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Longitudinal and Transverse Wake Fields in Flat Vacuum Chambers

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Abstract: Longitudinal and transverse wake fields are investigated which are excited by a bunch between two parallel plates representing the flat vacuum chamber in the bending magnets. The walls have a finite conductivity and a thin dielectric layer. Impedances, bunch lengthening, power losses and shifts of synchrotron and betatron frequencies are calculated.

1. Introduction and Summary.

A shift of the coherent betatron frequencies is observed in PETRA [1,2,3]) which has changed several times since the first measurements in 1979. The coherent betatron frequencies, which are determined by resonance excitation, are always decreasing with increasing bunch current (not with total current), whereas the incoherent frequencies, which are determined with help of satellite resonances, do not vary with currents [2]).

In particular, the measurements showed that the coherent vertical tune shift is always larger than the coherent horizontal tune shift. With a small number of cavities (4 to 15) the vertical tune shift was larger by nearly one order of magnitude whereas with a large number of cavities (up to 136) the ratio was reduced to about a factor of two.

The cavities certainly contribute to the coherent tune shift, but the cavities are round and their contributions to the horizontal and vertical frequency shifts should be equal, or more precisely, proportional to the horizontal and vertical amplitude functions in the cavity sections, which are very similar in most cases. A variation of the amplitude function and the increase of the number of cavities during the last five years have not shown a clear dependence of the tune shift on these parameters. But it is difficult to determine the dependence on the amplitude function since the amplitude function cannot be varied only in a short section; it is changed also at other positions, especially if the total tune is kept constant in order to avoid resonances.

The separating plates, which also should contribute to the tune shift, have not shown a significant influence, i.e. a variation of the betas did not show a considerable change of the coherent tune shift. The contribution most recently investigated comes from the shieldings of the bellows [4]). They give a vertical contribution, but they also explain only a part of the observed tune shift.

The following investigation shows the contribution of the flat aluminium vacuum chamber in the arcs. The stainless steel chamber in the straight sections has a circular and larger cross section and about half the length of the flat chamber. The contribution of the round chamber is smaller and about the same for the horizontal and vertical direction.

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The calculation gives for the vertical coherent tune shift $\Delta Q_{coh,v}$ of a bunch between two parallel plates with the finite conductivity σ and with an oxide layer of thickness Δh the expression

$$\Delta Q_{coh,v} = -\frac{r_e N_b \beta_v L}{2\gamma h_p^3} \left(\frac{\Gamma(1/4)}{\sqrt{2 Z_o \sigma \sigma_s}} + \frac{\sqrt{\pi}(\epsilon_r - 1)\Delta h}{\epsilon_r \sigma_s} \right) \tag{1}$$

with r_e = classical particle radius, N_b = number of particles per bunch, β_v = average vertical amplitude function in the aluminium chamber, L = chamber length, γ = relative particle energy,, $\Gamma(1/4) = 3.6256...$, h_p = distance between the plates, $Z_o = 377\,Ohm$, σ_s = standard deviation of longitudinal Gaussian distribution, ϵ_r = relative dielectric constant.

The finite conductivity ($\sigma(Al\,Mg\,Si) = 28m/Ohm/mm^2$) of the $1760\,m$ long aluminium chamber gives at 7 GeV with $\beta_v = 22\,m$, $\sigma_s = 5.3\,mm$ ($f_s = 12\,kHz$) and $h_p = 56\,mm$ a tune shift of $-0.0004/mA$ or $-0.05\,kHz/mA$, i.e. about 10% of the total observed tune shift. However, if the aluminium surface is rough the effective conductivity is reduced and the contribution to the impedance becomes larger.

If one assumes that the aluminium chamber has an oxide layer with a thickness of e.g. $1\,\mu m$ one obtains with $\epsilon_r = 6$ for the coherent vertical tune shift $-0.0003/mA$ or $-0.04\,kHz/mA$, which is about the same as the contribution of the finite conductivity. To determine quantitatively the contribution of the oxide layer, precise measurements of the layer thickness in the PETRA aluminium chamber are required.

There are, in addition, ceramic vacuum chambers for the injection kickers and current monitors. These ceramic chambers have the same cross section as the aluminium chambers and they have a coating of stainless steel with a thickness of about $1\mu m$, which is sufficient to shield the electromagnetic field of the bunch[5,6]. Without the coating these chambers with a total length of 4.5m and a ceramic thickness of 5mm have approximately the same effect as an oxide layer with a thickness of $15\mu m$. Therefore, the ceramic chambers can also increase the tune shift, if the metallic layer is partly lost or destroyed.

