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VERTICAL DISPERSION GENERATED BY CORRELATED CLOSED ORBIT DEVIATIONS

by

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(1) Introduction

Vertical displacement of quadrupole magnets is one of the main causes of a vertical dispersion in a flat storage ring and thus a major contributor to the height of an electron beam. A small beam height is usually desirable for achieving high luminosity of colliding electron-positron beams. Moreover in the electron-proton collider HERA, a small beam height is also desirable for achieving high longitudinal spin polarization at the interaction points /BAR85/.

Computer simulations of the beam height in the HERA electron ring using the PETROS optics code /KEW76/ give a value of the ratio ϵ_z / ϵ_x of more than 10 percent. This large value occurs even for an rms value of the quadrupole vertical displacements Δz as small as 0.01 mm.

Such a vertical emittance is much larger than one expects on the base of the theoretical estimate given by A. Piwinski /PIW83/ and it is clearly necessary to investigate the origin of the disagreements especially since the beam height has such an important influence on the machine performance.

The key to the understanding of this discrepancy lies in the correlations of the closed orbit deviations at different position of the machine. This will be investigated in the next section and in the section which follows we will derive the expression for the rms- value of dispersion and the vertical emittance. Finally the theoretical results will be compared with computer simulations.

(2) Correlations of Closed Orbit Deviations

The basic assumption leading to the expression for the vertical emittance due to a vertical orbit distortion in reference /PIW83/ is that closed orbit distortions at different positions are uncorrelated. It is obvious that this is not true if the closed orbit is generated by one or a few sources of error. However we will also demonstrate that it is also incorrect for an orbit generated by many small random kicks.

In order to estimate the vertical emittance generated by a large number of vertical error kicks one needs an expression for

$$A_z(s) = \frac{D_z^2(s) + [\alpha(s)D(s) + \beta(s)D'(s)]^2}{\beta(s)} \tag{1}$$

since the average of $A_z(s)$ in the bendings determines the generation of vertical emittance.

A. Piwinski derived the following expression /PIW83/ using the thin lens approximation for dipole errors:

$$\begin{aligned} A_z(s_k) &= \frac{1}{4\sin^2 \pi Q} \sum_{m=k}^{N+k-1} \sum_{n=k}^{N+k-1} \frac{l_n l_m}{\rho_n \rho_m} \sqrt{\beta_n \beta_m} \cos(\Phi_n - \Phi_m) \\ &= \frac{1}{4\sin^2 \pi Q} \sum_n \sum_m A_{nm} \end{aligned} \tag{2}$$

where β and Φ are the vertical envelope function and betatron phase advance, $\frac{l}{\rho}$ are the error kicks, Q is the vertical machine tune, N is the total number of error kicks and where the numbering of m and n starts at the first vertical error kick after s_k.

Since we are interested in the average effect of an ensemble of distorted machines we need

$$A_z(s_k) = \frac{1}{4\sin^2 \pi Q} \sum_{m=k}^{N+k-1} \sum_{n=k}^{N+k-1} < A_{nm} > \tag{3}$$

Because the optical functions $\beta^{n,m}$ and $\Phi_{n,m}$ are constant when the ensemble average is performed, we have

$$< A_{nm} > = \left\langle \frac{l_n l_m}{\rho_n \rho_m} \right\rangle \sqrt{\beta_n \beta_m} \cos(\Phi_n - \Phi_m) \tag{4}$$

If the error kicks are truely randomly distributed, all off diagonal elements of $< A_{nm} >$ vanish because the correlation $\left\langle \frac{l_n l_m}{\rho_n \rho_m} \right\rangle$ of randomly distributed quantities is zero, and the approximations for ϵ_z given e.g. in /PIW83/ really apply.

If however the error kicks are generated by the vertical closed orbit displacements in the quadrupoles eq. 4 becomes:

$$\begin{aligned} < A_{nm} > &= < k_n l_n z_n \cdot k_m l_m z_m > \sqrt{\beta_n \beta_m} \cos(\Phi_n - \Phi_m) \\ &= k_n l_n k_m l_m < z_n z_m > \sqrt{\beta_n \beta_m} \cos(\Phi_n - \Phi_m) \end{aligned} \tag{5}$$

(in general $k_{n,m} = k_{quad} + m_{sext} D_x$).

