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by

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ABSTRACT

Finite volume effects on the mass spectrum of massive quantum field theories (including QCD) are shown to be related to forward scattering amplitudes of the infinite volume theory. Using theoretical and experimental information on the latter, the finite size effects can be estimated. For the pion and the nucleon they are found to be small, provided $m_\pi L \geqslant 1$ (m_π: pion mass, L: box size).

1. INTRODUCTION

Consider, for example, an SU(2) gauge theory on an TxLxLxL periodic space-time lattice, where the time-like extent T is supposed to be big, i.e. ideally $T = \infty$. The mass gap M_o of the theory can then be determined from the exponential fall-off of the plaquette two-point function C(t) at large times t:

* Lecture given at the Nato Advanced Study Institute in Cargèse (September 1983)

(1) $$C(t) \propto t^{-3/2}\, e^{-M_o t}$$

Here and below the lattice spacing is set equal to one so that L is an integer and M_o is dimensionless. The energy gap M_o not only depends on the bare coupling constant g_o, but also on the box size L. If one attempts to extract the infinite volume mass gap

(2) $$m_o = \lim_{L \to \infty} M_o \qquad (g_o \text{ fixed})$$

from a finite volume calculation [1)3)], the relative deviation

(3) $$\delta_o = (M_o - m_o)/m_o$$

must therefore be accounted for as a systematic error. The size dependence of the energy gap is not a lattice artefact. One rather expects that δ_o becomes a universal function of

(4) $$\zeta = m_o L$$

in the finite volume continuum limit, which is obtained by taking $g_o \to 0$ and simultaneously $L \to \infty$ while keeping ζ fixed. *

Of course, finite size effects on the mass spectrum are a common feature of Lagrangian quantum field theories including QCD and lattice spin models. This lecture is devoted to the question of how δ_o typically varies with respect to ζ and in particular of how big ζ must be in order that (say) $|\delta_o| < 0.1$. The general experience is

* ζ should not be confused with the parameter $z = M_o L$, which I have used earlier as a weak coupling expansion parameter [2)3)]. The two parameters are, however, related by $z = \zeta(1 + \delta_o)$ so that $z \simeq \zeta$ at large ζ.

that δ_o is small for $\gamma \gtrsim 3$, but this statement is presumably not always true and in any case needs proof. Here I shall establish a formula for δ_o valid at large γ by which δ_o can be accurately estimated provided something is known about the forward elastic scattering amplitude of the infinite volume theory. Conversely, when δ_o has already been computed by other means, the formula allows (at least in principle) to extract the complete forward elastic scattering amplitude.

2. ISING MODEL

To warm up, let us first discuss the soluble case of the 2-dimensional Ising model. Thus, pick a T x L square lattice Λ with periodic boundary conditions and attach to each site n of Λ an Ising spin $\sigma(n)$ taking values ± 1. The action S of a spin field at inverse temperature β is given by

$$S = -\beta \sum_{n\in\Lambda} \sum_{\mu=0,1} \sigma(n)\,\sigma(n+\hat{\mu}),$$

where $\hat{\mu}$ denotes the unit vector in the positive μ-direction ($\mu = 0$ for time, $\mu = 1$ for space). In what follows, we shall only consider the high temperature (small β) phase of the model. As in the gauge theory case, the energy gap $M_o(\beta,L)$ can then be read off from the exponential decay at large times of the spin correlation function, which is defined by

$$\langle \sigma(n)\,\sigma(0)\rangle = \lim_{T\to\infty} \sum_{\text{fields}} \sigma(n)\sigma(0)\,e^{-S} \Big/ \sum_{\text{fields}} e^{-S}$$

Alternatively, we may note that e^{-M_o} is equal to the ratio of the next to highest to highest eigenvalue of the transfer matrix. These eigenvalues have already been computed by Schultz, Mattis and Lieb [4) almost twenty years ago so that here we only have to recall their beautiful exact results.