2. Calculation of the Longitudinal and Transverse Wake Fields.

2.1 Self-field of the bunch

As a model for the vacuum chamber in the bending magnets we consider two infinite parallel plates. In the rectangular coordinate system $\{x, y, s\}$ the metallic plates are placed at $y = \pm h$, and the dielectric layer has a thickness of Δh (see Fig. 1). The line current with a Gaussian distribution in the longitudinal direction has at $x = 0$ and $y = h_o$ the velocity v parallel to the s-axis. The space charge density of the bunch can be described by

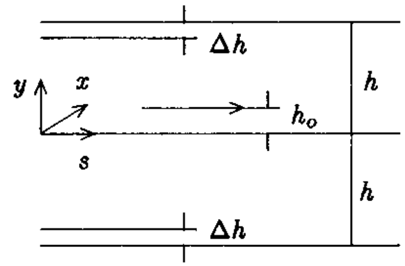

Fig.1

$$\rho(x,y,s) = \frac{eN_b}{\sqrt{2\pi}\sigma_s} \delta(x)\,\delta(y - h_o)\,\exp\left\{-\frac{(s - vt)^2}{2\sigma_s^2}\right\} \tag{2}$$

where e is the elementary charge and δ is the Dirac function. A decomposition into harmonic waves in x and s gives

$$\rho(x,y,s) = \delta(y - h_o) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{\rho}(\omega)\, \cos(qx)\, \exp\left\{ i\omega(\frac{s}{v} - t)\right\} dq\, d\omega \tag{3}$$

with

$$\tilde{\rho}(\omega) = \frac{eN_b}{4\pi^2 v} \exp\left\{-\frac{\omega^2 \sigma_s^2}{2v^2}\right\} \tag{4}$$

The electromagnetic field can be represented for each ω and each q with help of the two vectors

$$\vec{e}_1(a) = \{\, \cos(qx), -\frac{q}{a}\sin(qx),\ 0\,\}\ \exp\{-ay + i(ks - \omega t)\} \tag{5}$$

$$\vec{e}_2(a) = \{\ \sin(qx),\ \ \frac{q}{a}\cos(qx)\ ,\ 0\ \}\ \exp\{-ay + i(ks - \omega t)\} \tag{6}$$

with

$$a^2 = q^2 + \frac{k^2}{\gamma^2} \tag{7}$$

$$k = \frac{\omega}{v}$$

$$\beta = \frac{v}{c}, \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

where c is the velocity of light. The vectors $\vec{e}_{1,2}$ satisfy the equation

$$curl\ curl\ \vec{e}_{1,2}(a) = (q^2 + k^2 - a^2)\ \vec{e}_{1,2}(a) \tag{8}$$

The self-field of the bunch can be represented in the two regions above and below $y = h_o$ by

$$\pm h_o\ <\ \pm y$$

$$\tilde{\vec{E}} = A_{1,2}\ curl\,\vec{e}_1(\pm a) \tag{9}$$

$$i\omega\tilde{\vec{B}} = \beta^2\,k^2\,A_{1,2}\ \vec{e}_1(\pm a) \tag{10}$$

The conditions of continuity for $\tilde{E}_x$, $\tilde{E}_s$, $\tilde{B}_y$ and $\tilde{B}_s$, and the conditions for $\tilde{E}_y$ giving the space charge density $\tilde{\rho}$ and for $\tilde{B}_x$ giving the current density $v\tilde{\rho}$ at $y = h_o$ yield

$$A_{1,2} = \pm\frac{\tilde{\rho}(\omega)}{2ik\epsilon_o}\exp\{\pm a h_o\} \tag{11}$$

where ϵ_o is the dielectric constant of vacuum.

Now we can calculate for each ω and for each q the field which is produced by the vacuum chamber by adding appropriate combinations of $\vec{e}_{1,2}$ and $curl\,\vec{e}_{1,2}$ and by satisfying the conditions of continuity of the field on the wall surface. The total field produced by the bunch is then given by

$$\left\{\vec{E}, \vec{B}\right\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\tilde{\vec{E}}, \tilde{\vec{B}}\right\} dq\,d\omega \tag{12}$$

2.2) Wake fields due to the finite conductivity of the chamber wall.