Although the vertical closed orbit displacement is generated by a large number of *random* vertical kicks, the closed orbit deviations are not totally uncorrelated especially if the kick positions are not too far apart. It must be expected that ϵ_z is underestimated in /PIW83/ if the correlation length $d = n - m >> 1$.

In the following paragraph we investigate the properties of $< z_n \cdot z_m >$.

We assume that the closed orbit distortion z_n in the quadrupole magnets is generated by N random kicks $\Theta_i = k_i l_i \Delta z_i$ due to misaligned quadrupoles. Then

$$< z_n \cdot z_m >=< \frac{\sqrt{\beta_n\beta_m}}{4\sin^2\pi Q} \sum_{i=n}^{N+n-1} \sum_{j=m}^{N+m-1} \Theta_i\Theta_j\sqrt{\beta_i\beta_j} \cos(\Phi_n - \Phi_i + \pi Q)\cos(\Phi_m - \Phi_j + \pi Q) > \tag{6}$$

$$< z_n \cdot z_m >=< \frac{\sqrt{\beta_n\beta_m}}{8\sin^2\pi Q} \sum_{i=n}^{N+n-1} \sum_{j=m}^{N+m-1} \Theta_i\Theta_j\sqrt{\beta_i\beta_j} [\cos(\Phi_j - \Phi_i + \Phi_n - \Phi_m) + \cos(\Phi_n + \Phi_m - \Phi_i - \Phi_j + 2\pi Q)] >$$

Again all optical parameters are constant with respect to ensemble averaging:

$$< z_n \cdot z_m >= \frac{\sqrt{\beta_n\beta_m}}{8\sin^2\pi Q} \sum_{i=n}^{N+n-1} \sum_{j=m}^{N+m-1} < \Delta z_i \cdot \Delta z_j > (lk\sqrt{\beta})_i(lk\sqrt{\beta})_j [\cos(\Phi_j - \Phi_i + \Phi_n - \Phi_m) + \cos(\Phi_n + \Phi_m - \Phi_i - \Phi_j + 2\pi Q)] \tag{7}$$

Since the Δz_i are random we may write

$$< \Delta z_i \Delta z_j >= (\delta_{i,j} + \delta_{i-N,j} + \delta_{i+N,j}) < (\Delta z_i)^2 > \tag{8}$$

Here we assume that only self correlation occurs, and it is taken care of Δz_i and Δz_j being periodic with N. The latter results in additional delta functions for $|i-j| > N$, which may happen since i and j start running from different n and m, respectively. With $< (\Delta z_i)^2 >= \sigma_z^2$ we get

$$< z_n \cdot z_m >= \frac{\sqrt{\beta_n\beta_m}}{8\sin^2\pi Q}\sigma_z^2 \times \sum_{i=n}^{N+n-1} \sum_{j=m}^{N+m-1} (\delta_{i,j} + \delta_{i-N,j} + \delta_{i+N,j})(lk\sqrt{\beta})_i(lk\sqrt{\beta})_j \times [\cos(\Phi_j - \Phi_i + \Phi_n - \Phi_m) + \cos(\Phi_n + \Phi_m - \Phi_i - \Phi_j + 2\pi Q)] \tag{9}$$

In order to be able to perform the second sum, we want to split the first sum in two parts. Therefore, we distinguish three cases:

a) $\quad n < m : \quad \begin{pmatrix} i - N = j & \textit{is impossible,} \\ i + N = j & \textit{occurs for } i < m \textit{ only} \\ i = j & \textit{occurs for } i \geq m \textit{ only} \end{pmatrix}$

$$< z_n \cdot z_m >= \frac{\sqrt{\beta_n\beta_m}}{8\sin^2\pi Q}\sigma_z^2 \times \left\{ \sum_{i=m}^{N+n-1} (l^2k^2\beta)_i [\cos(\Phi_m - \Phi_n) + \cos(2\Phi_i - (\Phi_n + \Phi_m) - 2\pi Q)] + \sum_{i=n}^{m-1} (l^2k^2\beta)_i [\cos(-2\pi Q + \Phi_m - \Phi_n) + \cos(2\Phi_i - (\Phi_n + \Phi_m))] \right\} \tag{10a}$$

