quanta of the fundamental electron and photon fields are manifested as free particles. More generally, the spectrum of QCD consists only of colour neutral states. Any excitation with non-zero colour can only exist confined to a small region (of hadronic size) within colour neutral objects.

B. MASS GAP: the mass of the lowest state in QCD is positive (there is a gap between the zero energy vacuum state and the lowest particle state). Even in a (hypothetical) theory without quarks ("pure" non-Abelian gauge theory) the lowest state would have a positive mass. This is again contrary to QED, where the lowest state is the massless photon.

The masses of the quarks are free external parameters in QCD. In the Lagrangean (1.3) m is, in fact, a diagonal matrix in flavour:

$$m = \begin{pmatrix} m_u & & & & & \\ & m_d & & & & \\ & & m_c & & & \\ & & & m_s & & \\ & & & & \ddots & \end{pmatrix} \tag{1.4}$$

The approximate values of the quark masses are: $m_u \cong 4$ MeV, $m_d \cong 7$ MeV, $m_s \cong 150$ MeV (these are the "light quarks"), $m_c \cong 1200$ MeV, $m_b \cong 4500$ MeV (these are the known "heavy quarks"). The light quarks can, in many cases, be considered to a good approximation as massless.

The number of all quark flavours is exactly conserved in QCD. This corresponds (in the case of 6 quark flavours) to an exact global $U(1)^6$ symmetry. If we assume equal masses for the three light quarks: $m_u = m_d = m_s$ then the corresponding global $U(1)^3$ symmetry is extended to a global $U(3) = U(1) \otimes SU(3)$ symmetry. (Here U(1) stands for the total number of light quarks and SU(3) is the well known global symmetry of the hadron spectrum, which is in nature only approximate due to the small mass differences of light quarks.) If, in a further approximation, the common mass of the light quarks is neglected: $m_u = m_d = m_s = 0$, then the above SU(3) symmetry of the Lagrangean is enhanced to a chiral $SU(3) \otimes SU(3)$, which is generated by the vector and axialvector flavour charges of u, d and s quarks.

C. CHIRAL SYMMETRY BREAKING: the global chiral $SU(3) \otimes SU(3)$ symmetry is explicitly broken in the Lagrangean (1.3) by the non-zero and non-equal quark mass (m_u, m_d, $m_s \neq 0$). This is, however, not the only symmetry breaking mechanism. Chiral $SU(3) \otimes SU(3)$ is namely, also broken by the non-zero vacuum expectation values of the quark field "condensates":

$$\langle 0|\bar{\psi}_f(x)\psi_f(x)|0\rangle \neq 0 \quad (f = u, d, s, \dots). \tag{1.5}$$

This implies that the true symmetry for zero light quark masses is only SU(3) generated by the vector flavour charges. The Goldstone-theorem tells that the spontaneous breaking of SU(3) ⊗ SU(3) is accompanied by the existence of 8 massless Goldstone-bosons. These can phenomenologically be identified by the members of the pseudoscalar octet (π, K, η). The ninth pseudoscalar meson η' is not a (pseudo-) Goldstone boson. It would have non-zero mass even if the light quarks would be exactly massless. This comes from a quantum effect represented in renormalized perturbation theory by the colour "triangle-anomaly", which breaks the chiral U(3) ⊗ u(3) symmetry of the classical Lagrangean down to the above mentioned U(1) ⊗ SU(3) ⊗ SU(3).

In reality the pseudoscalar octet is not massless (although m_π, m_K and m_η are small compared to the other hadronic masses). This comes from the explicit chiral symmetry breaking caused by the light quark masses. Therefore, the pion, kaon and η-meson are only called "pseudo-Goldstone bosons" (instead of real, massless Goldstone bosons).

The existence of the quark-condensate (1.5) in the vacuum has a great influence on the hadron dynamics. A similarly important rôle is played also by other condensates, for instance by the bilinear "gluon-condensate":

$$\langle 0|F_a(x)_{\mu\nu}F_a(x)^{\mu\nu}|0\rangle \neq 0\,. \tag{1.6}$$

In fact, basic features like colour confinement can be also qualitatively understood as a consequence of the dynamics of the vacuum. The condensation of some gauge field configurations like monopoles and vortices [7] produces a large "disorder" in the vacuum compressing external colour fields to localized tubes (Fig. 1). This is the chromoelectric analogue of the Meissner-effect for magnetic fields in superconductors. The result is a quark-antiquark potential which rises linearly with the distance and hence implies quark confinement.

D. MELTING OF THE VACUUM: the characteristics of the vacuum change as the temperature increases. It is generally assumed that at some critical temperature $T = T_c$ the confining gauge field configurations in the vacuum "melt" and the hadronic matter undergoes a "deconfinement phase transition". The hadronic confining phase below T_c goes over into a quark-gluon plasma in which the colour-charges are not confined but Debye-screened, similarly to the electromagnetic plasma.

The essential difficulty in QCD is that the above mentioned basic facts A. - D. cannot be understood and explained in perturbation theory, therefore new, non-perturbative methods are needed. It follows from asymtotic freedom and renormalization group that e.g. the mass-gap M_G behaves for small coupling $g \to 0$ like

$$M_G \to \exp\left\{-\frac{\text{const}}{g^2}\right\}. \tag{1.7}$$

Obviously, this cannot be reproduced by a power series in g.

LATTICE REGULARIZATION. An important step towards the solution of the non-perturbative problems in QCD was the proposal of K. Wilson [2] to regularize QCD by the introduction of a discrete lattice in space-time. The lattice regularization of QCD

- allows for non-perturbative calculations and
- is explicitly gauge-invariant.

It is formulated in terms of the "path-dependent phase factor" U[C] (or "parallel-transporter" along C), which is built from the gauge-field matrix

$$A(x)_\mu \equiv \sum_{a=1}^{8} \frac{\lambda_a}{2} A_a(x)_\mu \tag{1.8}$$

like

$$U[C] = P\exp\left\{-ig\int_C A(x)_\mu\, dx^\mu\right\}. \tag{1.9}$$

Here C is a curve in space-time: $C = \{ x_\mu(t);\ 0 \le t \le 1 \}$, and P implies an ordering of the matrices $A(x)_\mu$ along the path C (i.e. an ordering according to t). It follows from the definition that $U[C]$ is an element of the gauge group $G = SU(N_c)$. (N_c is, in general, the number of colours. Physically we have $N_c = 3$).

It can be easily shown, that for a closed parallelogram $C_{\square} \equiv (x \to x + a \to x + a + b \to x + b \to x)$, in the limit $a, b \to 0$, we have

$$U[C_{\square}] = 1 + ig a^\mu b^\nu F(x)_{\mu\nu} - \frac{g^2}{2}\left(a^\mu b^\nu F(x)_{\mu\nu}\right)^2 + \cdots , \qquad (1.10)$$

where $F(x)_{\mu\nu}$ is the matrix field-strength tensor corresponding to (1.1):

$$F(x)_{\mu\nu} = \frac{\partial A(x)_\mu}{\partial x^\nu} - \frac{\partial A(x)_\nu}{\partial x^\mu} + ig\left[A(x)_\mu, A(x)_\nu\right] . \qquad (1.11)$$

If the curve $C_{A\to B}$ connects the points A and B, then the local (point-dependent) gauge transformation of $U[C_{A\to B}]$ is

$$U[C_{A\to B}] \longrightarrow U(B)^{-1} U[C_{A\to B}]\, U(A) ; \qquad (1.12)$$

$$U(A), U(B) \in SU(N_c) .$$

Therefore, examples of gauge invariant quantities are, with some closed curve C_o:

$$\mathrm{Tr}\{U[C_o]\} ; \quad \bar\psi(B)\, U[C_{A\to B}]\, \psi(A) . \qquad (1.13)$$

(Here $\bar\psi(P)$ and $\psi(P)$ denote the quark fields in the point P.) We shall see, that the lattice action of QCD is built up from quantities like (1.13), hence it is explicitly gauge invariant.

The discrete points in space-time are usually choosen to be the points of a (four-dimensional) hypercubical lattice. (See Figs. 2a-b.) The quark fields ψ_x, $\bar\psi_x$ are put on the lattice sites, the gauge field variables $U[x+\hat\mu, x]$ are on links connecting two neighbouring sites x and $x+\hat\mu$ ($\mu = 1,2,3,4$). The quark-fields ψ_x, $\bar\psi_x$ are anticommuting Grassmann-variables, the gauge field link variable $U[x+\hat\mu, x]$ is the parallel-transporter along the link, hence it is an element of the gauge group $SU(N_c)$.

<u>WILSON-ACTION OF QCD</u>. The lattice-action is defined in the four-dimensional Euclidean space, i.e. the time is imaginary (the fourth coordinate is $x_4 = it$). Let us denote the product of the four link-variables around a plaquette $\square$ by $U_\square$. Then, for a general colour gauge group $SU(N_c)$, the lattice action of Wilson [2] is

$$S = g^{-2} \sum_{\square} \left(N_c - \mathrm{Tr}(U_\square)\right) +$$
$$+ \sum_x \Big\{ \bar\psi_x \psi_x - \sum_\mu \bar\psi_{x+\hat\mu} K\, U[x+\hat\mu, x]\,(r+\gamma_\mu)\,\psi_x \Big\} . \qquad (1.14)$$

Here $\sum_{\square}$ denotes a summation over all <u>oriented</u> plaquettes, $\sum_x$ is a summation over all sites and $\sum_\mu$ means a sum over all (positive and negative) directions. r is an arbitrary, positive parameter, choosen in most cases to be $r = 1$, and $\gamma_\mu = -\gamma_{-\mu}$ is a Hermitean Dirac-matrix satisfying the anticommutation relations

$$\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu} . \qquad (1.15)$$

The lattice cuts off the high momenta (see below), therefore the coupling constant $g \equiv g(a)$ is a cut-off dependent bare coupling

constant (a is the lattice spacing, the cut-off is proportional to a^{-1}). The other "constant" in the Wilson action is the "hopping-parameter" $K \triangleq K(a)$. It is connected to the bare quark mass by

$$K_f = \frac{1}{8r + 2am_f} \qquad (f = u, d, c, s, \ldots) \; . \tag{1.16}$$

K has different values for different flavours, therefore it is in (1.14) a diagonal matrix in flavour, in the same way as the quark mass m in Eq. (1.4) is.

In the lattice theory one has to calculate integrals like e.g. the "partition function" (or "sum over states")

$$Z = \int \prod_\ell dU_\ell \prod_x d\bar{\psi}_x d\psi_x \, e^{-S} \; . \tag{1.17}$$

Here dU_ℓ is the invariant Haar-measure over the gauge group ($U_\ell \in SU(N_c)$ is the link variable), and $d\bar{\psi}_x d\psi_x$ is an integration over Grassmann-variables. The expectation value of a physical quantity F is

$$\langle F \rangle = Z^{-1} \int \prod_\ell dU_\ell \prod_x d\bar{\psi}_x d\psi_x \, e^{-S} F \; . \tag{1.18}$$

Our (more or less well-founded) theoretical prejudices tell us to expect that

- the Lorentz-invariance (broken by the discrete lattice) will be restored in the continuum-limit $a \to 0$;
- the number of colours $N_c = 3$ is not very essential for the dynamics (an SU(2) or SU(25) colour theory would be similar in many respects to SU(3) colour);

- the essential features of the dynamics are decided on the pure gauge theory level, the quarks are only small (but very interesting) perturbations. The above stated basic properties A, B and D can obviously be formulated and studied in the pure gauge theory without quarks. (C refers, of course, specifically to quarks.)

CONTINUUM LIMIT. In the limit of zero lattice spacing $a \to 0$ the Wilson-action (1.14) tends to the integral of the classical Lagrangean (1.3). This reassuring fact can be easily shown by using Eqs. (1.8 - 1.11). Of course, we are not only interested in the limit of the action but in the limit of the whole theory as defined by the expectation values (1.18). What we expect is, that in the limit $a \to 0$ lattice QCD describes the strong interactions of hadrons in the continuum. This means that whenever we calculate some physical quantity on the lattice we have to investigate its continuum limit for $a \to 0$.

In order to be able to do the continuum limit we have to know the behaviour of the parameters g(a) and K(a) as $a \to 0$. General renormalization theory tells (for a recent concise review and further references see [8]) that for $a \to 0$

$$a \frac{dg(a)}{da} \equiv \bar{\beta}(g) = \beta_0 g^3 + \beta_1 g^5 + O(g^7) \; . \tag{1.19}$$

The constants $\beta_{0,1}$ are universal, i.e. independent of the specific form of the lattice action, whereas the higher order terms $O(g^7)$ are lattice action dependent. ($\beta_{0,1}$ appear also in the Callan-Symanzik β-function $\beta = -\mu \, dg_R/d\mu$.) In the case of N_c colours and N_f massless flavours we have

$$\beta_0 = \frac{1}{16\pi^2}\left(\frac{11N_c}{3} - \frac{2N_f}{3}\right) ,$$
$$\beta_1 = \frac{1}{256\pi^4}\left(\frac{34N_c^2}{3} - \frac{10}{3}N_c N_f - \frac{(N_c^2-1)}{N_c}N_f\right) . \tag{1.20}$$

Quadrature and expansion gives from (1.19):

$$a\Lambda_{latt} = (\beta_0 g^2)^{-\beta_1/(2\beta_0^2)} \exp\left\{-\frac{1}{2\beta_0 g^2} + o(g^2)\right\} . \qquad (1.21)$$

Here Λ_{latt} is an integration constant. The consequence of this equation is that for $a \to 0$ we have $g \to 0$ (and vice versa). This fact is the manifestation of asymtotic freedom on the lattice (a closely related feature is the asymptotic vanishing of the renormalized coupling constant g_R as the renormalization scale μ grows). It follows from Eq. (1.16) that for $a,g \to 0$ we have for the hopping parameter $K \to 1/8$.

The renormalization group invariant quantities, like for instance any physical mass M, satisfy for $a \to 0$ the equation

$$\left\{a\frac{\partial}{\partial a} - \beta(g)\frac{\partial}{\partial g}\right\} M(a,g) = o(a^2 \ln a) \to 0 \qquad (1.22)$$

For $a,g \to 0$ this has a solution (using (1.21)):

$$M = c_M \Lambda_{latt} = \frac{c_M}{a}(\beta_0 g^2)^{-\beta_1/(2\beta_0^2)} \exp\left\{-\frac{1}{2\beta_0 g^2} + o(g^2)\right\} . \qquad (1.23)$$

In a numerical calculation one has to show this behaviour as a function of the coupling constant g and one has to extract the value of the constant c_M characterizing M. c_M is the value of the physical quantity M in "lattice units".

