DEUTSCHES ELEKTRONEN-SYNCHROTRON DESY

DESY 84-119
December 1984

A STRONG COUPLING SIMULATION OF EUCLIDEAN QUANTUM GRAVITY

by

B. Berg

CERN, Geneva

and

II. Institut für Theoretische Physik, Universität Hamburg

ISSN 0418-9833

NOTKESTRASSE 85 · 2 HAMBURG 52

DESY 84-119
December 1984

ISSN 0418-9833

A STRONG COUPLING SIMULATION OF EUCLIDEAN QUANTUM GRAVITY

by

BERND BERG*)
CERN - Geneva
and
II. Institut für Theoretische Physik, Hamburg

ABSTRACT

Relying on Regge calculus a systematic numerical investigation of models of 4d Euclidean gravity is proposed. The scale $a = l_o$ is set by fixing the expectation value of a length. Possible universality of such models is discussed. The strong coupling limit is defined by taking Planck mass $m_p \rightarrow 0$ (in units of l_o^{-1}). The zero order approximation $m_p = 0$ is called "fluctuating space" and investigated numerically in two 4d models. Canonical dimensions are realized and both models give a negative expectation value for the scalar curvature density.

*) In part supported by DFG, contract BE 915/2-1.

Classical gravity can be formulated as field theory and in a geometric way. One may therefore regard quantum gravity as a problem of formulating a quantum field theory or one may try a geometric approach. The "or" is not exclusive. Most work is done in the first direction, as it allows to rely on the established calculus of perturbative field theory. At distances of the order of the Planck length violent fluctuations in geometry are expected and a perturbative quantum field theory does shed little light on the understanding of quantum space and quantum time. The a-priori starting point is always a smooth and flat space-time continuum. The present paper attempts an approach from a very opposite limit, namely a heavily fluctuating space-time.

In a geommetric approach to quantum gravity one could imagine to extend the functional integral of the field theory to

$$Z = \int \mathcal{D}[\text{space}] \left\{ \int \mathcal{D}[\text{fields}]\, e^{-\int_V d^4x \sqrt{g}\, \text{Action}(\text{space, fields})} \right\}. \quad (1)$$

"Space" stand for space-time manifold and may be discrete. In equation (1) we weight over some class of (curved) spaces and define on each space a conventional quantum field theory coupling gravity and matter fields. The metric tensor g and the curvature R, for example, would get their values from $\int \mathcal{D}[\text{space}]$, whereas fermion- and gauge fields etc. would get their values from $\int \mathcal{D}[\text{fields}]$. To make the functional integral convergent /1/ the theory is formulated in the Euclidean. Minkowskian results are conjectured to be obtained by analytic continuation.

Using Regge calculus /2/ I shall illustrate these remarks by working out a 4d Euclidean example. Let us consider a decomposition of a torus into N_p <u>pentahedra p</u>: (The torus implies periodic boundary conditions and a pentahedron is a 4-simplex connecting five sites by ten links.) To each link l a link length x_l is assigned and l is contained in a number of pentahedra. We now reasign all link length under the constraint that each pentahedron remains constructable in flat Euclidean space. This defines a <u>Regge sceleton</u> space. The interior of each pentahedron is considered to be flat. Curvature is concentrated on <u>triangles</u> (= 2-simplexes) t and involves <u>deficit angles</u> α_t (which can be obtained by calculating the parallel transport around each triangle t). The Regge-Einstein action is given by

$$S_{RE} = \sum_t \alpha_t A_t \quad \left(A_t \text{ area of triangle } t, \; \alpha_t = 2\pi - \sum \text{angles}(t) \right) \quad (2a)$$

It goes over into the Einstein action

$$S_E = \int d^4x \sqrt{g}\ R \,, \tag{2.b}$$

when the Regge sceleton space tends to the corresponding continuum space and vice versa; see Ref. /2-4/.