In this case no dielectric layer is assumed, and the field between the plates can be be written in the form

$$\pm h_o\ <\ \pm y\ <\ h$$

$$\begin{aligned}\tilde{\vec{E}} = &(C_1 + C_2)\,\vec{e}_2(a)\ +\ (C_1 - C_2)\,\vec{e}_2(-a)\\ &+ curl\ (A_{1,2}\,\vec{e}_1(\pm a)\ + (G_1 + G_2)\,\vec{e}_1(a)\ +\ (G_1 - G_2)\,\vec{e}_1(-a))\end{aligned} \tag{13}$$

$$\begin{aligned}i\omega\tilde{\vec{B}} = &curl\ ((C_1 + C_2)\,\vec{e}_2(a)\ +\ (C_1 - C_2)\,\vec{e}_2(-a))\\ &+ \beta^2 k^2\ (A_{1,2}\,\vec{e}_1(\pm a)\ +\ (G_1 + G_2)\,\vec{e}_1(a)\ +\ (G_1 - G_2)\,\vec{e}_1(-a))\end{aligned} \tag{14}$$

The field within the chamber wall may be written as

$$h\ <\ \pm y$$

$$\tilde{\vec{E}} = (P_1 \pm P_2)\ \vec{e}_2(\pm p)\ e^{ph}\ +\ curl\ \left((Q_1 \pm Q_2)\ \vec{e}_1(\pm p)\ e^{ph}\right) \tag{15}$$

$$i\omega\tilde{\vec{B}} =\ curl\ \left((P_1 \pm P_2)\vec{e}_2(\pm p)e^{ph}\right) + (k^2 + i\omega\mu\sigma)(Q_1 \pm Q_2)\ \vec{e}_1 k(\pm p)\ e^{ph} \tag{16}$$

with

$$p^2 = q^2 + k^2/\gamma^2 - ikv\mu\sigma$$

and

$$Re\{p\} > 0$$

where μ is the permeability and σ is the conductivity (The definition of σ is given by Maxwells Equation: $curl\,\vec{B}/\mu = \epsilon\,\partial\vec{E}/\partial t + \sigma\vec{E}$). Here we have assumed that the thickness of the chamber wall is large as compared to the skin depth

$$d_s = \sqrt{\frac{2}{|\omega|\,\mu\sigma}}$$

of almost all freqencies of the bunch spectrum.

The conditions of continuity for $\tilde{E}_x$, $\tilde{E}_s$, $\tilde{B}_x/\mu$, $\tilde{B}_y$ and $\tilde{B}_s/\mu$ at $y = \pm h$ give the 8 equations

$$C_{1,2}S_{1,2} + ik\frac{q}{a}(G_{2,1}S_{1,2} + T_{1,2}) = P_{1,2} + ik\frac{q}{p}Q_{2,1} \quad (17)$$

$$G_{2,1}S_{1,2} + T_{1,2} = \frac{a}{p}\frac{p^2 - q^2}{a^2 - q^2}\,Q_{2,1} \quad (18)$$

$$C_{1,2}S_{2,1} + ik\beta^2\frac{a}{q}(G_{2,1}S_{2,1} + T_{1,2}) = \frac{a}{p}P_{1,2} + \frac{ia}{kq}(k^2 + q^2 - p^2)Q_{2,1} \quad (19)$$

$$C_{1,2}S_{2,1} = \frac{a}{p}\frac{p^2 - q^2}{a^2 - q^2}P_{1,2} \quad (20)$$

with

$$S_{1,2} = e^{-ah} \pm e^{ah}$$

$$T_{1,2} = \tfrac{1}{2}(A_1 \mp A_2)\,e^{-ah}$$

The longitudinal force at $x = 0$ and $y = h_o$ is given by

$$\begin{aligned}\tilde{F}_s &= e\,\tilde{E}_s \\ &= \frac{2\,e\,k^2}{a\,\gamma^2}\,(G_2\,cosh\,ah_o - G_1\,sinh\,ah_o)\,\exp\{i(ks - \omega t)\}\end{aligned} \quad (21)$$

and the vertical force at $x = 0$, $y = h_o$ is given by

$$\begin{aligned}\tilde{F}_y &= e(\,\tilde{E}_y + v\,\tilde{B}_x) \\ &= \frac{2\,i\,e\,k}{\gamma^2}(\,G_1 cosh\,ah_p - G_2\,sinh\,ah_o)\,\exp\{i(ks - \omega t)\}\end{aligned} \quad (22)$$

