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THEORY OF SPIN-ORBIT MOTION IN ELECTRON-POSITRON STORAGE RINGS

SUMMARY OF RESULTS

by

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Abstract

In the present report we present a highly condensed summary of our previous works [1], [2] and [3] in which we investigated the spin-orbit motion of particles in a storage ring. The starting points of these investigations are the Lorentz equation and the BMT equation. Having defined a suitable coordinate system in chapter 2 and having described the external fields in chapter 3, the linearized equations of motion are derived in chapters 4-7. In the discussion of the orbital motion (which includes radiation damping and quantum excitations) the betatron oscillations and synchrotron oscillations are treated simultaneously (coupled synchro-betatron oscillations; six dimensional phase space vector $(x, p_x, z, p_z, \sigma, p_\sigma = \frac{\delta p}{p})$). Explicit expressions for the transfer matrices of the orbital motion are given in chapter 5.2 for the most important beam line elements (i.e. quadrupoles, drift spaces, rotated quadrupoles, solenoids, cavities, electric and magnetic dipole fields). In the appendix a numerical scheme is described for calculating the spin transfer matrices. Using Bogoliubov's averaging method the following physical quantities are calculated:

i) damping constants of the synchro-betatron oscillations (α_k);

ii) beam emittance matrix ($\langle y_m(s)\, y_n(s) \rangle_{\delta c}^{stat}$);

iii) depolarization time (τ_D) due to quantum fluctuations of the radiation.

Finally, as an extension of the investigations [1, 2, 3], we show how one can express these quantities in terms of the dispersion.

Table of contents

0. Introduction

In the present report we describe a general linear theory of particle dynamics in electron-positron storage rings. This theory treats synchrotron and betatron oscillations simultaneously and it includes radiation effects and the spin motion.

Various methods and techniques are described in the literature to calculate the orbital motion of a particle in a storage ring [4, 5, 6, 7, 8].

For example, in investigating the synchrotron and betatron oscillations one usually assumes that the transverse oscillation amplitude of the circulating particle, measured with respect to a given closed reference trajectory, can be separated into two parts (y=x,z)

$$y(s) = \eta(s)\, D_y(s) + y_\beta(s) \qquad \eta \equiv \frac{\Delta p}{p}. \tag{0.1}$$

The first part takes into account the dispersion of the machine and the second part describes the free betatron oscillations about the instantaneous orbit $\eta\, D_y(s)$. This decomposition is well justified if $\eta(s)$ is a weakly varying function of s (arc length along the reference trajectory) compared with $D_y(s)$ and $y_\beta(s)$. Even in the case that synchrotron and betatron oscillations are coupled, this separation allows one to write down analytical expressions for the complete revolution matrices of the transverse and longitudinal oscillations. Knowing this revolution matrix one can study problems like the stability of the particle motion or synchro-betatron resonances [9, 10]. However this decomposition, although appropriate for the investigation of many questions has the disadvantage that the symplectic structure of the transfer matrices for the transverse and longitudinal oscillation amplitudes is lost if these two degrees of freedom are coupled. On the other hand, this symplectic structure of the transfer matrix allows one in a straightforward manner to extend both the linear theory of Courant, Snyder [6] and the theoretical treatment of radiative processes (influence of the synchrotron radiation on the particle motion) to the general case of multidimensional coupled systems. The problem of the coupled betatron oscillations (which requires the use of only the 4x4 symplectic matrix formalism) has been treated already in this way [8], [11], [12].

In order to include synchro-betatron oscillations in a consistently symplectic manner without decomposing the oscillation amplitude according to (0.1), it is necessary to introduce a full 6x6 matrix formalism which handles all six degrees of freedom simultaneously. This is the approach proposed by A. Chao.

Using this method and a suitable set of variables for the description of the particle motion, he has investigated the influence of the synchrotron radiation on the particle dynamics in the thin lens approximation for the general case of a coupling between the synchrotron and betatron oscillations [7].
The purpose of the present work is to demonstrate that the equation of motion approach [4, 5] allows a unified and systematic treatment of an arbitrarily coupled linear machine including the influence of radiation effects, and it is also demonstrated that this approach contains as special cases both A. Chao's result and the results of [11], [12].
In particular, it is shown

1) how to extend the theory of Courant and Snyder to the multidimensional case of coupled synchro-betatron oscillations.
2) how to extend the investigations of A. Chao systematically to thick lenses (for a separated function machine or a combined function machine).
   (Explicit expressions are given for the most important beam line elements such as quadrupoles, skew quadrupoles, synchrotron magnets, solenoids and cavities)

Furthermore, analytical expressions are given for the damping constants and the beam emittances for the most general case and a general proof of Robinson's theorem [13] is outlined. These orbital results can be used for six dimensional tracking programs.
With regard to the spin motion in storage rings, Sokolov and Ternov [14] have shown that, as a consequence of spin-flip synchrotron radiation, the electron beam becomes polarized antiparallel to the direction of the bending field. Various depolarizing mechanisms have been discussed and investigated extensively by the Novosibirsk group [15], [16]. Our purpose in this report is to show how one can include certain depolarization calculations in the equation of motion approach by just adding the BMT-equation to the LORENTZ-equation. In particular, we calculate the depolarizing effect caused by random changes in the orbital motion of the particle due to the stochastic emission of synchrotron light (spin diffusion). The expression for the depolarization time τ_D is equivalent to a result of A. Chao [17] derived from an extension of the 6x6 matrix theory (8x8 matrix theory). Finally we give a new and general expression for τ_D containing the dispersion. This expression can be used to derive general spin transparency conditions and spin matching conditions for more complicated storage rings [18].

1. Spin-orbit motion under the influence of an electromagnetic field

The LORENTZ-equation and the BMT-equation are the starting point for investigating the spin-orbit motion in electron-positron storage rings.

1.1 Orbital motion (LORENTZ-equation), (CGS-units)

$$e\vec{\varepsilon} + \frac{e}{c}\dot{\vec{r}} \times \vec{B} + \vec{R} = \frac{d}{dt}\left(\frac{E}{c^2}\dot{\vec{r}}\right) \tag{1.1}$$

with

$$E = \frac{m_o c^2}{\sqrt{1 - \frac{1}{c^2}(\dot{\vec{r}})^2}} \qquad \text{(energy of the particle)}$$

and the following definitions

e = elementary charge

m_o = rest mass of the particle

c = velocity of light

$\vec{\varepsilon}$ = electric field

$\vec{B}$ = magnetic field

$\vec{R}$ = radiation reaction force

$\vec{r}$ = radius vector of the particle

The radiation reaction force in (1.1) can be separated into two parts:

$$\vec{R} = \vec{R}^D + \delta\vec{R} \tag{1.2}$$

a continuous part $\vec{R}^D$ describing the radiation damping and a discontinuous part $\delta\vec{R}$ caused by the quantum fluctuations. The explicit expression for $\vec{R}^D$ is given by [4]

$$\vec{R}^D = -\frac{2}{3}\frac{e^2}{c^5}\gamma^4\dot{\vec{r}}\{(\ddot{\vec{r}})^2 + \frac{\gamma^2}{c^2}(\dot{\vec{r}}\,\ddot{\vec{r}})^2\} \tag{1.2a}$$

and $\delta\vec{R}$ is a Gaussian white noise process with [5]

$$< \delta\vec{R} > = 0 ; \tag{1.2b}$$

$$< \delta R_i(t)\delta R_j(t') > = C_{ij}(t) \cdot \delta(t - t') ; \tag{1.2c}$$

$C_{ij}(t)$ will be specified later (see (4.6b)).

1.2 Spin motion (BMT equation)

$$\frac{d}{dt}\vec{\xi} = \vec{\Omega}_o \times \vec{\xi} \tag{1.3a}$$

$$\vec{\Omega}_o = -\frac{e}{m_o\gamma c}\{(1+\gamma a)\vec{B} - \frac{a\gamma^2}{1+\gamma}\frac{1}{c^2}(\dot{\vec{r}}\,\vec{B})\cdot\dot{\vec{r}} - (a\gamma + \frac{\gamma}{1+\gamma})\,\dot{\vec{r}} \times \frac{\vec{\varepsilon}}{c}\}. \tag{1.3b}$$

The following abbreviations have been used in (1.3b):

$\vec{\xi}$ = spin expectation value in the particle's rest frame (effective spin of an ensemble of particles)

a = anomalous magnetic moment

$\gamma = E/m_o c^2$

2. Reference trajectory and coordinate frame

Eqs. (1.1) and (1.3) are expressed in terms of the laboratory coordinates. In accelerator theory, in order to simplify the equations of motion for the circulating particle, one usually introduces a new coordinate system, which is comoving with the particle under consideration.

For this purpose we shall assume, that an ideal closed orbit exists for a particle with fixed energy E_o, if we neglect the variation of energy caused by the radiation losses and the accelerating fields. We assume that this closed orbit consists of piece-wise plane curves either in the horizontal plane or vertical plane, so that there is no torsion.

Vectors lying on the reference trajectory will be called $\vec{r}_o(s)$ where s designates the arc length along this orbit.

In the well known manner we can now define a coordinate frame moving along the reference trajectory and consisting of

the normal unit vector $\vec{\nu}(s)$

the tangent unit vector $\vec{\tau}(s)$

and the binormal unit vector $\vec{\beta}(s) = \vec{\tau}(s) \times \vec{\nu}(s)$.

We require that the vector $\vec{\nu}(s)$ is directed outwards if the motion takes place in the horizontal plane and upwards if the motion takes place in the vertical plane.

Choosing the direction of $\vec{\nu}(s)$ in this way, implies that the curvature K(s), appearing in the Frenet formulae

$$\vec{\tau}(s) = \frac{d}{ds}\vec{r}_o(s) \equiv \vec{r}_o'(s); \tag{2.1}$$

$$\begin{cases} \frac{d\vec{\tau}}{ds} = -K(s)\,\vec{\nu}(s) \\ \frac{d\vec{\nu}}{ds} = K(s)\,\vec{\tau}(s) \\ \frac{d\vec{\beta}}{ds} = 0 \end{cases} \tag{2.2}$$

is always positive in the horizontal plane and negative in the vertical plane iff the centre of curvature lies above the reference trajectory.
In principle we can now expand each vector in (1.1) or (1.3) in terms of the unit vectors $\vec{\tau}$, $\vec{\nu}$ and $\vec{\beta}$.
However this representation has the disadvantage that the direction of the normal vector $\vec{\nu}(s)$ changes discontinuously if the particle trajectory is going over from the vertical plane to the horizontal plane and vice versa. Therefore it is advantageous to introduce new unit vectors $\vec{\tau}$, $\vec{e}_x$ and $\vec{e}_z$ which change their directions continuously. This is achieved by putting

$$\vec{e}_x(s) = \begin{cases} \vec{\nu}(s), \text{ if the orbit lies in the horizontal plane} \\ -\vec{\beta}(s), \text{ if the orbit lies in the vertical plane} \end{cases}$$

$$\vec{e}_z(s) = \begin{cases} \vec{\beta}(s), \text{ if the orbit lies in the horizontal plane} \\ \vec{\nu}(s), \text{ if the orbit lies in the vertical plane} \end{cases}$$

The Frenet formulae (2.2) now read

$$\begin{cases} \frac{d}{ds}\vec{e}_x(s) = K_x(s)\,\vec{\tau}(s) \\ \frac{d}{ds}\vec{e}_z(s) = K_z(s)\,\vec{\tau}(s) \\ \frac{d}{ds}\vec{\tau}(s) = -K_x\vec{e}_x(s) - K_z\vec{e}_z(s) \end{cases} \tag{2.3}$$

with

$$\vec{\tau}(s) = \vec{r}_0'(s);$$

$$K_x(s) \cdot K_z(s) = 0 \tag{2.4}$$

where $K_x(s)$, $K_z(s)$ designate the curvatures in x-direction and z-direction respectively.

Thus we can write

$$\vec{r}(s, x, z) = \vec{r}_0(s) + x(s)\,\vec{e}_x(s) + z(s)\,\vec{e}_z(s) \tag{2.5}$$

and generally

$$\vec{A}(s, x, z) = A_\tau(s, x, z)\,\vec{\tau}(s) + A_x(s, x, z)\,\vec{e}_x(s) + \\ + A_z(s, x, z)\,\vec{e}_z(s)\;; \tag{2.6}$$

$$(\vec{A} \equiv \vec{\xi}, \dot{\vec{r}}, \vec{B}, \vec{\varepsilon}\ldots)$$

Our next task will be to express (1.1) and (1.3) in terms of the $(\vec{\tau}, \vec{e}_x, \vec{e}_z)$ coordinate frame.

For that purpose we shall introduce, instead of time t, the arc length s as independent variable:

$$\frac{d}{dt} = \frac{ds}{d\ell} \cdot \frac{d\ell}{dt} \cdot \frac{d}{ds} = v \cdot \frac{1}{\ell'} \cdot \frac{d}{ds} \tag{2.7}$$

with

$$\ell' \equiv \frac{d\ell}{ds}\;;\quad d\ell^2 \equiv (d\vec{r})^2\,.$$

Then one gets

$$\vec{r}' \equiv \frac{d\vec{r}}{ds} = \vec{\tau}(s) + x'\vec{e}_x + z'\vec{e}_z + \vec{\tau}(K_x x + K_z z) \\ = \vec{\tau}(1 + K_x x + K_z z) + x'\vec{e}_x + z'\vec{e}_z\;; \tag{2.8}$$

$$(\ell')^2 \equiv \left(\frac{d\ell}{ds}\right)^2 = \left(\frac{d\vec{r}}{ds}\right)^2 = 1 + 2(K_x x + K_z z) + \ldots \tag{2.9}$$

$$\ell' = 1 + K_x x + K_z z + \ldots\;; \tag{2.10}$$

$$\vec{\xi}' = \xi_\tau'\vec{\tau} + \xi_x'\,\vec{e}_x + \xi_z'\,\vec{e}_z - \xi_\tau\cdot(K_x\vec{e}_x + K_z\vec{e}_z) + \xi_x K_x\vec{\tau} + \xi_z K_z\vec{\tau}\,. \tag{2.11}$$

3. Description of the electromagnetic fields

The electromagnetic fields appearing in (1.1) and (1.3) can be written in the form:

$$\varepsilon_\tau(s,\ x,\ z) = \varepsilon_c(s) + \Delta\varepsilon_\tau(s) \tag{3.1a}$$

with [1]

$$\varepsilon_c(s) = \hat{V}\{\sin\Phi + k\,\frac{2\pi}{L}\,\sigma(s)\,\cos\Phi\}\ \sum_\nu \delta(s - s_\nu)$$

$$\sigma = \int_o^s (ds - d\ell) = -\int_o^s ds\,[K_x x + K_z z]$$

ε_c = cavity field;

L = circumference of the equilibrium orbit

k = harmonic number

$$\varepsilon_x(s,\ x,\ z) = \Delta\varepsilon_x(s) \tag{3.1b}$$

$$\varepsilon_z(s,\ x,\ z) = \Delta\varepsilon_z(s) \tag{3.1c}$$

$$B_x(s,\ x,\ z) = B_x^{(o)}(s) + \Delta B_x(s) + x\left(\frac{\partial B_x}{\partial x}\right)_{x=z=o} + z\left(\frac{\partial B_x}{\partial z}\right)_{x=z=o} + \ldots \tag{3.2a}$$

$$B_z(s,\ x,\ z) = B_z^{(o)}(s) + \Delta B_z(s) + x\left(\frac{\partial B_z}{\partial x}\right)_{x=z=o} + z\left(\frac{\partial B_z}{\partial z}\right)_{x=z=o} + \ldots \tag{3.2b}$$

$$\begin{aligned} B_\tau(s,\ x,\ z) &= B_\tau(s,\ o,\ o) + (x^2 + z^2)\cdot D_1 + (x^2 + z^2)^2\cdot D_2 + \ldots \\ &= B_\tau^{(o)}(s) + \Delta B_\tau(s) + \cdots \end{aligned} \tag{3.2c}$$

[see for example [19]]

ΔB_x, ΔB_z, ΔB_τ and $\Delta\varepsilon_x$, $\Delta\varepsilon_z$, $\Delta\varepsilon_\tau$ designate external perturbing fields.

By Maxwell's equations

$$\left(\frac{\partial B_x}{\partial z}\right)_{x=z=o} = \left(\frac{\partial B_z}{\partial x}\right)_{x=z=o}$$

$$\left(\frac{\partial B_x}{\partial x} + \frac{\partial B_z}{\partial z} + \frac{\partial B_\tau}{\partial s}\right)_{x=z=o} = 0$$

and since the fields $B_x^{(o)}$ and $B_z^{(o)}$ satisfy the relations

$$\begin{aligned} K_x &= \frac{e}{E_o} B_z^{(o)} \\ K_z &= -\frac{e}{E_o} B_x^{(o)} \end{aligned} \tag{3.3}$$

equ's (3.2) can also be put into the form

$$\frac{e}{E_o} B_x(s, x, z) = -K_z + \frac{e}{E_o} \Delta B_x + x(N - H') + g\,z\ ; \tag{3.4a}$$

$$\frac{e}{E_o} B_z(s, x, z) = \quad K_x + \frac{e}{E_o} \Delta B_z - z(N + H') + g\,x\ ; \tag{3.4b}$$

$$\frac{e}{E_o} B_\tau(s, x, z) = 2H + \frac{e}{E_o} \Delta B_\tau \tag{3.4c}$$

with the following abbreviations

$$\begin{aligned} H &= \frac{1}{2}\ \frac{e}{E_o} B_\tau^{(o)}\ ;\ H' = \frac{1}{2} \frac{e}{E_o} \frac{\partial B_\tau^{(o)}}{\partial s} \\ N &= \frac{1}{2}\ \frac{e}{E_o}\left(\frac{\partial B_x}{\partial x} - \frac{\partial B_z}{\partial z}\right)_{x=z=o} \\ g &= \frac{e}{E_o}\left(\frac{\partial B_z}{\partial x}\right)_{x=z=o} \end{aligned} \tag{3.5}$$

4. The linearized equations of motion in the $(\vec{\tau}, \vec{e}_x, \vec{e}_z)$ coordinate frame

4.1 Orbital motion

Putting (2.7 - 2.10) and (3.1 - 3.3) into eq. (1.1) and (1.2) one obtains in linear approximation [1]:

$$\vec{y}\,' = (\underline{A} + \delta\underline{A}) \cdot \vec{y} + \vec{c}_0 + \vec{c}_1 + \delta\vec{c}\ ; \tag{4.1}$$

$$\vec{y}^{\,T} = (x,\ p_x,\ z,\ p_z,\ \sigma,\ \eta)\ ;\quad \eta = \frac{E - E_o}{E_o}; \tag{4.2a}$$

$$\vec{c}_o^{\,T} = (o,\ o,\ o,\ o,\ o,\ \frac{e\hat{V}}{E_o} \sin\Phi \sum_\nu \delta(s - s_\nu) - C_1(K_x^2 + K_z^2)); \tag{4.2b}$$

$$\begin{aligned} \vec{c}_1^{\,T} = (o,\ &\frac{e}{E_o}(\Delta\varepsilon_x - \Delta B_z),\ o,\ \frac{e}{E_o}(\Delta\varepsilon_z + \Delta B_x),\ o, \\ &\frac{e}{E_o}\Delta\varepsilon_\tau + 2C_1 \cdot \frac{e}{E_o}[K_x(\Delta\varepsilon_x - \Delta B_z) + K_z(\Delta\varepsilon_z + \Delta B_x)]) \end{aligned} \tag{4.2c}$$

with

$$C_1 = \frac{2}{3} e^2 \frac{\gamma_o^4}{E_o} , \tag{4.2c}$$

$$A = \begin{pmatrix} 0 & 1 & H & 0 & 0 & 0 \\ -(G_1 + H^2) & 0 & N & H & 0 & K_x \\ -H & 0 & 0 & 1 & 0 & 0 \\ N & -H & -(G_2 + H^2) & 0 & 0 & K_z \\ -K_x & 0 & -K_z & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{e\hat{V}}{E_o} \cdot k \cdot \frac{2\pi}{L} \cos\Phi \cdot \sum_\nu \delta(s - s_\nu) & 0 \end{pmatrix} \tag{4.3}$$

and

$$G_1 = K_x^2 + g ;$$

$$G_2 = K_z^2 - g ;$$

$$\delta\underline{A} = ((\delta A_{ik})) ;$$

$$\delta A_{22} = -\frac{e\hat{V}}{E_o} \sin\Phi \cdot \sum_\nu \delta(s - s_\nu) ;$$

$$\delta A_{44} = \delta A_{22} ;$$

$$\delta A_{61} = - C_1 \cdot [(K_x^2 + K_z^2) \cdot K_x + 2K_x \cdot \frac{e}{E_o} \left(\frac{\partial B_z}{\partial x}\right)_{x=z=o} + 4 K_x \cdot H^2 - 2K_z \cdot (N - H')] ;$$

$$\delta A_{62} = 4 C_1 \cdot K_z \cdot H ;$$

$$\delta A_{63} = - C_1 \cdot [(K_x^2 + K_z^2) \cdot K_z - 2K_z \cdot \frac{e}{E_o} \left(\frac{\partial B_z}{\partial x}\right)_{x=z=o} + 4 K_z \cdot H^2 - 2K_x \cdot (N + H')] ;$$

$$\delta A_{64} = - 4C_1 \cdot K_x \cdot H ;$$

$$\delta A_{66} = - 2C_1 \cdot (K_x^2 + K_z^2) ;$$

$$\delta A_{ik} = 0 \quad \text{(otherwise)} ; \tag{4.4}$$

$$\delta\vec{c}^{\,T} = (o,\ o,\ o,\ o,\ o,\ \delta c)\ ; \tag{4.5}$$

$$\delta c = \frac{\delta R_\tau}{E_o}\ ;$$

$$<\delta c(s)> = 0\ ; \tag{4.6a}$$

$$<\delta c(s)\ \delta c(s')> = \omega(s) \cdot \delta(s - s')\ ; \tag{4.6b}$$

$$\omega(s) = |K^3(s)| \cdot \frac{55\ r_e\ \hbar\ \gamma_o^5}{24\sqrt{3}\ m_o\ c}\ ;$$

$$r_e = \frac{e^2}{m_o c^2}\ .$$

4.2 Spin motion

Equ's (1.3) and (2.7) imply that ($v \approx c$)

$$\frac{d}{ds}\vec{\xi} = \frac{1}{c}\,\ell'\,\vec{\Omega}_o \times \vec{\xi}$$

or taking into account (2.10) and (2.11) this equation can be rewritten as

$$\xi'_\tau\,\vec{\tau} + \xi'_x\,\vec{e}_x + \xi'_z\,\vec{e}_z = \vec{\Omega} \times \vec{\xi} \tag{4.7}$$

with

$$\vec{\Omega} = (1 + K_x x + K_z z + \ldots)\,\frac{1}{c}\,\vec{\Omega}_o - K_z\,\vec{e}_x + K_x\,\vec{e}_z\ .$$

