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ABSTRACT

We investigate a class of Abelian models which lead to a linear potential between opposite charges. For uniformly moving charges with opposite velocities, the electromagnetic fields and the confining domains are obtained asymptotically for large separations.

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In high-energy e^+e^- colliding beam acccelerators the majority of the hadronic events, the two-jet events, are naturally interpreted as due to $e^+e^- \to q\bar{q}$. The hadronization of the quark pair involves the creation of additional quark pairs from the color field and the recombination of the various quarks and antiquarks into primordial mesons and baryons [1].

As a first step in understanding such a process in the context of the classical quasi-Abelian model [2], it is necessary to generalize from the previously treated static case to that of uniformly moving charges. Remarkably, the zeroth-order nonlinear partial differential equation for large separations can still be solved exactly in this more general situation.

The models considered here are the time-dependent versions of those studied previously [3], and may be viewed as Lorentz invariant nonlinear electrodynamics. In other words, the fields $\vec{D}$, $\vec{H}$, $\vec{E}$, and $\vec{B}$ satisfy Maxwell's equations

$$\nabla \cdot \vec{D} = \rho, \tag{1}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \tag{2}$$

$$\nabla \cdot \vec{B} = 0, \tag{3}$$

$$\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J}, \tag{4}$$

where the constitutive relations are

$$\vec{D} = \epsilon(E^2 - B^2)\vec{E} \tag{5}$$

and

$$\vec{H} = \epsilon(E^2 - B^2)\vec{B}, \tag{6}$$

with $E = |\vec{E}|$ and $B = |\vec{B}|$. Note that the same $\epsilon(E^2 - B^2)$ must appear in (5) and (6) in order to preserve Lorentz invariance. The special feature of this class of models is that

$$\epsilon(E^2 - B^2) = 0, \qquad \text{for } -B_0^2 \le E^2 - B^2 \le E_0^2. \tag{7}$$

It is useful to formulate these models in terms of a variational principle,

$$\delta S = 0, \tag{8}$$

with

$$S = \int d^3\vec{x}\, dt\, \{\mathcal{L}[E(\vec{x},t)^2 - B(\vec{x},t)^2] + \vec{A}(\vec{x},t)\cdot\vec{J}(\vec{x},t) - \varphi(\vec{x},t)\rho(\vec{x},t)\}, \tag{9}$$

where, as usual,

$$\vec{E}(\vec{x},t) = -\frac{\partial\vec{A}(\vec{x},t)}{\partial t} - \nabla\varphi(\vec{x},t), \tag{10}$$

$$\vec{B}(\vec{x},t) = \nabla\times\vec{A}(\vec{x},t). \tag{11}$$

In order to obtain (5) and (6) from (8), we choose

$$\mathcal{L}(\xi) = \tfrac{1}{2}\int_0^\xi d\xi'\,\epsilon(\xi'). \tag{12}$$

Throughout this note, we are interested in the case of two uniformly moving charges of opposite sign. For $t > 0$,

$$\rho(\vec{x},t) = Q[\delta(z-vt) - \delta(z+vt)]\delta(x)\delta(y), \tag{13}$$

which implies

$$\vec{J}(\vec{x},t) = \vec{e}_z vQ[\delta(z-vt) + \delta(z+vt)]\delta(x)\delta(y). \tag{14}$$

Because of Lorentz invariance, there is no loss of generality in choosing the velocities of the two charges to be opposite to each other.

The substitution of (9) and (12) into (8) gives

$$\delta S = \int d^3\vec{x}\, dt\, [\epsilon(E^2-B^2)(\vec{E}\cdot\delta\vec{E} - \vec{B}\cdot\delta\vec{B}) + \vec{J}\cdot\delta\vec{A} - \rho\delta\varphi] = 0. \tag{15}$$

Consequently, there are two domains. In domain (I), which contains the two charges, (1)–(6) hold with $E^2 - B^2 > E_0^2$. In domain (II), $\epsilon(E^2-B^2) = 0$, and hence, $\vec{D} = \vec{H} = 0$.