$$(\textit{with } \Phi_{i+N} = \Phi_i + 2\pi Q)$$

b) $\quad n > m : \quad \begin{pmatrix} i - N = j & \textit{occurs for } i \geq m + N \textit{ only} \\ i + N = j & \textit{is impossible} \\ i = j & \textit{occurs for } i < m + N \textit{ only} \end{pmatrix}$

$$< z_n \cdot z_m >= \frac{\sqrt{\beta_n\beta_m}}{8\sin^2\pi Q}\sigma_z^2 \times \left\{ \sum_{i=m+N}^{N+n-1} (l^2k^2\beta)_i [\cos(2\pi Q + \Phi_m - \Phi_n) + \cos(2\Phi_i - (\Phi_n + \Phi_m) - 4\pi Q)] + \sum_{i=n}^{N+m-1} (l^2k^2\beta)_i [\cos(\Phi_m - \Phi_n) + \cos(2\Phi_i - (\Phi_n + \Phi_m) - 2\pi Q)] \right\}$$

$$(\textit{with } \Phi_{i-N} = \Phi_i - 2\pi Q) \tag{10b}$$

c) $n = m$: $\begin{pmatrix} i - N = j & \textit{is impossible} \\ i + N = j & \textit{is impossible} \end{pmatrix}$

$$< z_n \cdot z_m >= \frac{\sqrt{\beta_n \beta_m}}{8 \sin^2 \pi Q} \sigma_z^2 \times$$
$$\sum_{i=n}^{N+n-1} (l^2 k^2 \beta)_i [\cos(\Phi_m - \Phi_n) + \cos(2\Phi_i - (\Phi_n + \Phi_m) - 2\pi Q)] \tag{10c}$$

So far the expressions are exact.

If we want to get a better understanding of $< z_n z_m >$, we should introduce approximations. Firstly it must be realized that $l_i^2 k_i^2 \beta_i$ is positive definite whereas the second cos-terms (containing Φ_i) in a) - c) oscillate when the sum is performed. If we assume $\Phi_i \approx \Phi_0 + i\Delta\Phi$ ($\Delta\Phi$ is the phase advance between subsequent error kicks), we may neglect all cos-terms in a) - c) containing Φ_i as long as $\Delta\Phi \gg \frac{\pi}{N}$, which can be assumed in all realistic large storage rings. We now get

a) $n < m$:

$$< z_n \cdot z_m > \simeq \frac{\sqrt{\beta_n \beta_m}}{8 \sin^2 \pi Q} \sigma_z^2 \times$$
$$\left\{ \sum_{i=m}^{N+n-1} (l^2 k^2 \beta)_i \cos(\Phi_m - \Phi_n) + \sum_{i=n}^{m-1} (l^2 k^2 \beta)_i \cos(-2\pi Q + \Phi_m - \Phi_n) \right\} \tag{11a}$$

b) $n > m$:

$$< z_n \cdot z_m > \simeq \frac{\sqrt{\beta_n \beta_m}}{8 \sin^2 \pi Q} \sigma_z^2 \times$$
$$\left\{ \sum_{i=m+N}^{N+n-1} (l^2 k^2 \beta)_i \cos(2\pi Q + \Phi_m - \Phi_n) + \sum_{i=n}^{N+m-1} (l^2 k^2 \beta)_i \cos(\Phi_m - \Phi_n) \right\} \tag{11b}$$

c) $n = m$:

$$< z_n \cdot z_m > \simeq \frac{\sqrt{\beta_n \beta_m}}{8 \sin^2 \pi Q} \sigma_z^2 \sum_{i=n}^{N+n-1} (l^2 k^2 \beta)_i \tag{11c}$$

In the longest part of large storage rings $l_i^2 k_i^2 \beta_i$ are nearly the same (periodic arc structure). Therefore we approximate $l_i^2 k_i^2 \beta_i \approx l^2 k^2 \overline{\beta}$. Then

a) $n < m$: $$< z_n \cdot z_m > \simeq \frac{\sqrt{\beta_n \beta_m}}{8 \sin^2 \pi Q} \sigma_z^2 l^2 k^2 \overline{\beta} \times$$
$$[(N + n - m) \cos(\Phi_m - \Phi_n) + (m - n) \cos(\Phi_m - \Phi_n) - 2\pi Q)] \tag{12a}$$

b) $n > m$: $$< z_n \cdot z_m > \simeq \frac{\sqrt{\beta_n \beta_m}}{8 \sin^2 \pi Q} \sigma_z^2 l^2 k^2 \overline{\beta} \times$$
$$[(N + m - n) \cos(\Phi_m - \Phi_n) + (n - m) \cos(\Phi_n - \Phi_m) - 2\pi Q)] \tag{12b}$$