In linear approximation $\vec{\Omega}$ is given by [2]:

$$\vec{\Omega}_\tau = -\,2H\,[1 + a\,\frac{\gamma_o}{1+\gamma_o}] - \frac{e}{E_o}\,\Delta B_\tau\,[1 + a\,\frac{\gamma_o}{1+\gamma_o}] -$$

$$-\,2H\,(K_x x + K_z z)\,[1 + a\gamma_o\,\frac{1}{1+\gamma_o}] +$$

$$+\,2H\,\eta\,[1 + a\,\frac{\gamma_o^2}{(1+\gamma_o)^2}] - a\gamma_o\,\frac{\gamma_o}{1+\gamma_o}\,(x'\,K_z - z'\,K_x)\ ; \tag{4.8a}$$

$$\Omega_x = K_z\, a\gamma_o + (1 + a\gamma_o)\, K_z^2\, z - K_z\, \eta -$$
$$- (1 + a\gamma_o)\, [(N - H')\cdot x + g\, z] + a\gamma_o \frac{\gamma_o}{1+\gamma_o} 2Hx' +$$
$$+ (a\gamma_o + \frac{\gamma_o}{1+\gamma_o})\cdot \frac{e}{E_o} \hat{V} \sin\Phi \sum_\nu \delta(s - s_\nu)\cdot z' -$$
$$- (1 + a\gamma_o)\cdot \frac{e}{E_o} \Delta B_x - (a\gamma_o + \frac{\gamma_o}{1+\gamma_o})\cdot \Delta\varepsilon_z \; ; \qquad (4.8b)$$

$$\Omega_z = - K_x\, a\gamma_o - (1 + a\gamma_o)\cdot K_x^2\, x + K_x\eta +$$
$$+ (1 + a\gamma_o)\, [(N + H')\cdot z - g\, x] +$$
$$+ a\gamma_o \frac{\gamma_o}{1+\gamma_o} 2H\cdot z' -$$
$$- (a\gamma_o + \frac{\gamma_o}{1+\gamma_o})\cdot \frac{e}{E_o} \hat{V} \sin\Phi \sum_\nu \delta(s - s_\nu)\cdot x' -$$
$$- (1 + a\gamma_o) \frac{e}{E_o} \Delta B_z + (a\gamma_o + \frac{\gamma_o}{1+\gamma_o})\cdot \Delta\varepsilon_x \, . \qquad (4.8c)$$

5 Introduction of a new reference orbit (closed orbit)

5.1 The equations for the new reference orbit

The equations (4.1) form a system of linear and inhomogeneous differential equations with the inhomogeneous parts $\delta\vec{c}$, $\vec{c}_o$ and $\vec{c}_1$. The term $\delta\vec{c}$ is due to quantum fluctuations of the radiation field and $\vec{c}_o$ is due to the variation of the energy of the circulating particle because of radiation losses and the presence of accelerating fields. The vector $\vec{c}_1$ originates from fields $\Delta\vec{B}$ and $\Delta\vec{\varepsilon}$ which can be interpreted as field errors or perturbing external fields.
The term $\delta\underline{A}$ which contains the accelerating fields and the radiation losses (see eq. (4.4)) will be treated with perturbation theory. From a physical point of view this term causes a damping of the particle motion.

For our later discussions of the orbital motion and the spin motion it is advantageous to eliminate the inhomogeneous parts $\vec{c}_o$ and $\vec{c}_1$ in eq. (4.1). This is achieved in the well-known manner by looking for the (unique) periodic solution $\vec{y}_o$ of the inhomogeneous equation

$$\vec{y}' = (\underline{A} + \delta\underline{A})\,\vec{y} + \vec{c}_0 + \vec{c}_1 \tag{5.1}$$

namely

$$\vec{y}_0' = (\underline{A} + \delta\underline{A})\,\vec{y}_0 + \vec{c}_0 + \vec{c}_1 \tag{5.2a}$$

$$\vec{y}_0(s_0 + L) = \vec{y}_0(s_0) \qquad \text{(condition of periodicity).} \tag{5.2b}$$

Then the general solution of (4.1) can be separated into

$$\vec{y} = \vec{y}_0 + \vec{\tilde{y}} \tag{5.3}$$

where the vector $\vec{\tilde{y}}$ describes the synchro-betatron oscillations about the new closed equilibrium trajectory $\vec{y}_0$, which is called "six-dimensional closed orbit" in the following.

For an approximate calculation of this new reference trajectory we are allowed to neglect the perturbation matrix $\delta\underline{A}$ in equ. (5.2) and thus equ. (5.2a) or (5.1) reduces to the simpler form

$$\vec{y}' = \underline{A}\vec{y} + \vec{c}_0 + \vec{c}_1 \tag{5.4}$$

of the undamped synchro-betatron oscillations.

The solution of (5.4) can be written in the form

$$\begin{pmatrix} \vec{y}(s) \\ 1 \end{pmatrix} = \hat{\underline{M}}(s, s_0) \begin{pmatrix} \vec{y}(s_0) \\ 1 \end{pmatrix} \tag{5.5}$$

where we have used the "enlarged" transfer matrix

$$\hat{\underline{M}}(s, s_0) = \begin{pmatrix} \underline{M}(s, s_0) & \vec{h}(s) \\ 0 & 1 \end{pmatrix}. \tag{5.6}$$

$\underline{M}(s, s_0)$ represents the (simple) transfer matrix belonging to the homogeneous equation

$$\vec{y}' = \underline{A}\vec{y}$$

and $\underline{M}(s, s_0)$ satisfies the following conditions

$$\frac{d}{ds}\underline{M}(s, s_o) = \underline{A}(s)\,\underline{M}(s, s_o); \tag{5.7a}$$

$$\underline{M}(s_o, s_o) = \underline{1}\,. \tag{5.7b}$$

The vector $\vec{h}(s)$ in (5.6) is a special solution of eq. (5.4)

$$\frac{d}{ds}\vec{h}(s) = \underline{A}(s)\,\vec{h}(s) + \vec{c}_o + \vec{c}_1 \tag{5.8a}$$

with the initial value

$$\vec{h}(s_o) = 0\,. \tag{5.8b}$$

Making use of (5.4) and (5.5) the condition of periodicity (5.2b) then takes the form

$$\begin{pmatrix} \underline{M}(s_o+L, s_o) & \vec{h}(s_o+L) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \vec{y}_o(s_o) \\ 1 \end{pmatrix} = \begin{pmatrix} \vec{y}_o(s_o) \\ 1 \end{pmatrix}$$

from which one can calculate the "initial vector" $\vec{y}_o(s_o)$ of the closed orbit:

$$\vec{y}_o(s_o) = [\underline{1} - \underline{M}(s_o + L, s_o)]^{-1}\,\vec{h}(s_o + L). \tag{5.9}$$

5.2 Calculation of the transfer matrices [1]

5.2.1 The transfer matrices for the miscellaneous beam line elements

5.2.1.1 Synchrotron magnet

$\Delta\vec{B} = \Delta\vec{\varepsilon} = 0$;

$N = H = H' = \hat{V} = 0$;

$K_x^2 + K_z^2 = \text{const} \neq 0$;

$K_x \cdot K_z = 0$;

$$\begin{cases} K_x \neq 0 & \text{curvature in x-direction;} \\ K_z \neq 0 & \text{curvature in z-direction;} \end{cases}$$

$G_1 = K_x^2 + g$; $G_2 = K_z^2 - g$; $g = \text{const} \neq 0$.

In this case the equations of motion (5.4) and (4.2) read ($p_\sigma \equiv \eta$)

$$
\begin{aligned}
x' &= p_x \\
p_x' &= - G_1 x + K_x \eta \\
z' &= p_z \\
p_z' &= - G_2 z + K_z \eta \\
\sigma' &= - K_x x - K_z z \\
\eta' &= - C_1 \, (K_x^2 + K_z^2)
\end{aligned}
\tag{5.10}
$$

The elements of the (enlarged) transfer matrix $\underline{\hat{M}}(s, s_o)$ are given by

$$\hat{M}_{11} = \cos\,[\sqrt{G_1}(s - s_o)] \quad ;$$

$$\hat{M}_{12} = \frac{1}{\sqrt{G_1}} \sin\,[\sqrt{G_1}(s - s_o)] \;;$$

$$\hat{M}_{16} = \frac{K_x}{G_1} \cdot \{1 - \cos\,[\sqrt{G_1}(s - s_o)\} \quad ;$$

$$\hat{M}_{17} = - \frac{1}{G_1} \cdot C_1 \cdot K_x (K_x^2 + K_z^2) \cdot \{(s - s_o) - \frac{1}{\sqrt{G_1}} \sin[\sqrt{G_1}(s - s_o)]\} ;$$

$$\hat{M}_{21} = - \sqrt{G_1} \cdot \sin\,[\sqrt{G_1}(s - s_o)] \;;$$

$$\hat{M}_{22} = \cos\,[\sqrt{G_1}(s - s_o)\,] \quad ;$$

$$\hat{M}_{26} = \frac{K_x}{\sqrt{G_1}} \cdot \sin\,[\sqrt{G_1}(s - s_o)] \quad ;$$

$$\hat{M}_{27} = - \frac{1}{G_1} \cdot C_1 \cdot K_x (K_x^2 + K_z^2) \cdot \{1 - \cos\,[\sqrt{G_1}(s - s_o)]\} \quad ;$$

$$\hat{M}_{33} = \cos\,[\sqrt{G_2}(s - s_o)] \quad ;$$

$$\hat{M}_{34} = \frac{1}{\sqrt{G_2}} \sin\,[\sqrt{G_2}(s - s_o)] \quad ;$$

$$\hat{M}_{36} = \frac{K_z}{G_2} \cdot \{1 - \cos\,[\sqrt{G_2}(s - s_o)]\} \quad ;$$

$$\hat{M}_{37} = - \frac{1}{G_2} \cdot C_1 \cdot K_z (K_x^2 + K_z^2) \cdot \{(s - s_o) - \frac{1}{\sqrt{G_2}} \cdot \sin\,[\sqrt{G_2}(s - s_o)]\} ;$$

$$\hat{M}_{43} = -\sqrt{G_2}\cdot\sin[\sqrt{G_2}(s-s_o)] \quad ;$$

$$\hat{M}_{44} = \cos[\sqrt{G_2}(s-s_o)] \quad ;$$

$$\hat{M}_{46} = \frac{K_z}{\sqrt{G_2}}\cdot\sin[\sqrt{G_2}(s-s_o)] \quad ;$$

$$\hat{M}_{47} = -\frac{1}{G_2}\cdot C_1\cdot K_z(K_x^2+K_z^2)\cdot\{1-\cos[\sqrt{G_2}(s-s_o)]\} \; ;$$

$$\hat{M}_{51} = -\frac{K_x}{\sqrt{G_1}}\sin[\sqrt{G_1}(s-s_o)] \quad ;$$

$$\hat{M}_{52} = -\frac{K_x}{G_1}\{1-\cos[\sqrt{G_1}(s-s_o)]\} \quad ;$$

$$\hat{M}_{53} = -\frac{K_z}{\sqrt{G_2}}\sin[\sqrt{G_2}(s-s_o)] \quad ;$$

$$\hat{M}_{54} = -\frac{K_z}{G_2}\{1-\cos[\sqrt{G_2}(s-s_o)]\} \quad ;$$

$$\hat{M}_{55} = 1 \; ;$$

$$\hat{M}_{56} = -\frac{K_x^2}{G_1}\{(s-s_o)-\frac{1}{\sqrt{G_1}}\sin[\sqrt{G_1}(s-s_o)]\} - \\ -\frac{K_z^2}{G_2}\{(s-s_o)-\frac{1}{\sqrt{G_2}}\sin[\sqrt{G_2}(s-s_o)]\} \quad ;$$

$$\hat{M}_{57} = \frac{1}{G_1}\cdot C_1\cdot K_x^2(K_x^2+K_z^2)\cdot\{\frac{1}{2}(s-s_o)^2+\frac{1}{G_1}\cos[\sqrt{G_1}(s-s_o)]-\frac{1}{G_1}\} + \\ +\frac{1}{G_2}\cdot C_1\cdot K_z^2(K_x^2+K_z^2)\cdot\{\frac{1}{2}(s-s_o)^2+\frac{1}{G_2}\cos[\sqrt{G_2}(s-s_o)]-\frac{1}{G_2}\};$$

$$\hat{M}_{66} = 1 = \hat{M}_{77} \; ;$$

$$\hat{M}_{67} = -C_1\cdot(K_x^2+K_z^2)\cdot(s-s_o) \quad ;$$

$$\hat{M}_{ik} = 0 \qquad \text{otherwise .} \qquad (5.11)$$

If the quantities G_1 and G_2 are negative, which may be the case, we can make use of the following relationships

$$\cos[\sqrt{-|G|}(s-s_o)] = \cosh[\sqrt{|G|}(s-s_o)] \; ;$$
$$\frac{1}{\sqrt{-|G|}}\sin[\sqrt{-|G|}(s-s_o)] = \frac{1}{\sqrt{|G|}}\sinh[\sqrt{|G|}(s-s_o)]. \qquad (5.12)$$

5.2.1.2 Quadrupole

$$N = H = H' = \hat{V} = K_x = K_z = 0$$

$$\vec{\Delta B} = \Delta\vec{\varepsilon} = 0$$

$$\left.\begin{aligned} G_1 &= +g \\ G_2 &= -g \end{aligned}\right\} \text{ with } g = \frac{e}{E_o}\left(\frac{\partial B_z}{\partial x}\right)_{x=z=o} = \text{const} \neq 0 \; .$$

The equations of motion now read

$$\begin{aligned} x' &= p_x \\ p_x' &= -g\,x \\ z' &= p_z \\ p_z' &= g\,z \\ \sigma' &= 0 \\ \eta' &= 0 \end{aligned} \tag{5.13}$$

with the transfer matrix

$$\underline{\hat{M}} = ((\hat{M}_{ik})) \;;$$

$$\hat{M}_{11} = \cos\left[\sqrt{g}(s - s_o)\right] \;;$$

$$\hat{M}_{12} = \frac{1}{\sqrt{g}} \sin\left[\sqrt{g}(s - s_o)\right] \;;$$

$$\hat{M}_{21} = -\sqrt{g}\,\sin\left[\sqrt{g}(s - s_o)\right] \;;$$

$$\hat{M}_{22} = \cos\left[\sqrt{g}(s - s_o)\right] \;;$$

$$\hat{M}_{33} = \cos\left[\sqrt{-g}(s - s_o)\right] \;;$$

$$\hat{M}_{34} = \frac{1}{\sqrt{-g}} \sin\left[\sqrt{-g}(s - s_o)\right] \;;$$

$$\hat{M}_{43} = -\sqrt{-g}\,\sin\left[\sqrt{-g}(s - s_o)\right] \;;$$

$$\hat{M}_{44} = \cos\left[\sqrt{-g}(s - s_o)\right] \;;$$

$$\hat{M}_{55} = 1 \;;$$

$$\hat{M}_{66} = 1 \;;$$

$\hat{M}_{77} = 1$;

$\hat{M}_{ik} = 0$ otherwise (and see eq. (5.12)) . (5.14)

5.2.1.3 Skew quadrupole

$$G_1 = G_2 = H = H' = K_x = K_z = \hat{V} = 0 \; ; \; \Delta\vec{\varepsilon} = \Delta\vec{B} = 0;$$

$$N \neq 0 \; .$$

The equations of motion are given by (see eq. (5.4), (4.2) and (4.3))

$$\begin{aligned} x' &= p_x \\ p_x' &= N \cdot z \\ z' &= p_z \\ p_z' &= N \cdot x \\ \sigma' &= 0 \\ \eta' &= 0 \end{aligned} \tag{5.15}$$

which means, that the betatron oscillations in x- and z-direction are coupled:

$$\begin{aligned} x'' &= N \cdot z \\ z'' &= N \cdot x \, . \end{aligned} \tag{5.16}$$