In order to connect the results obtained on the lattice with other physical quantities calculated in continuum perturbation theory, one has to investigate the bahaviour of some Green's functions at large momenta. Comparing the result obtained with lattice cut-off to the one in some conventional regularization scheme it is possible to deduce the connection between Λ_{latt} and some other Λ -parameter [9]. The result is:

$$\Lambda^{(Feynman)}_{MOM} / \Lambda_{latt} = \begin{cases} 83.5 & (N_c = 3,\ N_f = 0) \\ 57.5 & (N_c = 2,\ N_f = 0) \\ 105.7 & (N_c = 3,\ N_f = 3) \end{cases} \qquad (1.24)$$

FINITE LATTICE SIZE EFFECTS. The numerical calculations are done on finite size lattices. Therefore, besides the effects of the finite lattice spacing $a \neq 0$, one has also to worry about the influence of the finite lattice extension on the results. For instance, an N^4 lattice has the volume

$$V = L^3 = (aN)^3 , \qquad (1.25)$$

and inverse temperature

$$\beta = \frac{1}{T} = aN . \qquad (1.26)$$

We are usually interested in physical quantities at $V \to \infty$ and $\beta \to \infty$ $(T \to 0)$. (For exceptions see Chapter VII.) This means that, in principle, the infrared cut-off L^{-1} has to be removed first by taking the $N \to \infty$ limit, and then one can make the continuum limit $g \to 0$.

In practice, in order to keep both finite lattice spacing and finite lattice size effects small, one should always do the calculation in a range where

$$a \ll \xi \ll L , \qquad (1.27)$$

ξ being some characteristic correlation length of the system. This is, in general, difficult to fulfil with the presently available Monte Carlo calculations.

Very often periodic boundary conditions are imposed on the lattice configuration. This is believed to make the finite size effects smaller as compared to other possible choices like e.g. fixed or free boundaries. (For a discussion und use of the different boundary conditions see [10].) The effect of periodicity can be simply expressed

also in momentum-space. Denoting the (integer) coordinates of a lattice point by (n_1,n_2,n_3,n_4) $(1 \leq n_i \leq N_i;\ i = 1,\ldots,4)$, the periodicity of some amplitude A means

$$A(\ldots n_i \ldots) = A(\ldots n_i \pm N_i \ldots) \quad ; \quad (i = 1,\ldots,4) \quad . \tag{1.28}$$

A is defined on a four-dimensional torus, where there is more possibility for signal propagation then in the infinite space. Therefore, the connection to the corresponding infinite space amplitude A has to be taken as

$$A(n_1 \ldots n_4) = \sum_{l_1 \ldots l_4 = -\infty}^{+\infty} \mathcal{A}(n_1 + l_1 N_1, \ldots, n_4 + l_4 N_4) \, . \tag{1.29}$$

The terms on the right hand side with $l_i \neq 0$ describe propagations involving several windings around the torus. The Fourier-transform

$$\tilde{A}(p_1 \ldots p_4) = \sum_{n=1}^{N} e^{i(p_1 n_1 + \ldots + p_4 n_4)} A(n_1 \ldots n_4) \tag{1.30}$$

goes over in the continuum limit into the Fourier-transform of A if the momentum components obey

$$p_i = \frac{2\pi}{N_i} l_i \quad ; \quad (l_i = B_i, B_i + 1, \ldots, B_i + N_i - 1) \quad , \quad (i = 1,\ldots,4) \quad . \tag{1.31}$$

Here the integer number B_i is arbitrary. (Usually it is taken to be zero or $-\,\mathrm{int}(N_i/2)$.) In dimensional units, where the position-coordinates are $x_i = a n_i$, the momentum components are

$$k_i \equiv \frac{p_i}{a} = \frac{2\pi l_i}{a N_i} \qquad (i = 1,\ldots,4) \, . \tag{1.32}$$

For a given lattice spacing a the effect of taking more and more lattice points amounts to a denser and denser population of the Brillouin-zone

$$-\frac{\pi}{a} < k_i \leq \frac{\pi}{a} \qquad (i = 1,\ldots,4) \tag{1.33}$$

by the discrete momentum points in (1.32). After the limit $N_i \to \infty$ $(i = 1,2,3,4)$, i.e. infinite volume $V \to \infty$ and zero temperature $T \to 0$, has been taken the high momenta outside the Brillouin-zone (1.33) are still cut-off. This ultraviolet cut-off is removed in the zero lattice spacing $a \to 0$ limit which, according to Eq. (1.21), is achieved by taking $g \to 0$.

II. Hamiltonian formulation

TEMPORAL GAUGE. The quantum mechanics of lattice QCD is simplest to formulate in the "temporal gauge" where the time-component of the gauge field is zero:

$$A_a(x)_o = 0 \quad (a = 1,2,\ldots,8) . \tag{2.1}$$

The colour-electric field strength is in this gauge, according to Eq. (1.1),

$$E_a(x)_k \equiv F_a(x)_{ok} = -\frac{\partial A_a(x)_k}{\partial x_o} \qquad (k = 1,2,3) , \tag{2.2}$$

and the colour-magnetic field strength is:

$$H_a(x)_k \equiv \frac{1}{2}\,\varepsilon_{klm}\,F_a(x)_{lm} = \tag{2.3}$$

$$= \varepsilon_{klm}\,\nabla_l\,A_a(x)_m + \frac{g}{2}\,\varepsilon_{klm}\,f_{abc}\,A_b(x)_l\,A_c(x)_m .$$

The classical Lagrangean in the temporal gauge corresponding to (1.3) is

$$\mathcal{L} = -\frac{1}{2}H_a(x)_k\,H_a(x)_k + \frac{1}{2}E_a(x)_l\,E_a(x)_l +$$

$$+\bar{\psi}(x)\left[\frac{i}{2}\left(\vec{\partial}-\overleftarrow{\partial}\right)\cdot\gamma - m\right]\psi(x) + g\,A_a(x)_k\,\bar{\psi}(x)\frac{\lambda_a}{2}\gamma_k\,\psi(x) . \tag{2.4}$$

This has a similar from as the QED Lagrangean. The last term describes the colour interaction of quarks through the colour current density of quarks

$$J_a(x)_k = \bar{\psi}(x)\frac{\lambda_a}{2}\gamma_k\,\psi(x) . \tag{2.5}$$

The equations of motion, analogous to the Maxwell-equation, are

$$\nabla_m F_a(x)_{mn} - \frac{\partial E_a(x)_n}{\partial x_o} + g\,f_{abc}\,A_b(x)_m\,F_c(x)_{mn} = g\,J_a(x)_n . \tag{2.6}$$

This has to be supplemented by the Gauss-law for colour charge

$$\nabla_n E_a(x)_n = g\left[\bar{\psi}(x)\frac{\lambda_a}{2}\gamma_o\,\psi(x) + f_{abc}\,E_b(x)_n\,A_c(x)_n\right] . \tag{2.7}$$

The Hamilton-density corresponding to the Lagrangean (2.4) is

$$\mathcal{H} = \frac{1}{2}E_a(x)_k\,E_a(x)_l + \frac{1}{2}H_a(x)_l\,H_a(x)_k + m\,\bar{\psi}(x)\psi(x) - \tag{2.8}$$

$$-\frac{i}{2}\,\bar{\psi}(x)\left(\vec{\nabla}_n-\overleftarrow{\nabla}_n\right)\gamma_n\,\psi(x) - g\,A_a(x)_n\,J_a(x)_n .$$

The gauge condition in Eq. (2.1) does not fix the gauge completely, as it is still left invariant by the time-independent gauge transformations

$$A(x)_k \to U^{-1}(\underset{\sim}{x})\,A(x)_k\,U(\underset{\sim}{x}) - \frac{i}{g}\left(\nabla_k\,U(x)^{-1}\right)U(\underset{\sim}{x}) . \tag{2.9}$$

The $SU(N_c)$ element $U(\underset{\sim}{x})$ can depend here only on the space coordinates $\underset{\sim}{x} \equiv (x_1x_2x_3)$ but not on the time $x_o \equiv t$. As it can be easily shown, the temporal gauge Lagrangean (2.4) and the subsidiary condition (2.7) are invariant under the transformations (2.9). Therefore, the time-independent gauge transformations are exact (local) symmetries in the temporal gauge.

CONTINUOUS TIME LIMIT. Let us now deduce the (Hamiltonian) quantum mechanics of lattice QCD from the Euclidean lattice formulation given in the introduction. For simplicity, let us consider now the pure gauge theory only.

In order to make the Euclidean time $\tau \equiv x_4$ in it continuous one has to make the lattice spacing in the fourth direction (a_τ) different from the lattice spacing in space-directions (a). Then the discrete values of the Euclidean time are $\tau = Ka_\tau$ (K integer). Half of the plaquettes are "space-like" and belong to a single hyperplane $\tau = Ka_\tau$, the other half are "time-like" and connect two neighbouring τ-values with $\tau = Ka_\tau$ and $(K+1)a_\tau$ (see Fig. 3)- In the temporal gauge the links in the fourth direction are all put equal to unity, therefore the pure gauge action corresponding to (1.14) on the asymmetric lattice can be written as

$$S[U] = -\frac{1}{g^2}\sum_K \Big\{ \frac{a_\tau}{a} \sum_{k_K[rs]_\omega} \mathrm{Tr}\, U(k_K[rs]_\omega) + \frac{a}{a_\tau} \sum_{k_K \vec{r}} \mathrm{Tr}\Big[U[k_{K+1}\vec{r}]^+ U[k_K \vec{r}] + U[k_{K+1}\vec{r}]\, U[k_K \vec{r}]^+ \Big] \Big\}. \tag{2.10}$$

If there are N lattice points in the time direction, then the partition function in (1.17) has the form

$$Z = \int dU\, e^{-S[U]} = \mathrm{Tr}\,(\hat{T}^N_{a_\tau}) = \int dU_1 \dots dU_N \langle U_1 | \hat{T}_{a_\tau} | U_2 \rangle \dots \langle U_{N-1} | \hat{T}_{a_\tau} | U_N \rangle \langle U_N | \hat{T}_{a_\tau} | U_1 \rangle. \tag{2.11}$$

Here dU stands for integration over all link variables and dU_K (K = 1,...,N) denotes an integration over the links at given time $\tau = Ka_\tau$. The operator $\hat{T}_{a_\tau}$ is called "transfer operator". For $a_\tau \to 0$ it has the limit

$$\hat{T}_{a_\tau} = e^{-a_\tau \hat{H} + \sigma(a_\tau^2)} \tag{2.12}$$

This defines the Hamilton-operator $\hat{H}$.

Let us remark, that the above formulae have to be slightly corrected in order to assure Lorentz-invariance also in the Hamiltonian formulation. Namely, on an asymmetric lattice one has to choose different coupling constants in front of space-like and time-like plaquettes [9]. (This is something like the "renormalization of the speed of light" [11].) In Eq. (2.10) this amounts to the substitutions

$$\frac{1}{g^2}\frac{a_\tau}{a} \to \frac{1}{g_s^2}\frac{a_\tau}{a} \quad ; \quad \frac{1}{g^2}\frac{a}{a_\tau} \to \frac{1}{g_\tau^2}\frac{a}{a_\tau}. \tag{2.13}$$

In the continuum limit $g \to 0$ the connection between the different coupling constants is

$$\frac{1}{g_{s,\tau}^2} = \frac{1}{g^2} + c_{s,\tau} + \sigma(g^2) \tag{2.14}$$

with some calculable constants c_s and c_τ depending on the asymmetry a_τ/a (see also Chapter VII.).

HILBERT SPACE OF STATES. The transfer operator is defined in the Hilbert-space of square-integrable functions of link variables (at a given time). The set of link variables (let us denote them briefly by $U \equiv \{U[k\vec{r}]\}$) plays the same rôle here as the coordinates in ordinary quantum mechanics. The link operator $\hat{u}[k\vec{r}]$ acts on the state by multiplying it with the value of the link variable:

$$\hat{u}[k\vec{r}]\,|U\rangle = U[k\vec{r}]\,|U\rangle. \tag{2.15}$$

An element of a local (time independent) gauge transformation is the multiplication of a link variable (for instance, from the right). This is described by the operator

$$\hat{T}\{U_\alpha[\vec{k}]\}|U\rangle = |\ldots U[\vec{k}]\,U_\alpha[\vec{k}]\ldots\rangle , \qquad (2.16)$$

which we write in the form

$$\hat{T}\{U_\alpha[\vec{k}]\} = \exp\Big\{-i\sum_a \alpha_a[\vec{k}]\hat{E}_a[\vec{k}]\Big\} , \qquad (2.17)$$

if the $SU(N_c)$ element $U_\alpha[\vec{k}]$ is given by the parameters α_a like

$$U_\alpha[\vec{k}] = \exp\Big\{-i\sum_a \alpha_a[\vec{k}]\frac{\lambda_a}{2}\Big\} \in SU(N_c) . \qquad (2.18)$$

The operator $\hat{E}_a[\vec{k}]$ is something like the momentum operator in ordinary quantum mechanics. Here it is called the "colour electric flux operator" on the link $[\vec{k}]$. (In the continuum theory it is more usual to define the colour electric operator in a point and not on a link. This is also possible on the lattice [12], but there the link operators are more conventional.)

Taking the limit $a \to 0$ in Eqs. (2.11-12) gives the following Hamilton-operator:

$$\hat{H} = \hat{H}_E + \hat{H}_M = \frac{g^2}{2a}\sum_{\vec{k}}\hat{E}_a[\vec{k}]\hat{E}_a[\vec{k}] - \frac{1}{ag^2}\sum_{\vec{k}[rs]_\omega} \mathrm{Tr}\,\hat{U}(\vec{k}[rs]_\omega) . \qquad (2.19)$$

The first part here is the colour-electric energy, the second part the colour-magnetic one.

III. Strong coupling expansion

HAMILTONIAN FORMALISM. In the strong coupling limit the colour electric part $\hat{H}_E$ in the Hamilton-operator (2.19) dominates over the colour magnetic part $\hat{H}_M$. This is the basis of the strong coupling perturbation theory. The "free Hamiltonian" is $\hat{H}_E$ and the effect of the "interaction" $\hat{H}_M$ is expanded in powers of g^{-4}.

The lowest energy eigenstate of $\hat{H}_E$ is called the "strong coupling vacuum" state. It is the constant function in the Hilbert-space of states and satisfies

$$\hat{H}_E|0\rangle = 0 . \qquad (3.1)$$

(This follows easily from Eqs. (2.16-17).) All the physical states have to satisfy the lattice analogue of the Gauss-law (2.7), which is equivalent to the requirement of the invariance with respect to time-independent gauge transformations. Namely, it can be shown that the operator corresponding to the difference of the two sides of equation (2.7) is the generator of time-independent gauge transformations. The most general physical state can, therefore, be constructed by acting on the strong coupling vacuum with the trace of the product of link operators along an arbitrary closed curve C_i:

$$|\psi\rangle = \Big\{\prod_i \mathrm{Tr}\,\hat{U}[C_i]\Big\}|0\rangle . \qquad (3.2)$$

These are the physical states without colour charges which span out a subspace in the Hilbert-space of all square integrable functions. The other subspaces belong to the physical states with some external colour charges. For instance, a general state with an external quark in the point A and an external antiquark in the point B is

$$\hat{U}[C_{A\to B}]\Big\{\prod_i \mathrm{Tr}\,\hat{U}[C_i]\Big\}|0\rangle , \qquad (3.3)$$

with an arbitrary curve $C_{A\to B}$ connecting A and B.