Pure gravity is concerned with quantities, which have as dimensions powers of a length. For instance the total volume V, the Planck mass m_p (length l_p, $m_p = l_p^{-1}$) and the expectation values of the curvature densities. In the present paper I fix the scale by keeping the total volume finite and fixed. Vacuum expectation values are calculated with respect to the partition function

$$Z = \int_V \mathcal{D}[\text{space}]\ e^{+m_p^2\, S_{RE}} \,. \tag{3}$$

Here $\mathcal{D}[\text{space}]$ is a measure on a 4d Regge sceleton space. Weighted by the Boltzmann factor the link length are allowed to change within a range as determined by the pentahedra constraints. A numerical algorithm consists in proposing single new links and accepting or rejecting them according to the Metropolis procedure. Changing a single link length $x_l \rightarrow x_l'$ will in general change the volume $V \rightarrow V'$. The volume is kept constant by rescaling all links with the factor

$$\lambda = \sqrt[4]{\frac{V}{V'}} \,. \tag{4}$$

The Planck mass m_p is measured in system units and fixed to a constant. As the connection to Minkowskian gravity is not well understood one should consider the whole range $-\infty < m_p^2 < \infty$. The rescaling is consistent with detailed balance /5/ if and only if $\mathcal{D}[\text{space}]$ is scale invariant. Assuming $\mathcal{D}[\text{space}] = \prod_l d f(x_l) \prod_p F_\Theta(x_{p_1}, \ldots, x_{p_{10}})$ the measure is fixed to be

$$\mathcal{D}[\text{space}] = \prod_l d \ln x_l \prod_p F_\Theta(x_{p_1}, \ldots, x_{p_{10}}) \,. \tag{5}$$

Here $\prod_l$ goes over all links l and x_l is the length of link l. The product $\prod_p$ is over all pentahedra and the function $F_\Theta(x_1, \ldots, x_{10})$ takes care of the pentahedra constraints: $F_\Theta = 1$ if a flat pentahedron can be made from the (ordered) links of length $x_1, \ldots, x_{10}$ and otherwise $F_\Theta = 0$. Note: for small fluctuations around a fixed lenk length x_l^o the measure $d\ln x_l$ is equivalent to $x_l^{o-1} dx_l$.

Let us consider the limit of larger and larger volumes. At present it is not clear to me whether the Régge-Einstein action as used in equation (3) will lead to a finite action density or not. It is remarkable that the arguments of Ref. /1/, demonstrating the unboundedness of the Einstein action, do not apply directly to be present theory. The formulation of conformal transformations for the discrete system and their possible compatibility with my fixed volume constraint is not clear. The easiest way is to decide the question numerically. Even if the Regge-Einstein action is unbounded from below an exponentially decreasing entropy factor, coming from $\mathcal{D}[\text{space}]$, may save us.

We like to carry out Monte Carlo (MC) simulations. In general MC is good for getting a qualitative understanding. If precise quantitative results are needed, for instance of critical systems, MC has problems. In case of quantum gravity even simple qualitative questions are not well-understood and therefore MC seems to be the ideal method.

In lattice gauge theories it is convenient to express numerical results in units of the lattice spacing a. But in discrete gravity no lattice exists, because the link length are dynamical variables. A length scale in the spirit of a lattice spacing is best defined as an expectation value. Let v_p be the volume of pentahedron p. The expectation value of the volume of a pentahedron is

$$\langle V_p \rangle := \int \mathcal{D}[\text{space}] \frac{1}{N_p} \sum_p V_p \cdot e^{+m_p^2\, S_{RE}} \,. \tag{6}$$

The total volume V is kept fixed. Let $V = N_p v_o$, then we have configuration by

$$\langle V_p \rangle = v_o$$

and the system length scale is best defined as

$$l_o = \sqrt[4]{v_o} \tag{7}$$

We now like to calculate other expectation values in the limit $N_p \to \infty$ numerically. For this task one has to define a model. The only feasible way, I can see at present, is to define in flat space a Regge sceleton by a prescription of glueing links ar sites together, such that the torus becomes completely filled up by non-overlapping pentahedra. The model is defined by using the partition function (3) and integrating with the measure (5). Possible sceletons in flat space are the random lattice of Christ, Friedberg and Lee /6/, the hypercubic model as investigated by Rocek and Williams /7/ and a simplicial variant of the hypercubic model as introduced below.

A crucial question is of course: To what extent are physical results expected to be model dependent or universal. It has been emphasized by Feinberg et al. /3/ that the Regge-Einstein action (2.a) on any sceleton is invariant under general coordinate transformations. Therefore the present models of quantum gravity are all very satisfactory with respect to the essential invariance of general relativity. A non-universal feature is only the special prescription of glueing links at sites together. The random lattice /3/ is most natural. As it is (presently) constructed in flat space it is, however, still a very special initial configurations. A far reaching hope is that the details of glueing links at sites together may turn out to be irrelevant at all, because the links are dynamical variables. The invariance under general coordinate transformations is an argument in favour of this universality conjecture.