It follows from (5.16)

$$(x + z)'' = N(x + z) \; ;$$

$$(x - z)'' = - N(x - z).$$

In this form the differential equations are decoupled and they can be integrated easily. Thus we obtain the following expressions for the matrix elements of the transfer matrix $\underline{\hat{M}}(s_o, s)$:

$$\hat{M}_{11} = \frac{1}{2} \{\cos [\sqrt{-N}(s - s_o)] + \cos [\sqrt{+N}(s - s_o)]\} \; ;$$

$$\hat{M}_{12} = \frac{1}{2} \{\frac{1}{\sqrt{-N}} \sin [\sqrt{-N}(s - s_o)] + \frac{1}{\sqrt{+N}} \sin [\sqrt{+N}(s - s_o)]\} \; ;$$

$$\hat{M}_{13} = \frac{1}{2}\{\cos[\sqrt{-N}(s - s_o)] - \cos[\sqrt{+N}(s - s_o)]\} ;$$

$$\hat{M}_{14} = \frac{1}{2}\{\frac{1}{\sqrt{-N}} \sin[\sqrt{-N}(s - s_o)] - \frac{1}{\sqrt{+N}} \sin[\sqrt{+N}(s - s_o)]\} ;$$

$$\hat{M}_{21} = -\frac{1}{2}\{\sqrt{-N} \sin[\sqrt{-N}(s - s_o)] + \sqrt{+N} \sin[\sqrt{+N}(s - s_o)]\};$$

$$\hat{M}_{22} = \hat{M}_{11} ;$$

$$\hat{M}_{23} = -\frac{1}{2}\{\sqrt{-N} \sin[\sqrt{-N}(s - s_o)] - \sqrt{+N} \sin[\sqrt{+N}(s - s_o)]\} ;$$

$$\hat{M}_{24} = \hat{M}_{13} ;$$

$$\hat{M}_{31} = \hat{M}_{13} ;$$

$$\hat{M}_{32} = \hat{M}_{14} ;$$

$$\hat{M}_{33} = \hat{M}_{11} ;$$

$$\hat{M}_{34} = \hat{M}_{12} ;$$

$$\hat{M}_{41} = \hat{M}_{23} ;$$

$$\hat{M}_{42} = \hat{M}_{21} ;$$

$$\hat{M}_{43} = \hat{M}_{13} ;$$

$$\hat{M}_{44} = \hat{M}_{11} ;$$

$$\hat{M}_{55} = 1 ;$$

$$\hat{M}_{66} = 1 ;$$

$$\hat{M}_{77} = 1 ;$$

$$\hat{M}_{ik} = 0 \text{ otherwise} . \qquad (5.17)$$

5.2.1.4 Solenoid

$$G_1 = G_2 = N = K_x = K_z = \hat{V} = 0 ;$$

$$H = \text{const} \neq 0 ;$$

$$\Delta\vec{\varepsilon} = \Delta\vec{B} = 0 .$$

The equations of motion read

$$\begin{aligned}
x' &= p_x + H \cdot z \\
p_x' &= - H^2 x + H p_z \\
z' &= - Hx + p_z \\
p_z' &= - H p_x - H^2 z
\end{aligned} \tag{5.18}$$

and the tranfer matrix is given by

$$\hat{M}_{11} = \frac{1}{2} \cdot (1 + \cos 2\theta);$$

$$\hat{M}_{12} = \frac{1}{2H} \cdot \sin 2\theta \quad ;$$

$$\hat{M}_{13} = \frac{1}{2} \sin 2\theta \quad ;$$

$$\hat{M}_{14} = \frac{1}{2H} \cdot (1 - \cos 2\theta);$$

$$\hat{M}_{21} = - H \cdot \frac{1}{2} \sin 2\theta \quad ;$$

$$\hat{M}_{22} = \hat{M}_{11} \; ;$$

$$\hat{M}_{23} = - H \cdot \frac{1}{2} (1 - \cos 2\theta);$$

$$\hat{M}_{24} = \hat{M}_{13} \; ;$$

$$\hat{M}_{31} = - \hat{M}_{13} \; ;$$

$$\hat{M}_{32} = - \hat{M}_{14} \; ;$$

$$\hat{M}_{33} = \hat{M}_{11} \; ;$$

$$\hat{M}_{34} = \hat{M}_{12} \; ;$$

$$\hat{M}_{41} = - \hat{M}_{23} \; ;$$

$$\hat{M}_{42} = - \hat{M}_{13} \; ;$$

$$\hat{M}_{43} = \hat{M}_{21} \quad ;$$

$$\hat{M}_{44} = \hat{M}_{11} ;$$

$$\hat{M}_{55} = 1 ;$$

$$\hat{M}_{66} = 1 ;$$

$$\hat{M}_{77} = 1 ;$$

$$\hat{M}_{ik} = 0 \quad \text{otherwise} \tag{5.19}$$

with

$$\theta(s) = H\cdot(s - s_o) . \tag{5.20}$$

5.2.1.5 Cavity

$$G_1 = G_2 = N = H = H' = K_x = K_z = 0$$

$$\hat{V} \neq 0$$

$$\Delta\vec{\varepsilon} = \Delta\vec{B} = 0$$

The equations of motion take the form

$$x' = p_x$$

$$p_x' = 0$$

$$z' = p_z$$

$$p_z' = 0$$

$$\sigma' = 0$$

$$\eta' = \sigma\frac{e\hat{V}}{E_o} k \cdot \frac{2\pi}{L} \cos\Phi \sum_\nu \delta(s - s_\nu) + \frac{e\hat{V}}{E_o} \sin\Phi \sum_\nu \delta(s - s_\nu) \tag{5.21}$$

and the following expressions are found for the matrix elements of $\underline{\hat{M}}(s_\nu+o,s_\nu-o)$:

$$\hat{M}_{kk} = 1 \quad \text{for } k = 1,2...7$$

$$\hat{M}_{65} = \frac{e\hat{V}}{E_o} k \frac{2\pi}{L} \cos\Phi$$

$$\hat{M}_{67} = \frac{e\hat{V}}{E_o} \sin\Phi$$

$$\hat{M}_{ik} = 0 \quad \text{otherwise} . \tag{5.22}$$

5.2.1.6 Electric and magnetic dipole fields

$$G_1 = G_2 = N = H = H' = \hat{V} = K_x = K_z = 0$$
$$\Delta\vec{B} = \Delta\hat{\vec{B}} \cdot \delta(s - s_o)$$
$$\Delta\vec{\varepsilon} = \Delta\hat{\vec{\varepsilon}} \cdot \delta(s - s_o)$$

The equations of motion read

$$x' = p_x$$
$$p_x' = \frac{e}{E_o} (\Delta\hat{\varepsilon}_x - \Delta\hat{B}_z)\ \delta(s - s_o)$$
$$z' = p_z$$
$$p_z' = \frac{e}{E_o} (\Delta\hat{\varepsilon}_z + \Delta\hat{B}_x)\ \delta(s - s_o)$$
$$\sigma' = 0$$
$$\eta' = \frac{e}{E_o} \Delta\hat{\varepsilon}_\tau\ \delta(s - s_o) \tag{5.23}$$

with the transfer matrix:

$$\hat{M}_{kk} = 1 \quad \text{for } k = 1,2...7$$
$$\hat{M}_{17} = 0$$
$$\hat{M}_{27} = \frac{e}{E_o} (\Delta\hat{\varepsilon}_x - \Delta\hat{B}_z)$$
$$\hat{M}_{37} = 0$$
$$\hat{M}_{47} = \frac{e}{E_o} (\Delta\hat{\varepsilon}_z + \Delta\hat{B}_x)$$
$$\hat{M}_{57} = 0$$
$$\hat{M}_{67} = \frac{e}{E_o} \Delta\hat{\varepsilon}_\tau$$
$$\hat{M}_{ik} = 0 \quad \text{otherwise} \tag{5.24}$$

5.2.2 Approximation schemes

In the foregoing chapters we have given explicit expressions for the enlarged transfer matrices of miscellaneous beam line elements. In more complicated cases one has to apply suitable approximation schemes for calculating the transfer matrices. Now we want to describe two simple schemes of calculation.

5.2.2.1 Series expansion

Since the equations of motion for each beam line element are linear differential equations with constant coefficients

$$\underline{A}(s) = \text{const} \tag{5.25a}$$

$$\vec{c}_o(s) = \text{const} \tag{5.25b}$$

$$\vec{c}_1(s) = \text{const} \tag{5.25c}$$

one can write down the following expressions for the simple transfer matrix $\underline{M}(s, s_o)$ and the special solution $\vec{h}(s)$ defined in (5.6)

$$\underline{M}(s, s_o) = e^{\underline{A}(s-s_o)} \equiv \sum_{n=o}^{\infty} \frac{1}{n!} \underline{A}^n \cdot (s - s_o)^n \tag{5.26a}$$

$$\vec{h}(s) = (\sum_{n=1}^{\infty} \frac{1}{n!} \underline{A}^{n-1}(s - s_o)^n)(\vec{c}_o + \vec{c}_1) \tag{5.26b}$$

which can easily be verified by putting (5.26a, b) into the equations for $\underline{M}$ (eq. (5.7a, b)) and $\vec{h}$ (eq. (5.8a, b) respectively. Thus we have obtained a series expansion allowing an approximate calculation of $\underline{M}$ and $\vec{h}$, and hence of the enlarged transfer matrix $\hat{\underline{M}}$, if we truncate the expansion after a finite number of terms. The terms taken into account determine the accuracy of the approximation. It is worthwhile mentioning that the vector $\vec{h}(s)$ can be put into the form

$$\vec{h}(s) = [\underline{M}(s, s_o) - \underline{1}\ ]\ \underline{A}^{-1}\ (\vec{c}_o + \vec{c}_1)$$

if $\det(\underline{A}) \neq 0$ (existence of $\underline{A}^{-1}$). In this case one only needs the matrix $\underline{M}$ for a calculation of $\vec{h}$.

5.2.2.2 Decomposition of a beam line element into thin lenses (thin lens approximation)

If the conditions (5.25a, b, c) do not hold, for example, if we want to include the perturbation matrix $\delta\underline{A}$ in the calculation, we can divide the given beam line element into small segments and according to (5.4) one can calculate the infinitesimal transfer matrix $\hat{\underline{M}}(s + \Delta s, s)$

$$\hat{\underline{M}}(s + \Delta s, s) = \begin{pmatrix} [\underline{1} + \underline{A}(s) \cdot \Delta s] & [\vec{c}_o(s) + \vec{c}_1(s)]\Delta s \\ 0 & 1 \end{pmatrix}, \tag{5.27}$$

Multiplying the single infinitesimal transfer matrices we obtain the matrix $\hat{\underline{M}}(s_o + \ell, s_o)$ for the whole beam line element of length ℓ.

5.3 Equations of motion of the free synchro-betatron oscillations

Having set up the transfer matrices for the different types of lenses of a storage ring we are able to determine the six-dimensional closed orbit $\vec{y}_o(s)$. This means that we know the first component of the oscillation amplitude $\vec{y}(s)$ which has been decomposed according to eq. (5.3).
Inserting (5.3) into (4.1) and taking into account (5.2a) we obtain the following equation for $\vec{\tilde{y}}(s)$

$$\vec{\tilde{y}}\,' = (\underline{A} + \delta\underline{A})\,\vec{\tilde{y}} + \delta\vec{c} \tag{5.28}$$

where the inhomogeneous parts $\vec{c}_o$ and $\vec{c}_1$ have indeed disappeared as required. Eq. (5.28) describes the free synchro-betatron oscillations of a particle about the new reference trajectory $\vec{y}_o(s)$.

6. Spin motion

6.1 Perturbation theory

In analogy to the separation of the oscillation amplitude $\vec{y}$ into two parts (see eq. (4.7)) we can divide the vector $\vec{\Omega}$ into two components, namely:

$$\vec{\Omega}(\vec{y}) = \vec{\Omega}^{(o)} + \vec{\omega} \tag{6.1a}$$

with

$$\vec{\Omega}^{(o)} = \vec{\Omega}(\vec{y}_o), \qquad \vec{\omega} = \vec{\omega}(\vec{\tilde{y}}) = \vec{\Omega} - \vec{\Omega}^{(o)}. \tag{6.1b}$$

Then eq. (4.8) reduces to

$$\begin{aligned}\Omega_\tau^{(o)} = &- 2H\left[1 + a\frac{\gamma_o}{1+\gamma_o}\right] - 2H(K_x x_o + K_z z_o)\left[1 + a\gamma_o\frac{1}{1+\gamma_o}\right] - \\ &- a\gamma_o\frac{\gamma_o}{1+\gamma_o}\left[x_o' K_z - z_o' K_x\right] + 2H\eta_o\left[1 + a\frac{\gamma_o^2}{(1+\gamma_o)^2}\right] - \\ &- \frac{e}{E_o}\Delta B_\tau\left[1 + a\frac{\gamma_o}{1+\gamma_o}\right];\end{aligned} \tag{6.2a}$$

$$\Omega_x^{(o)} = K_z\, a\gamma_o + (1 + a\gamma_o) \cdot K_z^2 \cdot z_o - K_z\eta_o -$$

$$- (1 + a\gamma_o) \cdot [(N - H') \cdot x_o + g\, z_o] +$$

$$+ a\gamma_o \frac{\gamma_o}{1+\gamma_o}\, 2H \cdot x_o' - (1 + a\gamma_o) \frac{e}{E_o} \Delta B_x -$$

$$- (a\gamma_o + \frac{\gamma_o}{1+\gamma_o}) \frac{e}{E_o} \Delta\varepsilon_z +$$

$$+ (a\gamma_o + \frac{\gamma_o}{1+\gamma_o}) \cdot z_o' \cdot \frac{e\hat{V}}{E_o} \sin\Phi \sum_\nu \delta(s - s_\nu); \qquad (6.2b)$$

$$\Omega_z^{(o)} = - K_x\, a\gamma_o - (1 + a\gamma_o)\, K_x^2 x_o + K_x\eta_o +$$

$$+ (1 + a\gamma_o) \cdot [(N + H') \cdot z_o - g\, x_o] +$$

$$+ a\gamma_o \frac{\gamma_o}{1+\gamma_o}\, 2H\, z_o' - (a\gamma_o + \frac{\gamma_o}{1+\gamma_o}) \cdot x_o' \frac{e\hat{V}}{E_o} \sin\Phi \sum_\nu \delta(s - s_\nu) -$$

$$- (1 + a\gamma_o) \frac{e}{E_o} \Delta B_z + (a\gamma_o + \frac{\gamma_o}{1+\gamma_o}) \frac{e}{E_o} \Delta\varepsilon_x\, ; \qquad (6.2c)$$

$$\omega_\tau = - 2H\, (K_x\tilde{x} + K_z\tilde{z})\, [1 + a\gamma_o \frac{1}{1+\gamma_o}] -$$

$$- a\gamma_o \frac{\gamma_o}{1+\gamma_o} (\hat{x}' \cdot K_z - \tilde{z}' \cdot K_x) + 2H\tilde{\eta}\, [1 + a \frac{\gamma_o^2}{(1+\gamma_o)^2}]\; ; \qquad (6.3a)$$

$$\omega_x = (1 + a\gamma_o) \cdot K_z^2\tilde{z} - K_z\tilde{\eta} - (1 + a\gamma_o) \cdot [(N - H')\tilde{x} + g\, \tilde{z}] +$$

$$+ a\gamma_o \frac{\gamma_o}{1+\gamma_o}\, 2H\, \hat{x}' + [a\gamma_o + \frac{\gamma_o}{1+\gamma_o}] \cdot \hat{z}' \frac{e\hat{V}}{E_o} \sin\Phi \sum_\nu \delta(s - s_\nu)\; ; \qquad (6.3b)$$

$$\omega_z = - (1 + a\gamma_o)\, K_x^2\tilde{x} + K_x\tilde{\eta} + (1 + a\gamma_o) \cdot [(N + H')\tilde{z} - g\, \tilde{x}] +$$

$$+ a\gamma_o \frac{\gamma_o}{1+\gamma_o}\, 2H\, \hat{z}' - [a\gamma_o + \frac{\gamma_o}{1+\gamma_o}] \cdot \hat{x}' \frac{e\hat{V}}{E_o} \sin\Phi \sum_\nu \delta(s - s_\nu) \qquad (6.3c)$$

where we have also used the relations [1]:

$$\tilde{p}_x = \tilde{x}' - H \cdot \tilde{z}; \qquad (6.4a)$$

$$\tilde{p}_z = \tilde{z}' + H \cdot \tilde{x}; \qquad (6.4b)$$

(see eq. (4.1)).

Thus, $\vec{\Omega}$ is determined by the miscellaneous field components. In particular, we consider the following special cases:

a) Synchrotron magnet

$$N = H = H' = \hat{V} = 0$$

$$\Delta\vec{B} = \Delta\vec{\varepsilon} = 0$$

$$g \neq 0;\ K_x^2 + K_z^2 \neq 0;\ K_x \cdot K_z = 0$$

$$\Omega_\tau^{(o)} = - a\gamma_o \frac{\gamma_o}{1+\gamma_o} (x_o' K_z - z_o' K_x)$$

$$\Omega_x^{(o)} = K_z a\gamma_o + (1 + a\gamma_o) z_o \cdot G_2 - K_z \eta_o$$

$$\Omega_z^{(o)} = - K_x a\gamma_o - (1 + a\gamma_o) x_o \cdot G_1 + K_x \eta_o \qquad (6.5a)$$

$$\omega_\tau = - a\gamma_o \frac{\gamma_o}{1+\gamma_o} (\tilde{x}' K_z - \tilde{z}' K_x)$$

$$\omega_x = (1 + a\gamma_o)\tilde{z} \cdot G_2 - K_z \tilde{\eta}$$

$$\omega_z = - (1 + a\gamma_o)\tilde{x} \cdot G_1 + K_x \tilde{\eta} \qquad (6.5b)$$

b) Quadrupole

$$N = H = H' = \hat{V} = K_x = K_z = 0$$

$$\Delta\vec{B} = \Delta\vec{\varepsilon} = 0$$

$$g \neq 0$$

$$\Omega_\tau^{(o)} = 0$$

$$\Omega_x^{(o)} = - (1 + a\gamma_o)\, g\, z_o$$

$$\Omega_z^{(o)} = - (1 + a\gamma_o)\, g\, x_o \qquad (6.6a)$$

$$\omega_\tau = 0$$

$$\omega_x = - (1 + a\gamma_o)\, g\, \tilde{z}$$

$$\omega_z = - (1 + a\gamma_o)\, g\, \tilde{x} \qquad (6.6b)$$

c) Skew quadrupole

$$g = H = H' = \hat{V} = K_x = K_z = 0$$

$$\Delta\vec{B} = \Delta\vec{\varepsilon} = 0$$

$$N \neq 0$$

$$\Omega_\tau^{(o)} = 0$$

$$\Omega_x^{(o)} = -(1 + a\gamma_o) \cdot N\, x_o$$

$$\Omega_z^{(o)} = +(1 + a\gamma_o) \cdot N\, z_o \qquad (6.7a)$$

$$\omega_\tau = 0$$

$$\omega_x = -(1 + a\gamma_o) \cdot N\, \tilde{x}$$

$$\omega_z = +(1 + a\gamma_o) \cdot N\, \tilde{z} \qquad (6.7b)$$

d) Solenoid

$$G_1 = G_2 = N = K_x = K_z = \hat{V} = 0$$

$$\Delta\vec{B} = \Delta\vec{\varepsilon} = 0$$

$$H \neq 0$$

$$\Omega_\tau^{(o)} = -2H\,[1 + a\,\frac{\gamma_o}{1+\gamma_o}] + 2H\eta_o\,[1 + a\,\frac{\gamma_o^2}{(1+\gamma_o)^2}]$$

$$\Omega_x^{(o)} = (1 + a\gamma_o)\, H'\, x_o + a\gamma_o\,\frac{\gamma_o}{1+\gamma_o}\, 2H\, x_o'$$

$$\Omega_z^{(o)} = (1 + a\gamma_o)\, H'\, z_o + a\gamma_o\,\frac{\gamma_o}{1+\gamma_o}\, 2H\, z_o' \qquad (6.8a)$$

$$\omega_\tau = 2H\tilde{\eta}\,[1 + a\,\frac{\gamma_o^2}{(1+\gamma_o)^2}]$$

$$\omega_x = (1 + a\gamma_o)\, H'\, \tilde{x} + a\gamma_o\,\frac{\gamma_o}{1+\gamma_o}\, 2H\, \tilde{x}'$$

$$\omega_z = (1 + a\gamma_o)\, H'\, \tilde{z} + a\gamma_o\,\frac{\gamma_o}{1+\gamma_o}\, 2H\, \tilde{z}' \qquad (6.8b)$$

e) Cavity

$$G_1 = G_2 = N = H = K_x = K_z = 0$$

$$\Delta\vec{B} = \Delta\vec{\varepsilon} = 0$$

$$\hat{V} \neq 0$$

$$\Omega_\tau^{(o)} = 0$$

$$\Omega_x^{(o)} = (a\gamma_o + \frac{\gamma_o}{1+\gamma_o}) \; z_o' \; \frac{e\hat{V}}{E_o} \sin\Phi \sum_\nu \delta(s - s_\nu)$$

$$\Omega_z^{(o)} = - (a\gamma_o + \frac{\gamma_o}{1+\gamma_o}) \; x_o' \; \frac{e\hat{V}}{E_o} \sin\Phi \sum_\nu \delta(s - s_\nu) \qquad (6.9a)$$

$$\omega_\tau = 0$$

$$\omega_x = (a\gamma_o + \frac{\gamma_o}{1+\gamma_o}) \; \tilde{z}' \; \frac{e\hat{V}}{E_o} \sin\Phi \sum_\nu \delta(s - s_\nu)$$

$$\omega_z = - (a\gamma_o + \frac{\gamma_o}{1+\gamma_o}) \; \tilde{x}' \; \frac{e\hat{V}}{E_o} \sin\Phi \sum_\nu \delta(s - s_\nu) \qquad (6.9b)$$

f) Perturbing electric fields and magnetic dipole fields

$$G_1 = G_2 = N = H = \hat{V} = K_x = K_z = 0$$

$$\Delta\vec{B} = \Delta\vec{\hat{B}} \cdot \delta(s - s_o)$$

$$\Delta\vec{\varepsilon} = \Delta\vec{\hat{\varepsilon}} \cdot \delta(s - s_o)$$

$$\Omega_\tau^{(o)} = - \frac{e}{E_o} \; \Delta\hat{B}_\tau \; [1 + a \frac{\gamma_o}{1+\gamma_o}] \cdot \delta(s - s_o)$$

$$\Omega_x^{(o)} = - (1 + a\gamma_o) \frac{e}{E_o} \Delta\hat{B}_x \cdot \delta(s - s_o) - (a\gamma_o + \frac{\gamma_o}{1+\gamma_o}) \; \frac{e}{E_o} \Delta\hat{\varepsilon}_z \cdot \delta(s - s_o)$$

$$\Omega_z^{(o)} = - (1 + a\gamma_o) \frac{e}{E_o} \Delta\hat{B}_z \cdot \delta(s - s_o) + (a\gamma_o + \frac{\gamma_o}{1+\gamma_o}) \cdot \frac{e}{E_o} \Delta\hat{\varepsilon}_x \cdot \delta(s - s_o) \qquad (6.10a)$$

$$\omega_\tau = 0$$

$$\omega_x = 0$$

$$\omega_z = 0 \qquad (6.10b)$$

In the following $\vec{\omega}$ will be considered as a small perturbation. Making the "ansatz"

$$\vec{\xi} = \vec{\xi}^{(o)} + \vec{\xi}^{(1)} \tag{6.11}$$

$$\vec{\xi}^{(o)} = \xi_\tau^{(o)}\vec{\tau} + \xi_x^{(o)}\vec{e}_x + \xi_z^{(o)}\vec{e}_z \tag{6.12a}$$

$$\vec{\xi}^{(1)} = \xi_\tau^{(1)}\vec{\tau} + \xi_x^{(1)}\vec{e}_x + \xi_z^{(1)}\vec{e}_z \tag{6.12b}$$

and using (see eq. (4.7))

$$\vec{\tau} \cdot \frac{d}{ds}\xi_\tau^{(o)} + \vec{e}_x \cdot \frac{d}{ds}\xi_x^{(o)} + \vec{e}_z \cdot \frac{d}{ds}\xi_z^{(o)} = \vec{\Omega}^{(o)} \times \vec{\xi}^{(o)} \tag{6.13}$$

we obtain the following expression for $\vec{\xi}^{(1)}$ in linear order of perturbation theory

$$\vec{\tau} \cdot \frac{d}{ds}\xi_\tau^{(1)} + \vec{e}_x \frac{d}{ds}\xi_x^{(1)} + \vec{e}_z \frac{d}{ds}\xi_z^{(1)} = \vec{\Omega}^{(o)} \times \vec{\xi}^{(1)} + \vec{\omega} \times \vec{\xi}^{(o)} . \tag{6.14}$$

A. Chao [17] has shown that eq. (6.13) can be used to define a new system of orthogonal unit vectors which considerably simplify the spin motion determined by (6.14).