In both cases the forces due to the self-field, which are very small, are neglected. It is sufficient to solve the Eqs.(17) to (20) only for G_1 and G_2:

$$G_{2,1} = -\frac{T_{1,2}}{S_{1,2}^2}\left(\pm 2\gamma^2\frac{q}{p}e^{qh} + S_{1,2}\right) \quad (23)$$

Here we have assumed that the particle energy is large and that the skin depth is small as compared to the bunch length and as compared to the distance of the plates, i.e.

$$\gamma^2 \gg 1$$

$$|p| \gg k \approx \frac{1}{\sigma_s} \qquad |p| \gg q \approx \frac{1}{h}$$

with

$$p \approx \frac{1 \mp i}{d_s} \qquad for \pm \omega > 0 \quad (24)$$

The second term in Eq.(23) is due to the image charge on the wall surface. For large electron or positron energies ($E \approx 5GeV$) and conductivities of $\sigma \approx 1m/\Omega mm^2$ it is smaller than the first term by more than four orders of magnitude and will, therefore, be neglected in the following calculation.

The longitudinal and the vertical forces are then given by

$$\tilde{F}_s = \frac{i\,e\,\tilde{\rho}\,k}{2\,\epsilon_o\,p}\left(\frac{cosh^2 qh_o}{cosh^2 qh} + \frac{sinh^2 qh_o}{sinh^2 qh}\right)\exp\{i(ks-\omega t)\} \qquad (25)$$

$$\tilde{F}_y = \frac{e\,\tilde{\rho}\,q}{\epsilon_o\,p}\frac{cosh\,2qh}{sinh^2 qh}\,sinh\,2qh_o\,\exp\{i(ks-\omega t)\} \qquad (26)$$

The longitudinal force $\tilde{F}_s$ varies quadratically with h_o, and it is sufficient to calculate $\tilde{F}_s$ only for $h_o = 0$. The vertical force $\tilde{F}_y$ vanishes linearly with h_o, and it is sufficient to calculate the derivative $\partial\tilde{F}_y/\partial h_o$ at $h_o = 0$. With Eqs.(12) and (24) one obtains finally after several transformations:

$$F_s = \frac{r_e E_o N_b}{2h\sqrt{2Z_o\sigma}}\left|\frac{u}{\sigma_s}\right|^{3/2}\left(I_{1/4} - I_{-3/4} \mp I_{-1/4} \pm I_{3/4}\right)\exp\left\{-\frac{u^2}{4}\right\} \qquad (27)$$

$$\frac{\partial F_y}{\partial h_o} = \frac{\pi^2 r_e E_o N_b}{4h^3\sqrt{2Z_o\sigma}}\left|\frac{u}{\sigma_s}\right|^{1/2}\left(I_{-1/4} \mp I_{1/4}\right)\exp\left\{-\frac{u^2}{4}\right\} \qquad (28)$$

$$for \qquad \pm u > 0$$

with

$$u = \frac{s - vt}{\sigma_s} \qquad (29)$$

E_o is the rest energy of the electron, and the argument of the Besselfunctions $I_{\pm 1/4}$ and $I_{\pm 3/4}$ is $u^2/4$. The longitudinal and the transverse wake field forces are plotted in Figs.2a and 2b.

Eqs.(27) and (28) give immediately a connection between the longitudinal and the transverse wake field forces:

$$F_s = \frac{4h^2}{\pi^2}\frac{\partial}{\partial s}\frac{\partial F_y}{\partial h_o} \qquad (30)$$

The longitudinal force is exactly the same as calculated in [5,6]) for a round metallic vacuum chamber, if we replace h by the radius of the vacuum chamber. Therefore, also the power losses will be the same, which are obtained by multiplying the longitudinal force (Eq.(27)) with the charge density (Eq.(2)) and the particle velocity and by integrating the product with respect to s. The result is

$$P = \frac{\Gamma(3/4) r_e E_o N_b^2 v}{\pi h \sigma_s^{3/2}\sqrt{2Z_o\sigma}} \qquad (31)$$

where Γ is the Gammafunction ($\Gamma(3/4) = 1.2254...$).

