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COMMENT ON "PATH INTEGRAL SOLUTION FOR A MIE-TYPE POTENTIAL"

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Comment on "Path integral solution for a Mie-type potential"

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In a recent paper Erkoç and Sever¹ (hereafter referred to as ES) studied the Feynman path integral for a particle moving in the one-dimensional Mie-Lennard-Jones potential $((2\gamma, \gamma)$ -potential)^{2,3}

$$V(x) = \frac{V_0}{2\gamma} \left[\left(\frac{x}{a}\right)^{2\gamma} - 2 \left(\frac{x}{a}\right)^\gamma \right], \quad x > 0 \quad (1)$$

($V_0, \sigma, \gamma > 0$). For a particle described by the one-dimensional Hamiltonian $H = p^2/2 + V(x)$ ($\hbar = m = 1$) the time evolution from "state a" at time t_a to "state b" at time t_b is determined by the Feynman kernel K , which has the following phase-space path integral representation ($T = t_b - t_a > 0$, $\dot{x} = dx/dt$)⁴

$$K(T; x_b, x_a | V) = \int \mathcal{D}(x, p) \exp \left\{ i \int_{t_a}^{t_b} dt \left[p \dot{x} - \frac{p^2}{2} - V(x) \right] \right\} \quad (2)$$

with boundary conditions $x(t_a) = x_a, x(t_b) = x_b$. Applying the point canonical transformation

$$x = \sigma Q^{1/\gamma}, \quad p = \frac{\gamma}{\sigma} Q^{1-1/\gamma} P, \quad Q > 0 \quad (3)$$

and introducing the new path-dependent "time" s defined by

$$ds = \left(\frac{\gamma}{\sigma}\right)^2 Q^{-2-2/\gamma} dt \quad (4)$$

ES obtained the relation ($\dot{q} = dq/ds$)⁵

$$K^{(ES)}(T; x_b, x_a | V) = \frac{\sigma}{\gamma} \int_{-\infty}^{\infty} \frac{dE}{2\pi} e^{-iET} \int_0^{\infty} ds \left(Q_b^{1/\gamma-1} Q_a^{-1/\gamma-1} \right) \cdot \quad (5)$$

$$\cdot \int \mathcal{D}(Q, P) \exp \left\{ i \int_{t_a}^{t_b} ds \left[P \dot{Q} - \frac{P^2}{2} - \frac{V_0}{\gamma} \left(\frac{\sigma}{\gamma}\right)^2 \left(\frac{1}{2} Q^{2/\gamma-4} - Q^{-2/\gamma-3}\right) + E \left(\frac{\sigma}{\gamma}\right)^2 Q^{2/\gamma-2} \right] \right\}.$$

Here the path integration has to be carried out with the boundary conditions $Q(t_a) = Q_a = (x_a/\sigma)^\gamma, \quad Q(t_b) = Q_b = (x_b/\sigma)^\gamma$.

Abstract

We comment on several incorrect results given in a recent paper by Erkoç and Sever (ES). In particular, it is pointed out that their path integral formula for the one-dimensional Mie-Lennard-Jones potential is wrong, since a quantum correction proportional to \hbar^2 - which is a consequence of the stochastic nature of the Feynman paths - has been overlooked. The correct expression can be obtained from a general path integral formula, which we have derived in a previous paper. For the particular potential discussed in detail by ES, we give a complete path integral treatment, which allows us to derive the energies and normalized wave functions of the discrete spectrum.

Performing the trivial Gaussian path integration over the momentum p in eq. (2), we obtain for the one-dimensional kernel the configuration-space path integral representation

$$K(T; x_b, x_a | V) = \int_{x(t_a)=x_a}^{x(t_b)=x_b} \mathcal{D}x \exp \left\{ i \int_{t_a}^{t_b} dt \left[\frac{\dot{x}^2}{2} - V(x) \right] \right\}. \quad (6)$$

