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Abstract

Strong coupling and weak coupling expansions for the low-lying mass spectrum in pure gauge theories are reviewed.

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1. Introduction

In the beginning I should perhaps explain the title of this talk. QCD, quantum chromodynamics, refers to quantized non-abelian gauge field theory in 3+1 dimensions without quark fields, the gauge group being SU(2) or SU(3). Low-energy means the low-lying spectrum of particles, the so-called glueballs. The expression quantitative indicates that the values of masses of these particles are considered. The masses m depend on the renormalized coupling g and the normalization mass μ, which enters QCD via renormalization.

$$m(g,\mu) = \mu f(g) \tag{1}$$

The arbitrariness in the choice of μ reflects itself in the renormalization group equation

$$\left\{ \mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g} \right\} m = 0 \tag{2}$$

As a consequence of this any mass can be written as a multiple of a scale parameter Λ, e.g.

$$m_i = c_i \Lambda_{\overline{MS}} \tag{3}$$

where the c_i are constant numbers, not depending on μ or g. In particular mass ratios m_i/m_j are fixed numbers. There is no adjustable parameter and hence no expansion available (except possibly the 1/N expansion). The only remaining procedure, appropriate for this congress, is to solve QCD exactly. Unfortunately this has not yet been achieved. Meanwhile we may be allowed to try to introduce suitable expansion parameters. I would like to talk about two different approaches, each related to a certain cutoff, which may be imposed.

a) Ultraviolet-cutoff:

An UV-cutoff can be achieved by putting QCD on a lattice with lattice spacing a. The strong coupling expansion on the lattice amounts to an expansion for small 1/a.

b) Infrared-cutoff:

An IR-cutoff is introduced through a finite spatial box of volume L^3 (in the continuum). A weak coupling expansion can be derived, which is valid for small L. Both expansions start at the undesired side, of course.

Usually the term perturbative QCD refers to Feynman graph calculations resulting in expansions of e.g. scattering cross-sections in powers of a running coupling $g^2(Q^2)$, which is small at high energies Q^2. On the other hand low energy phenomena are the scope of non-perturbative methods, in particular strong coupling expansions and Monte Carlo calculations on a lattice. The approach b), mentioned above, which is due to Lüscher [8,9], does not quite fit into this picture. It is concerned with the low-lying spectrum; however, it is based on a weak coupling expansion in powers of $g^{2/3}$ in the continuum. I shall discuss it in the second part of this talk.

2. Strong coupling expansions

In both the Euclidean and the Hamiltonian approach to lattice gauge theories strong coupling expansions for masses in powers of g_o^{-2} can be derived, where g_o is the bare (unrenormalized) coupling constant [1-4]. In the Euclidean framework they are analogous to high temperature expansions in statistical mechanics, whereas in the Hamiltonian formulation they amount to quantum mechanical perturbation expansion, the chromomagnetic term in the Hamiltonian

being the perturbation.

Let us consider Euclidean lattice gauge theory. For sufficiently strong coupling correlation functions decay exponentially. The fall-off is determined by the low-lying masses m_i, corresponding to the highest eigenvalues

$$\lambda_i = \exp(-\, am_i) \tag{4}$$

of the transfer matrix. They are classified according to representations R of the cubic group. For simplicity we label them by the smallest spin of a representation of the rotation group SO(3), to which R can be extended. The lowest masses have strong coupling expansions of the form

$$am = -\,4 \log \beta + \sum_{k=0}^{\infty} c_k \, \beta^k \tag{5}$$

$$\beta = \frac{2N}{g_o^2} \quad \text{for gauge group SU(N)} \tag{6}$$

They have been calculated up to β^8 for spin 0, 1 and 2 states [2-4].