In the infinite volume limit ($L = \infty$), the mass gap is given by

$$m_o = -2\beta - \ln \operatorname{tgh}\beta \tag{5}$$

As β rises, m_o is monotonically decreasing and eventually vanishes at the critical point

$$\beta_c = \tfrac{1}{2}\ln(1+\sqrt{2}) \tag{6}$$

At $L = \infty$, there is actually a whole "mass shell" of 1-particle states with momentum q, $-\pi \leq q \leq \pi$, and energy ϵ_q given by

$$\operatorname{ch}\epsilon_q = \operatorname{ch} m_o + 1 - \cos q, \quad \epsilon_q \geq m_o. \tag{7}$$

Near β_c and at small q, this formula reduces to the relativistic energy momentum relation

$$\epsilon_q^2 = m_o^2 + q^2 \tag{8}$$

The finite volume energy gap M_o, which has also been obtained by Schultz et al., reads

$$M_o = m_o + \frac{1}{2}\sum_{\nu=0}^{L-1} \epsilon_{\frac{\pi}{L}(2\nu+1)} - \frac{1}{2}\sum_{\nu=0}^{L-1} \epsilon_{\frac{\pi}{L}2\nu} \tag{9}$$

The first sum here represents the zero point energy of the vacuum state and the second sum the zero point energy of the zero momentum 1-particle state. These frequency sums come with the "wrong" sign, because of the fermionic character of the harmonic oscillators in terms of which the diagonalization of the transfer matrix is achieved.

For further analysis, it is useful to rewrite eq. (9) as follows. First, using the Poisson summation formula, we have

$$M_o = m_o - L \sum_{\nu=-\infty}^{\infty} \int_{-\pi}^{\pi} \frac{dq}{2\pi} \, e^{iqL(2\nu+1)} \epsilon_q$$

Then, shifting the contour of integration to imaginary q and performing a number of substitutions, one arrives at

$$(10) \quad M_o = m_o + \sum_{\nu=0}^{\infty} \frac{2}{2\nu+1} \int_{-\pi}^{\pi} \frac{dq}{2\pi} \, e^{-(2\nu+1)L\epsilon_q}$$

It follows immediately that for fixed $\beta < \beta_c$ and growing L, the energy gap M_o is monotonically decreasing towards m_o. At large L, δ_o eventually vanishes exponentially fast:

$$(11) \qquad \delta_o \underset{L\to\infty}{\propto} L^{-1/2} e^{-m_o L}$$

We shall see later that this exponential falling off is typical for massive quantum field theories, although the decay rate is not always equal to one.

The continuum limit of δ_o at β_c is also easily obtained from eq. (10). Thus, letting β approach β_c and sending L to infinity in such a way that $\zeta = m_o L$ remains fixed, leads to

$$(12) \quad \delta_o = \sum_{\nu=0}^{\infty} \frac{2}{2\nu+1} \int_{-\infty}^{\infty} \frac{dp}{2\pi} \, e^{-(2\nu+1)\zeta p_o} , \qquad p_o = \sqrt{1+p^2}$$

Evaluating this expression numerically, yields the table

ζ	δ_o
1	0.392
2	0.089
3	0.026
4	0.008

As expected, δ_o quickly becomes small when ζ grows, so that beyond $\zeta = 3$ the finite size effects shift the mass gap by at most a few percent.

Before we turn to the general case, let us briefly consider the d+1-dimensional Ising model. No exact solution is available here, of course, but we may instead rely on the high temperature cluster expansion. Recently I found that the graphs contributing to δ_o can be arranged in such a way that to all orders

$$(13) \quad \delta_o \underset{\zeta\to\infty}{\sim} \zeta^{-d/2} e^{-\zeta} \sum_{\nu=0}^{\infty} c_\nu \zeta^{-\nu}$$

with perturbatively computable coefficients $c_\nu(\beta)$ [5]. While this result is interesting, its proof is mainly a book-keeping task. The following two features are, however, remarkable:

- High temperature graphs contributing to δ_o must wind around the world in such a way that they cannot be unwrapped. This fact alone already explains why δ_o is exponentially small at large L, because every such graph has a weight proportional to β^L.
- The coefficients c_ν in eq. (13) derive from graphs, which also contribute to the four point vertex function at certain special momenta, thus suggesting a connection with the elastic scattering amplitude. This is the topic, to which we turn now.