In a previous paper⁶ I derived a relation similar to eq. (5) for a class of nonlinear transformations (equivalent to the transformations (3), (4)) in the radial path integral. For three-dimensional quantum systems described by the Hamiltonian $H = \vec{p}^2/2 + V$ with spherically symmetric potentials $V = V(r)$, $r = |\vec{x}| > 0$, the Feynman kernel can be expanded into "partial waves" (see eq. (1) in Ref. 6), where the kernel K_{ℓ} (with fixed angular momentum ℓ) is given by the radial path integral ($r_a = |\vec{x}(t_a)|$, $r_b = |\vec{x}(t_b)|$)

$$K_{\ell}(T; r_b, r_a | V) = \int_{r(t_a)=r_a}^{r(t_b)=r_b} \mathcal{D}r \exp \left\{ i \int_{t_a}^{t_b} dt \left[\frac{\dot{r}^2}{2} - \frac{\ell(\ell+1)}{2r^2} - V(r) \right] \right\}. \quad (7)$$

Here the "measure" $\mathcal{D}r$ is identical to the "measure" $\mathcal{D}x$ of eq. (6), since the one-dimensional motion is restricted to the half-line $x > 0$. A comparison of eqs. (6) and (7) shows that the one-dimensional kernel K is identical to the S-wave ($\ell = 0$) radial kernel K_0 (with the trivial identifications $V(x) \rightarrow V(r)$, $(x_a, x_b) \rightarrow (r_a, r_b)$). This implies that relation (5) derived by ES must be a special case of our more general transformation formula derived in Ref. 6.

In this comment I would like to point out that relation (5) is incorrect for $\gamma \neq 1$, since a quantum correction proportional to \hbar^2 - which has to be added to the action in eq. (5) - has not been taken into account by ES. In Ref. 6 I derived a closed expression for this quantum correction from a careful treatment of the path integral using Feynman's time lattice subdivision process. The additional term is a direct consequence of the stochastic nature of the Feynman paths. The correct version of eq. (5) will be given below.

Since for general γ no path integral solution for the path integral in eq. (5) is known, ES considered the particular case $\gamma = 1$ as a check of relation (5). Fortunately enough, eq. (5) is correct for $\gamma = 1$, since the missing quantum correction vanishes in this case (being proportional to $(1 - \gamma)$). Unfortunately, ES rely in their evaluation on an incorrect identification of the eigenfunctions and of the quantum number n, such that the resulting energy eigenvalues and wave functions (eqs. (29-31) of ES) are wrong. Moreover, we shall see that relation (5) for $\gamma = 1$ is nothing else than a complicated rewriting of a simple scaling relation which relates the kernel K to itself. Thus relation (5) cannot be used in the case $\gamma = 1$ for a genuine path integral determination of the eigenfunctions and eigenvalues.

A complete path integral treatment of potential (1) with $\gamma = 1$ in the three-dimensional case has been carried out for arbitrary angular momentum in our recent paper⁷ on the path integral treatment of the hydrogen atom. Specializing to S-waves, we shall obtain the complete solution for the one-dimensional potential (1).

Let us start by casting the rather complicated expression (5), derived by ES, in a more compact form by following the procedure already used in our earlier paper⁶. For this purpose we introduce the time-independent Feynman kernel⁸

$$k(E; x_b, x_a | V) = i \int_0^{\infty} dt e^{iEt} K(T; x_b, x_a | V) \quad (8)$$

and the new potential

$$\hat{V}(Q) = A \left[\frac{1}{Q^a} - \frac{2}{Q^{a-1}} \right] - \frac{B}{Q^{a-2}} \quad (9)$$

$$a = 4 - \frac{2}{\gamma}, \quad A = \frac{\sigma^2 V_0}{2 \gamma^3}, \quad B = \frac{\sigma^2 E}{\gamma^2}.$$

We then obtain from (5) the equivalent relation

$$k^{(ES)}(E; x_b, x_a | V) = \frac{i}{\gamma} \sigma^2 (x_b x_a)^{\frac{1-\gamma}{2}} \int_0^{\infty} dS K(S; \frac{x_b}{\sigma}, \frac{x_a}{\sigma}) \left(\frac{x_b}{\sigma} \right)^{\gamma} \left(\frac{x_a}{\sigma} \right)^{\gamma} | \hat{V} \rangle, \quad (10)$$

which connects the time-independent kernel of the original system with potential $V(x)$ (given by (1)) to the time-dependent kernel of a new system with potential $\hat{V}(Q)$ (given by (9)).