Ultimately we are interested in the continuum limit of the lattice theory, which should be defined as a limit towards a critical point. The conjecture shared by almost all lattice gaugers is: $g_o^2 = 0$ is a critical point, and it is governed by the perturbative renormalization group, which implies

$$g_o^2(a) = -\,\frac{1}{b_o \log a^2\Lambda_L^2} + \ldots \,, \text{ as } a \to 0 \tag{7}$$

$$b_o = \frac{11\,N}{48\pi^2} \tag{8}$$

The expected behaviour of a m for small values of g_o^2 is therefore

$$am = C_m \exp\left(-\frac{1}{2b_o g_o^2}\right) (b_o g_o^2)^{-\frac{b_1}{2b_o^2}} \left\{1 + O(g_o^2)\right\} \tag{9}$$

$$C_m = \frac{m}{\Lambda_L} \tag{10}$$

The weak coupling behaviour (9) is separated from the strong coupling behaviour (5) by a crossover region, which is rather narrow according to Monte Carlo calculations. The strong coupling expansion ceases to converge at and beyond the crossover. Therefore different extrapolation methods have been tried. Particularly interesting are Smits results [5]. He reexpands the series in terms of the internal energy and applies a Padé-Borel extrapolation. The resulting function shows approximate scaling behaviour (equ. (9)) over a relatively large region, and the numerical result for the constant C_m (10) is consistent with Monte Carlo results in the case of the lowest spin 0 state.

Both strong coupling and Monte Carlo calculations for the absolute scales C_m of masses are, however, affected by the unknown factor $\left\{1 + O(g_o^2)\right\}$ in (9), which may well be large in the region under consideration. Therefore let us consider mass ratios m_i/m_j. Their approach to the continuum limit is much more rapid

$$\frac{m_i}{m_j} = \text{const.} \left\{1 + O(a^2 m^2)\right\} \text{ as } a \to 0 \tag{11}$$

the correction term being exponentially small in g_o^2. Furthermore mass ratios probably behave more smoothly as functions of g_o^2. Mass ratios have been studied in the Hamiltonian [1] as well as in the Euclidean [4,5] framework by means of Padé extrapolations. In [4] a more physical expansion parameter s has been chosen, which represents a measure of the correlation length ξ for large values of ξ.

$$S \sim (am(0^+))^{-1} \equiv \xi \quad \text{as } s \to \infty \tag{12}$$

$m(0^+)$: mass gap in the spin 0, parity +1 sector

In the Hamiltonian lattice gauge theory the correlation length ξ itself can be used as a strong coupling expansion parameter.

All these different strong coupling methods yield the following results. For gauge group SU(3)

$$\frac{m(2^+)}{m(0^+)} \approx 1 \quad , \quad \frac{m(1^+)}{m(0^+)} = 1.8 \pm 0.3 \tag{13}$$

and for gauge group SU(2)

$$\frac{m(2^+)}{m(0^+)} \approx 1.25 \tag{14}$$

Available Monte Carlo results for SU(2) used to point towards higher values [6], but recent calculations have a downward tendency [7].

Of course, the strong coupling expansions have severe problems: i) the series are rather short, ii) it is difficult to estimate the reliability. Therefore the numerical results above have to be considered as rough estimates at most.

3. Weak coupling expansion for the SU(2) Yang-Mills spectrum

This part is based on joint work with M. Lüscher [10].

3.1 Idea

Consider continuum Yang-Mills theory in a periodic spatial box of linear extent L, while time remains unrestricted. The allowed values of momenta

$$\vec{p} = \frac{2\pi}{L}\vec{\nu} \quad , \quad \vec{\nu} \in \mathbb{Z}^3 \tag{15}$$

as well as the eigenvalues of the Hamiltonian H are discrete. One particle mass shells are represented by discrete arrays of energy values $E_i(\vec{p})$. (The vacuum energy has been subtracted.) The masses

$$M_i = E_i(\vec{0}) \tag{16}$$

depend on L. We expect that

$$\lim_{L\to\infty} M_i \equiv m_i \tag{17}$$

exist and yield the mass spectrum of the infinite volume theory. How does M_i depend on L? For small L

$$M_i \underset{L\to 0}{\sim} \frac{\text{const.}}{L}(-\log L)^{-p} \tag{18}$$

with p = 0 or + 1/3. This will be discussed below. On the other hand for large L

$$M_i \xrightarrow[L\to\infty]{} m_i \tag{19}$$

such that [11]

$$\frac{M_i}{m_i} \underset{L\to\infty}{\sim} 1 + C(m_o L)^{-p'} \exp(-\alpha m_o L) \tag{20}$$

where m_o is the mass gap in the infinite volume theory and

$$\frac{\sqrt{3}}{2} \leq \alpha \leq 1. \tag{21}$$

The qualitative behaviour of M(L) is illustrated in fig. 1. The limiting cases (18) and (20) indicate a sharp crossover from small volume to large volume behaviour.