All the states (3.2-3) are eigenstates of the colour electric operator $\hat{H}_E$. This follows from the commutation relations of the colour electric flux operator (following from (2.16-18)):

$$[\hat{E}_a[k\vec{\imath}], \hat{E}_b[l\vec{\jmath}]] = \delta_{kl}\delta_{rs} i f_{abc} \hat{E}_c[k\vec{\imath}] \quad , \tag{3.4}$$

and

$$[\hat{U}[i\vec{\jmath}]_{\alpha\beta}, \hat{E}_a[k\vec{\imath}l]] =$$

$$= \delta_{ki}\delta_{rs}\delta_{lj}\left(\hat{U}[i\vec{\jmath}]\frac{\lambda_a}{2}\right)_{\alpha\beta} - \delta_{kj}\delta_{rs}\delta_{li}\left(\frac{\lambda_a}{2}\hat{U}[i\vec{\jmath}]\right)_{\alpha\beta} \quad . \tag{3.5}$$

The colour electric energy of a single non self-intersecting curve is obviously proportional to the length of the curve measured in lattice unit (i.e. the number of links in it). Therefore, the potential energy between a quark-antiquark pair in the infinitely strong coupling limit $g \to \infty$ is proportional to the distance. In other words there is confinement on the lattice for $g \to \infty$. Of course, the question is whether the confinement survives in the $g \to 0$ (weak coupling) continuum limit. For this it is sufficient (but not necessary) that there exists no singularity in the bare coupling constant g between $g = \infty$ and $g = 0$, where the confinement could be lost.

EUCLIDEAN FORMALISM. Technically it is easier to do the strong coupling expansion to higher order in the Euclidean formulation than in the Hamiltonian formulation. The aim is to calculate expectation values like

$$\langle F \rangle = \int dU\, e^{-S[u]} F \Big/ \int dU\, e^{-S[u]} \quad . \tag{3.6}$$

(Now dU means an integral over all-space like and time like - links of the lattice.) The pure gauge field action has the general form of a sum over plaquettes P, like

$$S[U] \equiv -\sum_P L_P = -\sum_P \beta\, d_\chi^{-1}\, \chi[U(P)] \quad , \tag{3.7}$$

where χ is an arbitrary character of $SU(N_c)$ belonging to a representation of dimension d_χ and $\beta \equiv g^{-2}$ is the inverse coupling constant squared. (In Eq. (1.14) the simplest character $\chi(U) = \mathrm{Tr}(U)$ was taken.)

An important step in the strong coupling expansion is the character expansion

$$\exp L_P = \sum_\nu c_\nu(\beta)\, \chi_\nu[U(P)] \, . \tag{3.8}$$

The expansion coefficients c_ν can usually be calculated. For instance in the simplest case of SU(2) and $L_p = \frac{\beta}{2}\,\mathrm{Tr}U(P)$ we have [13]

$$c_j(\beta) = 2(2j+1)\beta^{-1} I_{2j+1}(\beta) \qquad (j = 0, \tfrac{1}{2}, 1, \ldots). \tag{3.9}$$

Writing Eq. (3.8) like

$$\exp L_P = c_0(\beta)\left[1 + f_P\right] \quad , \tag{3.10}$$

the normalization factor $c_0(\beta)$ can be omitted (since it cancels in expectation values), and we have

$$\exp \sum_P L_P = \prod_P (1 + f_P) = \sum_{\mathcal{P}} \prod_{P \in \mathcal{P}} f_P \quad . \tag{3.11}$$

Here $P = \{P_1, P_2, \ldots\}$ is a subset of plaquettes, which can be written as

$$\mathcal{P} = \sum_i X_i \tag{3.12}$$

with disjoint $X_1, X_2, \ldots$ The partition function, for instance, (the denominator in (3.6)) can then be written like

$$Z = \sum_{\mathcal{P} = \sum X_i} \prod_i \phi(X_i) \,, \tag{3.13}$$

where the integral ϕ is

$$\phi(X_i) = \int \prod_l dU_l \prod_{P \in X_i} f_P \,. \tag{3.14}$$

It is easy to show that $\phi(X_i)$ vanishes for all sets X_i having at least one link which occurs only once ("free link"). Calling a set without free links a "polymer", the result for Z is

$$Z = \sum_{\mathfrak{D}} \phi(\mathfrak{D}) \,; \qquad \phi(\mathfrak{D}) = \prod_{X \in \mathfrak{D}} \phi(X) \,. \tag{3.15}$$

Here the sum $\sum_{\mathfrak{D}}$ goes over all sets $\mathfrak{D}$ of disjoint polymers.

STRING CONSTANT. The first quantity which was calculated in the strong coupling expansion to high order is the "string constant" for confinement. (A typical result [13] obtained in the Euclidean formalism is shown in Fig. 4 together with the earlier Monte Carlo results of Creutz. For the Hamiltonian calculation see [14] .) The string constant $\varkappa$ is the slope of the linear confining potential acting on an external quark-antiquark pair in pure gauge theory. In general, the potential acting on the quark-antiquark pair is

$$V(r) = -\lim_{T \to \infty} \frac{1}{T} \ln \langle \mathrm{Tr}\, U[C_W] \rangle \,. \tag{3.16}$$

The "Wilson-loop" C_W is a large rectangle with sides T and r (Fig. 5). Therefore, the string constant $\varkappa$ is

$$\lim_{r \to \infty} \frac{V(r)}{r} = -\lim_{r \to \infty} \lim_{T \to \infty} \frac{1}{rT} \ln \langle \mathrm{Tr}\, U[C_W] \rangle \,. \tag{3.17}$$

If there is (linear) confinement, then according to (3.17), the expectation value of a large Wilson-loop C_W goes to zero exponentially with the area of C_W ("area law"):

$$\langle \mathrm{Tr}\, U[C_W] \rangle \to e^{-\varkappa r T} \,. \tag{3.18}$$

The numerical value of $\varkappa$ deduced from strong coupling or Monte Carlo results (e.g. Fig. 4) is in reasonable agreement with the slope of the phenomenological $q\bar{q}$-potential. (By comparing the numbers to phenomenological ones the factors in Eq. (1.24) have to be taken into account.) The main difficulty of the strong coupling calculation is, of course, the extrapolation to the physically interesting weak coupling region. The extrapolation is usually done by the Padé technique. A rather unpleasant feature is, however, the presence of the "roughening" transition [16] which causes a singularity in the string tension. Beyond the roughening transion point the strong coupling extrapolation cannot be trusted. The Monte Carlo method of the next Chapter is able to go beyond this point (however, only numerically).

IV. Monte Carlo method.

The numerical Monte Carlo method of performing the integrals like (1.18) or (3.6) is a powerful tool for the study of the properties of quantum field theory. Let us restrict ourselves now to the case of a pure $SU(N_c)$ gauge field theory (QCD without quarks).

A typical integral to be evaluated is

$$\langle F \rangle \sim \int \{\prod_{links} dU_\ell\} F[U]\, e^{-S[U]} \tag{4.1}$$

with some function of the link variables $F[U]$, and the pure gauge Wilson-action

$$S[U] = g^{-2} \sum_{\square} \left(N_c - \mathrm{Tr}(U_\square) \right) > 0 \,. \tag{4.2}$$

Since $S[U]$ is non-negative, $e^{-S[U]}$ in (4.1) plays the rôle of the "Boltzmann-factor" in a (classical) statistical mechanics of fields defined on the (Euclidean) lattice. Similar integrals occur also in ordinary statistical mechanics and the methods of numerical evaluation can be taken over from there [17].

The number of integration variables in (4.1) is quite large. For $SU(N_c)$ colour it is (N_c^2-1)-times the number of links in the lattice. This can be as large as 10^5 - 10^6, therefore the only possibility is to use Monte Carlo integration. One has to produce a series of gauge field configurations distributed according to e^{-S} ("importance sampling") and the averaging over the series gives then an approximation to $\langle F \rangle$.

As an example, let us consider SU(3) colour on a (rather large) 10^4 lattice. This has $4 \cdot 10^4$ links, and to specify the SU(3) element one has to give at least 8 real numbers, therefore one has to store in the computer at least $3{,}2 \cdot 10^5$ numbers. In practice this looks even worse, because a reasonable representation of an SU(3) matrix is

$$U = \begin{pmatrix} U_{11} & U_{12} & (U_{21}U_{32} - U_{22}U_{31})^* \\ U_{21} & U_{22} & (U_{31}U_{12} - U_{32}U_{11})^* \\ U_{31} & U_{32} & (U_{11}U_{22} - U_{21}U_{12})^* \end{pmatrix} , \tag{4.3}$$

with

$$\begin{aligned} |U_{11}|^2 + |U_{21}|^2 + |U_{31}|^2 &= |U_{12}|^2 + |U_{22}|^2 + |U_{32}|^2 = 1 \,; \\ U_{11}U_{12}^* + U_{21}U_{22}^* + U_{31}U_{32}^* &= 0 \,. \end{aligned} \tag{4.4}$$

Storing all the 6 complex numbers $U_{11}, \ldots, U_{32}$ means that we have already $4{,}8 \cdot 10^5$ real numbers. Using 4 byte numbers this is $\sim$ 2 Mbyte. (Of course, it is more comfortable to store all the 9 complex numbers, which is already $\sim$ 3 Mbyte!)

The series of gauge field configurations is produced by a Markov-process: having the configuration C, there is a given transition probability $P(c',c)$ for the next one to be C'. It can be shown, that if $P(c',c)$ fulfils the "detailed balance" condition

$$P(c',c)e^{-S(c)} = P(c,c')e^{-S(c')} \,, \tag{4.5}$$

then the Markov-chain of configurations reproduces the distribution e^{-S}.

HEATH BATH METHOD. One way of generating the next configuration from some given one is to go through all links and replace the link variable U by a new one U' which is choosen according to the distribution

$$dP(U') = e^{-S(U',\ \text{rest fixed})} dU' \,. \tag{4.6}$$

S(U', rest fixed) means that in the action S all the other links, except for the picked out one, are fixed. The "heath bath" distribution e^{-S} in (4.6) is rather complicated even for a pure gauge theory, but for SU(2) [15] and SU(3) [18] it is possible to implement.

METROPOLIS METHOD: A generally applicable method for the updating of the configuration is due to Metropolis et al. [19]. It starts by choosing the new link variable U' randomly and calculating the change of the action S(U') - S(U) for the replacement of the link variable U by U'. The change $U \rightarrow U'$ is accepted if $\exp\left[S(U) - S(U')\right]$ is greater than a randomly choosen number in the interval $[0,1]$. (Note, that for decreasing action $S(U) - S(U') > 0$ the change is always performed.) In order to fulfil the detailed balance condition (4.5) the updated link has to be choosen randomly from all the links ("random updating"). In practice there is, however, little difference observed if instead of a random choice the links are taken in some definite order ("systematic updating"). The convergence to the equilibrium distribution e^{-S} is usually speeded up if instead of updating a link only once the above procedure $U \rightarrow U'$ is repeated several times on the same link.

To give an idea about the CPU-times needed for a complete "sweep" (i.e. one updating of all the links) let us take, as an example, an 8^4 lattice in SU(3). With 5 updatings per link a good program needs about 15 sec on the IBM 3081. Taking SU(2) instead of SU(3) gives about a factor of 5 increase in spead (and uses only a third of storage space). For most purposes SU(2) can be replaced by its icosahedron subgroup [20], which is another factor of $\sim$ 4 faster (and needs only little storage space as an element can be specified by an integer number between 1 and 120).

The main results obtained up to now by Monte Carlo methods in pure gauge theories are about

- the quark-antiquark potential;
- the deconfinement phase transition (see Chapter VII);
- the glueball spectrum.

In all these cases the effects of dynamical quarks are neglected. There are calculations both with SU(2) and SU(3) colour. The region where the renormalization group behaviour ("scaling") according to Eq. (1.23) is approximately observed is in SU(2) $\beta = 4/g^2 \gtrsim 2.0$ and in SU(3) $\beta = 6/g^2 \gtrsim 5.0$.

In the "physical" case of SU(3) colour the results are:

- for the square root of the string tension [21,18]: $\sqrt{\varkappa} = (170 \pm 50)\,\Lambda_{latt}$;
- for the deconfinement critical temperature [22,23]: $T_c = (80 \pm 5)\,\Lambda_{latt}$;
- for the mass of the lowest glueball state [24-26]: $m(0^{++}) = (280 \pm 50)\,\Lambda_{latt}$.

These numbers are quite reasonable, for instance, if the value of the lattice scale parameter (without the effects of quarks) is $\Lambda_{latt} = 2\text{-}3$ MeV. Using Eq. (1.24) this corresponds for $\Lambda_{MOM}^{(Feynman)}$ to a value of about 200 MeV.

Besides the value of the string tension there exist now also more detailed Monte Carlo studies of the $q\bar{q}$-potential in the SU(2) lattice gauge theory. A very interesting and important study [27] showed the rapid restoration of rotation symmetry for the $q\bar{q}$-potential (Fig. 6a-b). In a recent paper [28] it was also shown that the theoretically expected Coulomb + linear form fits the Monte Carlo data excellently (Fig. 7).

V. The spectrum of QCD.

The great challenge for lattice QCD is, of course, the calculation of the data in the Rosenfeld-table [29] about hadron masses, couplings etc.. This cannot be done without introducing also the quark fields on the lattice.