Let $\mathcal{B}$ denote the boundary between these two domains. The variation of S with respect to $\mathcal{B}$ shows that

$$\mathcal{L}[E(\vec{x},t)^2 - B(\vec{x},t)^2] \quad \text{is continuous across } \mathcal{B}. \tag{16}$$

Since $\epsilon(\xi)$ is taken to be positive for $\xi > E_0^2$, it follows from (16) with (12) that $D(\vec{x},t) = H(\vec{x},t) = 0$ for $(\vec{x},t)$ on $\mathcal{B}$.

For the charge and current as given by (13) and (14), rotational symmetry obtains and the nonzero field components in cylindrical coordinates are D_z, D_ρ, E_z, E_ρ, H_θ, and B_θ. For $\rho > 0$, (1) implies

$$\partial_z D_z + \frac{1}{\rho}\partial_\rho(\rho D_\rho) = 0. \tag{17}$$

Thus, D can be expressed in terms of a flux function [2] $\Psi(\rho, z, t)$ as

$$\rho D_z = \frac{1}{2\pi}\partial_\rho\Psi, \qquad \rho D_\rho = -\frac{1}{2\pi}\partial_z\Psi. \tag{18}$$

In terms of this Ψ, (4) implies, again for $\rho > 0$,

$$\partial_z(H_\theta - \frac{1}{2\pi\rho}\partial_t\Psi) = 0 \tag{19}$$

and

$$\partial_\rho\rho(H_\theta - \frac{1}{2\pi\rho}\partial_t\Psi) = 0. \tag{20}$$

Therefore,

$$\rho(H_\theta - \frac{1}{2\pi\rho}\partial_t\Psi) = \text{ function of } t.$$

But this function of t can be absorbed in Ψ. Therefore,

$$H_\theta = \frac{1}{2\pi\rho}\partial_t\Psi. \tag{21}$$

Thus, all field components can be expressed conveniently in terms of Ψ.

Since $D = H = 0$ in domain (II), Ψ is a constant there. Without loss of generality, this constant can be taken to be zero:

$$\Psi = 0 \tag{22}$$

in domain (II). It then follows from (16) that

$$\Psi = \partial_n\Psi = 0 \tag{23}$$

on the boundary $\mathcal{B}$. In other words, $\mathcal{B}$ is a free boundary.

For the present problem, since $E^2 - B^2$ is positive in domain (I), it is useful to rewrite the constitutive relations (5) and (6) in the form

$$(E^2 - B^2)^{1/2} = f\left(\sqrt{D^2 - H^2}\right). \tag{24}$$

In this notation, (7) implies that

$$f(0^+) = E_0. \tag{25}$$

Each model in the class is specified by this function of f.

Let us rewrite (2) in cylindrical coordinates:

$$\partial_z E_\rho - \partial_\rho E_z + \partial_t B_\theta = 0. \tag{26}$$

It follows from (5), (6) and (24) that

$$\frac{E_\rho}{D_\rho} = \frac{E_z}{D_z} = \frac{B_\theta}{H_\theta} = \frac{f\left(\sqrt{D^2 - H^2}\right)}{\sqrt{D^2 - H^2}}. \tag{27}$$

By (18) and (21), the substitution of (27) into (26) gives the partial differential equation for Ψ in domain (I):

$$\partial_\rho \left[\frac{\partial_\rho \Psi f\left(\sqrt{D^2 - H^2}\right)}{[(\partial_z\Psi)^2 - (\partial_t\Psi)^2 + (\partial_\rho\Psi)^2]^{1/2}} \right] + \partial_z \left[\frac{\partial_z \Psi f\left(\sqrt{D^2 - H^2}\right)}{[(\partial_z\Psi)^2 - (\partial_t\Psi)^2 + (\partial_\rho\Psi)^2]^{1/2}} \right]$$
$$- \partial_t \left[\frac{\partial_t \Psi f\left(\sqrt{D^2 - H^2}\right)}{[(\partial_z\Psi)^2 - (\partial_t\Psi)^2 + (\partial_\rho\Psi)^2]^{1/2}} \right] = 0. \tag{28}$$

The boundary conditions are (23) and

$$\Psi(0, z, t) = \begin{cases} Q, & |z| < vt \\ 0, & |z| > vt. \end{cases} \tag{29}$$