In any case one should avoid to take uncritically over possibly misleading concepts from lattice gauge theories. In my opinion discrete quantum gravity is very different. A main feature is that the coupling constant "β" has now dimension and is m_p^2. One should in the first goal take the models themselves seriously and explore the relation between Planck mass (length) and scalar curvature density. The physically interesting region is Planck length

$$l_p \ll \sqrt[4]{V}$$

with the subcases $l_p \gg l_o$ and $l_p \approx l_o$. For $l_p \approx l_o$ one would say that a fundamental length exists. $l_p \gg l_o$ reminds more on the conventional picture of lattice field theories and one may eventually think about a "scaling limit": $V \to \infty$ and then $l_o(l_p) \to 0$. One should be also aware of possible phase transtions from finite to infinite curvature density.

The outlined scenerious have to be explored systematically. In the present letter I present MC results for

$$m_p = 0. \tag{8}$$

In view of the partition function (3) this resembles the zero-order strong coupling limit of lattice gauge theories. On the other hand we have Planck length $l_p = \infty$. This resembles the spin wave limit, if we like the Planck length l_p to be proportional to a correlation length and illustrates that we have to be afraid about running into wrong analogies. I will call the system defined by partition function (3), measure (5) and Planck mass $m_p = 0$ "fluctuating space".

By equation (7) the length scale is introduced in a rather arbitrary way. This is satisfactory only if other ways are equivalent. Obvious possibilities are provided by the average link length

$$\langle L \rangle = \int \mathcal{D}[\text{space}] \frac{1}{N_l} \sum_l x_l \tag{9.a}$$

(N_l = number of links of the sceleton) and by the average area of a triangle

$$\langle A \rangle = \int \mathcal{D}[\text{space}] \frac{1}{N_t} \sum_t A_t . \tag{10.a}$$

Of course, also tetraeders may be used. For the simplicity of the approach it is very satisfying that indeed

$$\langle L \rangle = c_l \cdot l_o \tag{9.b}$$

and

$$\langle A \rangle = c_t \cdot l_o^2 \tag{10.b}$$

with small finite size effects for increasing N_p. Relations (9.b), (10.b) are not obvious, as one might expect Hausdorff dimensions as in the study of random surfaces.

Of major interest is the expectation value of the scalar curvature density

$$\left\langle \int d^4x \sqrt{g}\, R \Big/ \int d^4x \sqrt{g} \right\rangle$$

This is, up to a known proportionality constant, proportional to the expectation value of the Regge-Einstein action density

$$\langle S \rangle = \int \mathcal{D}[\text{space}] \frac{1}{N_t} \sum_t \alpha_t A_t \quad . \tag{11}$$

I now present the numerical results. By practical (computer) reasons I like to be able to handle the model using simple tables. To my knowledge the hypercubic model /4/, called model 1 henceforth, is the simplest 4d possibility. In this model one begins with a regular hypercubic lattice of

$$N = N_1 N_2 N_3 N_4 \tag{12}$$

sites. Each hypercubic is partitioned into $4! = 24$ pentahedra by drawing one appropriate diagonal for each square, cube and hypercube. At each site 30 links meet and we have to store 15 links per hypercubic in the computer memory. For these 15 links initial link length and number of pentahedra connected to each link are given in Table 1.

Model 2 is a simplicial variant of model 1. I omit the hypercube diagonal and add a new site at the center of the hypercube. The new site is connected by 16 new links with the sites at the corners of the hypercube. This partitions the hypercube into 48 pentahedra. The second model has two types of sites: Sites at which 44 links meet and new sites, where 16 links meet. We have to store 30 links per hypercube in the computer memory. The initial properties of these links are listed in Table 1. Two models are used to allow future checks of the universality conjecture.

The Monte Carlo (MC) calculation is done by scanning through all links of the lattice. For each link l a new link length $x_l' = e^{-\varepsilon} x_l$ is proposed, where ε is a uniformly distributed random number in the range $-0.4 \leq \varepsilon \leq 0.4$. The new link length is accepted if it is consistent with the (up to 36) pentahedra constraints. Otherwise it is rejected. Asymptotically we will sample configurations with the measure (5). Systems of $N = 2^4$ and $N = 2^3$ sites are used and measurements are performed after each sweep (a sweep is defined by applying the upgrading procedure once at each link).