6.2 The ($\vec{n}$, $\vec{m}$, $\vec{\ell}$) coordinate frame describing the spin motion [2]

In the following we shall introduce a compact matrix notation. Rewriting an arbitrary vector $\vec{A}$

$$\vec{A} = A_\tau\vec{\tau} + A_x\vec{e}_x + A_z\vec{e}_z$$

as a column vector with components A_τ, A_x, A_z

$$A_\tau\vec{\tau} + A_x\vec{e}_x + A_z\vec{e}_z \equiv \begin{pmatrix} A_\tau \\ A_x \\ A_z \end{pmatrix}$$

and defining the derivative of a column vector with respect to s as the derivative of the corresponding components A_i but not of the unit vectors

$$\frac{d}{ds}\begin{pmatrix} A_\tau \\ A_x \\ A_z \end{pmatrix} \equiv \vec{\tau}\frac{d}{ds}A_\tau + \vec{e}_x\frac{d}{ds}A_x + \vec{e}_z\frac{d}{ds}A_z$$

we get from (6.13)

$$\frac{d}{ds} \vec{\xi}^{(o)} = \underline{\Omega}^{(o)} \cdot \vec{\xi}^{(o)} \tag{6.15}$$

where we have set

$$\underline{\Omega}^{(o)} = \begin{pmatrix} 0 & -\Omega_z^{(o)} & \Omega_x^{(o)} \\ \Omega_z^{(o)} & 0 & -\Omega_\tau^{(o)} \\ -\Omega_x^{(o)} & \Omega_\tau^{(o)} & 0 \end{pmatrix} \tag{6.16a}$$

and

$$\vec{\xi}^{(o)} = \begin{pmatrix} \xi_\tau^{(o)} \\ \xi_x^{(o)} \\ \xi_z^{(o)} \end{pmatrix} . \tag{6.16b}$$

The infinitesimal transfer matrix $\underline{N}(s + \Delta s, s)$ defined by

$$\vec{\xi}^{(o)}(s + \Delta s) = \underline{N}(s + \Delta s, s)\, \vec{\xi}^{(o)}(s)$$

and given by

$$\underline{N}(s + \Delta s, s) = \underline{1} + \Delta s\, \underline{\Omega}^{(o)}(s)$$

satisfies the following relationships

$$\begin{aligned} \underline{N}^T(s + \Delta s, s) \cdot \underline{N}(s + \Delta s, s) &= \{\underline{1} + \Delta s\, \underline{\Omega}^{(o)}\}^T \{\underline{1} + \Delta s\, \underline{\Omega}^{(o)}\} \\ &= \underline{1} + \Delta s\, \{\underline{\Omega}^{(o)T} + \underline{\Omega}^{(o)}\} \\ &= \underline{1} \end{aligned}$$

and

$$\det\{\underline{N}(s + \Delta s, s)\} = 1.$$

(In deriving this expression we have used the fact that $\underline{\Omega}^{(o)}$ is an antisymmetric matrix ($\underline{\Omega}^{(o)T} = -\underline{\Omega}^{(o)}$)). These relations imply that $\underline{N}(s + \Delta s, s)$ is an orthogonal matrix with determinant 1. The matrix $\underline{N}(s_2, s_1)$ has the same properties

$$\underline{N}^T(s_2, s_1) \cdot \underline{N}(s_2, s_1) = \underline{1} \tag{6.17a}$$

$$\det\{\underline{N}(s_2, s_1)\} = +1 \tag{6.17b}$$

because a transfer matrix of finite length can be represented as a product of many infinitesimal transfer matrices and because a product of orthogonal matrices with determinant 1 is orthogonal with the same value for the determinant.

Let us now consider the eigenvalue problem for the revolution matrix $\underline{N}(s_o + L, s_o)$ with the eigenvalues α_μ and eigenvectors $\vec{r}_\mu(s_o)$:

$$\underline{N}(s_o + L, s_o)\, \vec{r}_\mu(s_o) = \alpha_\mu\, \vec{r}_\mu(s_o) \tag{6.18a}$$

$$(\mu = 1, 2, 3).$$

Because of (6.17a, b) we can write

$$\begin{cases} \alpha_1 = 1 \\ \alpha_2 = e^{i2\pi\nu} \\ \alpha_3 = e^{-i2\pi\nu} \end{cases} \tag{6.18b}$$

$$(\nu = \text{real})$$

and

$$\vec{r}_1(s_o) = \vec{n}_o(s_o) \tag{6.19a}$$

$$\vec{r}_2(s_o) = \vec{m}_o(s_o) + i\, \vec{\ell}_o(s_o) \tag{6.19b}$$

$$\vec{r}_3(s_o) = \vec{m}_o(s_o) - i\, \vec{\ell}_o(s_o) \tag{6.19c}$$

$$\vec{n}_o, \vec{m}_o, \vec{\ell}_o = \text{real vectors}.$$

If we require that

$$\vec{r}_1^{\,+} \cdot \vec{r}_1 = 1 \tag{6.20a}$$

$$\vec{r}_2^{\,+} \cdot \vec{r}_2 \equiv \vec{r}_3^{\,+} \cdot \vec{r}_3 = 2 \tag{6.20b}$$

(normalizing conditions)

we find

$$|\vec{n}_o(s_o)| = |\vec{m}_o(s_o)| = |\vec{\ell}_o(s_o)| = 1\,; \tag{6.21a}$$

$$\vec{n}_o(s_o) \perp \vec{m}_o(s_o) \perp \vec{\ell}_o(s_o)\,. \tag{6.21b}$$

This means that the vectors $\vec{n}_0(s_0)$, $\vec{m}_0(s_0)$ and $\vec{\ell}_0(s_0)$ form an orthogonal system of unit vectors and choosing the direction of $\vec{n}_0(s_0)$ such that

$$\vec{n}_0(s_0) = \vec{m}_0(s_0) \times \vec{\ell}_0(s_0) \tag{6.21c}$$

these vectors form a righthanded coordinate system.
In this way we have found a coordinate frame for the position $s = s_0$.
An orthogonal system of unit vectors at an arbitrary position s can be defined by applying the transfer matrix $\underline{N}(s, s_0)$ to the vectors $\vec{n}_0(s_0)$, $\vec{m}_0(s_0)$ and $\vec{\ell}_0(s_0)$:

$$\vec{n}_0(s) = \underline{N}(s, s_0)\, \vec{n}_0(s_0) \tag{6.22a}$$

$$\vec{m}_0(s) = \underline{N}(s, s_0)\, \vec{m}_0(s_0) \tag{6.22b}$$

$$\vec{\ell}_0(s) = \underline{N}(s, s_0)\, \vec{\ell}_0(s_0)\ . \tag{6.22c}$$

Because of eqs. (6.17a, b) the orthogonality relations remain unchanged

$$\begin{aligned} &\vec{n}_0(s) = \vec{m}_0(s) \times \vec{\ell}_0(s) \\ &\vec{m}_0(s) \perp \vec{\ell}_0(s) \\ &|\vec{n}_0(s)| = |\vec{m}_0(s)| = |\vec{\ell}_0(s)| = 1\ . \end{aligned} \tag{6.23}$$

The coordinate frame defined by $\vec{n}_0(s)$, $\vec{m}_0(s)$ and $\vec{\ell}_0(s)$ is not yet appropriate for a description of the spin motion, because it does not transform into itself after one revolution of the particle:

$$\begin{aligned} \vec{m}_0(s_0 + L) + i\, \vec{\ell}_0(s_0 + L) &= \underline{N}(s_0 + L, s_0)\, [\vec{m}_0(s_0) + i\, \vec{\ell}_0(s_0)] \\ &= e^{i2\pi\nu}[\vec{m}_0(s_0) + i\, \vec{\ell}_0(s_0)] \\ &\neq \vec{m}_0(s_0) + i\, \vec{\ell}_0(s_0) \end{aligned}$$

(if $\nu \neq$ integer).

But by introducing a phase function $\psi(s)$ and using another orthogonal matrix $\underline{D}(s, s_0)$

$$\underline{D}(s, s_0) = \begin{pmatrix} \cos[\psi(s) - \psi(s_0)] & \sin[\psi(s) - \psi(s_0)] \\ -\sin[\psi(s) - \psi(s_0)] & \cos[\psi(s) - \psi(s_0)] \end{pmatrix} \tag{6.24}$$

with

$$\underline{D}^T(s, s_o) \cdot \underline{D}(s, s_o) = \underline{1} \tag{6.25a}$$

$$\det\{\underline{D}(s, s_o)\} = 1 \tag{6.25b}$$

we can construct a periodic orthogonal system of unit vectors from $\vec{n}_o(s)$, $\vec{m}_o(s)$, $\vec{\ell}_o(s)$. Namely, if we put

$$\begin{pmatrix} \vec{m}(s) \\ \vec{\ell}(s) \end{pmatrix} = \underline{D}(s, s_o) \begin{pmatrix} \vec{m}_o(s) \\ \vec{\ell}_o(s) \end{pmatrix} \Longrightarrow$$

$$\Longrightarrow \vec{m}(s) + i\,\vec{\ell}(s) = e^{-i[\psi(s) - \psi(s_o)]}[\vec{m}_o(s) + i\,\vec{\ell}_o(s)] \tag{6.26a}$$

and

$$\vec{n}(s) = \vec{n}_o(s) \tag{6.26b}$$

we find

$$\vec{n}(s) = \vec{m}(s) \times \vec{\ell}(s) \tag{6.26c}$$

$$\vec{m}(s) \perp \vec{\ell}(s) \tag{6.26d}$$

$$|\vec{n}(s)| = |\vec{m}(s)| = |\vec{\ell}(s)| = 1 \tag{6.26e}$$

because of eqs. (6.25a, b). And since

$$\begin{aligned}\vec{m}(s_o + L) + i\,\vec{\ell}(s_o + L) &= e^{-i[\psi(s_o + L) - \psi(s_o)]} \times \\ &\times [\vec{m}_o(s_o + L) + i\,\vec{\ell}_o(s_o + L)] = \\ &= e^{-i[\psi(s_o + L) - \psi(s_o)]} e^{i2\pi\nu} [\vec{m}_o(s_o) + i\,\vec{\ell}_o(s_o)]\end{aligned}$$

it follows, that the condition of periodicity can indeed be fulfilled for $\vec{n}$, $\vec{m}$, $\vec{\ell}$

$$(\vec{n}, \vec{m}, \vec{\ell})_{s = s_o + L} = (\vec{n}, \vec{m}, \vec{\ell})_{s = s_o} \tag{6.27}$$

if the phase function $\psi(s)$ satisfies the following relationship

$$\psi(s_o + L) - \psi(s_o) = 2\pi\nu \quad . \tag{6.28}$$

Taking the derivatives of $\vec{m}(s)$ and $\vec{\ell}(s)$ with respect to s, and taking into account eqs. (6.26), (6.22) and (6.15) we get

$$\frac{d}{ds}\vec{m}(s) = \underline{\Omega}^{(o)} \cdot \vec{m}(s) + \psi'(s)\,\vec{\ell}(s) \tag{6.29a}$$

$$\frac{d}{ds}\vec{\ell}(s) = \underline{\Omega}^{(o)} \cdot \vec{\ell}(s) - \psi'(s)\,\vec{m}(s) \tag{6.29b}$$

and $\vec{n}(s)$ satisfies (see (6.22a))

$$\frac{d}{ds}\vec{n}(s) = \underline{\Omega}^{(o)} \cdot \vec{n}(s) \,. \tag{6.29c}$$

Finally we would like to mention, that the vectors

$$\vec{r}_1(s) = \vec{n}_o(s) \qquad\qquad \equiv \underline{N}(s, s_o) \cdot \vec{r}_1(s_o) \tag{6.30a}$$

$$\vec{r}_2(s) = \vec{m}_o(s) + i\,\vec{\ell}_o(s) \equiv \underline{N}(s, s_o) \cdot \vec{r}_2(s_o) \tag{6.30b}$$

$$\vec{r}_3(s) = \vec{m}_o(s) - i\,\vec{\ell}_o(s) \equiv \underline{N}(s, s_o) \cdot \vec{r}_3(s_o) \tag{6.30c}$$

are eigenvectors of the revolution matrix $\underline{N}(s + L, s)$ with the same eigenvalues as in (6.18b):

$$\underline{N}(s + L, s) \cdot \vec{r}_\mu(s) = \alpha_\mu\, \vec{r}_\mu(s) \,.$$

Thus, the eigenvalues α_μ and the quantity ν defined by eq. (6.18b) are independent of the chosen initial position s_o.

7. The general equations of motion of the spin-orbit motion

Following A. Chao [17] we make the following "ansatz"

$$\vec{\xi}^{(o)}(s) = \xi_o \cdot \vec{n}(s) \tag{7.1a}$$

$$\vec{\xi}^{(1)}(s) = \xi_o\,[\alpha(s)\,\vec{m}(s) + \beta(s)\,\vec{\ell}(s)] \tag{7.1b}$$

$$(|\alpha|^2 + |\beta|^2 \ll 1)$$

to solve the equations of motion (6.13) and (6.14).

Because of (6.29c) the expression (7.1a) is a solution of (6.13) and putting (7.1a, b) into eq. (6.14) we get

$$\alpha' = (\ell_\tau, \ell_x, \ell_z) \cdot \begin{pmatrix} \omega_\tau \\ \omega_x \\ \omega_z \end{pmatrix} + \beta \psi' \tag{7.2a}$$

$$\beta' = -(m_\tau, m_x, m_z) \begin{pmatrix} \omega_\tau \\ \omega_x \\ \omega_z \end{pmatrix} - \alpha \psi' \tag{7.2b}$$

where we have taken into account (6.29a, b).
Using (6.4a, b) equ's. (6.3) can be rewritten in the form

$$\begin{pmatrix} \omega_\tau \\ \omega_x \\ \omega_z \end{pmatrix} = \underline{F}_{(3x6)} \cdot \vec{\tilde{y}} \quad . \tag{7.3}$$

The matrix elements of $\underline{F}_{(3x6)} \equiv \underline{F}$ are defined by

$$F_{11} = -2H\,K_x[1 + a\gamma_o \frac{1}{1+\gamma_o}] - a\gamma_o \frac{\gamma_o}{1+\gamma_o} K_x\,H$$

$$F_{12} = -a\gamma_o \frac{\gamma_o}{1+\gamma_o} K_z$$

$$F_{13} = -2H\,K_z\,[1 + a\gamma_o \frac{1}{1+\gamma_o}] - a\gamma_o \frac{\gamma_o}{1+\gamma_o} K_z\,H$$

$$F_{14} = a\gamma_o \frac{\gamma_o}{1+\gamma_o} K_x$$

$$F_{15} = 0$$

$$F_{16} = 2H\,[1 + a \frac{\gamma_o^2}{(1+\gamma_o)^2}]$$

$$F_{21} = -(1 + a\gamma_o)\,(N - H') - (a\gamma_o + \frac{\gamma_o}{1+\gamma_o})H \cdot \frac{e\hat{V}}{E_o} \sin\Phi \sum_\nu \delta(s - s_\nu)$$

$$F_{22} = a\gamma_o \frac{\gamma_o}{1+\gamma_o} 2H$$

$$F_{23} = +(1 + a\gamma_o)G_2 + a\gamma_o \frac{\gamma_o}{1+\gamma_o} 2H^2$$

$$F_{24} = (a\gamma_o + \frac{\gamma_o}{1+\gamma_o}) \cdot \frac{e\hat{V}}{E_o} \sin\Phi \sum_\nu \delta(s - s_\nu)$$

$$F_{25} = 0$$

$$F_{26} = - K_z$$

$$F_{31} = - (1 + a\gamma_o)G_1 - a\gamma_o \frac{\gamma_o}{1+\gamma_o} 2H^2$$

$$F_{32} = - (a\gamma_o + \frac{\gamma_o}{1+\gamma_o}) \cdot \frac{e\hat{V}}{E_o} \sin\Phi \sum_\nu \delta(s - s_\nu)$$

$$F_{33} = (1 + a\gamma_o)(N + H') - (a\gamma_o + \frac{\gamma_o}{1+\gamma_o})H \frac{e\hat{V}}{E_o} \sin\Phi \sum_\nu \delta(s - s_\nu)$$

$$F_{34} = a\gamma_o \frac{\gamma_o}{1+\gamma_o} 2H$$

$$F_{35} = 0$$

$$F_{36} = K_x \qquad (7.4)$$

In particular one obtains the following expressions for

a) Synchrotron magnet

$$N = H = H' = \hat{V} = 0$$

$$g \neq 0 \; , \quad K_x^2 + K_z^2 \neq 0 \; ; \quad K_x \cdot K_z = 0$$

$$F_{11} = 0$$

$$F_{12} = - a\gamma_o \frac{\gamma_o}{1+\gamma_o} K_z$$

$$F_{13} = 0$$

$$F_{14} = a\gamma_o \frac{\gamma_o}{1+\gamma_o} K_x$$

$$F_{23} = (1 + a\gamma_o)G_2$$

$$F_{26} = - K_z$$

$$F_{31} = - (1 + a\gamma_o)G_1$$

$$F_{36} = K_x$$

$$F_{ik} = 0 \quad \text{otherwise} \qquad (7.5a)$$

b) Quadrupole

$$N = H = H' = \hat{V} = K_x = K_z = 0$$

$$G_1 = +g\ ,\quad G_2 = -g$$

$$F_{23} = -(1 + a\gamma_o)g$$

$$F_{31} = -(1 + a\gamma_o)g$$

$$F_{ik} = 0 \quad \text{otherwise} \tag{7.5b}$$

c) Skew quadrupole

$$G_1 = G_2 = H = H' = K_x = K_z = 0$$

$$N \neq 0$$

$$F_{21} = -(1 + a\gamma_o)N$$

$$F_{33} = (1 + a\gamma_o)N$$

$$F_{ik} = 0 \quad \text{otherwise} \tag{7.5c}$$

d) Solenoid

$$G_1 = G_2 = N = K_x = K_z = \hat{V} = 0$$

$$H \neq 0$$

$$F_{16} = 2H\left[1 + a\,\frac{\gamma_o^2}{(1+\gamma_o)^2}\right]$$

$$F_{21} = (1 + a\gamma_o)H'$$

$$F_{22} = a\gamma_o\,\frac{\gamma_o}{1+\gamma_o}\,2H$$

$$F_{23} = a\gamma_o\,\frac{\gamma_o}{1+\gamma_o}\,2H^2$$

$$F_{31} = -\,a\gamma_o\,\frac{\gamma_o}{1+\gamma_o}\,2H^2$$

$$F_{33} = (1 + a\gamma_o)H'$$

$$F_{34} = a\gamma_o \frac{\gamma_o}{1+\gamma_o} 2H$$

$$F_{ik} = 0 \quad \text{otherwise} \tag{7.5d}$$

e) Cavity

$$G_1 = G_2 = N = H = K_x = K_z = 0$$
$$\hat{V} \neq 0$$

$$F_{24} = (a\gamma_o + \frac{\gamma_o}{1+\gamma_o}) \cdot \frac{e\hat{V}}{E_o} \sin\Phi \sum_\nu \delta(s - s_\nu)$$

$$F_{32} = - F_{24}$$

$$F_{ik} = 0 \quad \text{otherwise} \tag{7.5e}$$

f) Perturbing electric field and magnetic dipole field

$$\Delta\vec{B} \neq 0 \,, \quad \Delta\vec{\varepsilon} \neq 0$$

$$F_{ik} = 0 \ . \tag{7.5f}$$

Taking into account (7.3) the spin equation (7.2) can be rewritten in the form

$$\frac{d}{ds}\vec{\zeta} = \underline{G}_o \vec{\tilde{y}} + \underline{D}_o \vec{\zeta} \tag{7.6}$$

with

$$\vec{\zeta} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \tag{7.7a}$$

$$\underline{G}_o = \begin{pmatrix} \ell_\tau & \ell_x & \ell_z \\ -m_\tau & -m_x & -m_z \end{pmatrix} \cdot \underline{F}_{(3x6)} \tag{7.7b}$$

$$\underline{D}_o = \begin{pmatrix} 0 & \Psi' \\ -\Psi' & 0 \end{pmatrix} . \tag{7.7c}$$

The solution of (7.6) is given by

$$\vec{\zeta}(s) = \underline{D}(s, s_o)\, \vec{\zeta}(s_o) + \int_{s_o}^{s} ds'\, \underline{D}(s, s')\, \underline{G}_o(s')\, \vec{\tilde{y}}(s') \tag{7.8}$$

where the matrix $\underline{D}(s, s_o)$ is defined in (6.24) and the vector $\vec{\tilde{y}}$ satisfies the orbital equation (5.28).