Relation (10) can be compared with our transformation formula for the radial kernel (eq. (22) in Ref. 6)⁹

$$k_{\ell}(E; x_b, x_a | V) = \frac{i}{\gamma} \sigma^{\gamma}(x_b, x_a) \int_0^{1-\gamma} dS K_{\ell}(S; \frac{x_b}{\sigma}, \frac{x_a}{\sigma}) | W \rangle, \quad (11)$$

which is valid for transformations (3), (4) with $\sigma, \gamma > 0$. Eq. (11) connects the time-independent radial kernel k_{ℓ} (with angular momentum ℓ) of the original quantum system with arbitrary potential $V(x)$ to the time-dependent radial kernel K_{ℓ} of a new quantum system with the new potential

$$W(Q) = \left(\frac{\sigma}{\gamma}\right)^2 Q^{2/\gamma-2} [V(\sigma Q^{1/\gamma}) - E] \quad (12)$$

and with effective angular momentum

$$L = L(\ell, \gamma) = \frac{2\ell + 1 - \gamma}{2\gamma}. \quad (13)$$

As already stated after eq. (7), the kernel k_{ℓ} coincides for $\ell = 0$ with the one-dimensional kernel k which we are looking for. For $\ell = 0$ we get for the radial kernel on the r.h.s. of eq. (11)

$$K_L(S; \frac{x_b}{\sigma}, \frac{x_a}{\sigma}) | W \rangle = K(S; \frac{x_b}{\sigma}, \frac{x_a}{\sigma}) | W + \mathcal{U} \rangle \quad (14)$$

($\hat{L} = L(0, \gamma) = (1 - \gamma)/2\gamma$), where the quantum correction is given by¹⁰

$$\mathcal{U}(Q) = \frac{\hat{L}(\hat{L} + 1)}{2Q^2} = \frac{1 - \gamma^2}{8\gamma^2} \cdot \frac{1}{Q^2}, \quad (15)$$

and K on the r.h.s. of (14) denotes the one-dimensional Feynman kernel. We thus obtain from eq. (11) the transformation formula for the one-dimensional Feynman kernel on the half-line

$$k(E; x_b, x_a | V) = \frac{i}{\gamma} \sigma^{\gamma}(x_b, x_a) \int_0^{1-\gamma} dS K(S; \frac{x_b}{\sigma}, \frac{x_a}{\sigma}) | W + \mathcal{U} \rangle. \quad (16)$$

Relation (16) holds for an arbitrary potential $V(x)$, $x > 0$. For potential (1) we derive from eq. (12) $W(Q) = \hat{V}(Q)$, where \hat{V} is defined in eq. (9). We are then led to the relation

$$k(E; x_b, x_a | V) = \frac{i}{\gamma} \sigma^{\gamma}(x_b, x_a) \int_0^{1-\gamma} dS K(S; \frac{x_b}{\sigma}, \frac{x_a}{\sigma}) | \hat{V} + \mathcal{U} \rangle. \quad (17)$$

Eq. (17) represents the correct transformation formula for potential (1) in the case of transformations (3), (4).

A comparison of our relation (17) with relation (10), derived by ES, shows that the two relations are different due to the quantum correction \mathcal{U} , which is missing in relation (10). Since the potential $U(Q)$, eq. (15), does not vanish for $\gamma \neq 1$, we conclude that relation (10) is not correct for $\gamma \neq 1$, i.e. $k \neq k^{(ES)}$ for $\gamma \neq 1$.