The approach to equilibrium is (for model 1 and 3^4 sites) depicted in Figure 1. $\langle L_i \rangle$ (i = 1, 2, ...) is the restriction of the expectation value $\langle L \rangle$ (9.a) to subclasses of links as defined in Table 1. After about 50 sweeps equilibrium is reached and the system has completely lost any memory of the original configuration in flat space. The results are entirely due to the entropy of the measure (5). In all subclasses the link average is nearly identical and a negative average curvature has developed.

After omitting 200 sweeps for reaching equilibrium precise measurements were performed. They are collected in Table 2. Finite size effects ($2^4 \rightarrow 3^4$) are small and both models give rather similar results. Particularly interesting is the negative curvature due to entropy of fluctuating space. In physical units it is negative infinite, because the Planck mass m_p is zero. Whether the result has applications, for instance to a very tiny and hot universe (big bang, black holes) remains to be explored.

In conclusion the present investigation is a starting point for future work on Euclidean quantum gravity. The next step should consist in investigating the Regge-Einstein action under relaxation and MC simulation with a finite Planck mass. Later one may study an asymptotically free theory on a fluctuating lattice and finally couple it to gravity. In particular one would like to understand how flat space eventually comes out. The main idea is to get a qualitative understanding of possible quantum effect. It would already be very satisfying if the present our similar models could once play the role of an Ising model of quantum gravity.

Acknowledgements

I would like to thank T. Banks, A. Billoire, D. Gross, H. Kamenski and E. Marinari for useful discussions. Computer calculations were done at CERN/Geneva and at DESY/Hamburg, and it is a pleasure to acknowledge help by B. Baumann. Finally I am indebted to M. Jacob and the CERN theory group for their hind kospitality.

References

/1/ S.W. Hawking in S.W. Hawking and W. Israel (eds.): "General relativity", Cambridge University Press 1979.

/2/ T. Regge, Nuovo Cim. 19 (1961) 558.

/3/ R. Friedberg and T.D. Lee, Nucl. Phys. B242 (1984) 145;
G. Feinberg, R. Friedberg, T.D. Lee and H.C. Ren, Preprint, Columbia University, CU-TP-281.

/4/ H. Cheeger, W. Müller and R. Schrader, Commun. Math. Phys. 92 (1984) 405.

/5/ For a review see: K. Binder, in: Phase transition and critical phenomena, Vol. 5B, eds. C. Domb and M.S. Green, Academic Press, New York, 1976.

/6/ N.H. Christ, R. Friedberg and T.D. Lee, Nucl. Phys. B202 (1982) 89.

/7/ M. Rocek and R.M. Williams, Z. Phys. C21 (1984) 171.

	LINKS	MODEL 1		MODEL 2	
		x_1	#p	x_1	#p
1.	1. - 4.	1	24	1	36
2.	5. - 10.	$\sqrt{2}$	12	$\sqrt{2}$	16
3.	11. - 14.	$\sqrt{3}$	12	$\sqrt{3}$	12
4.	15.	2	24		
5.	15. - 16.			1	24
6.	17. - 24.			1	12
7.	25. - 30.			1	8

Table 1

Initial link length x_1 and number of connected pentahedra #p for links in model 1 and 2.