For small L the running coupling constant

$$\bar{g}^2 = -\frac{1}{2b_o \log \Lambda_{MS} L} + \ldots \xrightarrow[L\to 0]{} 0 \tag{22}$$

is small. This suggests to compute M_i for small L in perturbation theory, i.e. to compute the left hand side of the curve in fig. 1. It has been shown by Lüscher [9] that this can be done. For the low-lying states the weak coupling expansion is of the form

$$M_i = \frac{1}{L}\sum_{k=0}^{\infty} \varepsilon_k \bar{\lambda}^k \tag{23}$$

$$\bar{\lambda} = \bar{g}^{2/3} \tag{24}$$

The idea is then to study this expansion near the crossover and possibly beyond, in order to obtain estimates for masses in the infinite volume limit.

In the case of the O(n) non-linear σ-model in two dimensions this method works quite well [8]. The weak coupling expansion is reliable up to $m_o L \approx 6$ far beyond the crossover and yields rather good results for m_o.

3.2 Lüscher's effective Hamiltonian [9]

Yang-Mills fields on a spatial torus $T^3 = S^1 \times S^1 \times S^1$ are described by Lie algebra valued vector fields $A_k(x)$ on T^3 since SU(n) principal bundles over T^3 are trivial. The Hamiltonian is

$$H = \int_0^L d^3x \left\{ \frac{1}{2} g_o^2 E_k^a E_k^a + \frac{1}{2g_o^2} B_k^a B_k^a \right\} \qquad (25)$$

$$E_k^a(x) = -i \frac{\delta}{\delta A_k^a(x)} \qquad (26)$$

and B_k^a is the usual magnetic field strength. In perturbation theory one considers regions in field space where B_k^a is small. Configurations with vanishing magnetic field are the so-called "torons". The only quantum mechanical stable torons in SU(2) Yang-Mills theory are the classical vacuum $A_k = 0$ and its central conjugate [9]

$$A_k = \frac{2\pi}{L} \frac{\sigma_3}{2i} \tau_k \quad ; \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ \tau_k \in \{0,1\} \qquad (27)$$

In perturbation theory it is sufficient to expand around

$$A_k = 0 \qquad (28)$$

If A_k is decomposed into a constant part in space and a fluctuating part, the lowest order Hamiltonian does not contain the constant field. This means that there is a degeneracy at $g = 0$, the degenerate subspace of wave functions

being the Hilbert space of constant gauge potentials c_k^a. This degeneracy is lifted in higher orders. According to the rules of degenerate perturbation theory the splitting is obtained by diagonalizing the Hamiltonian in the degenerate subspace. This yields an effective Hamiltonian in the space of constant gauge potentials.

Let us consider those states, which are lowest at $g = 0$, i.e. degenerate with the ground state. The perturbation expansion for their level splitting in higher orders of perturbation theory cannot be derived analytically; the effective Hamiltonian, however, can be calculated.

<u>Lüscher's lemma:</u>

The energies of these states have an expansion

$$E = \frac{1}{L} \sum_{k=1}^{\infty} \varepsilon_k \bar{\lambda}^k \qquad (29)$$

and they are <u>exactly equal</u> to the eigenvalues of an <u>effective quantum mechanical Hamiltonian</u> H', acting on wave functions on the space of constant gauge potentials

$$c_k^a \ ; \quad k = 1,2,3 \ ; \ a = 1,2,3 \qquad (30)$$

H' has an expansion

$$H' = \frac{\bar{\lambda}}{L} \sum_{\nu=0}^{\infty} \bar{\lambda}^{\nu} H'_{\nu} \qquad (31)$$

Up to one-loop order it reads

$$H_o' = -\frac{1}{2}\frac{\partial^2}{\partial c_k^a \, \partial c_k^a} + \frac{1}{4}\left(\varepsilon^{abc} c_k^b c_l^c\right)^2 \qquad (32)$$

$$H_1' = \varkappa_1 c_k^a c_k^a \;, \quad \varkappa_1 = -0.30104661 \ldots \qquad (33)$$

$$H_2' = 0$$

$$H_3' = \varkappa_2 H_o' + \varkappa_3 (c_k^a c_k^a c_1^b c_1^b + 2c_k^a c_1^a c_k^b c_1^b)$$
$$+ \varkappa_4 (5c_k^a c_k^a c_k^b c_k^b - c_k^a c_k^a c_1^b c_1^b - 2c_k^a c_1^a c_k^b c_1^b) \qquad (34)$$

with numerical coefficients $\varkappa_\nu$.