2.3) Wake fields due to a thin dielectric layer on the chamber wall

In this case we assume an infinite conductivity of the chamber wall. The field inside the chamber can again be represented by Eqs.(13) and (14). In the dielectric material the field can be written in the form:

$$h - \Delta h < \pm y < h$$

$$\begin{aligned}\tilde{\vec{E}} &= (K_1 \pm K_2)\vec{e}_2(\pm b)\,e^{bh} + (L_1 \pm L_2)\vec{e}_2(\mp b)\,e^{-bh} \\ &\quad + curl\left((M_1 \pm M_2)\,\vec{e}_1(\pm b)\,e^{bh} + (N_1 \pm N_2)\,\vec{e}_1(\mp b)\,e^{-bh}\right)\end{aligned} \qquad (32)$$

$$\begin{aligned}i\omega\tilde{\vec{B}} &= curl\left((K_1 \pm K_2)\,\vec{e}_2(\pm b)\,e^{bh} + (L_1 \pm L_2)\,\vec{e}_2(\mp b)\,e^{-bh}\right) \\ &\quad + \epsilon_r\mu_r\beta^2 k^2\left((M_1 \pm M_2)\,\vec{e}_1(\pm b)\,e^{bh} + (N_1 \pm N_2)\,\vec{e}_1(\mp b)\,e^{-bh}\right)\end{aligned} \qquad (33)$$

with

$$b^2 = q^2 + k^2(1-\beta^2\epsilon_r\mu_r)$$

where ϵ_r and μ_r are the relative dielectric constant and the relative permeability, respectively. The conditions of continuity for $\tilde{E}_x$, $\epsilon\tilde{E}_y$, $\tilde{E}_s$, $\tilde{B}_x/\mu$, $\tilde{B}_y$ and $\tilde{B}_s/\mu$ yield the four equations:

$$y = \pm h \mp \Delta h$$

$$\begin{aligned} C_{1,2}S_{1,2} + ik\frac{q}{a}(G_{2,1}S_{1,2}+T_{1,2}) &= K_{1,2}\,e^{b\Delta h} + L_{1,2}\,e^{-b\Delta h} \\ &\quad + ik\frac{q}{b}(M_{2,1}\,e^{b\Delta h} - N_{2,1}\,e^{-b\Delta h}) \end{aligned} \tag{34}$$

$$\begin{aligned} C_{1,2}S_{2,1} + ik\frac{a}{q}(G_{2,1}S_{2,1}+T_{1,2}) &= \epsilon_r\frac{a}{b}(K_{1,2}\,e^{b\Delta h} - L_{1,2}\,e^{-b\Delta h}) \\ &\quad + ik\epsilon_r\frac{a}{q}(M_{2,1}\,e^{b\Delta h} + N_{2,1}\,e^{-b\Delta h}) \end{aligned} \tag{35}$$

$$G_{2,1}S_{1,2} + T_{1,2} = \frac{a}{b}\frac{b^2-q^2}{a^2-q^2}(M_{2,1}\,e^{b\Delta h} - N_{2,1}e^{-b\Delta h}) \tag{36}$$

$$C_{1,2}S_{2,1} = \frac{a}{b\mu_r}\frac{b^2-q^2}{a^2-q^2}(K_{1,2}\,e^{b\Delta h} - L_{1,2}e^{-b\Delta h}) \tag{37}$$

The conditions at $y=\pm h$ for vanishing $\tilde{E}_x$, $\tilde{E}_s$, $\tilde{B}_y$ yield the two equations:

$$K_{1,2} + L_{1,2} + ik\frac{q}{b}(M_{2,1} - N_{2,1}) = 0 \tag{38}$$

$$M_{1,2} - N_{1,2} = 0 \tag{39}$$

With Eqs.(34) to (39) one obtains for $G_{2,1}$

$$G_{2,1} = \left(\pm\frac{1-\epsilon_r\mu_r}{\epsilon_r S_{1,2}}\frac{a}{b}\,e^{ah} - 1\right)\frac{T_{1,2}}{S_{1,2}} \tag{40}$$

Here we have made the following approximations:

$$\gamma^2 \gg 1$$

$$\Delta h \ll \frac{1}{b} \approx \sigma_s\sqrt{\epsilon_r\mu_r - 1}, \qquad \Delta h \ll h$$