WILSON LATTICE FERMIONS. There is an immediate problem for the introduction of fermion fields on the lattice, which is known under the name of "fermion doubling". Let us consider first a free fermion field. The "naive" lattice action can be obtained by the substitution

$$\frac{\partial}{\partial x^\mu}\psi(x) \longrightarrow \frac{1}{a}\left(\psi_{x+\hat{\mu}} - \psi_x\right) \tag{5.1}$$

from the continuum Euclidean action:

$$S_f(\bar{\psi},\psi) = \sum_x \left\{ m a^4 \bar{\psi}_x \psi_x - \frac{a^3}{2}\left(\bar{\psi}_{x+\hat{\mu}}\gamma_\mu \psi_x - \bar{\psi}_x \gamma_\mu \psi_{x+\hat{\mu}}\right)\right\} . \tag{5.2}$$

The corresponding free propagator $\tilde{G}(k)$ in momentum space satisfies the equation

$$\left[(am)^2 + \sum_{\mu=1}^{4} \sin^2(a k_\mu)\right] \tilde{G}(k) = 1 . \tag{5.3}$$

This shows that in the continuum limit $a \longrightarrow 0$ $\tilde{G}(k)$ has poles at $ak_\mu \cong 0, \pm\pi$. The poles near zero correspond to the wanted fermion state but the ones near $\pm\pi$ describe superfluous states not present in the original continuum theory. A simple way to get rid of these superfluous states is, following Wilson [30], to add a term which gives an infinite mass in the continuum limit to the superfluous states. The free fermion action becomes then ($r > 0$):

$$S_f(\bar{\psi},\psi) = \sum_x a^3 \left\{ (4r + am)\bar{\psi}_x \psi_x - \frac{1}{2}\sum_\mu \bar{\psi}_{x+\hat{\mu}}\,(r+\gamma_\mu)\psi_x \right\} . \tag{5.4}$$

(Here the sum $\sum_\mu$ runs also over negative directions: $\pm 1, \pm 2, \pm 3, \pm 4$.) With a convenient new normalization of the fermion fields and $K \equiv (2am + 8r)^{-1}$ this can be written as

$$S_f(\bar{\psi},\psi) = \sum_x \left\{ \bar{\psi}_x \psi_x - K \sum_\mu \bar{\psi}_{x+\hat{\mu}}\,(r+\gamma_\mu)\psi_x \right\} . \tag{5.5}$$

Going, as usual, to the momentum space, the equation for the lattice fermion propagator becomes

$$\left[1 - 2K \sum_{\mu>0}\left(r\cos\frac{2\pi l_\mu}{N_\mu} - i\gamma_\mu \sin\frac{2\pi l_\mu}{N_\mu}\right)\right]\tilde{G}_l = 1 . \tag{5.6}$$

Here we assumed a finite lattice with periodic boundary conditions, where the momentum values are given by Eq. (1.32). In the derivation of (5.6) a useful formula was

$$\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i \frac{ln}{N}} = \delta_{l,0} \qquad (l = 0, 1, \ldots, N-1) . \tag{5.7}$$

For the coordinate space propagator Eq. (5.6) gives

$$G_n = \frac{1}{N_1 N_2 N_3 N_4} \sum_{l=B}^{B+N-1} \exp\left\{-2\pi i\left(\frac{n_1 l_1}{N_1} + \cdots + \frac{n_4 l_4}{N_4}\right)\right\} \cdot$$
$$\cdot \left\{1 - 2K\sum_{\mu>0}\left[r\cos\frac{2\pi l_\mu}{N_\mu} + i\gamma_\mu \sin\frac{2\pi l_\mu}{N_\mu}\right]\right\} \cdot$$
$$\cdot \left\{\left[1 - 2K\sum_{\mu>0} r\cos\frac{2\pi l_\mu}{N_\mu}\right]^2 + 4K^2 \sum_{\mu>0}\sin^2\frac{2\pi l_\mu}{N_\mu}\right\}^{-1} . \tag{5.8}$$

Going to the Minkowski-space means (using (1.32)) to replace

$$\frac{2\pi l_4}{N_4} \longrightarrow -iaE , \tag{5.9}$$

where E is the energy. The pole position in the energy is, therefore, given by

$$\sinh^2(aE) = \Big[am + 4r - r\sum_{n=1}^{3}\cos(ak_n) - r\cosh(aE)\Big]^2 + \sum_{n=1}^{3}\sin^2(ak_n). \tag{5.10}$$

In the special case of zero three-momenta $k_n = 0$ (n = 1,2,3) we have

$$\pm\sinh(aE) = am + r - r\cosh(aE) \ . \tag{5.11}$$

For $a \to 0$ the only solution is $\pm E = m$, as it should be. In the general case the solution (for $0 < r \leq 1$) is:

$$\pm aE = \ln\left\{\frac{1}{1+r}\left[am + r + \sqrt{(am)^2 + 2ram + 1}\right]\right\} . \tag{5.12}$$

The price which has to be paid for the elimination of fermion doubling in the Wilson lattice fermion action is the loss of chiral invariance. The extra mass term $4r\bar{\psi}\psi$ breaks chiral symmetry explicitly even for zero bare mass am = 0. In the continuum limit this explicit breaking should go over in a spontaneous breaking.

After introducing the gauge interaction the fermion part of the QCD Wilson-action becomes (in accordance with (1.14)):

$$S_f = \sum_x \{\bar{\psi}_x\psi_x - K\sum_\mu \bar{\psi}_{x+\hat{\mu}}\, U[x+\hat{\mu}, x](r+\gamma_\mu)\psi_x\} \equiv$$

$$\equiv \sum_{x,y} \bar{\psi}_x Q_{xy} \psi_y \ . \tag{5.13}$$

This is bilinear in the quark fields with

$$Q = 1 - KM \tag{5.14}$$

The "hopping matrix" M is a sum over lattice sites and (8) directions:

$$M_{x_1x_2} = \sum_{x,\varrho} (r+\gamma_\varrho) U[x+\hat{\varrho}, x]\, \delta_{x_1, x+\hat{\varrho}}\ \delta_{x, x_2} \ . \tag{5.15}$$

The anticommuting quark variables $\bar{\psi}_x, \psi_x$ are not well suited for the computer, therefore it is convenient to perform the fermion integrations. For instance, the partition function becomes then

$$Z = \int dU\, d\bar{\psi}\, d\psi\, e^{-S[u]-S_f} = \tag{5.16}$$

$$= \int dU\, \det Q\, e^{-S[u]} \ ,$$

where $S[U]$ is the pure gauge action. The "quark determinant" detQ can also be written like

$$\det Q = \det(1-KM) \equiv \exp\{-S_{eff}[U]\} \ , \tag{5.17}$$

with the effective action $S_{eff}[U]$ depending only on the gauge field.

QUENCHED APPROXIMATION. The quark determinant in (5.16) takes into account the effect of closed virtual quark loops. In a numerical calculation it is not easy to evaluate detQ, because Q or M are very large matrices. For instance, on a 10^4 lattice in SU(3) the indices of Q or M can have $10^4 \cdot 3 \cdot 4 = 1{,}2 \cdot 10^5$ different values. This is by far too much for the conventional numerical ways of the evaluation of determinants. A radical solution is to neglect the quark determinant altogether in the calculation. This "quenched" approximation [31,32] can, in fact, be in many cases quite good. This is suggested by general theoretical arguments like the $1/N_c$ expansion and by phenomenological observations like the validity of the string picture and the Zweig-rule etc..

Some problems, however, cannot be treated without the quark determinant (examples are: the mixing of flavour neutral mesons, the mixing of glueball states with quark states etc.). In these cases one has to apply the Metropolis updating algorithm, where only the ratio of the two determinants (corresponding to the change $U \to U'$ on a link) is needed. The change in the effective action is

$$\Delta S_{eff} = -\ln\det\frac{1-KM(U')}{1-KM(U)} = \\ = -\ln\det\{1+K[M(U)-M(U')](1-KM(U))^{-1}\}. \qquad (5.18)$$

The matrix $[M(U) - M(U')]$ has only a few non-zero elements, therefore only a relatively small (24 x 24 in SU(3)) determinant has to be calculated. The difficulty is, of course, the reasonably fast evaluation of the matrix elements of $(1 - KM)^{-1}$.

OTHER LATTICE FERMIONS. Another way of diminishing the fermion degeneracy on the lattice was developped by Susskind [33]. The Susskind lattice action, in a simplified formulation [34,35], has the form

$$S_f = \sum_x \{am\,\tilde{\chi}_x\chi_x - \frac{1}{2}\sum_\mu \eta_{\mu,x}\,\tilde{\chi}_{x+\hat{\mu}}\,U[x+\hat{\mu},x]\,\chi_x\}, \qquad (5.19)$$

where χ_x, $\tilde{\chi}_x$ are single component Grassmann-fields and the phase factor $\eta_{\mu,x} = -\eta_{-\mu,x}$ can be choosen as

$$\eta_{1,x} = 1, \quad \eta_{2,x} = (-1)^{n_1}, \quad \eta_{3,x} = (-1)^{n_1+n_2}, \\ \eta_{4,x} = (-1)^{n_1+n_2+n_3}; \quad x = a(n_1,n_2,n_3,n_4). \qquad (5.20)$$

The above Susskind action describes 4 identical quark species ("flavours"). As χ is a single component field, the different spin and flavour components sit on different lattice sites. The reduction to the four components shows [36] that at finite lattice spacing the above action describes in the free case U = 1 flavour changing and parity violating transitions, which go away only at $a \to 0$. This restricts its applicability to the spectrum calculations. The great advantage of the Susskind

action is, however, that it preserves a discrete subgroup of the chiral symmetry for m = 0 also at finite lattice spacings. Therefore, it is well suited for the study of spontaneous chiral symmetry breaking.

From the theoretical point of view it is very interesting, that the action (5.19) can be considered as the lattice description of the Dirac-Kähler fermions. (The Dirac-Kähler equation in the continuum is the Dirac-equation written in terms of differential forms.) [37].

Another interesting question is whether it is possible to put chiral fermions on the lattice (for instance, in order to study the electro-weak interaction theory with lattice cut-off). The answer is , that under certain (rather weak) assumptions not [38]. This is a difficulty for putting the Glashow-Weinberg-Salam theory on the lattice.

HOPPING PARAMETER EXPANSION. In the case of the Wilson action the hopping parameter $K = (8r + 2am)^{-1}$, in the continuum limit $a \to 0$, goes to $K = 1/(8r)$. For r = 0(1) this is a small number, therefore it is worth to try an expansion in K. The hadron masses are extracted from the behaviour of some multi-quark Green's functions. For instance, the meson ground states appear in

$$T_{x_4} = \langle 0|T\{\sum_{\vec{x}} \tilde{\psi}_{\vec{x}x_4}\,\Gamma\,\psi_{\vec{x}x_4}\cdot\tilde{\psi}_{\vec{0}0}\,\Gamma\,\psi_{\vec{0}0}\}|0\rangle. \qquad (5.21)$$

The Dirac-matrix Γ is equal to γ_5 for the pseudoscalar mesons and $\gamma_{1,2,3}$ for the vector mesons. The sum over the space-points $\sum_{\vec{x}}$ ("time slice") assures that in Eq. (5.21) only the intermediate states with zero three-momentum contribute. In the flavour non-singlet channel the Green's function (5.21) is equal to the expectation value

$$T_{x_4} = \sum_{\vec{x}} \langle Tr_{CD}\{\Gamma(1-KM)^{-1}_{\vec{x}x_4,\vec{0}0}\,\Gamma\,(1-KM)^{-1}_{\vec{0}0,\vec{x}x_4}\}\rangle. \qquad (5.22)$$

Tr_{CD} denotes here the trace over colour and Dirac-indices.

The expansion of the quark propagator $(1 - KM)^{-1}$ in the hopping parameter is:

$$(1-KM)^{-1} = \sum_{j=0}^{\infty} K^j M^j \quad . \tag{5.23}$$

The effective action for the fermion determinant in (5.17) can also be expanded:

$$S_{eff}[U] = \sum_{j=1}^{\infty} \frac{K^j}{j} \mathrm{Tr}_{CDS}\left(M[U]^j\right) \quad . \tag{5.24}$$

Tr_{CDS} means a complete trace over all sorts of indices, that is also a sum over lattice sites. Using the explicit structure of the hopping matrix M in (5.15) we get

$$\mathrm{Tr}_{CDS}\left(M^j\right) = \sum_{x_1\varrho_1 \ldots x_j\varrho_j} \mathrm{Tr}_C\left\{U[x_1+\hat{\varrho}_{1}, x_1] \ldots U[x_j+\hat{\varrho}_j, x_j]\right\} \cdot \tag{5.25}$$

$$\cdot \mathrm{Tr}_D\left[(r+\gamma_{\varrho_1}) \ldots (r+\gamma_{\varrho_j})\right] \delta_{x_1+\hat{\varrho}_1, x_2} \delta_{x_2+\hat{\varrho}_2, x_3} \cdots \delta_{x_j+\hat{\varrho}_j, x_1} \quad .$$

The δ -functions on sites obviously define closed lines on the lattice, therefore

$$\mathrm{Tr}_{CDS}\left(M^j\right) = \sum_{\text{closed lines}} \mathrm{Tr}_C\{\ldots\}\,\mathrm{Tr}_D[\ldots] \quad , \tag{5.26}$$

that is, the complete trace is a sum over all closed lines of the product of colour and Dirac-traces. Similarly, the matrix elements of the quark propagator (5.23) can also be written as a sum over <u>open</u> lines, where the length of the line is equal to the order of the hopping parameter.

For large values of the time-distance x_4 the time slice amplitude (5.21) behaves like

$$T_{x_4} \xrightarrow[x_4 \to \infty]{} e^{-M_\Gamma x_4} \quad , \tag{5.27}$$

where M_Γ is the lowest mass in the given channel. This allows for the extraction of the mass M_Γ . In fact, for periodic boundary conditions Eq. (1.29) implies that (5.27) is modified to

$$T_{x_4} \to e^{-M_\Gamma a n_4} + e^{-M_\Gamma a (N_4 - n_4)} \qquad (n_4, N_4 - n_4 \gg 1), \tag{5.28}$$

with N_4 the number of lattice points in the time direction. (The higher terms in (1.29), corresponding to multiple windings around the torus, are neglected here.)

The hopping parameter expansion method [30,39] is based on the Monte Carlo evaluation of the K-expansion coefficients in pure gauge theory, after the fermion integration (5.16) is carried out. A possibility to determine the masses is to calculate the expansion coefficients of the time slices, to make an analytic continuation with Padé approximants to the required values (1.16) of the hopping parameter and to use Eq. (5.28). This procedure does not work very well, because only a few coefficients can be calculated for distant time slices. (The lowest non-vanishing coefficient in the K-expansion has the order $\min(n_4, N_4-n_4)$.) A better procedure is to first Fourier-transform to the momenta p_4 in (1.31) and do the Padé-extrapolation there. After the extrapolation is done one can either transform back to x-space und use Eq. (5.28), or stay in p-space and use the fact that a particle corresponds to a pole in the energy E (as defined in (5.9)):

$$\tilde{T}(E) \sim \frac{R}{E^2 - M_\Gamma(q, K)^2} \quad . \tag{5.29}$$

Another, rather advantageous, possibility for the hopping parameter expansion method is to consider the quark propagation without boundaries.

This is achieved by copying some periodic gauge field configuration so many times in the infinite lattice space that it covers all the curves for the quark propagation up to some given order (this can be called "copied gauge field" procedure in distinction to the "periodic box" case with periodic quark and gauge fields). In this case one usually obtains much more time slices, and there are not singled out momentum points like in Eq. (1.32). Therefore, the pole corresponding to (5.29) can be looked for in the hopping parameter variable K for any fixed value of E (and g). This is the easiest and best thing one can do with the Padé-approximants.