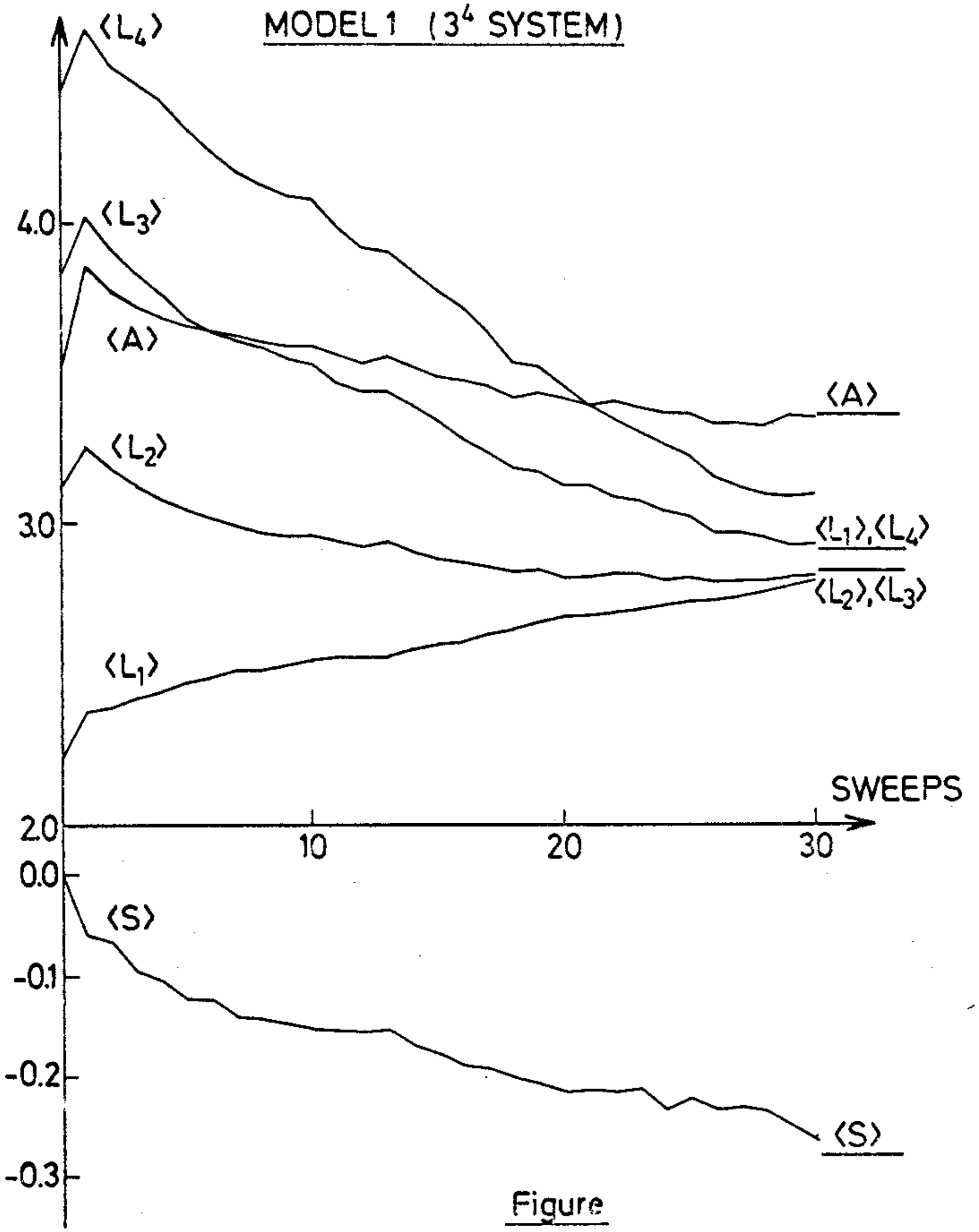
Figure

First 30 sweeps: Approach to equilibrium for various measured quantities (model 1, 3^4 system). Asymptotic averages are indicated by straight lines.

	MODEL 1, 2^4	MODEL 1, 3^4	MODEL 2, 2^4	MODEL 2, 3^4
STATISTICS	6 x 2000	5 x 400	8 x 1000	10 x 200
$\langle L \rangle$	2.8779 (17)	2.8651 (16)	2.8605 (16)	2.8557 (18)
$\langle A \rangle$	3.3814 (23)	3.3609 (23)	3.3571 (08)	3.3475 (11)
$\langle S \rangle$	$-.283 \pm .004$	$-.280 \pm .003$	$-.338 \pm .008$	$-.326 \pm .010$
$\langle L_1 \rangle$	2.9278 (10)	2.9133 (19)	2.9214 (51)	2.9142 (42)
$\langle L_2 \rangle$	2.8503 (34)	2.8417 (27)	2.8623 (59)	2.8570 (51)
$\langle L_3 \rangle$	2.8578 (28)	2.8396 (35)	2.8126 (71)	2.8049 (83)
$\langle L_4 \rangle$	2.9244 (56)	2.9148 (44)		
$\langle L_5 \rangle$			2.9306 (94)	2.9210 (82)
$\langle L_6 \rangle$			2.8831 (51)	2.8746 (70)
$\langle L_7 \rangle$			2.7965 (120)	2.8026 (95)

Table 2

Numerical results. The statistics is given in sweeps. Error bars are calculated with respect to the indicated number of bins. (The numbers in parenthesis are statistical errors in the last digits.)


MODEL 1 (3⁴ SYSTEM)
⟨L₄⟩
⟨L₃⟩
⟨A⟩
⟨L₂⟩
⟨L₁⟩
⟨S⟩
4.0
3.0
2.0
0.0
-0.1
-0.2
-0.3
10
20
30
SWEEPS
⟨A⟩
⟨L₁⟩,⟨L₄⟩
⟨L₂⟩,⟨L₃⟩
⟨S⟩

Figure