With these approximations $G_{2,1}$ has no poles as a function of ω or k. The exact calculation shows poles representing energy losses which are deposited in the layer. The poles also lead to a longitudinal shift of the field, which is, however, so small that it may be neglected completely. Neglecting also the term due to the image current (second term in Eq.(40)) we get, with Eqs.(21) and (22):

$$\tilde{F}_s = \frac{ie\tilde{\rho}(\epsilon_r\mu_r - 1)k\Delta h}{2\epsilon_o\epsilon_r}\left(\frac{cosh^2 qh_o}{cosh^2 qh} + \frac{sinh^2 qh_o}{sinh^2 qh}\right)\exp\{i(ks-\omega t)\} \tag{41}$$

$$\tilde{F}_y = \frac{e\tilde{\rho}(\epsilon_r\mu_r - 1)q\Delta h}{\epsilon_o\epsilon_r}\frac{cosh\,2qh}{sinh^2 2qh}\,sinh\,2qh_o\,\exp\{i(ks-\omega t)\} \tag{42}$$

Eqs.(41) and (42) show again that the longitudinal force varies quadratically with h_o and the transverse force vanishes linearly with h_o. The reason for this behaviour is, of course, the assumption that the layers on both plates have the same thickness. For $h_o = 0$ we obtain with Eq.(12) after integrating over q and ω for the forces produced by the bunch:

$$F_s = \sqrt{\frac{2}{\pi}}\,\frac{r_e N_b E_o(1-\epsilon_r\mu_r)\Delta h}{\epsilon_r\,h\,\sigma_s^2}\,u\exp\left\{-\frac{u^2}{2}\right\} \tag{43}$$

$$\frac{\partial F_y}{\partial h_o} = \left(\frac{\pi}{2}\right)^{\frac{3}{2}}\frac{r_e N_b E_o(\epsilon_r\mu_r - 1)\Delta h}{\epsilon_r h^3\sigma_s}\exp\left\{-\frac{u^2}{2}\right\} \tag{44}$$

where u is given by Eq.(29). The longitudinal distribution of $\partial F_y/\partial h_o$ is the same as the particle distribution, and the longitudinal distribution of

F_s is proportional to the derivative of $\partial F_y/\partial h_o$, i.e. Eq.(30) is satisfied also in this case. With the same parameters as before and with $5\,mA$ ($N_b = 2.4*10^{11}$) the maximum of F_s is $0.2\,keV/m$, and the maximum for $\partial F_y/\partial h_o$ is $5\,keV/m^2$.

3. Impedances.

Sometimes it is convenient to investigate instabilities with help of impedances. We can define a longitudinal impedance per unit length by [7])

$$Z_l' = -\frac{\tilde{U}_s}{L\,\tilde{I}} \qquad (45)$$

with

$$\frac{\tilde{U}_s}{L} = \int_{-\infty}^{\infty} \tilde{E}_s\,dq \qquad (46)$$

and

$$\tilde{I} = v\,\tilde{\rho} \qquad (47)$$

Here we have to take the field at $x = 0$, $y = h_o$ and $s = vt$. It is furthermore sufficient to consider only the case $y = h_o = 0$. With Eqs.(25) and (41) we obtain

$$Z_l' = \frac{1 \mp i}{h}\sqrt{\frac{|\omega|\,\mu}{2\sigma}} - \frac{i\omega Z_o(\epsilon_r\mu_r - 1)\Delta h}{h\epsilon_r v} \qquad for \pm\omega > 0 \qquad (48)$$

where the first term describes the finite conductivity and the second term the dielectric layer.

In a similar way we can define a transverse impedance according to [7]) by

$$Z_t' = \frac{1}{i}\,\frac{1}{L\tilde{I}}\,\frac{\partial \tilde{U}_y}{\partial h_o} \qquad (49)$$

with

$$\frac{1}{L}\frac{\partial \tilde{U}}{\partial h_o} = \frac{1}{e}\int_{-\infty}^{\infty} \frac{\partial \tilde{F}_y}{\partial h_o}\,dq \qquad (50)$$

and with $x = 0$, $y = h_o = 0$, $s = vt$, i.e. we consider here only the dipole mode. With Eqs.(26) and (42) we obtain

$$Z_t' = \frac{\pi^2}{4h^3}\left(\frac{\pm i - 1}{\sqrt{2\,|\omega|\,\epsilon_o\sigma}} + \frac{iZ_o(\epsilon_r\mu_r - 1)\Delta h}{\epsilon_r}\right) \qquad (51)$$