3. A FORMULA FOR δ_o IN TERMS OF THE ELASTIC SCATTERING AMPLITUDE

Let us consider a scalar field theory in d+1 dimensions describing a single self-interacting spinless particle of mass m_o. In the course of the discussion it will become clear how to generalize to more complicated situations. From now on I shall furthermore assume that the continuum limit has already been taken *, although I believe that, with appropriate modifications, the result obtained below (eq. (15)) also applies in lattice theories.

It is well known that the (infinite volume) forward scattering amplitude F extends to an analytic function of the "crossing variable" ν in the cut plane shown in Fig. 1 (F and ν are defined in the Appendix). In addition to the cuts starting at $\nu = \pm m_o$, $F(\nu)$ has simple poles at $\nu = \pm \frac{1}{2} m_o$, if one-particle exchange scattering processes are possible. In that case, an effective three particle coupling constant λ can be defined through

$$\lim_{\nu \to \pm \frac{1}{2} m_o} (\nu^2 - \tfrac{1}{4} m_o^2) F(\nu) = \tfrac{1}{2} \lambda^2 \tag{14}$$

Note that, due to crossing symmetry, $F(-\nu) = F(\nu)$, and the two limits in eq. (14) coincide.

Suppose now that the theory is enclosed in a d-dimensional periodic space box of size L and that δ_o is defined as before. The formula alluded to in the introduction then reads

* In particular, some physical units are henceforth used for dimensionful quantities like M_o, m_o and L.

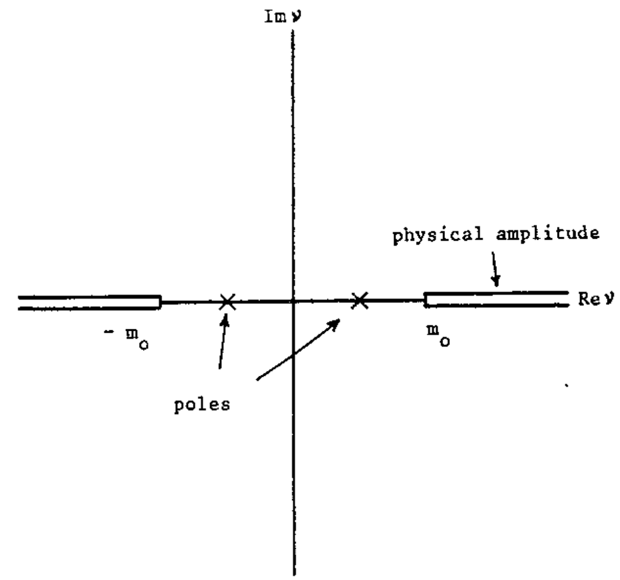

Fig. 1: Analyticity domain of $F(\nu)$.

$$\delta_o = -\lambda^2 m_o^{d-5} \frac{d}{8\pi} \left(\frac{4\pi}{\sqrt{3}} \zeta\right)^{1-\frac{d}{2}} K_{\frac{d}{2}-1}\left(\frac{\sqrt{3}}{2}\zeta\right) - \frac{d}{2m_o^2} \int d\mu(q)\, e^{-q_o L} F(iq_1) + O(e^{-\alpha\zeta}) \tag{15}$$

Here, K_1 denotes a modified Bessel function (Ref. 6, § 8.432) and, as usual, $\zeta = m_o L$. Furthermore,

$$d\mu(q) = \frac{d^d q}{(2\pi)^d 2q_o}, \qquad q_o = \sqrt{m_o^2 + \vec{q}^{\,2}}$$

is the Lorentz invariant measure on the one-particle mass shell. Note that the forward scattering amplitude F is integrated over along the imaginary axis in the ν-plane, far away from singularities. The error term in eq. (15) is dependent on dynamical details of the theory, but α never becomes smaller than $\sqrt{\frac{3}{2}}$ and may, on the other hand, be as large as 3.