(12a) and (12b) can be expressed simultaneously by

$$< z_n \cdot z_m >=< z_m \cdot z_n > \simeq \frac{\sqrt{\beta_n \beta_m}}{8 \sin^2 \pi Q} \sigma_z^2 l^2 k^2 \overline{\beta} \times$$
$$[(N - |n - m|) \cos(\Phi_m - \Phi_n) + |n - m| \cos(|\Phi_n - \Phi_m| - 2\pi Q)] \tag{12c}$$

which exhibits clearly the (physically necessary) property of symmetry $< z_n \cdot z_m >=< z_m \cdot z_n >$.

c) The self correlation (n=m) becomes:

$$< z_n \cdot z_n > \simeq \frac{\beta_n}{8 \sin^2 \pi Q} \sigma_z^2 l^2 k^2 \overline{\beta} N \tag{12d}$$

We now may calculate the coefficient of linear correlation R: (with $d = |n - m|$ and $\Phi_n - \Phi_m = d \cdot \Delta\Phi$)

$$R_d = \frac{< z_n \cdot z_{n+d} >}{\sqrt{< z_n^2 > \cdot < z_{n+d}^2 >}} \tag{13}$$

If we use

$$\sqrt{< z_n^2 >} = \sqrt{\frac{N}{2}} \sigma_z l k \frac{\sqrt{\beta_n \overline{\beta}}}{2 \sin \pi Q} \tag{14}$$

and

$$\sqrt{< z^2_{n+d} >} = \sqrt{\frac{N}{2}} \sigma_z l k \frac{\sqrt{\beta_{n+d}\bar{\beta}}}{2 \sin \pi Q} \tag{15}$$

(The factor 1/2 for N occurs because $< z_n^2 >$ is the mean square closed orbit (i.e. the self correlation of z_n) and *not* the mean square *amplitude* of closed orbit distortion as defined e.g. by Courant and Snyder /COU58/.) We get from eq. 12c:

$$R_d = \frac{N-d}{N} \cos(d\Delta\Phi) + \frac{d}{N} \cos(2\pi Q + d\Delta\Phi) \tag{16}$$

Besides the oscillating behaviour of R_d as d increases, it is seen that the amplitude of R_d depends only weakly on d, i.e. the correlation is large around the whole ring!

The amplitude of R_d is

$$\begin{aligned} a(d) &= \frac{1}{N}\sqrt{(N-d)^2 + d^2 + 2(N-d)\cos 2\pi Q} \\ &= \sqrt{N^2 \cos^2 \pi Q + (2d-N)^2 \sin^2 \pi Q} \end{aligned} \tag{17}$$

Eq 17 shows that $a(d)$ is symmetric with respect to $d = N/2$ where $a(d)$ is minimum:

$$a_{min} = a(d = N/2) = \cos \pi Q \tag{18}$$

The maximum is at $d \to 0$ and $d \to N$:

$$a_{max} = 1, \qquad i.e. \quad a_{min}/a_{max} = \cos \pi Q$$

Due to the large correlation it is most probable that application of the "most effective" correction kick[1] would considerably reduce $< z_n^2 >$ and the correlation length. This is in agreement with experience at existing storage rings. Such an approach would reduce the generated vertical dispersion very effectively. (see below)

Supression of the strongly correlated part of the orbit distortion is only possible, however, if the orbit monitors are sufficiently precise. If the accuracy

[1]The most effective correction kick is an orbit correction procedure used for example at PETRA, where the orbit deviations are minimized by using a single dipole correction kick

of the monitors is Δz it is very probable that after orbit corrections the remaining closed orbit contains a strongly correlated part (as described by eq.'s 11,12) of amplitude Δz resulting in correspondingly large contributions in ϵ_z.