Now let us consider the special value $\gamma = 1$. In this case the quantum correction \mathcal{U} vanishes identically, and relations (17) or (10) yield the transformation formula

$$k(E; x_b, x_a | V_1) = i \sigma \int_0^{\infty} dS K(S; \frac{x_b}{\sigma}, \frac{x_a}{\sigma}) | \hat{V}_1 \rangle. \quad (18)$$

Here the potentials V_1 and \hat{V}_1 are defined as follows

$$V_1(x) \equiv V(x) |_{\gamma=1} = \frac{\sigma^2 V_0}{2} \left[\frac{1}{x^2} - \frac{2}{\sigma x} \right], \quad x > 0, \quad (19)$$

$$\hat{V}_1(Q) \equiv \hat{V}(Q) |_{\gamma=1} = \frac{\sigma^2 V_0}{2} \left[\frac{1}{Q^2} - \frac{2}{Q} \right] - \sigma^2 E \quad (20)$$

$$\equiv V_1'(Q) - \sigma^2 E, \quad Q > 0.$$

Obviously, the new potential \hat{V}_1 is identical to the original potential V_1 - apart from trivial modifications. Thus nothing has been gained by applying transformations (3),

(4), since the kernel $K(\dots | \hat{V}_1)$ in eq. (18) has a path integral representation (2) or (6) which is as complicated as for the original kernel $K(\dots | V_1)$. Indeed, for $\gamma = 1$ relations (17) or (10) are essentially trivial identities. This is clear from transformations (3), (4), which for $\gamma = 1$ are mere scaling transformations

$$x = \sigma Q, \quad p = \frac{1}{\sigma} P, \quad ds = \frac{1}{\sigma^2} dt. \quad (21)$$

With (20) we obtain

$$K(s; \frac{x_1}{\sigma}, \frac{x_2}{\sigma} | \hat{V}_1) = e^{i\sigma^2 ES} K(s; \frac{x_1}{\sigma}, \frac{x_2}{\sigma} | V_1), \quad (22)$$

and relation (18) yields the scaling relation

$$k(E; x_0, x_1 | V_1) = \sigma k(\sigma^2 E; \frac{x_0}{\sigma}, \frac{x_1}{\sigma} | V_1). \quad (23)$$

(Here we used relation (8)). The inverse Fourier-transform of (8) gives for the time-dependent kernel

$$K(T; x_0, x_1 | V_1) = \frac{1}{\sigma} K(\frac{T}{\sigma^2}; \frac{x_0}{\sigma}, \frac{x_1}{\sigma} | V_1). \quad (24)$$

With the spectral decomposition¹¹

$$K(T; x_0, x_1 | V_1) = \sum_{n=1}^{\infty} \phi_n(x_0) \phi_n^*(x_1) e^{-iE_n T} \quad (25)$$

we obtain from (24) the scaling relations

$$E_n = \frac{1}{\sigma^2} E'_n \quad (26a)$$

$$\phi_n(x) = \frac{1}{\sqrt{\sigma}} \phi'_n(\frac{x}{\sigma}), \quad (26b)$$

where E'_n and $\phi'_n(Q)$ denote the eigenvalues and eigenfunctions, respectively, of the Hamiltonian $H'_1 \equiv p^2/2 + V'_1(Q)$. The scaling relations (26a,b) represent the essential content of eqs. (27a,b) of ES¹⁴.

If we rewrite potential V_1 , eq. (19), in the following form $(g \equiv \sigma^2 V_0/2, e \equiv (\sigma V_0)^{1/2})$

$$V_1(x) = \frac{g}{x^2} - \frac{e^2}{x}, \quad x > 0, \quad (27)$$

we observe from eq. (20) that potential $V'_1(Q)$ is obtained from $V_1(Q)$ by the simple transformation $g \rightarrow g, e \rightarrow \sigma^{1/2} e$, i.e. the two potentials differ only by the strength of the Coulomb potential. With $E_n \equiv f_n(g, e)$ this implies $E'_n = f_n(g, \sigma^{1/2} e)$, and we infer from (26a) $f_n(g, e) = \sigma^{-2} f_n(g, \sigma^{1/2} e)$, which yields for $\sigma = 1/e^2$

$$E_n = e^4 f_n(g, 1). \quad (28)$$

Similarly, we obtain with $\phi_n(x) \equiv g_n(x; g, e)$ and $\phi'_n(x) \equiv g_n(x; g, \sigma^{1/2} e)$ from (26b) $g_n(x; g, e) = \sigma^{-1/2} g_n(x/\sigma; g, \sigma^{1/2} e)$, which leads for $\sigma = 1/e^2$ to

$$\phi_n(x) = e g_n(e^2 x; g, 1). \quad (29)$$

Eq. (28) determines the complete e-dependence of the energy eigenvalues E_n , whereas eq. (29) tells us that - apart from a factor e in the normalization constant of the wave functions - the e-dependence of the eigenfunctions $\phi_n(x)$ enters only in the combination $\frac{1}{\sqrt{e}} \phi_n(x)$.