A most intriguing feature of eq. (15) is that it relates δ_o to an experimentally measurable quantity. Also, eq. (15) immediately implies that δ_o is exponentially decreasing at large ζ:

(16a) $$\delta_o \sim -\lambda^2 m_o^{d-5} \frac{d}{4\sqrt{3}} \left(\frac{4\pi}{\sqrt{3}} \zeta\right)^{\frac{1}{2}(1-d)} e^{-\frac{\sqrt{3}}{2}\zeta} \qquad (\text{if } \lambda \neq 0)$$

(16b) $$\delta_o \sim -F(0)\, m_o^{d-3} \frac{d}{4} (2\pi\zeta)^{-\frac{d}{2}} e^{-\zeta} \qquad (\text{if } \lambda = 0)$$

The actual size of δ_o at a given large ζ, however, depends on the strength of the interactions and may therefore vary considerably with respect to the parameters of the theory.

The best proof of eq. (15) I have so far been able to find goes as follows *. Let $\phi(x)$ be an interpolating scalar field so that at $L = \infty$

(17) $$\langle 0 | \phi(0) | p \rangle = 1$$

The finite L Euclidean 2-point function of ϕ is then written as

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* When this lecture was delivered, I sketched a different argument, which did not make use of Feynman diagrams and was therefore universally applicable. However, I later found some of the steps involved difficult to justify rigorously and now prefer the reasoning outlined here.

$$\langle \phi(x)\phi(0) \rangle_L = \int \frac{dp_o}{2\pi} L^{-d} \sum_{\vec{p}} e^{ipx} G_L(p)$$

$$G_L(p)^{-1} = m_o^2 + p^2 - \Sigma_L(p)$$

(Euclidean scalar products are used here, e.g. $p^2 = p_o^2 + \vec{p}^2$). The normalization (17) and the requirement that m_o be the physical mass at $L = \infty$ amounts to

$$\Sigma_\infty(p) = \frac{\partial}{\partial p^2} \Sigma_\infty(p) = 0 \qquad \text{for } p = (im_o, \vec{0}).$$

On the other hand, the finite L energy gap M_o is implicitly determined by

$$G_L(p)^{-1} = 0 \text{ for } p = (iM_o, \vec{0}).$$

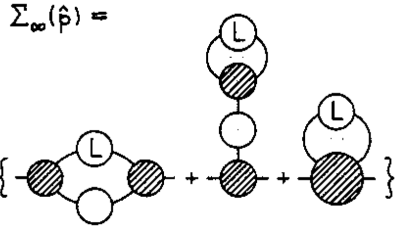
Writing $M_o = (1 + \delta_o)\, m_o$, it follows that

(18) $$\delta_o = -\frac{1}{2m_o^2} \left(\Sigma_L(\hat{p}) - \Sigma_\infty(\hat{p}) \right) + O(\delta_o^2)$$

where $\hat{p} = (im_o, \vec{0})$.

The second, more difficult step in the proof of eq. (15) consists in showing that

(19) $$\Sigma_L(\hat{p}) - \Sigma_\infty(\hat{p}) =$$

$\frac{1}{2}\{$  $\} + O(e^{-\alpha\zeta})$

The shaded bubbles here represent the $L = \infty$ Euclidean vertex functions (full propagator amputated, 1-particle irreducible n-point functions of ϕ), and

 $= G_\infty(q)$

(20)  $= 2 \sum_{i=1}^{d} \cos(q_i L)\, G_\infty(q)$

The L-dependence of the rhs of eq. (19) stems entirely from the modified propagator (20). All other factors and loop momentum integrations are exactly as in the infinite volume theory.

Eq. (19) can be understood heuristically by noting that in position space the modified propagator (20) is equal to the old propagator with the argument shifted by a period L in any of the 2d space directions. The rhs of eq. (19) may then be interpreted as the sum of all self-energy diagrams where exactly one line is allowed to wind around the world. A mathematical proof of eq. (19) can be given, within the framework of Feynman diagrams, for any massive, local Lagrangian theory of the field ϕ [5]. The form of the self-interactions of ϕ is irrelevant for the proof, as long as a perturbative treatment is meaningful. In view of this generality, it seems reasonable to expect eq. (19) to be valid also beyond perturbation theory.