The difficulty for the K-expansion method is to go to high enough order, especially for light quarks. (Note, that because of Eq. (1.16) the hopping parameter expansion is essentially an $^1/_m$ expansion in the inverse quark mass.) The first detailed calculations [40] were done up to 10^{th} order for mesons and 12^{th} order for baryons. The results are in the strong coupling region $\beta < 5.5$ similar to the analytic strong coupling results: baryon/meson mass ratio $\simeq 2$, too small spin splitting between N and Δ etc.. The short series could not reproduce scaling in the "transition region" $\beta > 5{,}5$. The recent higher order calculation [41] (24^{th} order in the "copied gauge field" case and 32^{nd} order in an 8^4 "periodic box") at $\beta = 5.7$ and 6.0 suffers from another disease (see below). The difficulties, however, seem to be not insurmountable, therefore the hopping parameter expansion method may give very valuable results for the hadron spectrum in the future.

GAUSS-SEIDEL METHOD. Another way to calculate the matrix elements of the quark propagator $(1 - KM)^{-1}$ is to use Gauss-Seidel iteration [31,32]. Choosing some initial vector i let us define the vector g by

$$g = (1-KM)^{-1} i \,. \tag{5.30}$$

This is equivalent to

$$g = i + KMg \,, \tag{5.31}$$

which can be used as an iterative equation:

$$g_n = i + KMg_{n-1} \quad (n = 1,2,\dots); \; g_0 = i \,. \tag{5.32}$$

Taking on the right hand side the already calculated elements of g_n instead of the old g_{n-1} elements (and hence also saving half of the storage space in the computer), this equation becomes

$$g_n = i + KM_{\triangleright} g_n + KM_{\triangleleft} g_{n-1} \,. \tag{5.33}$$

Here, in the decomposition $M = M_{\triangleright} + M_{\triangleleft}$, the part $M_{\triangleright}$ has non-zero elements only below the main diagonal. Eq. (5.33) can also be supplied by a relaxation parameter ω, in order to speed up the convergence:

$$g_n = (1-\omega) g_{n-1} + \omega \left[i + KM_{\triangleright} g_n + KM_{\triangleleft} g_{n-1} \right] . \tag{5.34}$$

The main problem in the Gauss-Seidel iteration method is the very slow convergence for the physically interesting values of the hopping parameter. In most cases there is, in fact, no convergence at the physical value [31], therefore the iteration has to be run at somewhat lower values of K, and the results (masses etc.) have to be (e.g. linearly) extrapolated in K. This introduces large errors and even with these errors the results are not completely satisfactory [31,32].

The present calculations of the hadron spectrum with the Gauss-Seidel method are restricted to the quenched approximation. Taking into account the quark determinant, for instance, by Eq. (5.18) would be extraordinarily slow (except for some extremely small lattices). To calculate the quark determinant in the hopping parameter expansion seems easier (and it was actually done in Ref. [40]).

A variant of the original iteration equation (5.32) ("Jacobi-iteration") can also be used for the calculation of the hopping parameter expansion coefficients of the quark propagator [41]. Defining i_n by

$$i_n \equiv M^n i \qquad (n = 0,1,2,\dots) \,, \tag{5.35}$$

we have the iterative equation

$$i_n = M\, i_{n-1} \qquad (n = 1,2,\dots). \tag{5.36}$$

This gives the required matrix elements for (5.23). A further possibility is to evaluate the matrix elements of M^n stochastically [42].

FINITE SIZE PROBLEM. Confinement is an infrared property of QCD, therefore on a finite lattice there is no exact confinement, because the infrared cut-off $L^{-1} = V^{-1/3}$ is still finite. Saying it differently, the confining potential is modified by the boundary conditions at distances of the order of the lattice size L. In the case of periodic boundary conditions the infinitely rising confining potential is replaced by a periodic one (Fig. 8). In the periodic potential there are no real bound states. The stationary states are the Bloch-waves which describe the more or less strong tunneling between the potential minima. Of course, for large lattices (which are much larger than a typical hadron size) the lowest energy bands are very narrow, the tunneling is very weak and these states approximate the bound states well enough for any practical purposes.

The problem with the existing hadron spectrum calculations is that the spatial size is usually even smaller than the expected physical size ($\gtrsim$ 2 fm) of a hadron. The recent high order hopping parameter expansion on an 8^4 lattice (which has a larger spatial size than the other recently considered lattices) shows [41], that the singularity of the multi-light-quark Green's functions looks very much more like a cut (corresponding to almost free quarks) than a pole or a sum of physical poles (corresponding to bound states). Even for heavy (charmed or bottom) quarks, where the tunneling is weaker, the first two bound states are very probably dissolved in rather broad bands. This is shown also by a potential model studied in [41]. (See Fig.9.)

This size problem can presumably be to a large extent overcome by taking lattices with 2- or 3-times larger linear sizes. This would mean 16^4 or 24^4 lattices which are certainly feasible by the presently known methods.

VI. Current amplitudes

The simplest, "naive" definition of the vector and axialvector current on the lattice is to take $\bar\psi_x \gamma_\mu \psi_x$ and $\bar\psi_x \gamma_\mu\gamma_5 \psi_x$, respectively. The correct normalization of the currents is, however, a nontrivial problem. (Conserved charges have to keep their values in spite of the gauge interaction.) In the continuum the conserved currents keep their normalization due to the Ward-identities. The same can be achieved also on the lattice with a current which satisfies a conservation equation on the lattice. In the case of the the Wilson-action (with r = 1) the correctly normalized conserved vector current is [43]:

$$V(x)_\mu \equiv V_{x,\mu} = Ka^{-3}\{\bar\psi_{x+\hat\mu}\, U[x+\hat\mu, x](1+\gamma_\mu)\psi_x - \bar\psi_x\, U[x, x+\hat\mu](1-\gamma_\mu)\psi_{x+\hat\mu}\}. \tag{6.1}$$

As a consequence of the equations of motion, this satisfies the conservation equation,

$$\sum_{\mu=1}^{4} \nabla_\mu V_{x-\hat\mu,\mu} \equiv \sum_{\mu=1}^{4} a^{-1}\left[V_{x,\mu} - V_{x-\hat\mu,x}\right] = 0. \tag{6.2}$$

The normalization of the axialvector current is more difficult, because it is not conserved even for zero masses. (This is the consequence of the explicit chiral symmetry breaking in the Wilson-action.)

A simple current amplitude, which can be calculated on the lattice, is the electromagnetic vacuum-polarization

$$A(x)_{\mu,\nu} = \langle 0|T\{J_{em}(x)_\mu\, J_{em}(0)_\nu\}|0\rangle. \tag{6.3}$$

The electromagnetic current $J_{em\,\mu}$ is built up from the vector currents $V_{q,\mu}$ of quark flavours q = u,d,c,s,... like

$$J_{em,\mu} = \tfrac{2}{3} V_{u,\mu} - \tfrac{1}{3} V_{d,\mu} + \tfrac{2}{3} V_{c,\mu} - \tfrac{1}{3} V_{s,\mu} + \dots \tag{6.4}$$

In a lattice Monte Carlo calculation one usually considers the "time-slices" of the spacelike components of the vacuum polarization amplitude

$$T_{rs}(x_4) = \int d^3x\, A(\underset{\sim}{x}, x_4)_{rs} \quad , \tag{6.5}$$

or its Fourier-transform

$$\tilde{T}_{rs}(p_4) = \int dx_4\, e^{-ip_4 x_4}\, T_{rs}(x_4) \, . \tag{6.6}$$

The Källen-Lehmann representation of T_{rs} and $\tilde{T}_{rs}$ is, respectively,

$$T_{rs}(x_4) = \delta_{rs} \int_0^\infty dm^2 \rho(m^2) \frac{e^{-x_4 m}}{2m} \quad , \tag{6.7}$$

$$\tilde{T}_{rs}(p_4) = \delta_{rs} \int_0^\infty dm^2 \rho(m^2) (p_4^2 + m^2)^{-1} \, .$$

The present lattice calculations are limited to moderate (or even small) values of momenta where, according to phenomenology, the vacuum polarization is dominated by the vector-meson poles V = $\rho, \omega, \phi, \psi \ldots$. In the pole dominance approximation the spectral function $\rho(m^2)$ is given by

$$\rho(m^2) = \sum_V m_V^4 f_V^{-2} \delta(m^2 - m_V^2) \, . \tag{6.8}$$

Measuring the time slices $T_{rs}(x_4)$ on the lattice one can extract the masses (m_V) and electromagnetic decay constants (f_V) of the vector-mesons.

In the quenched approximation the general structure of the two-current time slice amplitude on the lattice is

$$\Big\langle \sum_{\underset{\sim}{x}} V_q(\underset{\sim}{x}, x_4)_r \, V_q(\underset{\sim}{0}, 0)_s \Big\rangle = C_q(x_4)_{rs} - D_q(x_4)_{rs} \, , \tag{6.9}$$

where C_q is an amplitude with a single colour and Dirac-trace over quark propagators ("connected" in colour and spin) and D_q is a product of two traces ("disconnected" in colour and spin). The "disconnected" amplitude D_q is presumably small (for SU(2) it vanishes identically, because it is proportional to the imaginary part of the colour trace). Neglecting D_q and assuming exact isospin symmetry ($K_d = K_u$), the flavour structure of the vacuum polarization amplitude (following from (6.4)) is

$$\Big\langle \sum_{\underset{\sim}{x}} J_{em}(\underset{\sim}{x}, x_4)_r \, J_{em}(\underset{\sim}{0}, 0)_s \Big\rangle = C_{IV}(x_4)_{rs} + C_{IS}(x_4)_{rs} \, , \tag{6.10}$$

where the isovector (IV) and isoscalar (IS) parts are

$$C_{IV}(x_4)_{rs} = \frac{1}{2} C_u(x_4)_{rs} \, , \tag{6.11}$$

$$C_{IS}(x_4)_{rs} = \frac{1}{18} C_u(x_4)_{rs} + \frac{4}{9} C_c(x_4)_{rs} + \frac{1}{9} C_s(x_4)_{rs} \, .$$

The recent calculations of the vacuum polarization (with the "naive" current in [31,32] and with the point-split current (6.1) in [44]) still have the problem of large finite size effects, expecially for the light quarks. For the charmed quarks things work out reasonably well at $\beta = 5.7$ on an 8^4 lattice [44], where the result for $f_{J/\psi}$ is (in paranthesis the experimental value):

$$f_{J/\psi}^{-1} = 0.089 \quad (0.083) \tag{6.12}$$

if one assumes a^{-1} = 1 GeV at this β -value. There is, however, no renormalization group scaling between β = 5.7 and β = 6.0. This may be due in part to the larger finite size effects at β = 6.0, where the physical size of the lattice decreases by a factor of $\sim$ 1.4 (if renormalization group behaviour is assumed). The general tendency for the light quarks is that the vacuum polarization (and f_ρ^{-1} , f_ϕ^{-1}) comes out too big. This is due almost certainly to the finite size effects, which generally effect the wave functions (and hence $f_{\rho,\phi}^{-1}$) even more than the masses.

VII. Finite temperatures

According to the general theoretical expectation, at very high temperatures (in the early universeand hopefully also in relativistic heavy ion collisions), the hadronic matter is in a quark-gluon plasma phase. At low temperatures, there is the strongly interacting, confining hadronic phase. Between the two there is probably a singular point corresponding to the "deconfining phase transition" although, in principle, a continuous transition without singularity is also possible. (For a more detailed discussion and references see Ref. [45]).

Near the deconfining phase transition the non-perturbative effects are certainly very important. This region is not accesible by the asymptotically free perturbation theory. Fortunately, the lattice Monte Carlo calculations are rather powerful in this respect. The recent, first calculations already provided us with very interesting, new informations about the behaviour of the QCD-matter near the deconfinement transition.

Let us restrict the discussion now to the pure gauge theory without quarks. In order to determine the thermodynamical quantities let us consider an $N^3 \times N_\beta$ lattice. If the lattice spacing is a, the volume is $V = (aN)^3$, and the inverse temperature corresponding to the (assumed) periodicity in the fourth direction is $\beta = T^{-1} = aN_\beta$. The thermodynamical properties are defined in the "thermodynamical limit" $V \to \infty$ (T fixed), therefore we have to consider $N \gg N_\beta$. (In practical calculations, up to now, N went up to 10 and N_β was 2-5 [46,23].) The partition function ("sum over states") is

$$Z = \int dU\, e^{-S[U]} \quad . \tag{7.1}$$

The energy density (ε) and pressure (p) are given by

$$\varepsilon = -\frac{1}{V}\frac{\partial \ln Z}{\partial \beta} \; ; \quad p = \frac{1}{\beta}\frac{\partial \ln Z}{\partial V} \quad . \tag{7.2}$$

We have seen above that on the lattice β and V are discrete variables,

therefore it is advantageous to consider instead of $\partial/\partial\beta$ and $\partial/\partial V$ the derivative with respect to a continuous parameter like $\partial/\partial g$ or $\partial/\partial a$. If N and N_β are kept fixed we get by the variation of a, from the above formulae,

$$a\frac{\partial \ln Z}{\partial a} = -3V\beta\left(\frac{\varepsilon}{3}-P\right)\equiv -3V\beta\,\delta . \tag{7.3}$$

The quantity $\delta = \frac{1}{3}\varepsilon - P$ measures, in a way, the deviation from the relativistic ideal gas behaviour (for which $\delta = 0$). Denoting the set of all plaquettes by $\{p\}$ and the real part of the plaquette variable $\mathrm{ReTr}U_p$ by A_p we have from Eqs. (7.3), (1.19), (4.2):

$$\delta = \frac{4}{3a^4N^3N_\beta}\,\frac{\bar\beta(g)}{g^3}\left\langle \sum_{\{p\}}(A_p - N_c)\right\rangle . \tag{7.4}$$

The temperature measured in lattice scale units is according to Eqs. (1.20), (1.21):

$$T/\Lambda_{latt} = N_\beta^{-1}\left(\frac{11N_c g^2}{48\pi^2}\right)^{51/121}\exp\left\{\frac{24\pi^2}{11N_c g^2}\right\} , \tag{7.5}$$

therefore, the leading order for $\bar\beta(g)/g^3$ in Eq. (7.4) gives

$$\delta/T^4 = N_\beta^4\,\frac{11N_c}{6\pi^2}\left\langle A_p - N_c\right\rangle_{g,N_\beta} . \tag{7.6}$$

This still contains the (divergent) vacuum contributions, which have to be subtracted. The simplest way is to take the difference belonging to two values of the temperature T_1 and T_2 (as the vacuum part is T-independent). Keeping g fixed, we have, for simplicity at $N_\beta \gg 1$:

$$\frac{d\delta}{dT}T^{-3} = N_\beta^5\,\frac{11N_c}{6\pi^2}\left[\langle A_p\rangle_{g,N_\beta} - \langle A_p\rangle_{g,N_\beta+1}\right] . \tag{7.7}$$

The Monte Carlo measurement of the right hand side gives a huge peak for $d\delta/dT$ at the critical temperature $T_c \cong 80\,\Lambda_{latt}$ (in SU(3); see Ref. [23] and Fig. 10). This peak corresponds to the deconfinement phase transition, where the expectation value of the Wilson-line (closed by periodicity) in the fourth direction becomes non-zero (Fig. 11). This "order parameter" measures the free energy of a separated quark, hence detects non-confinement [47].