3.3 Study of the effective Hamiltonian

The lowest order Hamiltonian H_o' looks as if it might be solvable analytically. The corresponding classical system, however, is not completely integrable (12). Therefore there is little hope that the quantum mechanical one will be.

The following properties of H_o' are known (9)

i) it has a purely discrete spectrum. (This has also been proven by Simon in five different ways (13)).

ii) the ground state is nondegenerate, its wave function can be chosen positive. It is invariant under colour and space rotations, space reflections and charge conjugations.

iii) all eigenfunctions decay at least like $\exp(-\alpha |C|^{3/2})$, $\alpha > 0$.

Numerical study of H'

We studied the effective Hamiltonian H' numerically by two different methods.

1) Rayleigh-Ritz variational method

A harmonic oscillator basis with up to 100 states per J^P (spin, parity) sector was chosen. The variational calculation was done with the help of a computer, partly using algebraic manipulation programs. It turned out to be necessary to use extended precision (33 digits). States with J = 0 and 2 have been considered (J = 1 states do not exist in the Hilbert space under consideration). The computation required about 10 minutes CPU time per state.

2) Large-N expansion

If the index a of the variables c_k^a is allowed to run from 1 to N, one obtains a SO(N) symmetric vector model. This can be dealt with in a 1/N expansion. Energy values have expansions of the form

$$E = N^{4/3} \sum_{\nu=0}^{\infty} \omega_\nu N^{-\nu} \qquad (35)$$

We calculated them up to N^{-9} and N^{-8} for $J^P = 0^+$ and 2^+ states respectively. The coefficients are generated recursively with the help of a computer, using extended precision and a maximum of 4 Megabytes main storage. The computing time was 2 minutes CPU per state.

For one-dimensional systems the large-N method is known to yield rather accurate results. In our case the expansions evaluated at N=3 yield results for the

- two lowest states with $J^P = 0^+$
- lowest state with $J^P = 2^+$

These agree with the Rayleigh-Ritz method.

3.4 Results

For $J^P = 0^+$, 2^+ and 0^- states (J=1 states do not exist) we obtained the coefficients ε_k in (29) up to k=4. Let

$$M(0^+) = E(\text{first excited state}) - E(\text{ground state}) \quad (36)$$

be the mass gap in the 0^+ sector and define

$$z = M(0^+)\,L \equiv \frac{L}{\xi} = \text{box size in physical units}$$
$$= c_1\bar{\lambda} + c_2\bar{\lambda}^2 + c_3\bar{\lambda}^3 + c_4\bar{\lambda}^4 + \ldots \quad (37)$$

The quantity z serves as a universal expansion parameter: reexpanding in terms of z yields series whose coefficients are independent of the renormalization prescription.

A) Mas gap in the 0^+ sector

For the mass gap divided by the usual scale parameter $\Lambda_{\overline{MS}}$ we obtain

$$\log(M(0^+)/\Lambda_{\overline{MS}}) = 125.8\, z^{-3} - 58.4\, z^{-2}$$
$$- 10.3\, z^{-1} - 7.4 + \frac{274}{121}\log z + O(z) \quad (38)$$

This is plotted as the left curve in fig. 2. The right curve is obtained by using the corresponding expansion in terms of $\bar{\lambda}$ and calculating $\bar{\lambda}$ as a function of z through an inversion of (37). For small z both curves coincide, of course. The deviation for $z \gtrsim 1.5$ indicates the limitations of the small z expansion. It should not be extrapolated beyond $z \approx 1.5 - 2.0$.

Therefore the large L regime unfortunately cannot be reached and it is not possible to obtain reliable estimates for $m(0^+)/\Lambda_{\overline{MS}}$. This is quite different from the case of the σ-model, mentioned above.

One conclusion, however, can be drawn definitively. From Monte Carlo simulations and strong coupling expansions the infinite volume mass gap is estimated to be

$$m(0^+)/\Lambda_{\overline{MS}} \sim 10 \quad (39)$$

as indicated by the dashed line in fig. 2. Even if this number is wrong by an order of magnitude, to match it with the steeply falling weak coupling curves of fig. 2 requires a crossover to be located near $z \simeq 2$. Because the curve is so steep for $z \lesssim 1.5$ the crossover is probably very sharp.