Note that (28) is invariant under Lorentz transformations in the (z, t) plane. Carrying out the differentiations gives:

$$\frac{g(\eta)}{\eta}\{(\partial_\rho\Psi)^2[\partial_{zz}\Psi - \partial_{tt}\Psi] + [(\partial_z\Psi)^2 - (\partial_t\Psi)^2]\partial_{\rho\rho}\Psi - 2(\partial_\rho\Psi)[\partial_z\Psi\,\partial_{\rho z}\Psi - \partial_t\Psi\,\partial_{\rho t}\Psi]\}$$
$$+ [(\partial_\rho\Psi)^2 + (\partial_z\Psi)^2 - (\partial_t\Psi)^2][\partial_{\rho\rho}\Psi - \rho^{-1}\partial_\rho\Psi + \partial_{zz}\Psi - \partial_{tt}\Psi] = 0, \tag{30}$$

with

$$\eta = \frac{1}{2\pi\rho}[(\partial_\rho\Psi)^2 + (\partial_z\Psi)^2 - (\partial_t\Psi)^2]^{1/2} \tag{31}$$

and

$$g(\eta) = \frac{f(\eta)}{f'(\eta)} - \eta. \tag{32}$$

We limit ourselves to the class of functions $f(\eta)$ studied in reference [3], i.e.,

$$f(\eta) = 1 + \eta + o(\eta), \tag{33}$$

with suitable choice of units. We also take $Q = 1$.

From now on, we concentrate on the case of large separation of the charges, i.e., $vt \gg 1$. For the purpose of obtaining the zeroth-order solution, we follow the method of sect. 4 of ref. [3] and use $f(\eta) = 1 + \eta$ and hence

$$g(\eta) = 1. \tag{34}$$

Since we expect the transverse size of domain (I) to be small compared with its longitudinal size, $\partial_z\Psi$ and $\partial_t\Psi$ can be neglected compared with $\partial_\rho\Psi$. Therefore, it follows from (30) and (31) that

$$\eta = -\frac{\partial_\rho\Psi}{2\pi\rho}, \tag{35}$$

and Ψ satisfies

$$-2\pi\rho\{(\partial_\rho\Psi)^2[\partial_{zz}\Psi - \partial_{tt}\Psi] + [(\partial_z\Psi)^2 - (\partial_t\Psi)^2]\partial_{\rho\rho}\Psi$$
$$- 2(\partial_\rho\Psi)[\partial_z\Psi\,\partial_{\rho z}\Psi - \partial_t\Psi\,\partial_{\rho t}\Psi]\} + (\partial_\rho\Psi)^3[\partial_{\rho\rho}\Psi - \rho^{-1}\partial_\rho\Psi] = 0. \tag{36}$$

Let the free boundary $\mathcal{B}$ be given by $\rho = \rho_b(z, t)$ and let

$$\tau = \frac{\rho}{\rho_b(z,t)}. \tag{37}$$

In terms of the variables τ, z and t, (36) is

$$-2\pi\left\{[(\partial_z\Psi)^2 - (\partial_t\Psi)^2]\partial_{\tau\tau}\Psi + (\partial_\tau\Psi)^2\left[\frac{2}{\rho_b}(\partial_z\rho_b\partial_z\Psi - \partial_t\rho_b\partial_t\Psi)\right.\right.$$
$$\left.\left. - \frac{\tau\partial_\tau\Psi}{\rho_b}(\partial_{zz}\rho_b - \partial_{tt}\rho_b) + \partial_{zz}\Psi - \partial_{tt}\Psi\right] - 2\partial_\tau\Psi(\partial_z\Psi\partial_{\tau z}\Psi - \partial_t\Psi\partial_{\tau t}\Psi)\right\}$$
$$+ \frac{1}{\rho_b^4}(\partial_\tau\Psi)^3\frac{\partial}{\partial\tau}\left(\frac{\partial_\tau\Psi}{\tau}\right) = 0. \tag{38}$$

With the boundary conditions (23) and (29), Ψ is found to be

$$\Psi = (1-\tau^2)^2, \tag{39}$$

and the partial differential equation (38) reduces to

$$\rho_b^3(\partial_{zz}\rho_b - \partial_{tt}\rho_b) + 4/\pi = 0. \tag{40}$$

The appropriate solution of (40) is

$$\rho_b = \frac{2}{\pi^{1/4}v^{1/2}}\left[\sqrt{t^2-z^2} - \sqrt{1-v^2}\,t\right]^{1/2}. \tag{41}$$