Combining the orbital part $\vec{\tilde{y}}$ and the spin part $\vec{\zeta}$ into an eight-dimensional vector

$$\vec{u} = \begin{pmatrix} \vec{\tilde{y}} \\ \\ \vec{\zeta} \end{pmatrix} \tag{7.9}$$

we can rewrite the spin equation (7.6) and the orbital equation (5.28) in a compact matrix notation as follows

$$\frac{d}{ds}\vec{u} = (\hat{\underline{A}} + \delta\hat{\underline{A}})\,\vec{u} + \delta\vec{\hat{c}} \tag{7.10}$$

with

$$\hat{\underline{A}} = \begin{pmatrix} \underline{A} & \underline{0} \\ \\ \underline{G}_o & \underline{D}_o \end{pmatrix} \tag{7.11a}$$

$$\delta\hat{\underline{A}} = \begin{pmatrix} \delta\underline{A} & \underline{0} \\ \\ \underline{0} & \underline{0} \end{pmatrix} \tag{7.11b}$$

$$\delta\vec{\hat{c}} = \begin{pmatrix} \delta\vec{c} \\ 0 \\ 0 \end{pmatrix} \tag{7.11c}$$

($\underline{A}$, $\delta\underline{A}$, $\delta\vec{c}$ defined in (5.28))

Eqs. (7.11) describe the spin-orbit motion in an electron-positron storage ring. These equations are the starting point for further investigations.

8. The unperturbed problem

In order to investigate the spin-orbit motion it is reasonable to neglect in a first approximation the small terms $\delta\hat{\underline{A}}$ and $\delta\vec{\hat{c}}$ and to consider only the "unperturbed problem"

$$\frac{d}{ds}\vec{u} = \hat{\underline{A}}\,\underline{u} \tag{8.1}$$

with the orbital part

$$\frac{d}{ds}\vec{\tilde{y}} = \underline{A}\,\vec{\tilde{y}} \tag{8.2a}$$

and the spin part

$$\frac{d}{ds}\vec{\zeta} = \underline{G}_o\,\vec{\tilde{y}} + \underline{D}_o\,\vec{\zeta}\ . \tag{8.2b}$$

The radiative perturbations described by $\delta\hat{\underline{A}}$ and $\delta\vec{\hat{c}}$ will then be treated in a second step with perturbation theory.

8.1 The unperturbed orbital motion

8.1.1 Symplectic structure of the transfer matrices

The unperturbed orbital motion is described by (8.2a). The solution of this equation is given by

$$\vec{\tilde{y}}(s) = \underline{M}(s,\ s_o)\ \vec{\tilde{y}}(s_o) \tag{8.3}$$

with $\underline{M}(s,\ s_o)$ being the transfer matrix belonging to (8.2a). The elements of $\underline{M}(s,\ s_o)$ have been calculated already in chapter 5.1.2: $\underline{M}(s,\ s_o)$ is a submatrix of the enlarged transfer matrix $\hat{\underline{M}}(s,\ s_o)$ such that

$$M_{ik} = \hat{M}_{ik} \qquad (i,\ k = 1,\ 2.....6)\ . \tag{8.4}$$

It is important for our further considerations that the orbital equations can be written in canonical form

$$\tilde{x}' = \frac{\partial \mathcal{H}}{\partial \tilde{p}_x} \quad ; \quad \tilde{p}_x' = - \frac{\partial \mathcal{H}}{\partial \tilde{x}}$$

$$\tilde{z}' = \frac{\partial \mathcal{H}}{\partial \tilde{p}_z} \quad ; \quad \tilde{p}_z' = - \frac{\partial \mathcal{H}}{\partial \tilde{z}}$$

$$\sigma' = \frac{\partial \mathcal{H}}{\partial \tilde{p}_\sigma} \quad ; \quad p_\sigma' = - \frac{\partial \mathcal{H}}{\partial \tilde{\sigma}}$$

with the Hamiltonian ($\tilde{p}_\sigma \equiv \tilde{\eta}$)

$$\mathcal{H} = \frac{1}{2} \{G_1 \tilde{x}^2 + G_2 \tilde{z}^2 - 2N\tilde{x}\tilde{z} + (\tilde{p}_x + H\tilde{z})^2 + (\tilde{p}_z - H\tilde{x})^2\}$$
$$- \frac{1}{2} \tilde{\sigma}^2 \frac{e\hat{V}}{E_o} k \cdot \frac{2\pi}{L} \cos\Phi \sum_\nu \delta(s - s_\nu) - (K_x\tilde{x} + K_z\tilde{z})\tilde{p}_\sigma .$$

The canonical structure of the equations of motion then implies that the transfer matrices $\underline{M}(s, s_o)$ must be symplectic which means that the following relations are valid [1]

$$\underline{M}^T(s, s_o) \cdot \underline{S} \cdot \underline{M}(s, s_o) = \underline{S} \tag{8.5a}$$

with

$$\underline{S} = \begin{pmatrix} \underline{S}_2 & \underline{0} & \underline{0} \\ \underline{0} & \underline{S}_2 & \underline{0} \\ \underline{0} & \underline{0} & \underline{S}_2 \end{pmatrix} \quad ; \quad \underline{S}_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} . \tag{8.5b}$$

8.1.2 The eigenvalue spectrum of the revolution matrix $\underline{M}(s_o + L, s_o)$

The characteristic features of the synchro-betatron oscillations show up in the symplectic structure of the transfer matrices, in particular in the revolution matrix $\underline{M}(s_o + L, s_o)$.

The following statements are valid for the eigenvalue problem of $\underline{M}(s_o + L, s_o)$

$$\underline{M}(s_o + L, s_o) \cdot \vec{v}_\mu(s_o) = \lambda_\mu \vec{v}_\mu(s_o)$$

1) The eigenvectors can be divided into three groups

$$(\vec{v}_k(s_o), \ \vec{v}_{-k}(s_o)) \quad ; \quad k = I, II, III$$

with the properties

$$\underline{M}\,\vec{v}_k = \lambda_k\,\vec{v}_k \;;\;\; \underline{M}\,\vec{v}_{-k} = \lambda_{-k}\,\vec{v}_{-k} \;;\;\; \lambda_k\,\lambda_{-k} = 1 \qquad (8.6a)$$

$$\vec{v}_{-k}^{\,T}(s_o)\cdot\underline{S}\cdot\vec{v}_k(s_o) = -\vec{v}_k^{\,T}(s_o)\cdot\underline{S}\cdot\vec{v}_{-k}(s_o) \neq 0 \qquad (8.6b)$$

$$\vec{v}_\mu^{\,T}(s_o)\cdot\underline{S}\cdot\vec{v}_\nu(s_o) = 0 \qquad \text{otherwise}$$

$$(k = \text{I, II, III}).$$

In the following we shall put

$$\lambda_k = e^{-i2\pi Q_k}$$
$$\lambda_{-k} = e^{-i2\pi Q_{-k}} \qquad (8.7a)$$
$$(k = \text{I, II, III})$$

Using (8.6a) we get

$$Q_{-k} = -Q_k \qquad (8.7b)$$

where the quantity Q_k can be a real or complex number.

2) Eqs. (8.6a) or (8.7) imply that the eigenvalues of the matrix $\underline{M}(s_o + L, s_o)$ always appear in reciprocal pairs

$$(\lambda_k,\ \lambda_{-k} = 1/\lambda_k) \qquad (k = \text{I, II, III})$$

If λ is an eigenvalue, λ^* is also an eigenvalue because $\underline{M}(s_o + L, s_o)$ is a real matrix.
With these statements we find the following possibilities for the eigenvalue spectrum of the revolution matrix $\underline{M}(s_o + L, s_o)$ [1]:

a) All of the six eigenvalues are complex and lie on the unit circle in the complex plane

$$|\lambda_k| = |\lambda_{-k}| = 1 \qquad (8.8)$$
$$(k = \text{I, II, III})$$

and we have

$$Q_k = \text{real} \tag{8.9a}$$

$$\lambda_{-k} = \lambda_k^* \tag{8.9b}$$

$$\vec{\upsilon}_{-k} = \vec{\upsilon}_k^* \tag{8.9c}$$

b) One, two or three reciprocal pairs are real and the remaining eigenvalues lie on the unit circle.

c) One eigenvalue, for example λ_I, is complex but does not lie on the unit circle

$$|\lambda_I| \neq 1 \quad ; \quad \lambda_I^* \neq \lambda_I$$

Then the following condition must hold

$$\lambda_{-I} = 1/\lambda_I$$

and (with an appropriate choice of the eigenvalues)

$$\lambda_{II} = \lambda_I^*$$

$$\lambda_{-II} = 1/\lambda_I^*$$

or

$$\lambda_{II} = 1/\lambda_I^*$$

$$\lambda_{-II} = \lambda_I^*$$

The third, remaining pair must lie on the unit circle or on the real axis.

It will turn out, that the particle motion is only stable in case a).

3) If we define

$$\vec{\upsilon}_\mu(s) = \underline{M}(s, s_o)\, \vec{\upsilon}_\mu(s_o) \tag{8.10}$$

then $\vec{v}_\mu(s)$ is an eigenvector of the revolution matrix $\underline{M}(s + L, s)$ belonging to the same eigenvalue λ_μ [1]:

$$\underline{M}(s + L, s)\, \vec{v}_\mu(s) = \lambda_\mu\, \vec{v}_\mu(s) \tag{8.11}$$

Thus, the eigenvalue itself is independent of s:

$$\lambda_\mu(s) = \lambda_\mu(s_o)\,. \tag{8.12}$$

4) Defining

$$\begin{aligned} \vec{v}_\mu(s) &= \hat{\vec{v}}_\mu(s)\, e^{-i2\pi Q_\mu \cdot \frac{s}{L}} \\ \hat{\vec{v}}_\mu(s) &= \vec{v}_\mu\; e^{+i2\pi Q_\mu \cdot \frac{s}{L}} \end{aligned} \tag{8.13a}$$

we find

$$\hat{\vec{v}}_\mu(s + L) = \hat{\vec{v}}_\mu(s) \tag{8.13b}$$

which can be verified easily by putting (8.13a) into (8.11).

Eq. (8.13) is called the Floquet-theorem. It states that the vectors $\vec{v}_\mu(s)$, which are special solutions of the equations of motion (8.2a) can be written as the product of a periodic function $\hat{\vec{v}}_\mu(s)$ and a (generally aperiodic) harmonic function

$$e^{-i2\pi Q_\mu \cdot \frac{s}{L}}\,.$$

5) The general solution of the equation of motion (8.2a) is a linear combination of the special solutions (8.13a). Therefore it can be written in the form

$$\vec{\tilde{y}}(s) = \sum_{\substack{k=I,II\\ III}} \{ A_k\, \hat{\vec{v}}_k(s)\; e^{-i2\pi Q_k \cdot \frac{s}{L}} + A_{-k}\, \hat{\vec{v}}_{-k}(s)\; e^{-i2\pi Q_{-k} \cdot \frac{s}{L}} \}\,. \tag{8.14}$$

This equation implies that the amplitudes of the synchro-betatron oscillations remain bounded (stable motion) only if the quantities Q_k are real numbers, which also means, that the eigenvalues must lie on the unit circle, as mentioned already:

$$|\lambda_k| = |\lambda_{-k}| = 1 \qquad (k = I, II, III) \tag{8.15}$$

(criterion of orbital stability).

If at least one of the exponents Q_k is complex, Q_k, or Q_{-k} has a positive imaginary part. In this case the components of $\vec{y}(s)$ grow exponentially and the particle motion becomes unstable.

6) For the following discussions we shall always assume that the criterion of stability (8.15) is satisfied.
Then, it follows from (8.9c)

$$\vec{v}_{-k} = \vec{v}_k^{\,*} \qquad (k = I, II, III)$$

and eq. (8.6b) reduces to ($\vec{v}^{\,+} = (\vec{v}^{\,*})^T$)

$$\vec{v}_k^{\,+}(s_o) \cdot \underline{S}\, \vec{v}_k(s_o) = - \vec{v}_{-k}^{\,+}(s_o) \cdot \underline{S}\, \vec{v}_{-k}(s_o) \neq 0 \tag{8.16a}$$

$$\vec{v}_\mu^{\,+}(s_o) \cdot \underline{S}\, \vec{v}_\nu(s_o) = 0 \qquad \text{otherwise.} \tag{8.16b}$$

The terms

$$\vec{v}_\mu^{\,+}(s_o) \cdot \underline{S}\, \vec{v}_\mu(s_o)$$

appearing in (8.16a) are purely imaginary:

$$[\vec{v}_\mu^{\,+}(s_o) \cdot \underline{S}\, \vec{v}_\mu(s_o)]^+ = \vec{v}_\mu^{\,+}(s_o) \cdot \underline{S}^+\, \vec{v}_\mu(s_o) = - [\vec{v}_\mu^{\,+}(s_o) \cdot \underline{S}\, \vec{v}_\mu(s_o)]$$

$$(\text{since } \underline{S}^+ = - \underline{S})$$

so that the following normalizing conditions can be used for the vectors $\vec{v}_k(s_o)$ and $\vec{v}_{-k}(s_o)$ (k = I, II, III)

$$\vec{v}_k^{\,+}(s_o) \cdot \underline{S}\, \vec{v}_k(s_o) = - \vec{v}_{-k}^{\,+}(s_o) \cdot \underline{S}\, \vec{v}_{-k}(s_o) = i \tag{8.17}$$

$$(k = I, II, III)\,.$$

The validity of the symplectic condition (8.5a) then implies that the eigenvectors $\vec{v}_k(s)$ and $\vec{v}_{-k}(s)$ (k = I, II, III) at the position s satisfy the same conditions (8.16) and (8.17)

$$\begin{aligned} &\vec{v}_k^{\,+}(s)\, \underline{S}\, \vec{v}_k(s) = - \vec{v}_{-k}^{\,+}(s)\, \underline{S}\, \vec{v}_{-k}(s) = i\,; \\ &\vec{v}_\nu^{\,+}(s)\, \underline{S}\, \vec{v}_\mu(s) = 0 \qquad \text{otherwise}\,. \end{aligned} \tag{8.18}$$

8.2 The transfer matrix of the unperturbed spin-orbit motion

Taking into account (8.3) and (7.8) the solution of the spin equation (7.6) can be written in the form

$$\vec{\mathfrak{z}}(s) = \underline{D}(s, s_0)\, \vec{\mathfrak{z}}(s_0) + \int_{s_0}^{s} ds'\, \underline{D}(s, s')\, \underline{G}_0(s')\, \underline{M}(s', s_0)\, \vec{y}(s_0) \qquad (8.19)$$

and the spin-orbit vector $\vec{u}$ (see (8.1)) is given by

$$\vec{u}(s) = \underline{M}_{(8x8)}(s, s_0)\, \vec{u}(s_0) \qquad (8.20)$$

with

$$\underline{M}_{(8x8)}(s, s_0) = \begin{pmatrix} \underline{M}(s, s_0) & \underline{0}_{(6x2)} \\ \underline{G}(s, s_0) & \underline{D}(s, s_0) \end{pmatrix} . \qquad (8.21)$$

$\underline{G}(s, s_0)$ has been defined in (8.19)

$$\underline{G}(s, s_0) = \int_{s_0}^{s} ds'\, \underline{D}(s, s')\, \underline{G}_0(s')\, \underline{M}(s', s_0) \qquad (8.22)$$

and $\underline{M}_{(8x8)}(s, s_0)$ is the transfer matrix of the unperturbed spin-orbit motion.

In particular, one finds the following expression for the revolution matrix $\underline{M}_{(8x8)}(s_0 + L, s_0)$:

$$\underline{M}_{(8x8)}(s_0 + L, s_0) = \begin{pmatrix} \underline{M}(s_0 + L, s_0) & \underline{0}_{(6x2)} \\ \underline{G}(s_0 + L, s_0) & \underline{D}(s_0 + L, s_0) \end{pmatrix} \qquad (8.23)$$

with

$$\underline{D}(s_0 + L, s_0) = \begin{pmatrix} \cos 2\pi\nu & \sin 2\pi\nu \\ -\sin 2\pi\nu & \cos 2\pi\nu \end{pmatrix} \qquad (8.24)$$

(see eq's. (6.24) and (6.28)).

The eigenvectors $\vec{q}_\mu(s_0)$ of the matrix $\underline{M}_{(8x8)}(s_0 + L, s_0)$ defined by

$$\underline{M}_{(8x8)}(s_0 + L, s_0) \cdot \vec{q}_\mu(s_0) = \hat{\lambda}_\mu\, \vec{q}_\mu(s_0) \qquad (8.25)$$

satisfy the following relationships

$$\vec{q}_k(s_o) = \begin{pmatrix} \vec{\upsilon}_k(s_o) \\ \vec{w}_k(s_o) \end{pmatrix} ; \tag{8.26a}$$

$$\vec{q}_{-k}(s_o) = [\vec{q}_k(s_o)]^* \qquad \text{for } k = I,\ II,\ III \tag{8.26b}$$

and for k = IV

$$\vec{q}_{IV}(s_o) = \begin{pmatrix} \vec{0}_6 \\ \vec{w}_{IV}(s_o) \end{pmatrix} ; \tag{8.27a}$$

$$\vec{q}_{-IV}(s_o) = [\vec{q}_{IV}(s_o)]^* \ . \tag{8.27b}$$

The two-dimensional vectors $\vec{w}_k$ (k = I, II, III) and $\vec{w}_{IV}$ in (8.26a) and (8.27a) fulfill the following conditions [3]

$$\vec{w}_k(s_o) = - [\underline{D}(s_o + L,\ s_o) - \hat{\lambda}_k \cdot \underline{1}]^{-1}\ \underline{G}(s_o + L,\ s_o)\ \vec{\upsilon}_k(s_o) \tag{8.28a}$$

$$\vec{w}_{IV}(s_o) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{-i\psi(s_o)} \tag{8.28b}$$

and

$$\vec{w}_{-k}(s_o) = [\vec{w}_k(s_o)]^* \qquad (k = I,\ II,\ III,\ IV) \tag{8.29}$$

($\vec{\upsilon}_k(s_o)$ being defined in (8.6)).

The corresponding eigenvalues are

$$\hat{\lambda}_k = \lambda_k = e^{-i2\pi Q_k} \qquad (k = I,\ II,\ III) \tag{8.30a}$$

and

$$\hat{\lambda}_{IV} = e^{-i2\pi Q_{IV}} \qquad \text{with} \quad Q_{IV} = \nu. \tag{8.30b}$$

The following expressions can be obtained for the eigenvectors $\vec{q}_\mu(s)$ belonging to the transfer matrix $\underline{M}$(s + L, s) (initial position s)

$$\vec{q}_\mu(s) = \underline{M}_{(8\times8)}(s,\ s_o)\ \vec{q}_\mu(s_o). \tag{8.31}$$

In particular one gets

$$\vec{q}_{IV}(s) = \begin{pmatrix} \vec{0}_6 \\ \\ \vec{w}_{IV}(s) \end{pmatrix} \quad ; \quad \vec{q}_{-IV}(s) = [\vec{q}_{IV}(s)]^* \tag{8.32a}$$

with

$$\begin{aligned} \vec{w}_{IV}(s) &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{-i\psi(s)} ; \\ \vec{w}_{-IV}(s) &= [\vec{w}_{IV}(s)]^* . \end{aligned} \tag{8.32b}$$

The eigenvalues remain unchanged [3]:

$$\hat{\lambda}_\mu(s) = \hat{\lambda}_\mu(s_o) . \tag{8.33}$$

The following orthogonality relations for $\vec{w}_{IV}(s)$ are important for our later investigations:

$$\begin{aligned} \vec{w}^{+}_{IV}(s) \cdot \underline{S}_2 \vec{w}_{IV}(s) &= - \vec{w}^{+}_{-IV}(s) \cdot \underline{S}_2 \vec{w}_{-IV}(s) = i ; \\ \vec{w}^{+}_{-IV}(s) \cdot \underline{S}_2 \vec{w}_{IV}(s) &= \vec{w}^{+}_{IV}(s) \cdot \underline{S}_2 \vec{w}_{-IV}(s) = 0 . \end{aligned} \tag{8.34}$$

These are the same relations as for $\vec{v}_k(s)$ (see (8.16)).