The proof of eq. (15) is now easily completed along the following lines. First note that, for symmetry reasons, the modified propagator (20) may be replaced by

$$2d\, e^{iq_1 L}\, G_\infty(q)$$

without changing the rhs of eq. (19). In each of the diagrams, we then substitute $q_1 \rightarrow q_1 + is$, $s > 0$, so that

$$e^{iq_1 L} \longrightarrow e^{-sL}\, e^{iq_1 L}$$

In the course of this shift of the q_1 integration contour, singularities are met. Those closest to the real axis are the poles of the propagators at $q^2 = -m_o^2$ and $(q + \hat{p})^2 = -m_o^2$ (in the first diagram). Other singularities are further off and need not be considered, if we allow for an error of order $e^{-\alpha \varphi}$. The pole contributions, on the other hand, are easily evaluated by the residue theorem, which puts the momenta flowing through the propagators on the mass shell. We are then left with an amputated 4-point function with all legs on mass shell, i.e. with a scattering amplitude. In this way eq. (15) is obtained. In particular, the term proportional to λ^2 arises from a contribution of the first diagram on the rhs of eq. (19), where the momenta flowing through the propagators are both on the mass shell. *

4. APPLICATIONS

4.1 2-dimensional Ising model

This is one of the rare cases where δ_o is known exactly. Comparing the continuum limit result (12) with eq. (15), we can read off the scattering amplitude:

(21) $$F(\nu) = -8\, m_o \sqrt{m_o^2 - \nu^2}$$

In 1+1 dimensions, identical particles can only scatter forwardly so that F in fact determines the complete elastic scattering amplitude. Thus, the scattering operator $\mathbb{S}$ in the 2-particle sector can be calculated and comes out to be

* Note that $\lambda^2 = (\Gamma_3(k_1,k_2,k_3))^2$, where Γ_3 is the 3-point vertex function and $k_i^2 = -m_o^2$ for all i.

(22) $S = -1$

In view of eq. (12), the striking simplicity of this result is perhaps not unexpected. Anyhow, eq. (22) has been known before [7)8)], and we may therefore consider the present calculation a non-perturbative check of the new relation (15).

4.2 2-dimensional O(n) non-linear σ-model

There has been increasing evidence that the particle spectrum of this model consists of a massive O(n) vector multiplet. With a few structural assumptions (most of which were later justified), the brothers Zamolodchikov have been able to calculate the scattering matrix exactly [9)]. In particular, the forward amplitude $F(\nu)$, as defined in the Appendix, is known and can be inserted into eq. (15), which is also valid in the present case. For n = 3, the formula for the mass shift of the O(n) vector multiplet obtained in this way, assumes the following simple form:

$$\delta_0 = 4\pi \int_{-\infty}^{\infty} dt \; e^{-z \,\mathrm{ch}\, t} \frac{\mathrm{ch}\, t}{t^2 + \frac{9}{4}\pi^2} + O(e^{-\alpha z}) \qquad (23)$$

Evaluating the integral numerically and neglecting the error term, yields the table

z	δ_0	
1	0.650	
2	0.155	(n = 3)
3	0.045	
4	0.014	

As in the Ising model, δ_0 thus turns out to be small for $z \geq 3$.

Some time ago, I computed $m_0/\Lambda_{\overline{MS}}$ by a method, which was based on finite L perturbation theory [2)] ($\Lambda_{\overline{MS}}$ denotes the Λ-parameter in the modified minimal subtraction scheme). A working hypothesis was that finite size effects on the mass gap are small for (say) $z = 3 \ldots 4$, an assumption, which is now seen to be entirely justified. Furthermore, the news on δ_0 can be combined with the calculation of Ref. 2 to obtain the improved value

$$m_0/\Lambda_{\overline{MS}} = 1.6 \qquad (n = 3) \qquad (24)$$

I believe that this result is accurate to 10 %, but a reliable error estimation has to wait for the 2-loop mass formula [15)].