We have discussed in detail the consequences of the scaling relations (26a,b) in order to illustrate that these relations contain some useful information on the eigenvalue problem of the Hamiltonian H_1 . It is clear, however, that relations (26) do not allow to construct "a path integral solution for the Mie-type potential (27)" as claimed by ES, since a complete solution requires a full knowledge of E'_n and $\phi'_n(x)$ - but the latter are again solutions of the very same potential (27) with $g \rightarrow g, e \rightarrow \sigma^{1/2} e$! Of course, if one takes E'_n and $\phi'_n(x)$ from a solution of the Schrödinger equation - as done by ES -, one must obtain the correct results for E_n and $\phi_n(x)$. But one should be aware of the fact that the results obtained along these lines are nothing more than a reformulation of the scaling relations (28), (29). Unfortunately, the results given by ES (their

eqs. (29), (30) are wrong, because of several mistakes, which we shall enumerate at the end of this note. But before that we would like to point out that a genuine path integral treatment of potential $V_1(x)$ is possible and has already been presented by us in Ref. 7.

Actually, in Ref. 7 we treated the more general case of the radial kernel k_{ℓ} defined by the radial path integral (7) with potential $V_1(r)$. From our transformation formula (11) and eq. (13) we obtain for $\gamma = 1/2, \sigma = 1$

$$R_{\ell}(E; r_b, r_a | V_1) = 2i (r_b r_a)^{1/4} \int_0^{\infty} dS K_{2\ell+1/2}(S; \sqrt{r_b}, \sqrt{r_a} | W). \quad (30)$$

Eq. (12) gives for the new potential

$$W(Q) = \frac{4g}{Q^2} - 4e^2 + \frac{\Omega^2}{2} Q^2, \quad \Omega = 2\sqrt{-2E}, \quad (31)$$

which leads to a complete factorization of the dependence on the coupling constant e^2

$$K_{2\ell+1/2}(S; \sqrt{r_b}, \sqrt{r_a} | W) = e^{i4e^2 S} K_{\ell}^{osc}(S; \sqrt{r_b}, \sqrt{r_a}). \quad (32)$$

Here $\bar{\ell}$ is defined by $\bar{\ell}(\bar{\ell} + 1) = (2\ell + 1/2)(2\ell + 3/2) + 8g$, and OSC refers to a harmonic oscillator with frequency Ω . We thus obtain from (30) the relation

$$R_{\ell}(E; r_b, r_a | V_1) = 2 (r_b r_a)^{1/4} K_{\bar{\ell}}^{osc}(4e^2; \sqrt{r_b}, \sqrt{r_a}). \quad (33)$$

Since the radial path integral for the three-dimensional isotropic harmonic oscillator can be carried out, relation (33) offers a genuine path integral solution for potential $V_1(r)$. Inserting the path integral solution for the harmonic oscillator, we derived from eq. (33) in Ref. 7 the discrete spectrum

$$E_{\ell n_r} = - \frac{e^4}{2 [n_r + 1/2 + \mathcal{R}(\ell)]^2}$$

$$\chi_{\ell n_r}(r) = N_{\ell n_r} \int^{\mathcal{R}(\ell)+1/2} e^{-\rho/2} F(-n_r, 1+2\mathcal{R}(\ell); \rho)$$

$$N_{\ell n_r} = \frac{e}{n_r + 1/2 + \mathcal{R}(\ell)} \frac{1}{\Gamma(1+2\mathcal{R}(\ell))} \sqrt{\frac{\Gamma(n_r + 1 + 2\mathcal{R}(\ell))}{n_r!}} \quad (34)$$