To obtain the energy and pressure separately one has to go to a lattice with asymmetric spacing: $\alpha \equiv a_\beta/a \equiv a_\tau/a \neq 1$. One has to separate the space-like $\{p_s\}$ and time-like (or "temperature-like") $\{p_\beta\}$ plaquettes, and to take two different coupling constants g_s and g_β in front of them. The action looks like Eq. (2.10), with the substitution (2.13) (and everywhere changing τ to β for the sake of subject). In the limit of $a \to 0$ and $\alpha \to 1$ we have for the two couplings

$$\begin{aligned} g_s^{-2} &= g^{-2} + (\alpha-1)c'_s + O(g^2,(1-\alpha)^2) \\ g_\beta^{-2} &= g^{-2} + (\alpha-1)c'_\beta + O(g^2,(1-\alpha)^2) \end{aligned} \tag{7.8}$$

The values of the constants $c'_{s,\beta}$ in SU(3) are [48]: $c'_s = -0.2016$ and $c'_\beta = 0.1319$. Differentiating lnZ with respect to a and a_β we obtain for the energy density [46,23]:

$$\begin{aligned} \varepsilon &= \varepsilon_a + \varepsilon_g \quad ; \\ \varepsilon_a &= \frac{6}{g^2}(TN_\beta)^4\left\langle A_{p_\beta} - A_{p_s}\right\rangle_{g,N_\beta} \quad ; \\ \frac{d\varepsilon_g}{dT}T^{-3} &= 6N_\beta^5\left\{c'_s\left[\langle A_{p_s}\rangle_{g,N_\beta+1} - \langle A_{p_s}\rangle_{g,N_\beta}\right] + \right. \\ &\quad \left. + c'_\beta\left[\langle A_{p_\beta}\rangle_{g,N_\beta+1} - \langle A_{p_s}\rangle_{g,N_\beta}\right]\right\} . \end{aligned} \tag{7.9}$$

From the Monte Carlo data one can determine ε_a and ε_g. It turns out that ε_a dominates: $\varepsilon_a \gg |\varepsilon_g|$. The existing computations were done on lattices with N_β = 2-5, where the finite size effects are non-negligible. These can be corrected for, in an approximate way, by considering a free scalar field on the same lattice [49]. The calculated behaviour of the thermodynamical quantities ε, P and δ in the pure SU(3) gauge theory is shown in Fig. 12.

The qualitative conclusions drawn from the Monte Carlo studies are (in the pure gauge theory):

- there is a deconfining phase transition at $T_c \cong 80 \Lambda_{latt}$;
- at high temperature $T \gg T_c$ the behaviour is very well described by a free gluon gas (the Stefan-Boltzmann law is fulfilled and $\varepsilon = 3P$), in accordance with asymptotic freedom at high temperatures;
- the Stefan-Boltzmann limit $\varepsilon = (8\pi^2/15)T^4$ is reached very rapidly above T_c;
- the jump in the energy density near T_c is approximately given by $\Delta\varepsilon \cong 4B$, where for the bag-constant B one has to take $B \cong (0.9T_c)^4$.

A very interesting problem is to investigate the effect of the quarks on the behaviour of thermodynamical quantities. (The first steps in this direction were already done in Ref. [50].) Besides, calculations done on larger lattices could clean up the results from the finite lattice spacing and finite lattice size effects which are all the time lurking around our Monte Carlo's.

References

[1] G. 't Hooft, Nucl. Phys. B35 (1971) 167.

[2] K.G. Wilson, Phys. Rev. D14 (1974) 2455;
Phys. Reports 23C (1976) 331;
in New Developments in Quantum Field Theory and Statistical Mechanics, Cargèse 1976, eds. M. Levy and P. Mitter (Plenum Press, New York 1977).

[3] M. Creutz, Numerical studies of gauge field theories, lecture notes for the 19th International School of Subnuclear Physics, Erice 1981.

[4] P. Hasenfratz, Lattice gauge theories, lecture notes for the EPS International Conference on High-Energy Physics, Lisbon 1981.

[5] C. Rebbi, Lattice gauge theories and Monte Carlo simulations, lecture notes for the G.I.F.T. School, San Feliu de Guixols, Spain and for the Summer Workshop of High Energy Physics, ICTP 1981.

[6] R.C. Brower, Discrete Quantum Chromodynamics, Santa Cruz preprint UCSC-TH-147-81.

[7] G. 't Hooft, Nucl. Phys. B153 (1979) 141;
G. Mack, E. Pietarinen, Phys. Lett. 94B (1980) 397;
G. Mack, V.B. Petkova, Z. Phys. 12C (1982) 177.

[8] K. Symanzik, Some topics in quantum field theory, DESY preprint 81-068 (1981).

[9] A. Hasenfratz, P. Hasenfratz, Phys. Lett. 93B (1980) 165;
Nucl. Phys. B193 (1981) 210.

[10] K.H. Mütter, K. Schilling, Nucl. Phys. B200 [FS4] (1982) 362;
Phys. Lett. 117B (1982) 75.

[11] J. Kogut, J. Shigemitsu, Nucl. Phys. B193 [FS3] (1981) 365.

[12] J. Engels, I. Montvay, Z. Phys. C5 (1980) 247;
Z. Phys. C6 (1980) 217.

[13] G. Münster, Phys. Lett. 95B (1980) 59;
G. Münster, P. Weisz, Nucl. Phys. B180 (1981) 13;
G. Münster, Nucl. Phys. B180 (1981) 23.

[14] J.B. Kogut, R.B. Pearson, J. Shigemitsu,
Phys. Rev. Lett. 43 (1979) 484; Phys. Lett. 98B (1981) 63.

[15] M. Creutz, Phys. Rev. Lett. 43 (1979) 553;
Phys. Rev. D21 (1980) 2308.

[16] C. Itzykson, M. Peskin, Z.B. Zuber, Phys. Lett. B95 (1980) 259;
M. Lüscher, G. Münster, P. Weisz, Nucl. Phys. B180 (1980) 1;
A. Hasenfratz, E. Hasenfratz, P. Hasenfratz, Nucl. Phys. B181
(1981) 353.

[17] K. Binder, in Phase Transitions and Critical Phenomena,
eds. C. Domb, M.S. Green, New York 1976, Vol. 5B.

[18] E. Pietarinen, Nucl. Phys. B190 (1981) 349.

[19] J.M. Hammersley, D.C. Handcomb, Monte Carlo Methods, New York 1964;
N. Metropolis, A.W. Rosenbluth, M.N. Rosenbluth, A.H. Teller,
E. Teller, J. Chem. Phys. 21 (1953) 1087.

[20] G. Bhanot, C. Rebbi, Nucl. Phys. B180 [FS2] (1981) 469;
D. Petcher, D. Weingarten, Phys. Rev. D22 (1980) 2465.

[21] M. Creutz, Phys. Rev. Lett. 45 (1980) 313.

[22] K. Kajantie, C. Montonen, E. Pietarinen, Z. Phys. C9 (1981) 253.

[23] I. Montvay, E. Pietarinen, Phys. Lett. 110B (1982) 148;
Phys. Lett. 115B (1982) 151.

[24] B. Berg, A. Billoire, Phys. Lett. 113B (1982) 65;
Phys. Lett. 114B (1982) 324;
Glueball spectroscopy in 4-d SU(3) lattice gauge theory,
DESY preprint 1982.

[25] K. Ishikawa, G. Schierholz, M. Teper, Phys. Lett. 116B (1982) 429;
K. Ishikawa, A. Sato, G. Schierholz, M. Teper, Prediction of
low-lying oddballs in lattice QCD, DESY preprint 82-053 (1982).

[26] for a review and further references see:
B. Berg, Glueball calculation in 4d lattice gauge theories,
CERN preprint TH.3327 (1982)

[27] C.B. Lang, C. Rebbi, Phys. Lett. 115B (1982) 137.

[28] J.D. Stack, The heavy quark potential in SU(2) lattice gauge
theory, Urbana preprint ILL-(TH)-82-21 (1982).

[29] Particle Data Group, Review of Particle Properties,
Phys. Lett. Vol. 111B, April 1982.

[30] K.G. Wilson, in New Phenomena in Subnuclear Physics,
ed. A. Zichichi, Plenum Press, New York, 1977, Erice (1975).

[31] H. Hamber, G. Parisi, Phys. Rev. Lett. 47 (1981) 1792;
E. Marinari, G. Parisi, C. Rebbi, Phys. Rev. Lett. 47 (1981) 1795;
H. Hamber, E. Marinari, G. Parisi, C. Rebbi,
Phys. Lett. 108B (1982) 314;
H. Hamber, G. Parisi, Numerical estimates for the spectrum of
quantum chromodynamics, Brookhaven preprint BNL 31322 (1982);
F. Fucito, G. Martinelli, C. Omero, G. Parisi, R. Petronzio,
F. Rapuano, Nucl. Phys. B210 [FS6] (1982) 407.

[32] D.H. Weingarten, Phys. Lett. B109 (1982) 57;
Masses and decay constants in lattice QCD, Bloogminton preprint
IUHET-82 (1982).

[33] L. Susskind, Phys. Rev. D16 (1977) 3031.

[34] N. Kawamoto, J. Smit, Nucl. Phys. B192 (1981) 100.

[35] H. Sharatchandra, H.J. Thun, P. Weisz, Nucl. Phys. B192 (1981) 205.

[36] F. Gliozzi, Nucl. Phys. B204 (1982) 419.

[37] P. Becher, Phys. Lett. 104B (1981) 221;
J.M. Rabin, Nucl. Phys. B201 (1982) 315;
P. Becher, H. Joos, The Dirac-Kähler equation and fermions on
the lattice, DESY preprint 82-031 (1982).

[38] H.B. Nielsen, M. Ninomiya, Nucl. Phys. B185 (1981) 20;
Phys. Lett. 105B (1981) 219; Nucl. Phys. B193 (1981) 173;
L.H. Karsten, Phys. Lett. 104B (1981) 315.

[39] I.O. Stamatescu, München preprint MPI-PAE/PTH 60/80 (1980).

[40] A. Hasenfratz, P. Hasenfratz, Z. Kunszt, C.B. Lang, Phys. Lett. 110B (1982) 282; Phys. Lett. 117B (1982) 81.

[41] P. Hasenfratz, I. Montvay, Hadron spectrum: evidence for size problems on the lattice, Santa Barbara preprint NSF-ITP-82-135 (1982)

[42] J. Kuti, Phys. Rev. Lett. 49 (1982) 183.

[43] L.H. Karsten, J. Smit, Nucl. Phys. B183 (1981) 103.

[44] A. Ali, I. Montvay, A calculation of the electromagnetic vacuum polarization in lattice QCD, DESY preprint 82-81 (1982).

[45] Proceedings of the Workshop on Future Relativistic Heavy Ion Experiments, eds. R. Bock, R. Stock, GSI Darmstadt 1980; Proceedings of the Workshop on Quark Matter Formation and Heavy Ion Collisions, Bielefeld, May 1982; to be published.

[46] J. Engels, F. Karsch, I. Montvay, H. Satz, Phys. Lett. 101B (1981) 89; Phys. Lett. 102B (1981) 332; Nucl. Phys. B205 [FS5] (1982) 545.

[47] L.D. McLerran, B. Svetitsky, Phys. Lett. 98B (1981) 195; J. Kuti, J. Polônyi, K. Szlachányi, Phys. Lett. 98B (1981) 199.

[48] F. Karsch, Nucl. Phys. B205 (1982) 285.

[49] J. Engels, F. Karsch, H. Satz, Nucl. Phys. B205 (1982) 239.

[50] J. Kogut, M. Stone, H.W. Wyld, J. Shigemitsu, S.H. Shenker, D.K. Sinclair, Phys. Rev. Lett. 48 (1982) 1140; J. Engels, F. Karsch, H. Satz, Phys. Lett. 113B (1982) 398.

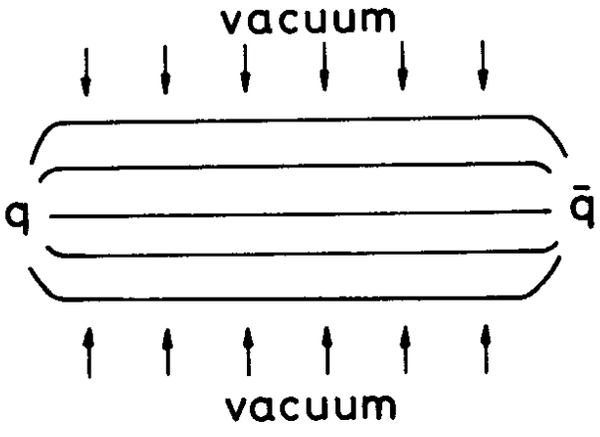

Fig. 1 The chromo-electric flux tube between a quark and an antiquark.

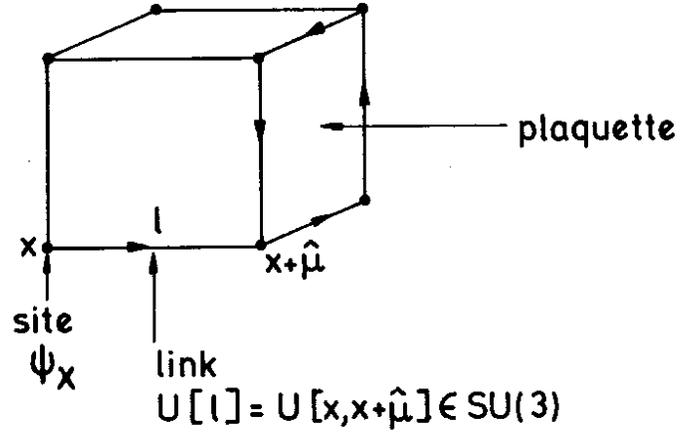

Fig. 2a The elements of a three-dimensional cube in the lattice and the corresponding field variables.