For $n > 3$, the situation is qualitatively the same as in the O(3) model, and at $n = \infty$ a further non-trivial check of eq. (15) is obtained, because δ_0 can also be computed by other means in this case [2)]. It is conceivable that eq. (23) is but the first term of a complete expansion of δ_0 as in the Ising model (eq. (12)), the higher terms being related to many particle scattering processes. Such a magic formula would eventually lead to an exact determination of $m_0/\Lambda_{\overline{MS}}$.

4.3 4-dimensional pure SU(n) gauge theory

Little is known about the particle spectrum of these models, but all the recent investigations point to the existence of a lowest lying $J^{PC} = 0^{++}$ particle and of a perhaps nearby 2^{++} particle. As an interpolating field for the 0^{++} particle one may take, for example,

$$\phi(x) = F^a_{\mu\nu}(x) F^a_{\mu\nu}(x),$$

where $F^a_{\mu\nu}$ is the gauge field tensor. From perturbation theory one easily shows that the 3-point function of ϕ does not vanish.

Barring an accidental zero on the mass shell, it follows that the effective coupling λ does not vanish, too. We therefore conclude that

$$\delta_0 \underset{\ell \to \infty}{\sim} -\left(\frac{\lambda}{m_0}\right)^2 \frac{3}{16\pi}\, \ell^{-1}\, e^{-\frac{\sqrt{3}}{2}\ell}, \tag{25}$$

where m_0 denotes the mass of the 0^{++} particle and δ_0 the corresponding finite volume mass shift. Since λ must be real, eq. (25) implies that

$$\delta_0 < 0 \text{ for large } \ell. \tag{26}$$

On the other hand, $\delta_0 \longrightarrow +\infty$ for $\ell \longrightarrow 0$ [3] so that there must be a value of ℓ, where M_0/m_0 is minimal. This behaviour is quite different from the situation in the non-linear σ-model (M_0 is monotonically decreasing there) and is perhaps related to the small radius of convergence of the perturbation expansion of M_0 derived in Ref. 3.

4.4 Realistic QCD *

Let us first consider the pion mass shift δ_π. Performing the integral over the momentum components q_2 and q_3 explicitly, eq. (15) reduces to

$$\delta_\pi = -\frac{3}{8\pi \ell} \int_{-\infty}^{\infty} \frac{dp}{2\pi}\, e^{-\ell\sqrt{1+p^2}}\, F_{\pi\pi}(i m_\pi p) + O(e^{-\alpha \ell}) \tag{27}$$

* Not the quenched approximation. Vacuum polarization is crucial for the validity of the formulae in this paragraph. Isospin breaking effects are neglected.

where m_π denotes the pion mass, $F_{\pi\pi}(\nu)$ the forward $\pi\pi$-scattering amplitude, and $\ell = m_\pi L$. To estimate $F_{\pi\pi}(\nu)$ near $\nu = 0$, we have current algebra results and experimental data at our disposal. Thus, denoting isospin indices by $\alpha, \beta, \ldots$, the elastic $\pi\pi$-scattering amplitude can be written as

$$T_{\alpha'\beta',\alpha\beta} = \delta_{\alpha\beta}\,\delta_{\alpha'\beta'}\, A(s,t,u) + \delta_{\alpha\alpha'}\,\delta_{\beta\beta'}\, A(t,u,s) + \delta_{\alpha\beta'}\,\delta_{\beta\alpha'}\, A(u,s,t) \tag{28}$$

(see the Appendix for unexplained notation). To lowest order of the chiral low energy expansion, the invariant amplitude A is then given by Weinberg's formula [10]:

$$A(s,t,u) = \frac{1}{F_\pi^2}\,(s - m_\pi^2) \tag{29}$$

F_π = pion decay constant.