$$\rho = \frac{2e^2 r}{n_r + 1/2 + \mathcal{R}(\ell)}$$

$$\mathcal{R}(\ell) = \sqrt{(\ell + 1/2)^2 + 2g}$$

($F(a, b; x)$ is the confluent hypergeometric function, and $n_r = 0, 1, 2, \dots$ denotes the radial quantum number). For the following discussion it is important to notice that $\chi(r)$ is the reduced radial wave function defined by $\chi(r) \equiv rR(r)$, if the radial wave function $R(r)$ is defined by the usual expression for the complete wave function in spherical coordinates, $\psi(r, \theta, \varphi) = R(r) Y_{\ell}^m(\theta, \varphi)$. $\chi_{\ell n_r}(r)$ is a solution of the reduced radial Schrödinger equation

$$\left[\frac{1}{2} \frac{d^2}{dr^2} + E_{\ell n_r} - \frac{\ell(\ell+1)}{2r^2} - V_1(r) \right] \chi_{\ell n_r}(r) = 0 \quad (35)$$

with boundary condition $\chi_{\ell n_r}(0) = 0$. From eq. (35) and the remarks after eq. (7) it is obvious, that the one-dimensional eigenvalue problem for potential $V_1(x)$ is a special case of the above problem. Putting $\ell = 0, n \equiv n_r + 1 = 1, 2, \dots$ we get

$$E_n = E_{0, n-1} \quad (36)$$

$$\phi_n(x) = \chi_{0, n-1}(x).$$

We then obtain from (34) the final result

$$E_n = - \frac{\sigma^2 V_0^2}{2 [n - 1/2 + \mathcal{R}]^2} \quad (37)$$

$$\phi_n(x) = N_n \sum_{j=1}^{n+1/2} e^{-x/2} F(-n+1, 1+2x; j; x)$$

$$N_n = \frac{\sqrt{\sigma V_0}}{n-1/2+x} \frac{1}{\Gamma(1+2x)} \sqrt{\frac{\Gamma(n+2x)}{(n-1)!}}$$

$$\sum = \frac{2\sigma V_0}{n-1/2+x} \times \quad (38)$$

$$x = x(0) = \frac{1}{2} \sqrt{1+4\sigma^2 V_0}$$

For the one-dimensional Feynman kernel on the half-line we derive from (33)

$$k(E; x_b, x_a | V_a) = 2(x_b x_a)^{1/4} k_{-1/2+2x}^{osc}(4\sigma V_0; \sqrt{x_b}, \sqrt{x_a}) \quad (39)$$

Various useful representations of the r.h.s. of (39) can be found in Ref. 7 and will not be given here. It should be clear, by now, why it is advantageous for a study of the one-dimensional path integral on the half-line to work with the more general radial path integral for arbitrary angular momentum.

Finally, let us comment on the results obtained by ES. A comparison between our eqs. (37-39) and eqs. (29-31) of ES shows that the results derived by ES are incorrect. In eq. (30) of ES we find the following expression for the energy eigenvalues ($\hbar = m = 1$)

$$E_n^{(ES)} = -\frac{\sigma^4 V_0^2}{2x^2} \quad (40)$$

The factor σ^4 in (40) seems to be a misprint, since in eq. (28) of ES we find the correct factor σ^2 in accordance with our result (37). However, the factor n^{-2} is wrong and should be replaced by the factor $[n - 1/2 + x]^{-2}$ of eq. (37). Without going into too much details, we give the following list of errata:

i) In eqs. (20), (22b) and (28) of ES there is a term $(2x + 1)^2$, which should be replaced by 1, since the one-dimensional problem corresponds to the S-wave case, $\ell = 0$, of the three-dimensional problem.

ii) The quantity n used by ES (see their eqs. (20), (21), (22c) and (28)) is not a natural number, but merely a convenient abbreviation, which is used in solving the Schrödinger equation. In the solution for the eigenfunctions, n enters the hypergeometric function in the form $F(-n + s + 1, \dots; \dots)$ (see eq. (21) of ES), where the quantity s is defined in eq. (22b) of ES. In order that the wave functions are normalizable, one must put $-n + s + 1 \equiv -n_r$ ($n_r = 0, 1, 2, \dots$), which leads to the correct wave functions given in eq. (34) above. The correct quantum number to count the energy levels is, therefore, $n - s = n_r + 1$, which is a natural number and has been named n by us (see eq. (36) above), but which is different from the quantity n of ES.

iii) ES identify the one-dimensional wave functions on the half-line, $\phi_n(x)$, with the radial wave function $R(x)$ rather than with the reduced wave function $\chi(x)$ as it should be done according to our eq. (36). Their wave functions show, therefore, a wrong "threshold behaviour" for $x \rightarrow 0$.