With Eqs.(48) and (51) one obtains a relation between the longitudinal and the transverse impedance:

$$Z_t' = -\frac{\pi^2 v}{4h^2\omega}Z_l' = -\frac{\pi^2}{4h^2 k}Z_l' \qquad (52)$$

4. Frequency Shifts and Bunch Lengthening.

4.1 Shift of the incoherent synchrotron frequency and bunch lengthening.

The longitudinal wake field produces an additional gradient for the synchrotron oscillation. Since this field travels with the bunch it causes a bunch lengthening and a shift of the incoherent synchrotron frequency, but no change of the coherent synchrotron frequency. We will determine the gradient at the center of the bunch and then calculate approximately the frequency shift and the bunch lengthening.

From Eqs.(27) and (43) follows the gradient due to the wake field

$$\frac{\partial E_{s,w}}{\partial s} = -\frac{r_e E_o N_b}{C} \oint \left(\frac{2^{1/4}}{2\Gamma(3/4)\sqrt{Z_o \sigma}\, h \sigma_s^{5/2}} + \sqrt{\frac{2}{\pi}} \frac{(\epsilon_r \mu_r - 1)\Delta h}{\epsilon_r h \sigma_s^3} \right) ds \tag{53}$$

The gradient due to the rf voltage is given by

$$\frac{\partial F_{s,rf}}{\partial s} = \frac{E Q_s^2}{\alpha_M R^2} \tag{54}$$

where α_M is the momentum compaction faktor and R is the mean machine radius. The bunch length is then given by

$$\frac{1}{\sigma_s^2} = \frac{1}{\sigma_E^2 \alpha_M E} \left(\frac{\partial F_{s,rf}}{\partial s} + \frac{\partial F_{s,w}}{\partial s} \right) \tag{55}$$

where σ_E is the relative energy spread. In order to determine the equilibrium bunch length we have to solve the following equation for σ_s:

$$\left(\frac{\sigma_{so}}{\sigma_s}\right)^2 = 1 + \left(\frac{\sigma_{so}}{\sigma_s}\right)^{5/2} \left(\frac{\partial F_{s,w}}{\partial s}\right)_{\sigma_s = \sigma_{so}} \Big/ \frac{\partial F_{s,rf}}{\partial s} \tag{56}$$

where we have considered only the contribution from the finite conductivity.

In an injection optics ($E = 7GeV, \alpha_M = 0.003$) on the central acceleration frequency with $Q_s = 0.088\,(f_s = 12kHz)$ the theoretical bunch length is $2\sigma_{so} = 10.6mm$. With $N_b = 2.4 * 10^{11}(\,I_b = 5mA)$, $\sigma = 28\,m/Ohm/mm^2$ and $h = 28\,mm$ one obtains an equilibrium bunch length of $11.4\,mm$, i.e. a bunch lengthening of 8%. With the maximum synchrotron frequency of $25\,kHz(\,Q_s = 0.19)$ the theoretical minimum bunch length is $2\sigma_{so} = 5\,mm$ and Eq.(56) gives an equilibrium bunch length of $5.6\,mm$, i.e. an increase of 12%. The incoherent synchrotron frequency is, at the same time, decreased by the same factor.

The measurement of the bunch length is, however, especially at such a short bunch length, not precise enough to show the change.

4.2 Shift of the coherent betatron frequency.

The dependence of the vertical force on the vertical bunch position causes a shift of the coherent vertical betatron frequency, but no shift of the incoherent frequency. The betatron tune shift is given by

$$\Delta Q_v = -\frac{1}{4\pi} \oint \beta\, K\, ds \tag{58}$$

where β_v is the vertical amplitude function and K is given by

$$K = \frac{1}{E} \frac{\partial F_y}{\partial h_o}$$

Since $\partial F_y / \partial h_o$ varies in longitudinal direction we have to average over the whole bunch:

$$\Delta Q_{coh,v} = -\frac{1}{4\pi\sqrt{2\pi}} \oint \beta_v \int_{-\infty}^{\infty} \frac{\partial F_y}{\partial y} \exp\left\{-\frac{u^2}{2}\right\} du\, ds \tag{59}$$