Recently, the next order has also been worked out [11], but, for simplicity, I shall stick to the lowest order expression. The forward amplitude is then easily calculated and eq. (27) becomes

$$\delta_\pi = \frac{3}{8\pi \ell}\,\frac{m_\pi^2}{F_\pi^2} \int_{-\infty}^{\infty} \frac{dp}{2\pi}\, e^{-\ell\sqrt{1+p^2}} + O(e^{-\alpha \ell}) \tag{30}$$

Inserting the experimental values m_π = 139 MeV, F_π = 93 MeV, we finally obtain the table

ℓ	δ_π
1.0	0.051
1.5	0.016
2.0	0.006
2.5	0.003

δ_π is thus roughly an order of magnitude smaller than the mass shift δ_o in the two-dimensional non-linear σ-model.

A modified form of eq. (15) also applies to the finite size mass shift δ_p of the proton, which is defined by

(31) $$\delta_p = (M_p - m_p)/m_p$$

(m_p is the physical proton mass and M_p the energy of the zero momentum proton state at finite L). The relevant scattering amplitude in this case is the pion-proton forward elastic scattering amplitude $F_{\pi p}(\nu)$, where, in the notation of the Appendix, the proton is the A-particle and the pion the B-particle. $F_{\pi p}(\nu)$ has poles at

(32) $$\nu = \pm \nu_B \,, \qquad \nu_B = \frac{m_\pi^2}{2 m_p} \,,$$

with residues

(33) $$\lim_{\nu \to \pm \nu_B} (\nu^2 - \nu_B^2)\, F_{\pi p}(\nu) = - 6\, g_{\pi N}^2\, \nu_B^2$$

$g_{\pi N}$: pion-nucleon coupling constant.

These poles come from nucleon exchange diagrams, and they give rise to the leading term in the proton mass shift formula

(34) $$\delta_p = \frac{9}{4}\left(\frac{m_\pi}{m_p}\right)^3 \frac{g_{\pi N}^2}{4\pi \xi}\, e^{-\xi\sqrt{1-\nu_B^2/m_\pi^2}} - \frac{3}{8\pi\xi}\left(\frac{m_\pi}{m_p}\right)^2 \int_{-\infty}^{\infty} \frac{dp}{2\pi}\, e^{-\xi\sqrt{1+p^2}}\, F_{\pi p}(i m_\pi p) + O(e^{-\alpha\xi})$$

(as before $\xi = m_\pi L$). Note, however, that for reasonable values of ξ, both terms are of the same order of magnitude, because $\nu_B^2/m_\pi^2 \ll 1$. The first term only dominates for academically large ξ.

The experimental pion-nucleon scattering data were fitted by Höhler et al. [12], and, using dispersion relations, they were able to determine the pion-nucleon coupling constant $g_{\pi N}$ and the forward scattering amplitude near $\nu = 0$ *. To display their results, we first subtract the pseudo-vector Born term from the scattering amplitude,

(35) $$F_{\pi p}(\nu) = 6\, g_{\pi N}^2\, \nu_B^2 / (\nu_B^2 - \nu^2) + R(\nu),$$

and expand the remainder $R(\nu)$ in a convergent power series:

(36) $$R(\nu) = \sum_{k=0}^{\infty} r_k\, (\nu/m_\pi)^{2k} \qquad (|\nu| < m_\pi).$$

From the analysis of Höhler et al., we then have

(37a) $$g_{\pi N}^2/4\pi = 14.3$$

(37b) $$r_0 = -60.7, \quad r_1 = 45.3, \quad r_2 = 8.1$$

(this order of magnitude of the coefficients r_i is also suggested from chiral perturbation theory [13]).

At first sight, $R(\nu)$ appears to be a rather large amplitude,

* $F_{\pi p}(\nu)$ is related to the amplitude $C^+(\nu)$ of Höhler et al. through $F_{\pi p}(\nu) = 6\, m_p\, C^+(\nu)$.

at least, when compared with $F_{\pi\pi}(0) \simeq -2$. In eq. (34), $R(\nu)$ is however multiplied by $(m_\pi/m_p)^2$ so that its contribution to the proton mass shift at say $\xi = 1$ comes out to be a few percent only. The same is true for the other two contributions, which, furthermore, come with opposite sign and almost cancel each other. The conclusion then is that

$$0 < \delta_p < 0.05 \text{ for } \xi = 1$$

$$0 < \delta_p < 0.01 \text{ for } \xi \geq 1.5$$

i.e. the proton mass shift is even a little smaller than the pion mass shift.