B) Mass ratios

As in the case of strong coupling expansions the situation for mass ratios is different from that for absolute values. Mass ratios do not have to undergo a crossover. They may behave smoothly as a function of z near $z \approx 2$. Their universal expansions read

$$\frac{M}{M(0^+)} = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + O(z^4) \quad (40)$$

This is plotted in fig. 3 for different states. Surprisingly the lowest state has spin 2. It is nearly degenerate with the lowest 0^+ state near the crossover $z \approx 2$. Another such degenerate pair of 2^+ and 0^+ states is seen at $M \approx 1.5\, M(0^+)$.

We do not know the behaviour of the curves in the crossover. If they behave smoothly there, the expansions (40) would be more reliable than (38).

The conclusion then would be:

1) There are two low states with $J^P = 0^+$ and 2^+, in agreement with strong coupling and some Monte Carlo results.

2) There are nearby excited states.

3.5 Questions and possible extensions

There are many open questions and possible ways to extend the obtained results. I would like to mention only some catchwords:

- direct comparison with Monte Carlo data
- incorporation of tunnelling between sectors with different electric flux (central sectors)
- J = 1 states
- two-loop calculations
- gauge group SU(3)
- introduction of heavy quarks

and refer to ref. 10 for further discussion.

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Figure Captions

Fig. 1: Qualitative behaviour of the mass M of a stable particle as a function of the box size L.

Fig. 2: Energy gap $M(0^+)$ in the 0^+ sector as a function of z as explained in section 3.4.

Fig. 3: Mass ratios as a function of z.