Footnotes and References

- 1 Ş. Erkoç and R. Sever, Phys. Rev. D30, 2117 (1984).
- 2 In ES χ was denoted by k . - It is understood that $V(x) = \infty$ for $x \ll 0$, and that the one-dimensional eigenfunctions satisfy the boundary condition $\phi_n(0) = 0$.
- 3 Potential (1) is used in the study of dynamical properties of rare gas solids with a preferred value of $\chi = 6$. See, for example: "Rare Gas Solids", ed. by M.L. Klein and J.A. Venables (Academic Press, London, 1976), Vol. 1, p. 32, 37 and 88. For $\chi = 1$, see ¹⁶.
- 4 We use the notation $\mathcal{D}(x,p)$ instead of $\frac{DxDP}{2\pi}$ for the phase-space path integral "measure".
- 5 In eq. (13) of ES $\chi^{(ES)}$ is denoted by \bar{k} to indicate that relation (5) has been obtained (in a rather heuristic fashion) by taking a kind of "arithmetic mean of the integrands". - In eq. (5) we made the substitution $E \rightarrow -E$ in order to agree with the standard definition (see eq. (8) below) of the time-independent Feynman kernel.
- 6 F. Steiner, Phys. Lett. 106A, 356 (1984).
- 7 F. Steiner, Phys. Lett. 106A, 363 (1984).
- 8 A small positive imaginary part has to be added to E .

- 9 In the notation of Ref. 6 we have $r = x$, $R = Q$, $f(r) = \sigma^{2\lambda} \chi^{-2} r^{2-2\lambda}$, $g(R) = \sigma R^{1/\lambda}$, $\nu = 2-2\lambda$. - In eq. (11) the radial kernel K_L is defined by the path integral (7), but with $\mathcal{L} \rightarrow L$.
- 10 In the three-dimensional case of the radial kernel, the potential U plays the rôle of an additional centrifugal barrier, which is present even for $\mathcal{L} = 0$.
- 11 Eq. (25) is valid for the discrete spectrum of the Hamiltonian $H_1 \equiv p^2/2 + V_1(x)$. Here (and in the following) we shall not discuss the continuous spectrum.
- 12 ES use the notation $\phi_n^1 = \psi_n$, $H_i^1 = H_{\text{eff}}^1$.
- 13 The scaling relations (26a,b) hold for any potential $V(x)$, where the quantities with a prime refer to a system described by the Hamiltonian $H' \equiv p^2/2 + \sigma^2 V(\sigma Q)$; for e.g. $V(x) = \lambda x^\beta$, $x > 0$, one obtains $E_n = \lambda^{2/(\beta+2)} a_n$, $\chi_n(x) = \lambda^{1/2(\beta+2)} b_n(\lambda^{1/(\beta+2)} x)$, where a_n and $b_n(x)$ are independent of λ . Scaling relations of this kind have been discussed by K. Symanzik (unpublished) and B. Simon, Ann. Phys. (NY) 58, 76 (1970).
- 14 Note that the argument of ϕ_n in eq. (27b) of ES must read x instead of $Q = x/\sigma$.
- 15 For the following considerations it is useful to regard g , e and σ as independent parameters.
- 16 Until recently, potential (27) did not seem to have any physical significance. In a recent paper¹⁷ it is argued that potentials of the type (27) (with g a function of \mathcal{L}) may be used to interrelate the spectra of different atoms and ions showing evidence for a phenomenological supersymmetry in atomic physics.
- 17 V.A. Kostelecký and M.M. Nieto, Phys. Rev. Lett. 53, 2285 (1984).