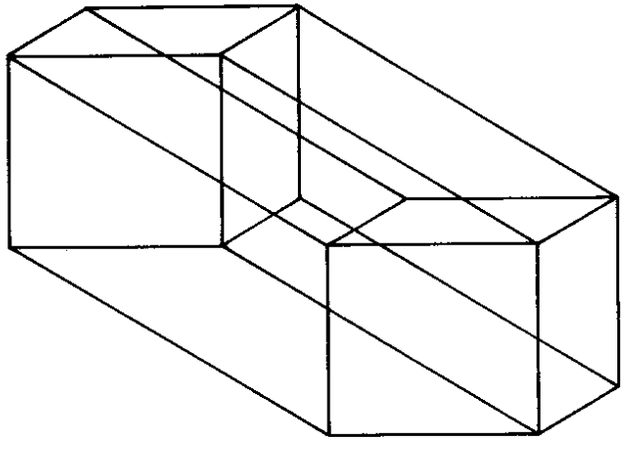

Fig. 2b The four-dimensional super-cube.

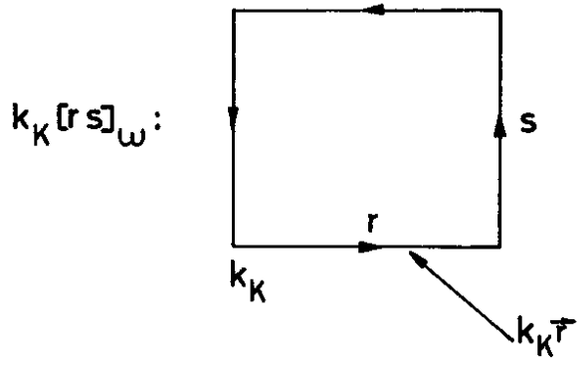

Fig. 3a Space-like plaquette ($\omega = \pm$ is the direction of links).

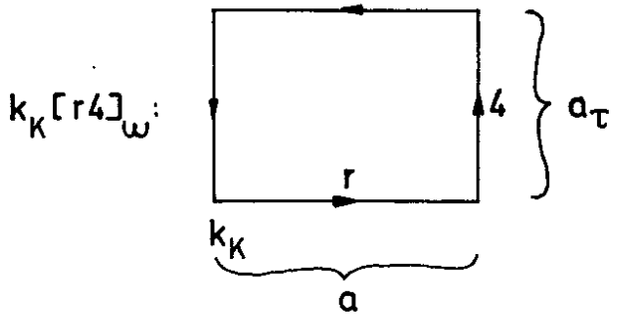

Fig. 3b Time-like plaquette ($\omega = \pm$ is the direction of links).

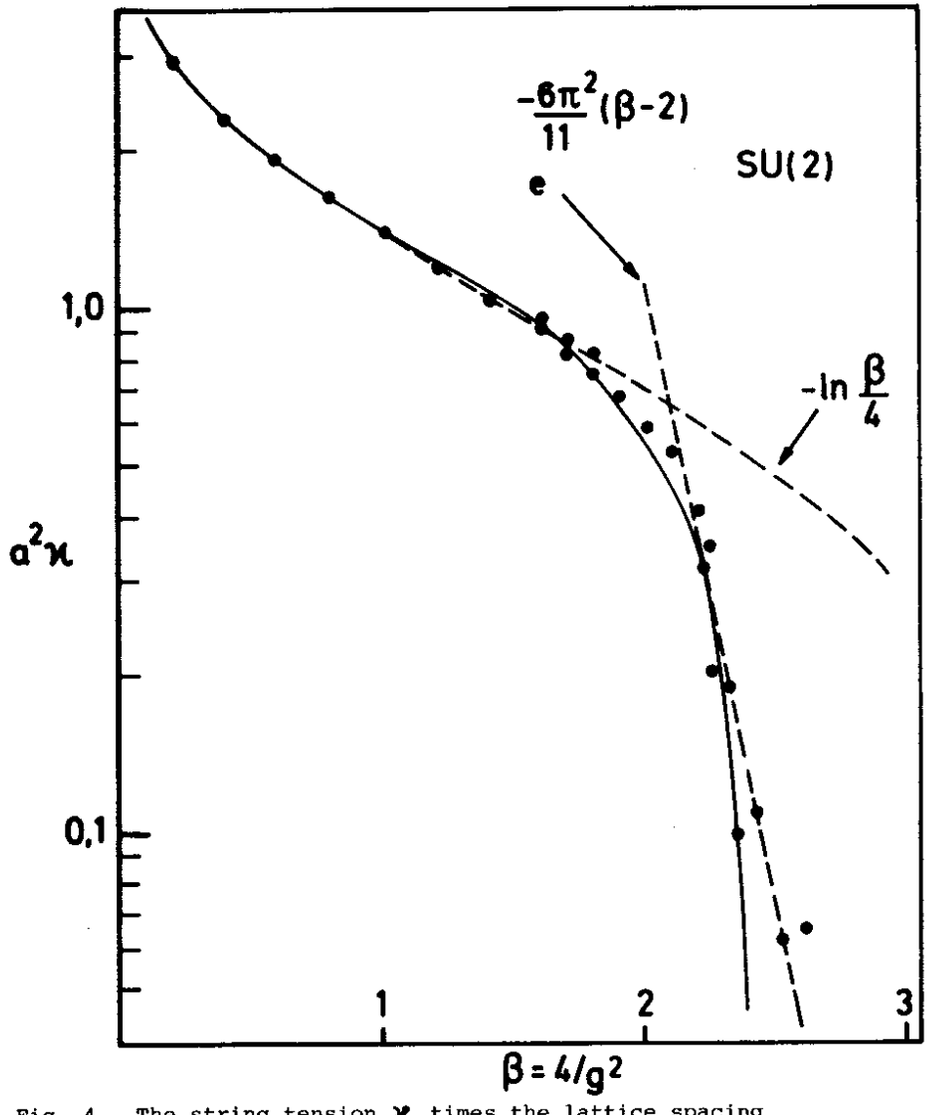

Fig. 4 The string tension $\varkappa$ times the lattice spacing squared a^2 as a function of $\beta = 4/g^2$ for SU(2) lattice gauge theory. The solid line represents the results of the strong coupling expansion [13] up to order 12. The dots are results of an earlier Monte Carlo calculation by Creutz [15]. The dashed lines are the lowest order strong coupling curve and the weak coupling fit of Ref. [15].

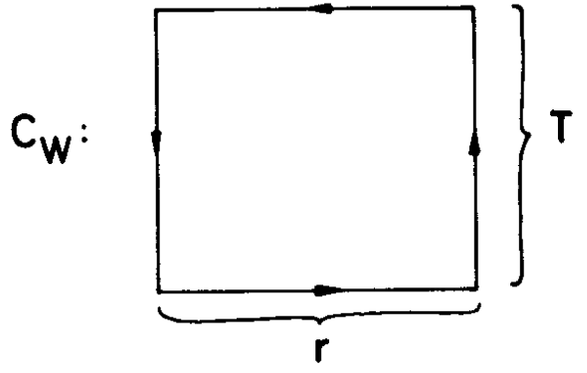

Fig. 5 The Wilson-loop C_W with sides T, r.

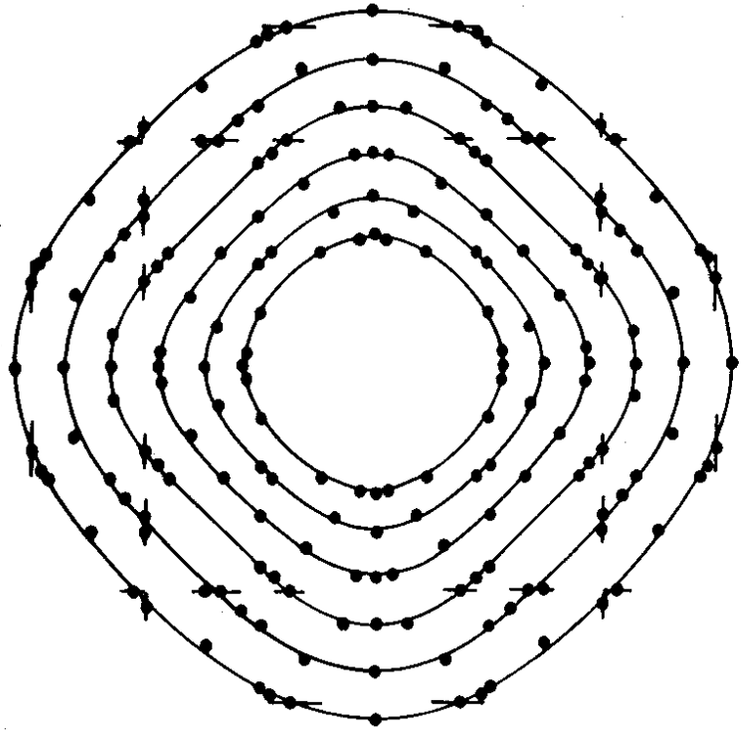

Fig. 6a Equipotential curves of the $q\bar{q}$-potential in SU(2) lattice gauge theory [27] for $\beta = 2$.

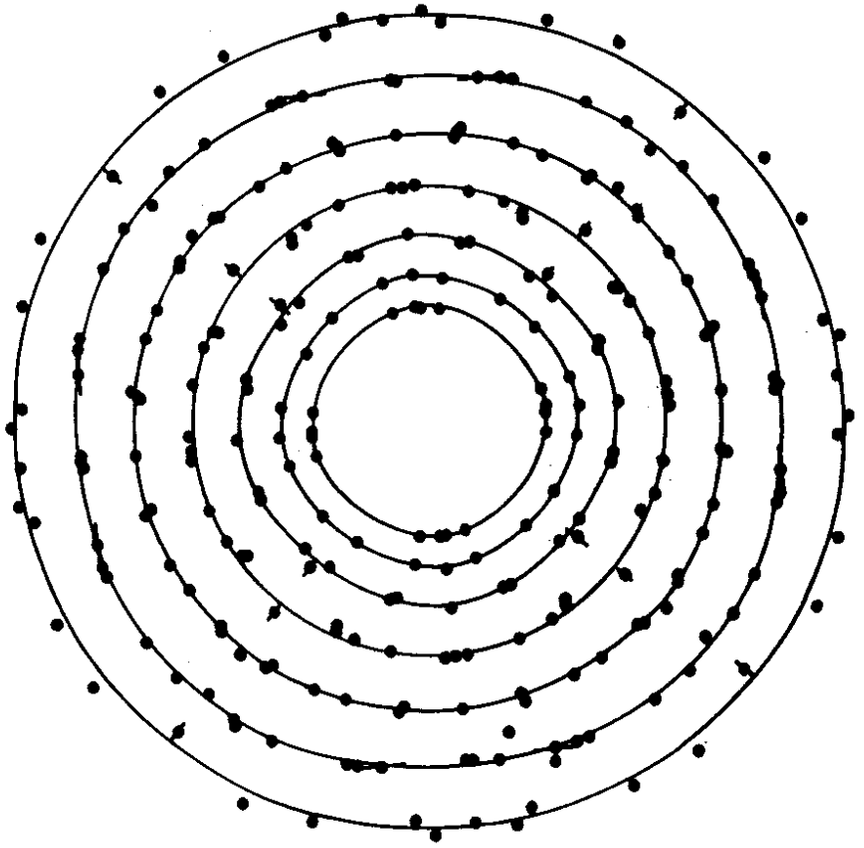

Fig. 6b The same as Fig. 6a for $\beta = 2{,}25$.

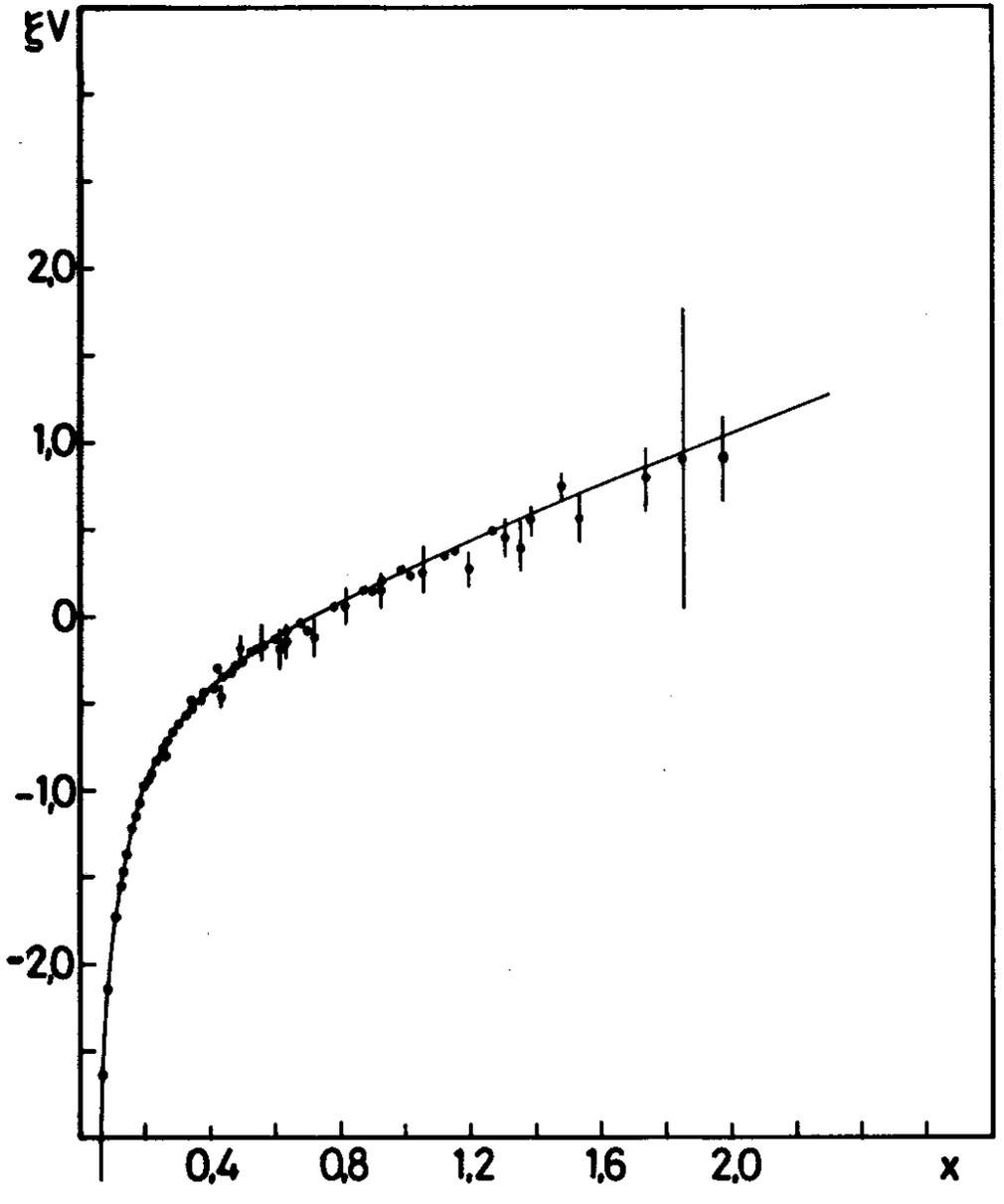

Fig. 7 The $q\bar{q}$-potential in SU(2) lattice gauge theory [28].
The continuous curve is a Coulomb + linear fit.