Defining

$$\vec{q}_\mu(s) = \hat{\vec{q}}_\mu(s) \quad e^{-i2\pi Q_\mu \frac{s}{L}}$$

we find

$$\hat{\vec{q}}_\mu(s + L) = \hat{\vec{q}}_\mu(s) . \tag{8.35}$$

Eq. (8.35) is the extension of the Floquet-theorem to the spin-orbit motion and it will play an important role in our further investigations.

9. The perturbed problem: Spin-orbit motion under the influence of the synchrotron radiation

9.1 "Ansatz" for solving the perturbed problem; Bogoliubov's averaging method

The general solution of the unperturbed equation of motion (8.1) can be written in the form (see (8.31))

$$\vec{u}(s) = \sum_{\substack{k=I,II,\\III,IV}} \{A_k\, \vec{q}_k(s) + A_{-k}\, \vec{q}_{-k}(s)\}$$

with A_k, A_{-k} being constants of integration (k = I, II, III, IV).

In order to solve the perturbed problem (7.10) we make the following "ansatz" (variation of constants):

$$\vec{u} = \sum_{\substack{k=I,II\\III,IV}} \{A_k(s)\, \vec{q}_k(s) + A_{-k}(s)\, \vec{q}_{-k}(s)\}. \qquad (9.1)$$

Inserting (9.1) into (7.10) one obtains

$$\sum_{\substack{k=I,II,\\III,IV}} \{A'_k(s)\vec{q}_k + A'_{-k}(s)\vec{q}_{-k}\} = \delta\underline{\hat{A}} \sum_{\substack{k=I,II,\\III,IV}} \{A_k(s)\vec{q}_k + A_{-k}(s)\vec{q}_{-k}\} + \delta\vec{\hat{c}}$$

or dividing this equation into its orbital part and spin part one finds

$$\sum_{\substack{k=I,II,\\III}} \{A'_k(s)\vec{\upsilon}_k + A'_{-k}(s)\vec{\upsilon}_{-k}\} = \delta\underline{A} \sum_{\substack{k=I,II,\\III}} \{A_k(s)\vec{\upsilon}_k + A_{-k}(s)\vec{\upsilon}_{-k}\} + \delta\vec{c}\,; \qquad (9.2a)$$

$$A'_{IV}(s)\vec{w}_{IV} + A'_{-IV}(s)\vec{w}_{-IV} = - \sum_{\substack{k=I,II,\\III}} \{A'_k(s)\vec{w}_k + A'_{-k}(s)\vec{w}_{-k}\}\,. \qquad (9.2b)$$

Using the normalizing conditions (8.18), (8.34) and Floquet's theorem (8.35) one gets

$$A'_k(s) = \sum_{\substack{\ell=I,II,\\III}} A_\ell(s)(-i)\vec{\hat{\upsilon}}_k^{+}(s)\, \underline{S}\, \delta\underline{A}\, \vec{\hat{\upsilon}}_\ell(s)\, e^{i2\pi(Q_k-Q_\ell)\cdot\frac{s}{L}} +$$

$$+ \sum_{\substack{\ell=I,II,\\III}} A_{-\ell}(s)(-i)\vec{\hat{\upsilon}}_k^{+}(s)\, \underline{S}\, \delta\underline{A}\, \vec{\hat{\upsilon}}_{-\ell}(s)\, e^{i2\pi(Q_k+Q_\ell)\cdot\frac{s}{L}} - i\, \vec{\upsilon}_k^{+}(s)\, \underline{S}\, \delta\underline{c}(s);$$

$$A'_{-k}(s) = [A'_k(s)]^* \;; \quad (k = I,\, II,\, III)\,.$$

Taking into account (4.5) and (8.5b) and using Bogoliubov's averaging technique [5] these equations reduce to

$$A_k'(s) = -\,i\,A_k(s)\cdot\frac{2\pi}{L}\,\delta Q_k + i\,\upsilon_{k5}^*(s)\cdot\delta c(s) \qquad (9.3a)$$

$$A_{-k}'(s) = [A_k'(s)]^* \qquad (9.3b)$$

with the solution

$$A_k(s) = e^{-i2\pi\frac{1}{L}\,\delta Q_k(s-s_o)}\cdot\Big\{A_k(s_o) + i\int_{s_o}^{s} ds'\, e^{i\frac{2\pi}{L}\,\delta Q_k(s'-s_o)}\cdot\vec{\upsilon}_{k5}^{\,*}(s')\,\delta c(s')\Big\}; \qquad (9.4a)$$

$$A_{-k}(s) = A_k^*(s) \qquad (9.4b)$$

where we have introduced the following abbreviation

$$\delta Q_k = \frac{1}{2\pi}\int_{s_o}^{s_o+L} ds\,\vec{\upsilon}_k^{\,+}(s)\,\underline{S}\,\delta\underline{A}(s)\cdot\vec{\upsilon}_k(s). \qquad (9.5)$$

In our previous report [3] we have shown, that δQ_k is just the (complex) Q-shift of the k-th oscillation mode caused by the perturbation $\delta\underline{A}$.
One further derives from (9.2b) with the use of (9.3)

$$\begin{aligned} A_{IV}'(s) = \sum_{k=I,II,III} \Big\{ & A_k(s)\,\frac{2\pi}{L}\,\delta Q_k\,\hat{\vec{w}}_{IV}^{\,+}\,\underline{S}_2\,\hat{\vec{w}}_k\,e^{i2\pi(Q_{IV}-Q_k)\frac{s}{L}} - \\ & - A_{-k}(s)\,\frac{2\pi}{L}\,\delta Q_k^*\,\hat{\vec{w}}_{IV}^{\,+}\,\underline{S}_2\,\hat{\vec{w}}_{-k}\,e^{i2\pi(Q_{IV}+Q_k)\frac{s}{L}}\Big\} - \\ & - \sum_{k=I,II,III}\Big\{\hat{\vec{\upsilon}}_{k5}^{\,*}\,\delta c\,\vec{w}_{IV}^{\,+}\,\underline{S}_2\,\vec{w}_k - \hat{\upsilon}_{k5}\,\delta c\,\vec{w}_{IV}^{\,+}\,\underline{S}_2\,\vec{w}_{-k}\Big\}\;; \end{aligned} \qquad (9.6a)$$

$$A_{-IV}'(s) = [A_{IV}'(s)]^*\;. \qquad (9.6b)$$

Neglecting the oscillating terms in (9.6a) which are proportional to δQ we finally obtain after integration

$$A_{IV}(s) = A_{IV}(s_o) - \sum_{k=I,II,III} \int_{s_o}^{s} ds' \{ \upsilon_{k5}^*(s')\, \delta c(s') \quad \times$$

$$\times \; \vec{w}_{IV}^{+}(s')\, \underline{S}_2\, \vec{w}_k(s') - \upsilon_{k5}(s')\, \delta c(s')\; \vec{w}_{IV}^{+}(s')\, \underline{S}_2\, \vec{w}_{-k}(s') \} \qquad (9.7a)$$

$$A_{-IV}(s) = [\, A_{IV}(s)]^* \qquad (9.7b)$$

Eq.'s (9.4) and (9.7) are the general solutions of the perturbed problem (7.10). Together with eq. (9.1), which contains the orbital part

$$\vec{\tilde{y}} = \sum_{k=I,II,III} \{ A_k(s)\, \vec{\upsilon}_k(s) + A_{-k}(s)\, \vec{\upsilon}_{-k}(s) \} \qquad (9.8a)$$

and the spin part

$$\vec{\zeta} = \sum_{k=I,II,III} \{ A_k(s)\, \vec{w}_k(s) + A_{-k}(s)\, \vec{w}_{-k}(s) \} \; +$$

$$+ \; A_{IV}(s)\, \vec{w}_{IV}(s) + A_{-IV}(s)\, \vec{w}_{-IV}(s) \;, \qquad (9.8b)$$

these equations describe the spin-orbit motion under the influence of the synchrotron radiation and they allow to calculate the finite beam dimensions, the damping constants and the depolarization time caused by the quantum fluctuations.

9.2 Influence of the synchrotron radiation on the orbital motion of the particle

Since the general equations of motion (7.10) contain stochastic terms $\delta\vec{c}$, the components of

$$\vec{u} = \begin{pmatrix} \vec{\tilde{y}} \\ \vec{\zeta} \end{pmatrix}$$

are also stochastic quantities. Therefore, the quantities of interest are correlation functions of the form

$$< f(\vec{u}(s)) \cdot g(\vec{u}(s')) >_{\delta c}$$

where the average $<\ldots>_{\delta c}$ has to be taken with respect to the statistical properties of the quantum fluctuations (Gaussian white noise process [5]).

We first consider the orbital motion, which is independent of the spin motion (see eq's (9.4) and (9.8a)) and we determine the following expressions (beam emittance matrix)

$$
\begin{aligned}
< \tilde{y}_m(s)\, \tilde{y}_n(s) >_{\delta c} = \sum_{\substack{k,\ell = I, II,\\ III}} \{ & A_{(k,\ell)}(s) \cdot v_{km}\, v_{\ell n} \; + \\
& + A_{(k,-\ell)}(s)\, v_{km}\, v_{-\ell n} \; + \\
& + A_{(-k,\ell)}(s)\, v_{-km}\, v_{\ell n} \; + \\
& + A_{(-k,-\ell)}(s)\, v_{-km}\, v_{-\ell n} \} ; \qquad (9.9)
\end{aligned}
$$

$(m,\ n = 1,\ 2 \ldots 6)$

with

$$A_{(k,\ell)}(s) = < A_k(s)\, A_\ell(s) >_{\delta c} = [A_{(-k,-\ell)}(s)]^* \qquad (9.10a)$$

$$A_{(k,-\ell)}(s) = < A_k(s)\, A_{-\ell}(s) >_{\delta c} = [A_{(-k,\ell)}(s)]^* . \qquad (9.10b)$$

Since we are mainly interested in the stationary (or equilibrium) values of the beam dimensions at an arbitrary position s in the storage ring we shall calculate

$$\overset{stat}{A}_{(k,\pm\ell)}(s) = \lim_{N \to \infty} A_{(k,\pm\ell)}(s + N \cdot L). \qquad (9.11)$$

Using the following definitions

$$\delta Q_k = \delta \hat{Q}_k - \frac{i}{2\pi}\, \alpha_k \qquad (9.12)$$

$$\delta \hat{Q}_k = R_e \{\delta Q_k\}$$

$$\alpha_k = -\, 2\pi\ \mathrm{Im}\ \{\delta Q_k\}$$

and taking into account (4.6) we get from (9.4)

$$A_{(k,-k)}(s + N \cdot L) \equiv \langle |A_k(s + N \cdot L)|^2 \rangle_{\delta c}$$

$$= \langle |A_k(s)|^2 \rangle_{\delta c} \cdot e^{-2\alpha_k \cdot N} +$$

$$+ \frac{1 - e^{-2\alpha_k \cdot N}}{e^{2\alpha_k} - 1} \cdot \int_s^{s+L} d\tilde{s}\; e^{-2\alpha_k \frac{1}{L}(s - \tilde{s})} \times$$

$$\times\; \omega(\tilde{s})\; |\upsilon_{k5}(\tilde{s})|^2 \; ; \qquad (9.13a)$$

$$A_{(k,\pm\ell)}(s + N \cdot L) \equiv \langle A_k(s + N \cdot L)\; A_{\pm\ell}(s + N \cdot L) \rangle_{\delta c}$$

$$= \langle A_k(s) \cdot A_{\pm\ell}(s) \rangle_{\delta c} \cdot e^{-i2\pi[\delta\hat{Q}_k + \delta\hat{Q}_\ell] \cdot N} \times$$

$$\times\; e^{-(\alpha_k + \alpha_\ell) \cdot N} \quad \text{otherwise.} \qquad (9.13b)$$

Eq.'s (9.13a, b) imply, that the particle motion is only stable under the influence of the synchrotron radiation if

$$\alpha_k > 0 \quad (k = I, II, III).$$

If this condition is satisfied, which is always assumed in the following, the first terms on the right hand side of (9.13a, b) are damped out after several revolutions because of the factors

$$e^{-2\alpha_k \cdot N} \quad \text{or} \quad e^{-(\alpha_k + \alpha_\ell) \cdot N} .$$

Thus we find as equilibrium (stationary) values ($N\alpha_k \gg 1$)

$$A^{stat}_{(k,-k)}(s) \equiv \langle |A_k(s)|^2 \rangle^{stat}_{\delta c}$$

$$= \frac{1}{e^{2\alpha_k} - 1} \int_s^{s+L} d\tilde{s}\; e^{2\alpha_k \frac{1}{L}(s - \tilde{s})}\; \omega(\tilde{s})\; |\upsilon_{k5}(\tilde{s})|^2 \; ; \qquad (9.14a)$$

$$A^{stat}_{(k,\pm\ell)}(s) = 0 \quad \text{otherwise.} \qquad (9.14b)$$

Using (9.9) and (9.10) the beam emittance matrix can be written as

$$<\tilde{y}_m(s) \cdot \tilde{y}_n(s)>^{stat}_{\delta c} = 2 \sum_{\substack{k=I,II,\\ III}} <|A_k(s)|^2>^{stat}_{\delta c} \cdot \mathrm{Re}\{v_{km}(s) v^*_{kn}(s)\}. \qquad (9.15)$$

Generally we have

$$\alpha_k \ll 1$$

and instead of (9.14a), we can write approximately

$$<|A_k(s)|^2>^{stat}_{\delta c} = \frac{1}{2\alpha_k} \int_s^{s+L} d\tilde{s}\; \omega(\tilde{s})\; |v_{k5}(\tilde{s})|^2 \; . \qquad (9.16)$$

In this case the integrand is periodic and $<|A_k(s)|^2>^{stat}_{\delta c}$ is independent of the initial position

$$<|A_k(s)|^2>^{stat}_{\delta c} = \text{const} \qquad (9.17)$$

(independent of s).

Using the general expressions (9.13) it was also shown in [3] that one can write approximately

$$<|A_k(s+L)|^2>_{\delta c} - <|A_k(s)|^2>_{\delta c} = -2\alpha_k <|A_k(s)|^2>_{\delta c} + \int_s^{s+L} d\tilde{s}\; \omega(\tilde{s}) |v_{k5}(\tilde{s})|^2 \quad (9.18)$$

It can be seen directly from this equation that the stochastic exitations of the synchro-betatron oscillations which are caused by the quantum fluctuations of the radiation field (function $\omega(s)$ in (9.16)), and the damping of the oscillations (α_k) caused by the continuous emission of synchrotron light can lead to the equilibrium state of $<|A_k(s)|^2>_{\delta c}$ given by (9.14a) or (9.16). The role of α_k as damping constant becomes obvious if we neglect the second term on the right hand side of (9.18) and if we write approximately

$$\frac{d}{ds} <|A_k(s)|^2>_{\delta c} \approx \frac{<|A_k(s+L)|^2>_{\delta c} - <|A_k(s)|^2>_{\delta c}}{L} = -\frac{2\alpha_k}{L} \cdot <|A_k(s)|^2>_{\delta c}$$

with (see eq.'s (9.5), (9.12))

$$\alpha_k = \frac{i}{2} \int_{s_0}^{s_0+L} d\tilde{s}\; \vec{v}_k^{\,+}(\tilde{s}) \left[\underline{S}\; \delta\underline{A}(\tilde{s}) + \delta\underline{A}^T(\tilde{s})\; \underline{S} \right] \vec{v}_k(\tilde{s}) \; . \qquad (9.19)$$

Putting (4.4) and (8.5b) into (9.19) one gets

$$\alpha_k = \frac{i}{2} \int_{s_o}^{s_o+L} d\tilde{s} \, \{\delta A_{22} \, [(v_{k_1} \cdot v^*_{k_2} - v^*_{k_1} \cdot v_{k_2}) + (v_{k_3} \cdot v^*_{k_4} - v^*_{k_3} \cdot v_{k_4})] +$$

$$+ \delta A_{61} \, (v_{k_5} \cdot v^*_{k_1} - v^*_{k_5} \cdot v_{k_1}) +$$

$$+ \delta A_{62} \, (v_{k_5} \cdot v^*_{k_2} - v^*_{k_5} \cdot v_{k_2}) +$$

$$+ \delta A_{63} \, (v_{k_5} \cdot v^*_{k_3} - v^*_{k_5} \cdot v_{k_3}) +$$

$$+ \delta A_{64} \, (v_{k_5} \cdot v^*_{k_4} - v^*_{k_5} \cdot v_{k_4}) +$$

$$+ \delta A_{66} \, (v_{k_5} \cdot v^*_{k_6} - v^*_{k_5} \cdot v_{k_6}) \} \qquad (9.20)$$

(with the matrix elements δA_{ik} from (4.4)).

Finally we want to mention that the damping constants α_k satisfy the following relation discovered by K.W. Robinson [13]

$$\alpha_I + \alpha_{II} + \alpha_{III} = 2 \frac{U_o}{E_o} \qquad (9.21)$$

where U_o is the mean energy supplied by the cavities or the mean radiation loss per revolution. If one knows for example two damping constants the third constant is then automatically fixed by this relation. A simple proof of this (Robinson-) theorem can be found in [3].

9.3 Spin depolarization

Using eq's (9.15) and (9.20) one can calculate the interesting quantities of the orbital motion of a particle under the influence of radiation effects, namely, the damping constants of the coupled synchro-betatron oscillations and the beam dimensions (mean fluctuations of the synchro-betatron oscillations about the six-dimensional closed orbit).