Note that this free boundary passes through the lines $z = \pm vt$, as should be the case for the zeroth-order solution. In the limit of fixed $vt = R$ and with $v \to 0$, (41) reduces to

$$\rho_b = \frac{\sqrt{2}}{\pi^{1/4}R^{1/2}}\sqrt{R^2-z^2}, \tag{42}$$

which is the zeroth-order solution in the static case [3]. More generally, with $R = vt$ being half the distance between the two charges, (41) can be rewritten as

$$\rho_b = \frac{2}{\pi^{1/4}}\left[\frac{R^2-z^2}{\sqrt{R^2-v^2z^2}+\sqrt{1-v^2}\,R}\right]^{1/2}, \tag{43}$$

which is an increasing function of v for all $|z| < R$. In particular, for $v \to 1$, ρ_b is simply

$$\rho_b = \frac{2}{\pi^{1/4}}(R^2-z^2)^{1/4}. \tag{44}$$

The shape of the confinement domain is illustrated in Fig. 1.

In the static case the electric energy is the quantity of main interest, since it gives the potential between the charges. In the present case, let U be the total electromagnetic energy at a given time. It is

$$U = \int d^3\vec{x}\left\{\int_0^{\sqrt{D^2-H^2}} d\eta\, f(\eta) + \vec{H}\cdot\vec{B}\right\}. \tag{45}$$

In the zeroth-order approximation this is simply

$$U = \int_{-R'}^{R'} dz \int_0^{\rho_b} d\rho \left\{-\partial_\rho\Psi - \frac{1}{2}\frac{(\partial_z\Psi)^2}{\partial_\rho\Psi} - \frac{1}{2}\frac{(\partial_t\Psi)^2}{\partial_\rho\Psi} + \frac{1}{4\pi\rho}(\partial_\rho\Psi)^2\right\}, \tag{46}$$

where R' is close, but not equal, to R (see sect. 4.3 of [3]).

In order to determine R', we note that, in deriving the zeroth-order solution, the ratio

$$\frac{(\partial_z\Psi)^2 - (\partial_t\Psi)^2}{(\partial_\rho\Psi)^2}$$

has been neglected when compared with 1. From (39), this ratio is found to be

$$\frac{1}{\sqrt{\pi}}\left[\frac{-2}{v\sqrt{t^2-z^2}} + \frac{v}{\sqrt{t^2-z^2}-\sqrt{1-v^2}\,t}\right].$$

Therefore, the zeroth-order solution fails when

$$\frac{1}{v}\left[\sqrt{t^2-z^2} - \sqrt{1-v^2}\,t\right] \tag{47}$$

is of order 1. Let this quantity (47) be denoted by C. Then, when $R = vt$ is large, the approximate solution of (47) is

$$z = R - \sqrt{1-v^2}\,C. \tag{48}$$

In other words, the limit of integration R' is given by the right-hand side of (48).

The integration of (46) gives for large $R\sqrt{1-v^2}$:

$$U = 2R\left[1 + \frac{1}{3\sqrt{\pi}}\frac{\ln\left(R\sqrt{1-v^2}\right)}{R\sqrt{1-v^2}}\right], \tag{49}$$

which reduces to the static potential for $v = 0$ [3]. For v small compared to 1, (49) gives a velocity-dependent correction to the static potential for large distances. For v near 1, the present zeroth-order solution is insufficient for a satisfactory description. We believe that a partial summation such as the one carried out in sect. 6 of [3] is needed.†

† In eq. (6.7) of [3] there is a sign mistake: $+2(\partial_u w/w)$ should be $-2(\partial_u w/w)$. Consequently, eq. (6.17) should be replaced by $h(\xi) = -(9-2b_2)\xi/2(1+A\xi)$, and $w_1(\xi)$ of eq. (6.21) is now given by $w_1(\xi) = -[A\ \xi\ /4\sqrt{1+A\xi}\,]\ln(1+A\xi)$.

With little additional complication, we have generalized in this note the zeroth-order solution from the static case to uniformly moving charges. Although not given here, this generalization has also been carried through for the first-order solution.

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FIGURE CAPTION

Fig. 1. Shape of the zeroth-order confinement domain for $v = 0,\ 0.8$ and 1.