With Eqs.(28) and (44) one obtains:

$$\Delta Q_{coh,v} = -\frac{r_e N_b}{16\gamma} \oint \left(\frac{\Gamma(1/4)}{\sqrt{2 Z_o \sigma_s \sigma}} + \frac{\sqrt{\pi}(\epsilon_r \mu_r - 1)\Delta h}{\epsilon_r \sigma_s} \right) \frac{\beta_v}{h^3} ds \tag{60}$$

with $\Gamma(1/4) = 3.6256...$

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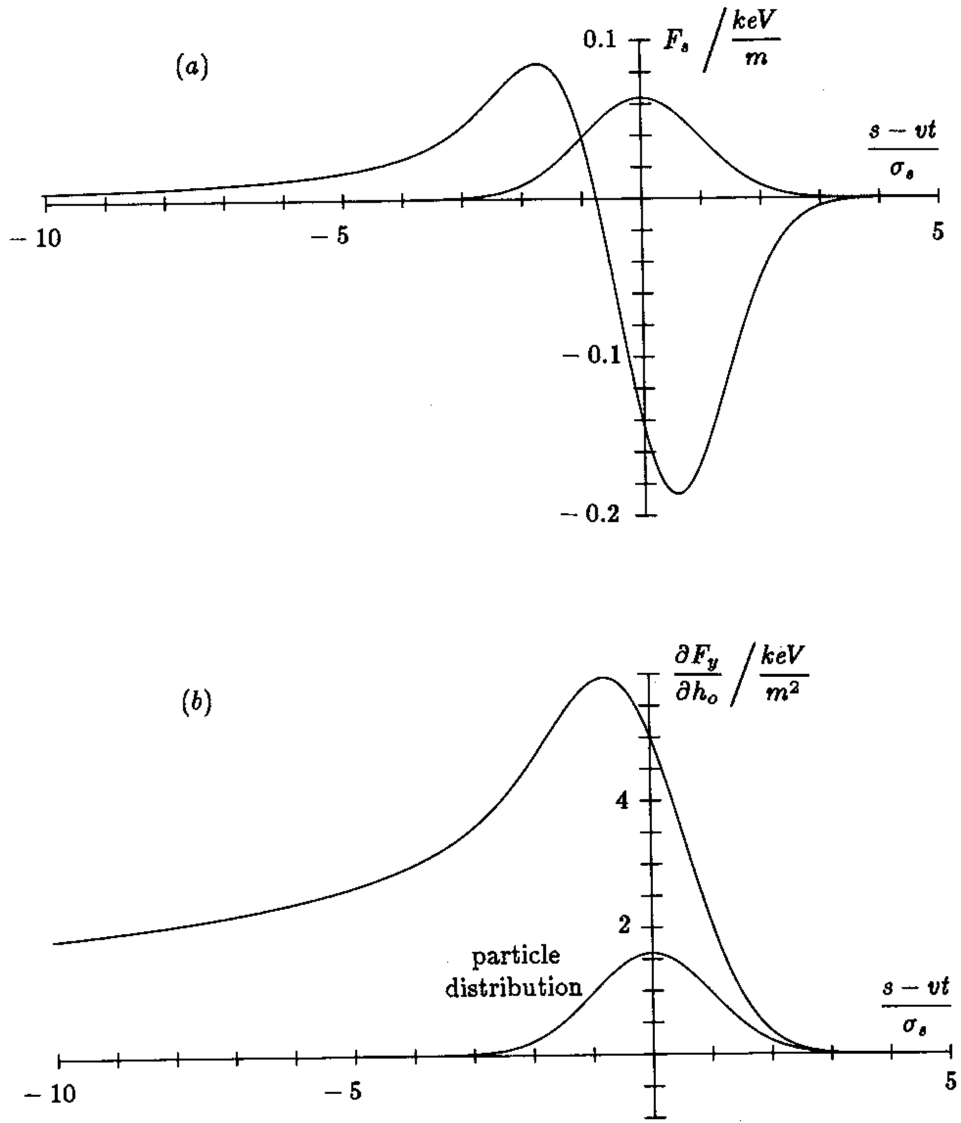

Fig.2 Longitudinal distribution of the longitudinal (a) and transverse (b) wake field forces ($N_b = 2.4 * 10^{11}$ ($I_b = 5mA$), $E = 7\,GeV$, $\sigma_s = 5.3\,mm$, $\sigma = 28\,m/Ohm/mm^2$, $h_p = 2h = 56\,mm$).