In order to investigate the spin motion under the influence of the synchrotron radiation we need, in addition to (9.13a, b) i.e.

$$A_{(k,\pm\ell)}(s) \equiv \langle A_k(s)\, A_{\pm\ell}(s) \rangle_{\delta c}$$

the expressions including $A_{\pm IV}$

$$A_{(IV,\nu)}(s) = \langle A_{IV}(s) \cdot A_\nu(s) \rangle_{\delta c}$$

$$(\nu = \pm I, \pm II, \pm III, \pm IV)$$

or explicitly (k = I, II, III)

$$A_{(IV,-IV)}(s) \equiv \langle A_{IV}(s)\, A^*_{IV}(s) \rangle_{\delta c} = A_{(-IV,IV)}(s)$$

$$A_{(IV,IV)}(s) \equiv \langle A_{IV}(s)\, A_{IV}(s) \rangle_{\delta c} = [A_{(-IV,-IV)}(s)]^*$$

$$A_{(IV,-k)}(s) \equiv \langle A_{IV}(s) A^*_k(s) \rangle_{\delta c} = [A_{(-IV,k)}(s)]^*$$

$$A_{(IV,k)}(s) \equiv \langle A_{IV}(s)\, A_k(s) \rangle_{\delta c} = [A_{(-IV,-k)}(s)]^* \,.$$

Using (8.5b) and (8.32b) we get from (9.7)

$$\begin{aligned} A_{(IV,-IV)}(s) &\equiv \langle A_{IV}(s)\, A^*_{IV}(s) \rangle_{\delta c} \\ &= \langle A_{IV}(s_o)\, A^*_{IV}(s_o) \rangle_{\delta c} + \\ &+ 2 \int_{s_o}^{s} d\tilde{s}\, \omega(\tilde{s}) \cdot \Big\{ \Big[\mathrm{Im} \sum_{k=I,II,III} (v^*_{k_5}\, w_{k_1}) \Big]^2 + \\ &+ \Big[\mathrm{Im} \sum_{k=I,II,III} (v^*_{k_5}\, w_{k_2}) \Big]^2 \Big\} \,. \end{aligned} \tag{9.22}$$

And if one neglects all integrals with oscillating functions in the expressions for the remaining terms $A_{(IV,IV)}$ and $A_{(IV,\pm k)}$ one finds [3]:

$$\langle A_{IV}(s_o + N\cdot L) \cdot A_{IV}(s_o + N\cdot L) \rangle_{\delta c} = \langle A_{IV}(s_o) \cdot A_{IV}(s_o) \rangle_{\delta c}\,; \tag{9.23}$$

$$\langle A_{IV}(s_o + N\cdot L) \cdot A_{\pm k}(s_o + N\cdot L) \rangle_{\delta c} = e^{-N\cdot\alpha_k}\, e^{\mp i 2\pi \delta\hat{Q}_k \cdot N} \cdot \langle A_{IV}(s_o) \cdot A_{\pm k}(s_o) \rangle_{\delta c}\,; \tag{9.24}$$

$$(k = I, II, III).$$

Thus one gets for the average spin components $<\alpha>$ and $<\beta>$

$$<\alpha(s)>_{\delta c} = \sum_{\substack{k=I,II,\\ III,IV}} \{<A_k(s)>_{\delta c}\cdot w_{k_1}(s) + <A_k^*(s)>_{\delta c}\cdot w_{k_1}^*(s)\} \tag{9.25a}$$

$$<\beta(s)>_{\delta c} = \sum_{\substack{k=I,II,\\ III,IV}} \{<A_k(s)>_{\delta c}\cdot w_{k_2}(s) + <A_k^*(s)>_{\delta c}\cdot w_{k_2}^*(s)\} \tag{9.25b}$$

$$<\alpha^2(s)+\beta^2(s)>_{\delta c} = 2\cdot \sum_{\substack{\ell,k=I,II,\\ III,IV}} \{Re[A_{(k,\ell)}(s)\cdot(w_{k_1}w_{\ell_1} + w_{k_2}w_{\ell_2}) + A_{(k,-\ell)}(s)\cdot(w_{k_1}w_{\ell_1}^* + w_{k_2}w_{\ell_2}^*)]\} . \tag{9.26}$$

Furthermore, we adopt the following initial values at $s = s_o$

$$< A_k(s_o) >_{\delta c} = 0 \qquad \text{for } k = I,\ II,\ III,\ IV \tag{9.27a}$$

$$\Longrightarrow\ < \alpha(s_o) >_{\delta c} = < \beta(s_o) >_{\delta c} = 0 \tag{9.27b}$$

and

$$< \alpha^2(s_o) + \beta^2(s_o) >_{\delta c} = 0 \ ; \tag{9.28}$$

$$A_{(IV,IV)}(s_o) = 0 \ . \tag{9.29}$$

Eq.'s (9.27) and (9.28) imply that the beam is polarized along the $\vec{n}$-axis at the position $s = s_o$ with a (given) polarization degree (expectation value of the spin)

$$P(s_o) = \xi_o \tag{9.30}$$

(see eq. (7.1)).
Taking into account (4.6a), (9.4), (9.7) and (9.23) we find after N particle revolutions

$$\left.\begin{array}{l} < A_k(s_o + N\,L) >_{\delta c} = 0 \\ \text{for } k = I,\ II,\ III,\ IV \end{array}\right\} \Longrightarrow \begin{array}{l} < \alpha(s_o + N\,L) >_{\delta c} = 0\ ; \\ < \beta(s_o + N\,L) >_{\delta c} = 0\ ; \end{array} \tag{9.31}$$

$$< A_{IV}(s_o + N\,L)\cdot A_{IV}(s_o + N\,L) >_{\delta c} = 0. \tag{9.32}$$

In the case that the orbital motion has reached its stationary value, i.e.

$$N \cdot \alpha_k \gg 1$$

and by taking into account (8.32b), (8.36), (9.13a, b), (9.24) and (9.32) we obtain

$$A_{(k,\ell)}(s_o + N{\cdot}L) = 0 ;$$

$$A_{(k,-\ell)}(s_o + N{\cdot}L) = 0 \qquad \text{for } k \neq \ell ;$$

$$A_{(k,-k)}(s_o + N{\cdot}L) = <|A_k(s_o)|^2 >_{\delta c}^{stat} \qquad (k \neq IV).$$

With these expressions $< \alpha^2 + \beta^2 >_{\delta c}$ reduces to (see (9.26))

$$\frac{1}{2} < \alpha^2(s_o + N{\cdot}L) + \beta^2(s_o + N{\cdot}L) >_{\delta c} = \sum_{\substack{k=I,II,\\ III}} <|A_k(s_o)|^2 >_{\delta c}^{stat} \times$$

$$\times \; [|w_{k_1}(s_o)|^2 + |w_{k_2}(s_o)|^2] + <|A_{IV}(s_o + N{\cdot}L)|^2 >_{\delta c} . \qquad (9.33)$$

Since we have used a perturbation theory for

$$\alpha^2(s) + \beta^2(s) \ll 1$$

we have to require that N does not become too big so that the following inequality still holds

$$< \alpha^2(s_o + N{\cdot}L) + \beta^2(s_o + N{\cdot}L) >_{\delta c} \ll 1 . \qquad (9.34)$$

Because of (9.31) the spin components $< \alpha >$ and $< \beta >$ disappear at the position $s = s_o + N\,L$ and thus the average polarization is again directed along the $\vec{n}$-axis but with a reduced polarization degree. This relative reduction of polarization after N revolutions of the particle, namely

$$\frac{P(s_o) - P(s_o + N{\cdot}L)}{P(s_o)}$$

is given by the right hand side of (9.33) so that we can write (see Fig. 1 ; $|\vec{\xi}|$ is a constant of motion, as can be seen from eq. (4.71)):

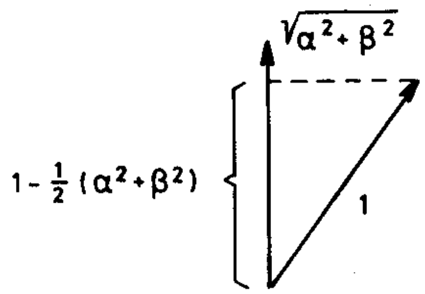

Fig. 1

$$\frac{P(s_o) - P(s_o + N\cdot L)}{P(s_o)} = \sum_{\substack{k=I,II,\\ III}} <|A_k(s_o)|^2>^{stat}_{\delta c} \times$$

$$\times \; (|w_{k_1}(s_o)|^2 + |w_{k_2}(s_o)|^2) + <|A_{IV}(s_o + N\cdot L)|^2>_{\delta c} . \tag{9.35}$$

The depolarization time τ_D defined by

$$\tau_D^{-1} = - \frac{1}{P}\frac{dP}{dt} \approx - \frac{c}{P}\frac{dP}{ds}$$

can be calculated according to

$$\tau_D^{-1} \approx - \frac{c}{P(s_o + N\cdot L)} \cdot \frac{P(s_o + (N+1)\cdot L) - P(s_o + N\cdot L)}{L}$$

$$\approx \frac{c}{L}\frac{1}{P(s_o)} \{ P(s_o + N\cdot L) - P(s_o + (N+1)\cdot L)\} \quad \text{(see (9.34))}$$

$$= \frac{c}{L}\frac{1}{P(s_o)} \{[P(s_o) - P(s_o + (N+1)\cdot L)] - [P(s_o) - P(s_o + N\,L)]\}$$

$$= \frac{c}{L} \{ <|A_{IV}(s_o + (N+1)\cdot L)|^2>_{\delta c} - <|A_{IV}(s_o + N\cdot L)|^2>_{\delta c}\}$$

and taking into account (9.22) we finally obtain

$$\tau_D^{-1} = c \cdot \frac{2}{L} \int_{s_o}^{s_o+L} d\tilde{s}\; \omega(\tilde{s}) \; \{[\mathrm{Im} \sum_{\substack{k=I,II,\\ III}} (v^*_{k_5}\, w_{k_1})]^2 +$$

$$+ \; [\mathrm{Im} \sum_{\substack{k=I,II,\\ III}} (v^*_{k_5}\, w_{k_2})]^2\} . \tag{9.36}$$

This result derived already by A. Chao [17] in a different manner describes the depolarizing influence of the synchrotron radiation on an initially polarized particle beam.

10. Introduction of the dispersion

10.1 Orbital motion

In this section we wish to show how the dispersion, a quantity that is often used in the theory of accelerators and storage rings, can be incorporated into the matrix formalism described in the previous chapters.

For that purpose we first consider the transverse part of the orbital motion (see eq. (5.28))

$$\frac{d}{ds}\begin{pmatrix}\tilde{x}\\ \tilde{p}_x\\ \tilde{z}\\ \tilde{p}_z\end{pmatrix} = (\underline{B} + \delta\underline{B})\begin{pmatrix}\tilde{x}\\ \tilde{p}_x\\ \tilde{z}\\ \tilde{p}_z\end{pmatrix} + \tilde{\eta}\cdot\vec{K} \qquad (10.1a)$$

with

$$\left.\begin{aligned} B_{ik} &= A_{ik} \\ \delta B_{ik} &= \delta A_{ik}\end{aligned}\right\} \quad \text{for } i, k = 1, 2, 3, 4\ ; \qquad (10.1b)$$

$$\vec{K}^T = (o,\ K_x,\ o,\ K_z).$$

Then the dispersion vector $\vec{D}$ is defined by

$$\frac{d}{ds}\vec{D} = (\underline{B} + \delta\underline{B})\,\vec{D} + \vec{K}\ ; \qquad (10.2a)$$

$$\vec{D}(s_o+L) = \vec{D}(s_o)\ ; \qquad (10.2b)$$

$$\vec{D}^T = (D_1,\ D_2,\ D_3,\ D_4). \qquad (10.2c)$$

Making the separation

$$\begin{pmatrix} \tilde{x} \\ \tilde{p}_x \\ \tilde{z} \\ \tilde{p}_z \end{pmatrix} = \vec{y}_\beta + \tilde{\eta} \cdot \vec{D} \tag{10.3}$$

and using (10.2a) eq. (10.1a) can be written in the form

$$\frac{d}{ds} \vec{y}_\beta = \underline{B}\, \vec{y}_\beta + \delta\underline{B}\, \vec{y}_\beta - \vec{D} \cdot \tilde{\eta}' \; . \tag{10.4}$$

The term $(-\vec{D}\tilde{\eta}')$ can be considered as a small perturbation since $\tilde{\eta}$ is a slowly varying quantity compared with $\vec{y}_\beta$.

The longitudinal components $\tilde{\sigma}$ and $\tilde{\eta}$ can be put into the form (see (4.4) and (10.3))

$$\frac{d}{ds}\tilde{\sigma} = -[K_x D_1 + K_z D_3]\tilde{\eta} - K_x\, y_1^{(\beta)} - K_z\, y_3^{(\beta)} \; ; \tag{10.5a}$$

$$\frac{d}{ds}\tilde{\eta} = \frac{e\hat{V}}{E_0} k \cdot \frac{2\pi}{L} \cos\Phi \sum_\nu \delta(s - s_\nu) \cdot \tilde{\sigma} + \sum_{\mu=1}^{4} \delta A_{6\mu}\, y_\mu^{(\beta)} + \tilde{\eta}\,[\delta A_{66} + \sum_{\mu=1}^{4} \delta A_{6\mu} D_\mu] + \delta c \; . \tag{10.5b}$$

Then, the equation of motion for the new orbit vector

$$\bar{\vec{y}} = \begin{pmatrix} y_1^{(\beta)} \\ y_2^{(\beta)} \\ y_3^{(\beta)} \\ y_4^{(\beta)} \\ \tilde{\sigma} \\ \tilde{\eta} \end{pmatrix} \tag{10.6}$$

takes the form

$$\frac{d}{ds}\bar{\vec{y}} = \underline{\bar{A}}\; \bar{\vec{y}} + \delta\underline{\bar{A}}\; \bar{\vec{y}} + \delta c \cdot \begin{pmatrix} -\vec{D} \\ 0 \\ 1 \end{pmatrix} \tag{10.7}$$

with

$$\underline{\bar{A}} = \begin{pmatrix} 0 & 1 & H & 0 & 0 & 0 \\ -(G_1 + H^2) & 0 & N & H & 0 & 0 \\ -H & 0 & 0 & 1 & 0 & 0 \\ N & -H & -(G_2 + H^2) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -[K_x D_1 + K_z D_3] \\ 0 & 0 & 0 & 0 & \frac{e\hat{V}}{E_o} k \frac{2\pi\cos\Phi}{L} \sum_\nu \delta(s-s_\nu) & 0 \end{pmatrix} \quad (10.8)$$

and

$$\delta\underline{\bar{A}} = ((\delta\bar{A}_{ik})) ;$$

$$\delta\bar{A}_{ik} = \delta A_{ik} - D_i \cdot \delta A_{6k} \qquad \text{for } i, k = 1, 2, 3, 4 ;$$

$$\left.\begin{aligned} \delta\bar{A}_{i5} &= - D_i \frac{e\hat{V}}{E_o} k \frac{2\pi}{L} \cos\Phi \sum_\nu \delta(s - s_\nu) \\ \delta\bar{A}_{i6} &= - D_i \left[\delta A_{66} + \sum_{\mu=1}^{4} \delta A_{6\mu} D_\mu\right] \end{aligned}\right\} \quad i = 1, 2, 3, 4 ;$$

$$\delta\bar{A}_{51} = - K_x ;$$

$$\delta\bar{A}_{53} = - K_z ;$$

$$\delta\bar{A}_{66} = \delta A_{66} + \sum_{\mu=1}^{4} \delta A_{6\mu} D_\mu ;$$

$$\delta\bar{A}_{ik} = \delta A_{ik} \qquad \text{otherwise .} \qquad (10.9)$$

The transfer matrix $\underline{\bar{M}}(s, s_o)$ of the unperturbed problem

$$\frac{d}{ds} \vec{y} = \underline{\bar{A}} \; \vec{y} \qquad (10.10)$$

fulfills the symplectic condition [8]

$$\underline{\bar{M}}^T \underline{S} \, \underline{\bar{M}} = \underline{S} \qquad (10.11)$$

and has the following structure

$$\bar{\underline{M}} = \begin{pmatrix} \underline{M}^{(\beta)}_{(4\times4)} & \underline{0}_{(4\times2)} \\ \underline{0}_{(2\times4)} & \underline{M}^{(\sigma)}_{(2\times2)} \end{pmatrix} . \tag{10.12}$$

In this approximation there is no coupling between synchrotron oscillations and betatron oscillations.

The eigenvectors $\bar{\vec{v}}_{\pm k}(s_0)$ belonging to the revolution matrix $\bar{\underline{M}}(s_o + L, s_o)$ satisfy the following relations

$$\left.\begin{aligned} \bar{\underline{M}}(s_o + L, s_o)\, \bar{\vec{v}}_{\pm k}(s_o) &= \bar{\lambda}_{\pm k}\, \bar{\vec{v}}_{\pm k}(s_o) \\ \bar{\vec{v}}_{\pm k}(s) &= \bar{\underline{M}}(s, s_o)\, \bar{\vec{v}}_{\pm k}(s_o) \end{aligned}\right\} \quad k = I, II, III \; ; \tag{10.13}$$

$$\bar{\vec{v}}_k = \begin{pmatrix} \vec{v}^{(\beta)}_k \\ \vec{0}_2 \end{pmatrix} \qquad \text{for} \quad k = I, II \; ; \tag{10.14a}$$

$$\bar{\vec{v}}_k = \begin{pmatrix} \vec{0}_4 \\ \vec{v}^{(\sigma)}_{III} \end{pmatrix} \qquad \text{for} \quad k = III \; ; \tag{10.14b}$$

$$\bar{\vec{v}}_{-k} = [\bar{\vec{v}}_k]^* \; ; \tag{10.14c}$$

$$\left.\begin{aligned} \bar{\lambda}_k &= e^{-i2\pi\bar{Q}_k} \; ; \\ \bar{\lambda}_{-k} &= (\bar{\lambda}_k)^* \; . \end{aligned}\right. \tag{10.15}$$

If we keep the same normalizing conditions for $\bar{\vec{v}}_k(s_o)$ as in (8.16):

$$\bar{\vec{v}}_k^{\,+}(s_o)\, \underline{S}\, \bar{\vec{v}}_k(s_o) = i \tag{10.16}$$

we get from (8.17)

$$\vec{\bar{\upsilon}}_k^+(s)\ \underline{S}\ \vec{\bar{\upsilon}}_k(s) = -\ \vec{\bar{\upsilon}}_{-k}^+(s)\ \underline{S}\ \vec{\bar{\upsilon}}_{-k}(s) = i;$$

$$\vec{\bar{\upsilon}}_\mu^+(s)\ \underline{S}\ \vec{\bar{\upsilon}}_\nu(s) = o \quad \text{otherwise.} \tag{10.17}$$

Making the "ansatz"

$$\vec{\bar{y}}(s) = \sum_{\substack{k=I,II\\III}} \{\bar{A}_k(s)\ \vec{\bar{\upsilon}}_k(s) + \bar{A}_{-k}(s)\ \vec{\bar{\upsilon}}_{-k}(s)$$

and using Bogoliubov's averaging technique the equations for the perturbed orbital motion (10.7) reduce to

$$\bar{A}_k'(s) = -\ i\ \bar{A}_k(s)\ \frac{2\pi}{L}\,\delta\bar{Q}_k - i\ f_k^* \cdot \delta c \tag{10.18}$$

with

$$\delta\bar{Q}_k = \frac{1}{2\pi} \int_{s_o}^{s_o+L} d\tilde{s}\ \vec{\bar{\upsilon}}_k^+(\tilde{s})\ \underline{S}\ \ \delta\underline{\bar{A}}(\tilde{s})\ \vec{\bar{\upsilon}}_k(\tilde{s}) \tag{10-19}$$

and

$$f_k^* = \vec{\bar{\upsilon}}_k^+\ \underline{S}\ \cdot \begin{pmatrix} -\vec{D} \\ 0 \\ 1 \end{pmatrix}$$

$$= \begin{cases} (\upsilon_{k1}^{(\beta)})^*\ D_2 - (\upsilon_{k2}^{(\beta)})^*\ D_1 + (\upsilon_{k3}^{(\beta)})^*\ D_4 - (\upsilon_{k4}^{(\beta)})^*\ D_3 \\ \qquad\qquad \text{for } k = I,\ II \\ -\ (\upsilon_{III,1}^{(\sigma)})^* \qquad \text{for } k = III\ . \end{cases} \tag{10.20}$$

Eq. (10.18) has the same structure as the corresponding eq. (9.3) in chapter 9. Therefore we can make the same manipulations as in this chapter for calculating the damping constants and the beam dimensions. The final result can be obtained immediately by making the following replacements

$$\delta\underline{A} \longrightarrow \delta\underline{\bar{A}}$$

$$\upsilon_{k_5}^* \longrightarrow f_k^* \qquad \text{(see (10.20))}\ .$$

Thus, one finds for the damping constants

$$\bar{\alpha}_k = -2\pi\ \mathrm{Im}(\delta\bar{Q}_k) = \frac{i}{2} \int_{s_0}^{s_0+L} d\tilde{s}\ \vec{\bar{\upsilon}}_k^{+}(\tilde{s})[\underline{S}\ \delta\bar{\underline{A}}(\tilde{s}) + \delta\bar{\underline{A}}^T(\tilde{s})\underline{S}] \cdot \vec{\bar{\upsilon}}_k(\tilde{s}) \qquad (10.21)$$

where $\delta\bar{Q}_k$ is the complex Q-shift caused by the perturbation $\delta\bar{\underline{A}}$. The beam emittance matrix now takes the form

$$< \bar{y}_m(s)\ \bar{y}_n(s) > = 2 \cdot \sum_{k=I,II,III} < |\bar{A}_k|^2 >_{\delta c}^{stat} \times \mathrm{Re}\{\bar{\upsilon}_{km}(s)\ \bar{\upsilon}_{kn}^{*}(s)\} \qquad (10.22a)$$

with

$$< |\bar{A}_k|^2 >_{\delta c}^{stat} = \frac{1}{2\bar{\alpha}_k} \int_{s_0}^{s_0+L} d\hat{s}\ \omega(\hat{s})\ |f_k(\hat{s})|^2 . \qquad (10.22b)$$

Taking into account (4.4), (10.9) and (10.14) eq. (10.21) can be reduced to

$$\bar{\alpha}_k = \frac{1}{2}\frac{U_0}{E_0} + \mathrm{Im}\ \{\int_{s_0}^{s_0+L} d\tilde{s}\ [-\upsilon_{k_1}^{(\beta)*} \cdot D_2$$

$$+ \upsilon_{k_2}^{(\beta)*} \cdot D_1 - \upsilon_{k_3}^{(\beta)*} \cdot D_4 + \upsilon_{k_4}^{(\beta)*} \cdot D_3] \times \sum_{\lambda=1}^{4} \delta A_{6\lambda}\ \upsilon_{k\lambda}^{(\beta)}\} \qquad (10.23a)$$

for k = I, II

and

$$\bar{\alpha}_{III} = \frac{U_0}{E_0} - \frac{1}{2} \int_{s_0}^{s_0+L} d\tilde{s} \sum_{\lambda=1}^{4} \delta A_{6\lambda}(\tilde{s}) \cdot D_\lambda(\tilde{s}) \qquad (10.23b)$$

for k = III

with

$$U_0 = \int_{s_0}^{s_0+L} d\tilde{s}\ e\ \hat{V}\ \sin\Phi \sum_{\nu} \delta(s - s_\nu) \text{ (≡ mean energy supplied by the cavities per particle revolution)}$$

$$= E_0 \int_{s_0}^{s_0+L} d\hat{s}\ C_1[K_x^2(\tilde{s}) + K_z^2(\tilde{s})] \text{(≡ mean radiation loss per revolution)}. \qquad (10.24)$$

For $H \equiv 0$ these equations have been derived already by Leleux and Piwinski [11], [12].