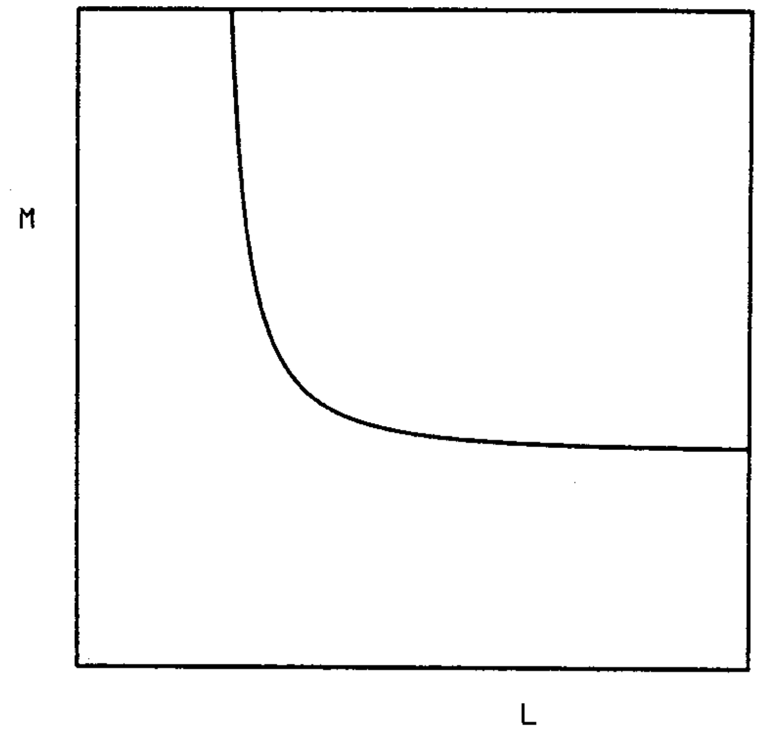

Fig. 1

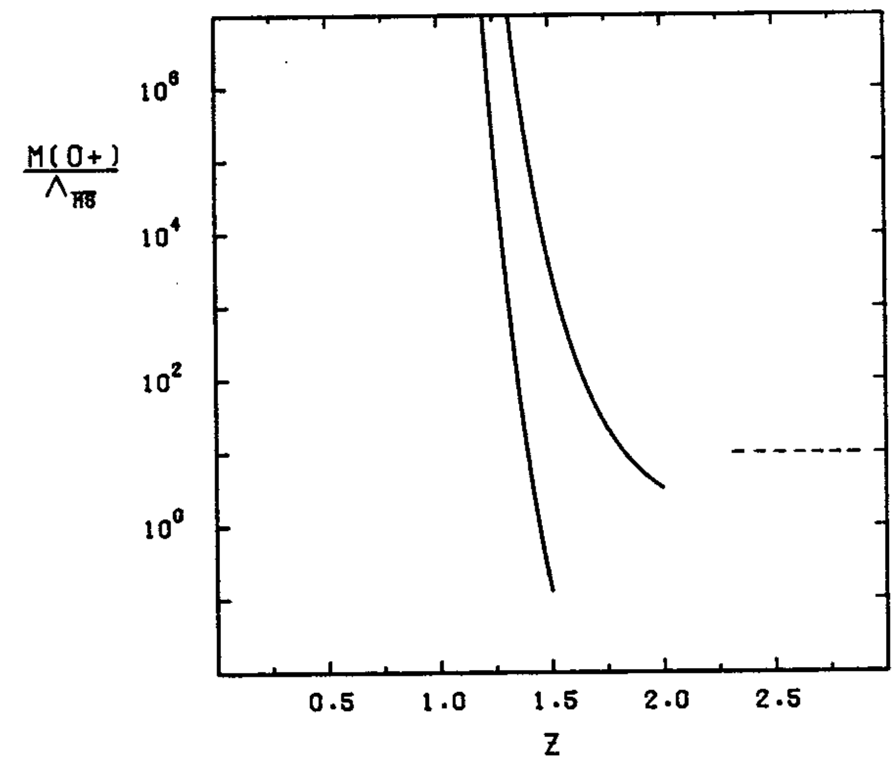
$\frac{M(0+)}{\Lambda_{\overline{MS}}}$
10^6
10^4
10^2
10^0
0.5
1.0
1.5
2.0
2.5
Z

Fig. 2

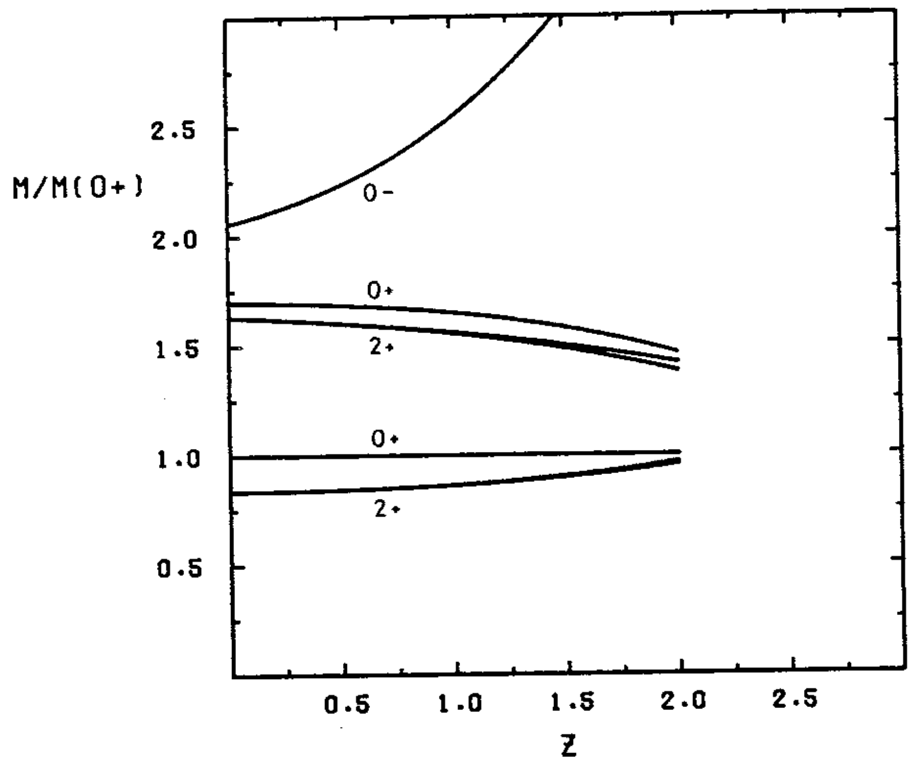
M/M(0+)
2.5
2.0
1.5
1.0
0.5
0-
0+
2+
0+
2+
0.5
1.0
1.5
2.0
2.5
Z

Fig. 3