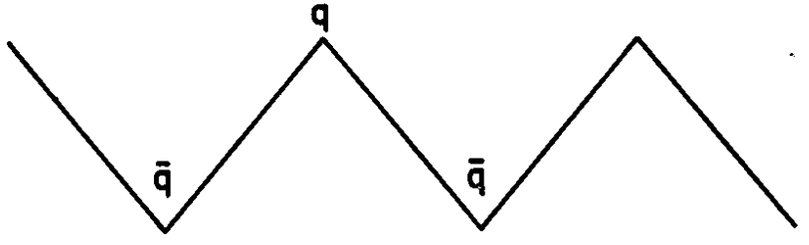

Fig. 8 The periodic $q\bar{q}$-potential on the periodic lattice.

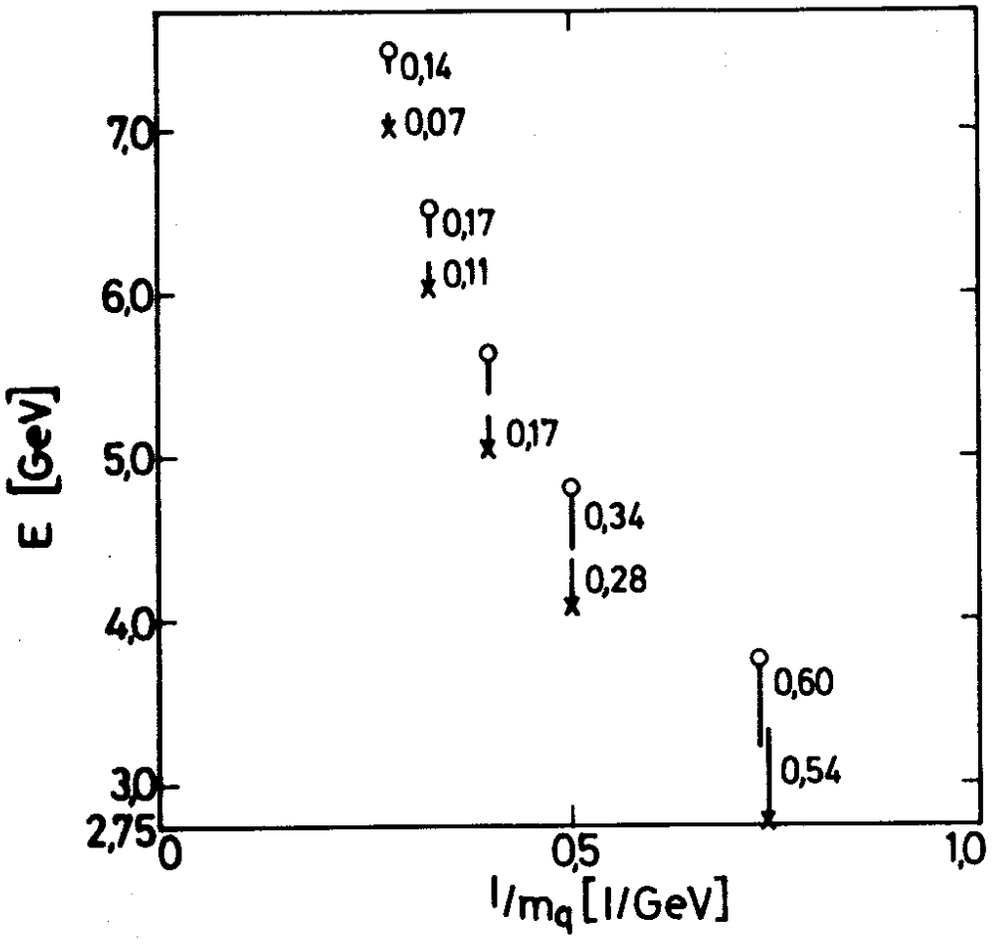

Fig. 9 The bandwidths of the 1S and 2S states as a function of the inverse quark mass as obtained by solving the Schrödinger equation on a lattice. The potential is periodic with a length period of 1 fm. Within a box it is given by $V(r) = -0.51/r + 0.17\ (GeV)^2 r$. The resolution is 0.167 fm. The numbers give the width in GeV. The crosses and circles show the discrete 1S and 2S levels, respectively, in a single box with periodic boundary conditions [41].

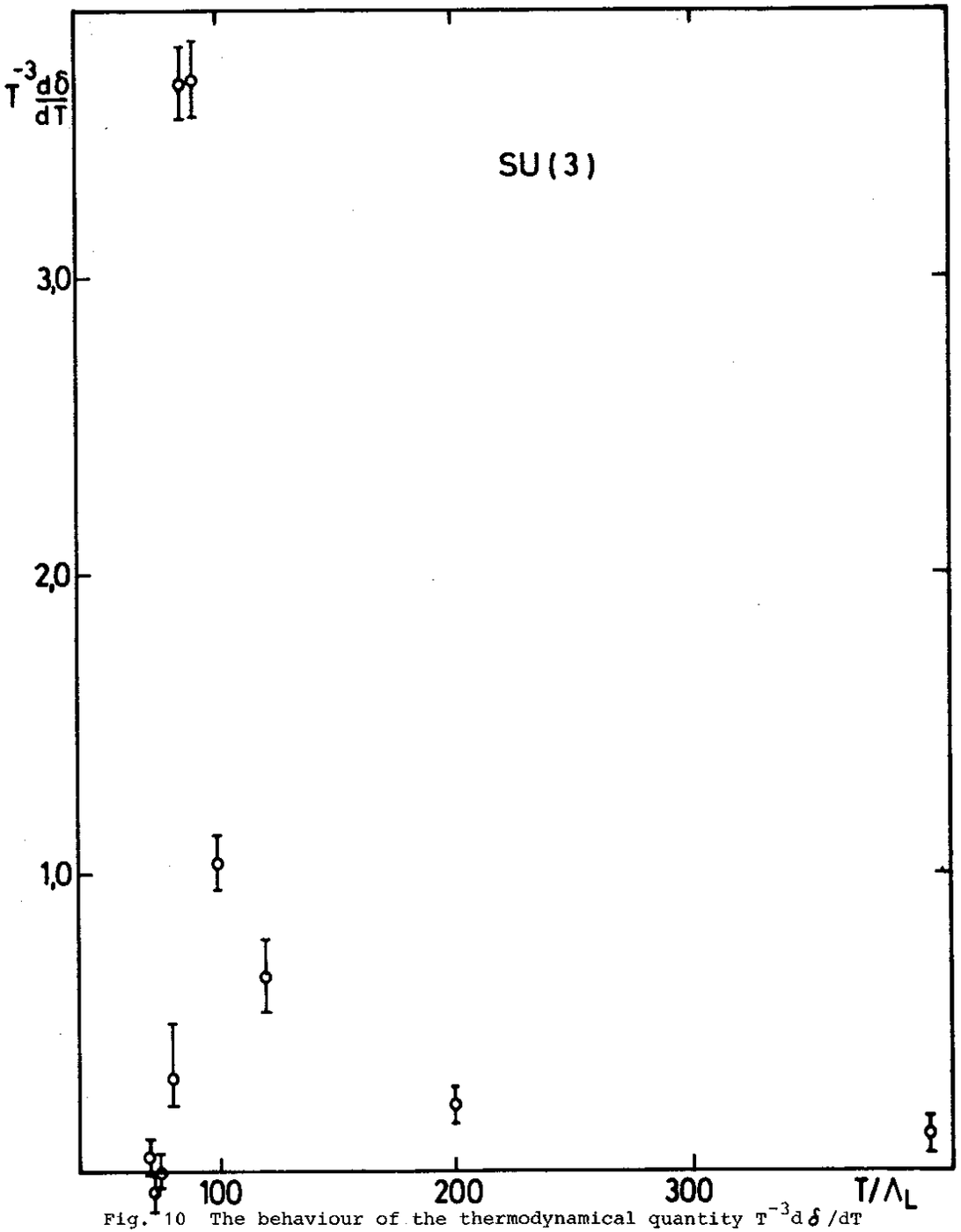

Fig. 10 The behaviour of the thermodynamical quantity $T^{-3} d\delta/dT$ in SU(3) as extracted from the Monte Carlo data on $9^3 \cdot 3$ and $9^3 \cdot 4$ lattices [23].

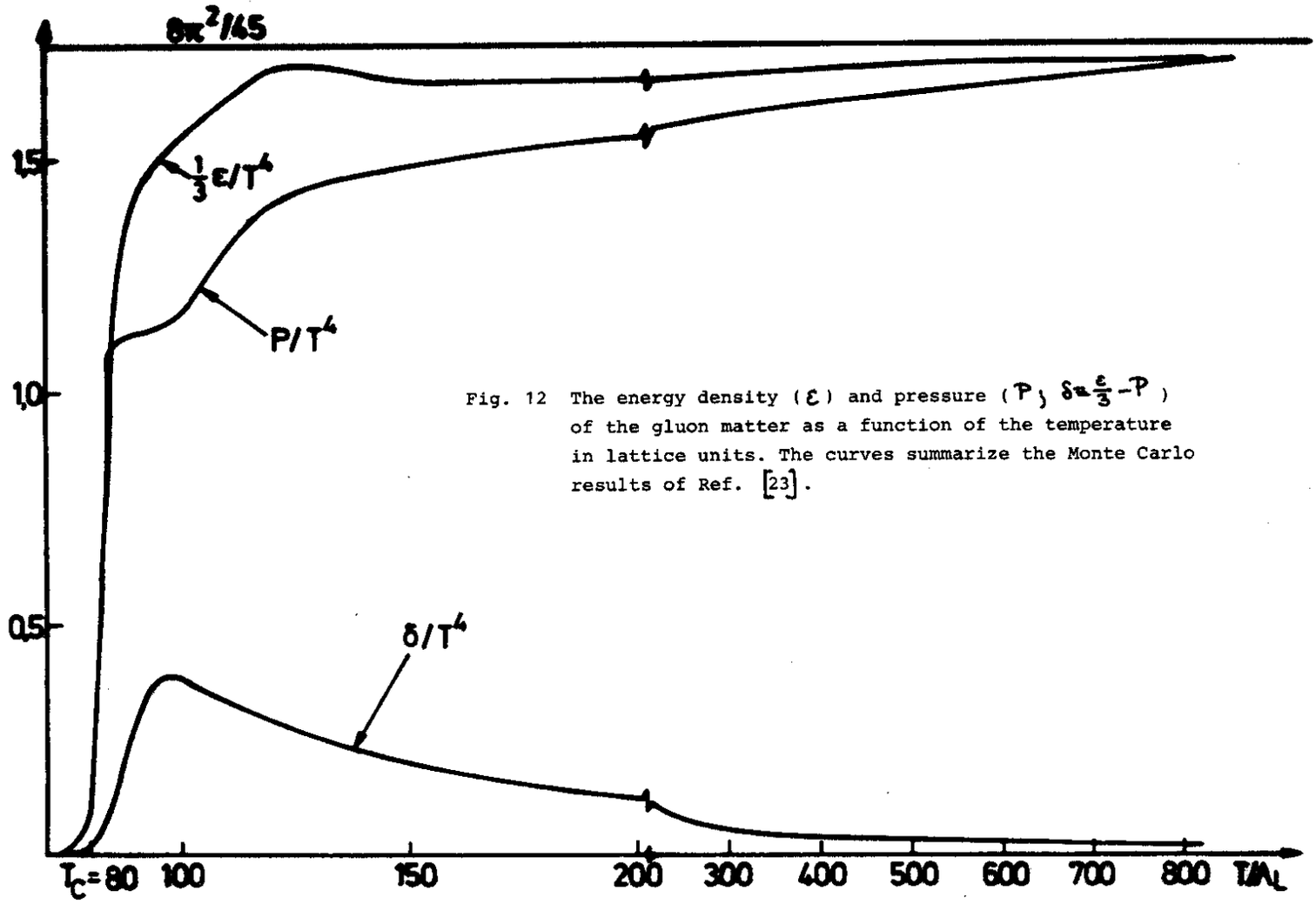

Fig. 12 The energy density (ε) and pressure (P; $\delta = \frac{\varepsilon}{3} - P$) of the gluon matter as a function of the temperature in lattice units. The curves summarize the Monte Carlo results of Ref. [23].

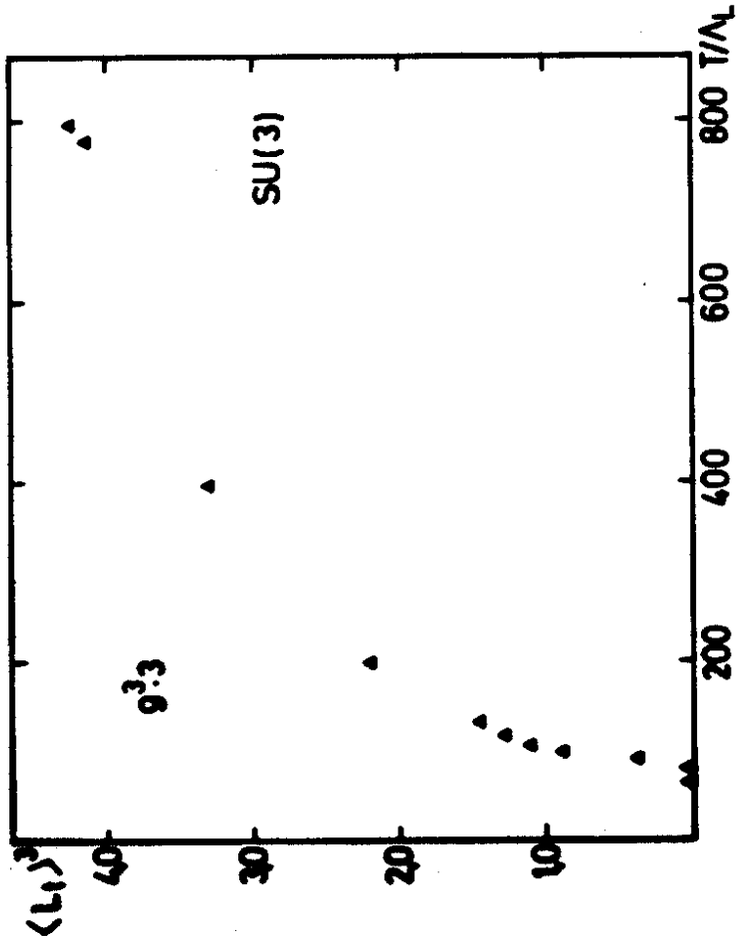

Fig. 11 The expectation value of the Wilson line $\langle L_t \rangle$ in SU(3) as a function of the temperature in lattice units, measured on a $9^3 \cdot 3$ lattice [23].