Performing the sum

$$\bar{\alpha}_I + \bar{\alpha}_{II} + \bar{\alpha}_{III}$$

we obtain from (10.23)

$$\bar{\alpha}_I + \bar{\alpha}_{II} + \bar{\alpha}_{III} = 2\,\frac{U_o}{E_o} + \frac{1}{2} \sum_{\lambda=1} \delta A_{6\lambda} \cdot \int_{s_o}^{s_o+L} d\tilde{s} \quad \times$$

$$\times \{- D_\lambda + \sum_{k=I,II} [- i(\vec{v}_k^{(\beta)+} \cdot \underline{S}_4 \cdot \vec{D}) v_{k\lambda}^{(\beta)} +$$

$$+ i\, (\vec{v}_k^{(\beta)+} \cdot \underline{S}_4 \cdot \vec{D})^* \, v_{k\lambda}^{(\beta)*}]\} \qquad (10.24)$$

with

$$\underline{S}_4 = \begin{pmatrix} \underline{S}_2 & \underline{0} \\ \underline{0} & \underline{S}_2 \end{pmatrix} .$$

Expanding the vector $\vec{D}$ in terms of the eigenvectors

$$\vec{v}_k^{(\beta)} \text{ and } \vec{v}_{-k}^{(\beta)} = (\vec{v}_k^{(\beta)})^* \qquad (k = I,\ II)$$

namely

$$\vec{D} = \sum_{k=I,II} \{c_k\, \vec{v}_k^{(\beta)} + c_{-k}\, \vec{v}_{-k}^{(\beta)}\} \qquad (10.25)$$

with the expansion coefficients c_k and c_{-k} given by (see (10.14a) and (8.18))

$$c_k = - i\, (\vec{v}_k^{(\beta)+} \cdot \underline{S}_4\, \vec{D})$$

$$c_{-k} = c_k^* \qquad (10.26)$$

we find that the second expression on the right hand side of (10.24) disappears and thus one gets the Robinson theorem

$$\bar{\alpha}_I + \bar{\alpha}_{II} + \bar{\alpha}_{III} = 2\, \frac{U_o}{E_o} . \qquad (10.27)$$

10.2 Spin Motion

Now, in order to calculate the depolarization time τ_D we also introduce the dispersion into the spin equation (7.6). Inserting (10.3) into (7.6) one obtains [18]

$$\frac{d}{ds}\vec{\zeta} = \underline{\bar{G}}_o\, \vec{\bar{y}} + \underline{D}_o\, \vec{\zeta} \tag{10.28}$$

with

$$\underline{\bar{G}}_o = ((\bar{G}^{(o)}_{\mu\nu})) ;$$

$$\bar{G}^{(o)}_{\mu\nu} = G^{(o)}_{\mu\nu} \qquad \text{for } \nu = 1, 2, 3, 4, 5 ;$$

$$\bar{G}^{(o)}_{\mu 6} = G^{(o)}_{\mu 6} + \sum_{\lambda=1}^{4} G^{(o)}_{\mu\lambda} \cdot D_\lambda . \tag{10.29}$$

The eight-dimensional transfer matrix $\underline{\bar{M}}_{(8x8)}$ of the spin-orbit vector

$$\vec{\bar{u}} = \begin{pmatrix} \vec{\bar{y}} \\ \vec{\zeta} \end{pmatrix} \tag{10.30}$$

can be calculated by making the following replacements in eq. (8.21)

$$\underline{M} \longrightarrow \underline{\bar{M}}$$

$$\underline{G} \longrightarrow \underline{\bar{G}}$$

with

$$\underline{\bar{G}}(s, s_o) = \int_{s_o}^{s} d\tilde{s}\; \underline{D}(s, \tilde{s})\, \underline{\bar{G}}_o(\tilde{s})\, \underline{\bar{M}}(\tilde{s}, s_o) . \tag{10.31}$$

Thus, $\underline{\bar{M}}_{(8x8)}(s, s_o)$ takes the form

$$\underline{\bar{M}}_{(8x8)}(s, s_o) = \begin{pmatrix} \underline{\bar{M}}(s, s_o) & \underline{O}_{(6x2)} \\ \underline{\bar{G}}(s, s_o) & \underline{D}(s, s_o) \end{pmatrix} . \tag{10.32}$$

The eigenvectors of the revolution matrix $\underline{\bar{M}}_{(8x8)}(s+L, s)$ satisfy the following relations

$$\underline{\bar{M}}_{(8x8)}(s+L, s)\, \vec{\bar{q}}_{\pm k}(s) = \hat{\bar{\lambda}}\; \vec{\bar{q}}_{\pm k}(s) \qquad k = I, II, III, IV ;$$

$$\vec{\bar{q}}_k = \begin{pmatrix} \vec{\bar{v}}_k \\ \vec{\bar{\omega}}_k \end{pmatrix} ; \qquad k = I, II, III \tag{10.33a}$$

with

$$\vec{\bar{\omega}}_k = -\,[\underline{D}(s+L,\ s) - \hat{\bar{\lambda}}_k\ \underline{1}]^{-1}\ \underline{\bar{G}}(s+L,\ s)\ \vec{\bar{\upsilon}}_k(s)\ ; \tag{10.33b}$$

$$\hat{\bar{\lambda}}_k = \bar{\lambda}_k \equiv e^{-i2\pi\bar{Q}_k}$$

and

$$\vec{\bar{q}}_{IV} = \begin{pmatrix} \vec{0}_6 \\ \vec{\omega}_{IV} \end{pmatrix}\ ;\quad k = IV \tag{10.34}$$

and

$$\vec{\bar{q}}_{-k} = [\vec{\bar{q}}_k]^* \qquad (k = I,\ II,\ III,\ IV)\ . \tag{10.35}$$

The following calculations are now the same as in chapter 9 and one finally gets

$$\tau_D^{-1} = \frac{c}{L}\cdot 2 \int_{s_0}^{s_0+L} d\hat{s}\ \ \omega(\tilde{s})\{[\mathrm{Im} \sum_{\substack{k=I,II\\III}} (f_k^*\ \bar{\omega}_{k1})]^2 + $$

$$+ [\mathrm{Im} \sum_{\substack{k=I,II,\\III}} (f_k^*\ \bar{\omega}_{k2})]^2\} \tag{10.36}$$

with f_k^* being defined in (10.20).

This expression can be used to derive general spin transparency conditions and spin matching condition [18].

<u>Remarks</u>:

1) In deriving eq. (10.30) we have assumed that the vectors $\vec{n}$, $\vec{\ell}$, and $\vec{m}$ which appear in the matrix $\underline{\bar{G}}_o$ (see eq. (10.29)) have been calculated with the help of the six-dimensional closed orbit $\vec{y}_o$. $\vec{y}_o$ can be expressed approximately in terms of the dispersion as we shall now show.

The starting point is the equation determining $\vec{y}_o$ (see (5.2a)

$$\vec{y}\,' = (\underline{A} + \delta A)\vec{y} + \vec{c}_o + \vec{c}_1 \tag{10.37}$$

$$\vec{y}^{\,T} = (y_1, y_2, y_3, y_4, y_5, y_6) ;$$

$$\vec{c}_o^{\,T} = (0, 0, 0, 0, 0, c_{16}) ;$$

$$\vec{c}_1^{\,T} = (0, c_{12}, 0, c_{14}, 0, c_{06}).$$

Following the considerations of chapter 10 we separate $\vec{y}$ into its transverse part and longitudinal part according to

$$\frac{d}{ds}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = (\underline{B} + \delta\underline{B})\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} + y_6\,\vec{K} + \begin{pmatrix} 0 \\ c_{12} \\ 0 \\ c_{14} \end{pmatrix} ; \tag{10.38}$$

$$\frac{d}{ds}\, y_5 = - K_x\, y_1 - K_z\, y_3 \;; \tag{10.39a}$$

$$\frac{d}{ds}\, y_6 = \frac{e\hat{V}}{E_o}\, k\, \frac{2\pi}{L} \cos\Phi \sum_{\nu} \delta(s - s_\nu) \cdot y_5 +$$

$$c_{06} + c_{16} + \sum_{k=1}^{4} \delta A_{6k} \cdot y_k + \delta A_{66}\, y_6 \;. \tag{10.39b}$$

Making the "ansatz"

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \vec{y}_\beta + y_6 \cdot \vec{D} \tag{10.40}$$

with $\vec{D}$ defined by eq. (10.2) one finds:

$$\frac{d}{ds}\,\vec{y}_\beta = - y_6'\,\vec{D} + (\underline{B} + \delta\underline{B})\,\vec{y}_\beta + \begin{pmatrix} 0 \\ c_{12} \\ 0 \\ c_{14} \end{pmatrix} ; \tag{10.41a}$$

$$\frac{d}{ds}\, y_5 = -\,(K_x\, D_1 + K_z\, D_3)\cdot y_6 + \delta\overline{A}_{51}\; y_1^{(\beta)} + \delta\overline{A}_{53}\; y_3^{(\beta)} \; ; \qquad (10.41b)$$

$$\frac{d}{ds}\, y_6 = \frac{e\hat{V}}{E_o}\, k \cdot \frac{2\pi}{L} \cos\Phi \sum_{\nu} \delta(s - s_\nu)\; y_5 + c_{06} +$$

$$+ c_{16} + \sum_{k=1}^{4} \delta\overline{A}_{6k}\; y_k^{(\beta)} + \delta\overline{A}_{66}\; y_6 \; . \qquad (10.41c)$$

The matrix elements $\delta\overline{A}_{ik}$ are given by (10.9) .

Neglecting the quantities $\delta\overline{A}_{ik}$, δB_{ik} (which are small) and the term $y_6'\, \vec{D}$ in eq. (10.41a) ($\vec{y}_6 \equiv \delta p/p$ is a slowly varying quantity compared with $y_k^{(\beta)}$), eq's (10.41a, b, c) can be rewritten as

$$\frac{d}{ds} \begin{pmatrix} \vec{y}_\beta \\ y_5 \\ y_6 \end{pmatrix} = \underline{\overline{A}} \begin{pmatrix} \vec{y}_\beta \\ y_5 \\ y_6 \end{pmatrix} + \vec{c}_o + \vec{c}_1 \qquad (10.42)$$

with $\underline{\overline{A}}$ defined by eq. (10.8).
In particular we are interested in the periodic solution of (10.42):

$$\frac{d}{ds} \begin{pmatrix} \vec{y}_\beta^{(o)} \\ y_5^{(o)} \\ y_6^{(o)} \end{pmatrix} = \underline{\overline{A}} \begin{pmatrix} \vec{y}_\beta^{(o)} \\ y_5^{(o)} \\ y_6^{(o)} \end{pmatrix} + \vec{c}_o + \vec{c}_1 \qquad (10.43a)$$

with

$$\begin{pmatrix} \vec{y}_\beta^{(o)} \\ y_5^{(o)} \\ y_6^{(o)} \end{pmatrix}_{s=s_o} = \begin{pmatrix} \vec{y}_\beta^{(o)} \\ y_5^{(o)} \\ y_6^{(o)} \end{pmatrix}_{s=s_o+L} . \qquad (10.43b)$$

Using eqs. (10.40) and (10.43) the closed orbit is given by

$$\vec{y}_o = \begin{pmatrix} \vec{y}_\beta^{(o)} + y_6^{(o)}\, \vec{D} \\ y_5^{(o)} \\ y_6^{(o)} \end{pmatrix} \equiv \begin{pmatrix} x_o \\ p_{xo} \\ z_o \\ p_{zo} \\ \sigma_o \\ \eta_o \end{pmatrix}$$

Thus we have expressed the closed orbit distortions caused by $\vec{c}_0$ (radiation losses are compensated in the cavity sections and not in the arcs where these losses occur) and $\vec{c}_1$ (perturbing external fields or field errors) in terms of the dispersion.

2) If one transforms the rotation matrix $\underline{D}(s+L, s)$

$$D(s+L,\ s) = \begin{pmatrix} \cos 2\pi\nu & \sin 2\pi\nu \\ -\sin 2\pi\nu & \cos 2\pi\nu \end{pmatrix}$$

into principal axes:

$$\underline{U}^{-1}\,\underline{D}\,\underline{U} = \underline{J} \quad ; \quad \underline{D} = \underline{U}\,\underline{J}\,U^{-1} \quad ;$$

$$\underline{U} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \quad ;$$

$$\underline{J} = \begin{pmatrix} e^{i\cdot 2\pi\nu} & 0 \\ 0 & e^{-i\cdot 2\pi\nu} \end{pmatrix}$$

the vector $\vec{\bar{\omega}}_k$ in eq. (10.33b) can be put into the form

$$\vec{\bar{\omega}}_k = -\left[\underline{U}\,\underline{J}\,\underline{U}^{-1} - \underline{1}\cdot e^{-i\cdot 2\pi\bar{Q}_k}\right]^{-1}\cdot \underline{G}(s+L,\ s)\ \vec{\bar{\upsilon}}_k(s)$$

$$= -\underline{U}\left[\underline{J} - \underline{1}\cdot e^{-i\cdot 2\pi\bar{Q}_k}\right]^{-1}\cdot \underline{U}^{-1}\cdot \underline{G}(s+L,\ s)\ \vec{\bar{\upsilon}}_k(s)$$

$$= \underline{U} \begin{pmatrix} \frac{i}{2} e^{i\pi(\bar{Q}_k-\nu)} \dfrac{1}{\sin \pi(\bar{Q}_k+\nu)} & 0 \\ 0 & \frac{i}{2} e^{i\pi(\bar{Q}_k+\nu)} \dfrac{1}{\sin \pi(\bar{Q}_k-\nu)} \end{pmatrix} \underline{U}^{-1}\cdot\underline{G}(s+L,s)\vec{\bar{\upsilon}}_k(s) \tag{10.44}$$

Eq. (10.44) shows that the components of $\vec{\bar{\omega}}_k$ (and thus, also τ_D^{-1}) become infinitely large for

$$\bar{Q}_k \pm \nu \longrightarrow \text{integer}\ .$$

Therefore at these well known "intrinsic resonances" no polarization can be expected.

Appendix

Calculation of the spin transfer matrix $\underline{N}(s, s_1)$

The equations determining the matrix $\underline{N}(s, s_1)$ read (see (6.14)):

$$\frac{d}{ds}\underline{N}(s, s_1) = \underline{\Omega}^{(o)}(s) \cdot \underline{N}(s, s_1) \; ; \qquad (1a)$$

$$\underline{N}(s_1, s_1) = \underline{1} \, . \qquad (1b)$$

In order to solve this equation we consider two special cases:

a) pointlike fields ;

b) fields of finite length .

a) Pointlike fields .

For pointlike fields the matrix $\underline{\Omega}^{(o)}(s)$ appearing in (1a) is given by

$$\underline{\Omega}^{(o)}(s) = \underline{P} \cdot \delta(s - s_1) \qquad (2a)$$

with

$$\underline{P} = \begin{pmatrix} 0 & -P_3 & P_2 \\ P_3 & 0 & -P_1 \\ -P_2 & P_1 & 0 \end{pmatrix} . \qquad (2b)$$

Examples:

α) end fields of a solenoid (see eq. (6.7a)) ;

β) cavity fields (see eq. (6.8a)) ;

γ) pointlike electric and magnetic fields (see eq. (6.9a)) .

The transfer matrix $\underline{N}(s_1 + 0, s_1 - 0)$ is given by [17]:

$$\underline{N}(s_1 + 0, s_1 - 0) = e^{\underline{P}} = \sum_{k=o}^{\infty} \frac{1}{k!} \underline{P}^k$$

or

$$\underline{N}(s_1+0, s_1-0) = \begin{pmatrix} \hat{P}_1^2(1-\cos P)+\cos P & \hat{P}_1\hat{P}_2(1-\cos P)-\hat{P}_3\sin P & \hat{P}_1\hat{P}_3(1-\cos P)+\hat{P}_2\sin P \\ \hat{P}_2\hat{P}_1(1-\cos P)+\hat{P}_3\sin P & \hat{P}_2^2(1-\cos P)+\cos P & \hat{P}_2\hat{P}_3(1-\cos P)-\hat{P}_1\sin P \\ \hat{P}_3\hat{P}_1(1-\cos P)-\hat{P}_2\sin P & \hat{P}_3\hat{P}_2(1-\cos P)+\hat{P}_1\sin P & \hat{P}_3^2(1-\cos P)+\cos P \end{pmatrix}$$

with

$$\hat{P}_\nu = \frac{1}{P} P_\nu \qquad (\nu = 1, 2, 3); \tag{4a}$$

$$P = \sqrt{P_1^2 + P_2^2 + P_3^2} \; . \tag{4b}$$

b) Fields of finite length

For fields of finite length the solution of eq. (1) can be written in the form of a power series, namely:

$$\underline{N}(s, s_1) = \sum_{k=o}^{\infty} \frac{1}{k!}(s - s_1)^k \; \{\frac{d}{ds^k} \underline{N}(s, s_1)\}_{s=s_1} \; . \tag{5}$$

Our task is to calculate the derivatives

$$\underline{N}_\nu(s_1) \equiv \{\frac{d}{ds^\nu} \underline{N}(s, s_1)\}_{s=s_1} \tag{6}$$

which determine (up to a factor 1/k!) the coefficients of the Taylor expansion (5).
Eq. (1b) implies that

$$\underline{N}_o(s_1) \equiv \underline{N}(s_1, s_1) = \underline{1} \; . \tag{7}$$

Inserting (7) into (1a) one gets $\underline{N}_1(s_1)$, namely

$$\underline{N}_1(s_1) \equiv \{ \frac{d}{ds} \underline{N}(s, s_1)\}_{s=s_1} = \underline{\Omega}^{(o)}(s_1). \tag{8}$$

Thus, we have calculated the first terms $\underline{N}_o$ and $\underline{N}_1$ in the expansion. The higher order terms

$$\underline{N}_{(n+1)}(s_1) \qquad (n = 1, 2 \ldots)$$

are found by taking the n-th derivative of eq. (1a):

$$\underline{N}_{(n+1)}(s_1) = \{ \frac{d^{n+1}}{ds^{n+1}} \underline{N}(s, s_1)\}_{s=s_1} =$$

$$= \sum_{\lambda=o}^{n} \binom{n}{\lambda} \{\frac{d^\lambda}{ds^\lambda} \underline{\Omega}^{(o)}(s)\}_{s=s_1} \cdot \{\frac{d^{(n-\lambda)}}{ds^{(n-\lambda)}} \underline{N}(s, s_1)\}_{s=s_1} \cdot \tag{9}$$

Writing the matrix $\underline{\Omega}^{(o)}(s)$ in the form (see eq. (6.2))

$$\underline{\Omega}^{(o)}(s) = \sum_{\nu=1}^{6} \underline{\Omega}_{\nu}^{(o)} \; y_{o\nu}(s) \tag{10}$$

with $y_{o\nu}$ being the components of the vector $\vec{y}_o(s)$ (six dimensional closed orbit):

$$y_{o1} \equiv x_o$$
$$y_{o2} \equiv P_{xo}$$
$$y_{o3} \equiv z_o$$
$$y_{o4} \equiv P_{zo}$$
$$y_{o5} \equiv \sigma_o$$
$$y_{o6} \equiv \eta_o$$

one gets

$$\left\{\frac{d^{\lambda}}{ds^{\lambda}} \underline{\Omega}^{(o)}(s)\right\}_{s=s_1} = \sum_{\nu} \underline{\Omega}_{\nu}^{(o)} \left\{ \frac{d^{\lambda}}{ds^{\lambda}} y_{o\nu}(s)\right\}_{s=s_1} \tag{11}$$

with

$$\left\{\frac{d^{o}}{ds^{o}} y_{o\nu}(s)\right\}_{s=s_1} \equiv y_{o\nu}(s_1) \tag{12a}$$

(as a given initial condition)

and

$$\left\{\frac{d^{\mu+1}}{ds^{\mu+1}} \vec{y}_o(s)\right\}_{s=s_1} = \underline{A} \left\{\frac{d^{\mu}}{ds^{\mu}} \vec{y}_o(s)\right\}_{s=s_1} . \tag{12b}$$

Taking into account eqs. (11) and (12), eq. (9) defines a recursion relation to calculate the coefficients $\underline{N}_{\nu}$ which are needed to evaluate the power series (5).

The Taylor series is an exact solution if $(s - s_1)$ is smaller than the corresponding convergence radius. By truncating this series after the k-th term one gets an approximate transfer matrix $\underline{N}(s, s_1)$.

Remark: The same scheme can be applied to calculate the matrix $\underline{G}$ (see (8.22)).

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