The smallness of the pion mass has always been a potential source of difficulties in hadron mass calculations. Here we have seen that it is balanced, to some extent, by the weakness of the interactions of the pion at low energies. This property is a consequence of spontaneous chiral symmetry breaking and current algebra. Because of the unphysical anomalies of the axial currents in lattice QCD, we must therefore be careful, when carrying over the results obtained above. The prospects for calculating the spectrum on small lattices are otherwise rather good, since the condition $m_\pi L \geq 1$ is not very restrictive.

4.5 Low temperature quantum field theory

It is well-known that the canonical ensemble describing a given quantum field theory at temperature T can be realized in Euclidean space by letting the time coordinate become an angular variable with period $L = 1/k_B T$ (k_B: Boltzmann constant). From a mathematical point of view, the situation is then not very different from the one we have studied so far, except that time and space are interchanged and that now only one coordinate is

compactified. Also, the physical meaning of M_o is not so much that of an energy, but rather of an inverse correlation length characterizing the falling off of correlation functions in space-like directions. In the present context, eq. (15) remains valid provided the rhs is divided by d. It represents the leading term of a low temperature expansion of M_o.

In most cases we have considered, δ_o is positive. This means that the correlation length decreases as the temperature rises, a behaviour, which is typical for systems where the low temperature phase is either unbounded or bounded by first order transitions. Especially, a phase transition in QCD in the region $0 \leq k_B T \leq m_\pi$ seems rather unlikely, because M_π and M_p are practically temperature independent there. In the pure Yang-Mills gauge theory, on the other hand, δ_o is negative and may eventually drop to -1 at some critical T. A phase transition is in fact known to exist in lattice gauge theories [14], although the order of the transition has not yet been pinned down.

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APPENDIX: Scattering amplitude notations

My conventions concerning (d+1)-vectors are:

$$p^\mu = (p^0, \vec{p}) \;, \quad \vec{p} = (p^1, \ldots, p^d)$$

$$p_o = p^o\ ,\quad p_i = -p^i \quad (i = 1,\dots,d)$$

$$p\cdot q = p_\mu q^\mu = p^o q^o - \vec{p}\cdot\vec{q}$$

One-particle states $|p\alpha\rangle$ with momentum p and quantum numbers α (spin, isospin) are always normalized such that

$$\langle p'\alpha' | p\alpha\rangle = \delta_{\alpha'\alpha}\, 2p^o\, (2\pi)^d\, \delta(\vec{p}\,' - \vec{p})$$

The T-matrix for elastic scattering A+B → A+B is defined by

$$\langle p'\alpha', q'\beta'\ \mathrm{out}\,|\, p\alpha, q\beta\ \mathrm{in}\rangle =$$

$$\delta_{fi} + i(2\pi)^{d+1}\,\delta(p'+q'-p-q)\, T(p'\alpha', q'\beta'\,|\,p\alpha, q\beta),$$

where $p\alpha$ and $p'\alpha'$ refer to particle A and

$$\delta_{fi} = \langle p'\alpha', q'\beta'\ \mathrm{in}\,|\,p\alpha, q\beta\ \mathrm{in}\rangle.$$

For spinless particles, we also write

$$T(p'\alpha', q'\beta'\,|\,p\alpha, q\beta) = T_{\alpha'\beta',\alpha\beta}(s,t,u)$$

$$s = (p+q)^2,\quad t = (q'-q)^2,\quad u = (q'-p)^2$$

The forward amplitude F, which enters the formula for δ_o, is defined through

$$F = \sum_\beta T(p\alpha, q\beta\,|\,p\alpha, q\beta).$$

In all cases considered here, F is a Lorentz scalar depending only on the "crossing variable"

$$\nu = \frac{s-u}{4m_A}\ ,\quad m_A\text{: mass of particle A.}$$

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