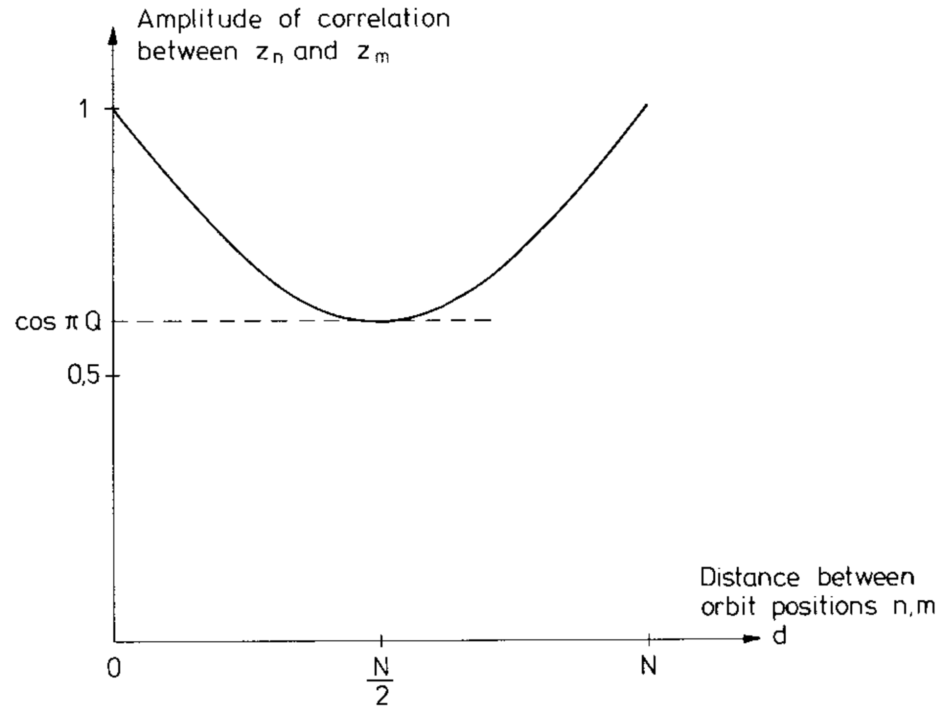

fig 1 Amplitude of Closed Orbit Correlations vs Distance between Orbit Positions

(3) Expectation Value of Vertical Dispersion Generated by Small Random Vertical Kicks

The considerations in the previous paragraph will be used to derive an expression for the emittance of the vertical dispersion A defined in eq (1). Although some considerations will be repeated, the argumentation will be more straight forward if we start again from the eq (2) taken from ref /PIW83/. For the error kick due to an orbit deviation z in quadrupoles and sextupoles we insert:

$$\frac{l_n}{\rho_n} = (kl - mD_x l)_n \cdot z_n \tag{19}$$

where the orbit deviation z_n is given by

$$z_n = \frac{\sqrt{\beta_n}}{2\sin\pi Q} \sum_{i=n}^{N+n-1} (kl\sqrt{\beta})_i \cdot \Delta z_i \cos(\Phi_i - \Phi_n - \pi Q) \tag{20}$$

The Δz_i are the displacements of the quadrupole magnets. Insertion of this expression in eq (2) for $A_z(s)$ yields

$$\begin{aligned} < A_z(s_k) >= \frac{1}{16\sin^4\pi Q} \\ < \sum_{n=k}^{k+N-1} \sum_{m=k}^{k+N-1} \sum_{i=n}^{n+N-1} \sum_{j=m}^{m+N-1} ((kl - mD_x l)\beta)_n ((kl - mD_x l)\beta)_m \\ (kl\sqrt{\beta})_i (kl\sqrt{\beta})_j \Delta z_i \cdot \Delta z_j \\ \cos(\Phi_n - \Phi_m) \cdot \cos(\Phi_i - \Phi_n - \pi Q) \cdot \cos(\Phi_j - \Phi_m - \pi Q) > \end{aligned} \tag{21}$$

The expression for $< A_z >$ does not depend on the index k and we can take $k = 1$. The quadrupole displacements Δz_i are assumed to be randomly distributed and the brackets in eq (21) mean that we have to average over an ensemble of many machines with different random quadrupole displacements. For that purpose it is more convenient to split the sums over i and j into two parts respectively: The first part extends from n or m to N and the second part from 1 to $n-1$ or $m-1$. The phase advances Φ_i and Φ_j are now counted from the same reference point. Therfore we have to add $2\pi Q$ to the arguments of the cosine functions with Φ_i and Φ_j in the second parts of the sum. The expression for $< A_z >$ then consists of four terms. Using the

abrevation $\xi_n = \beta_n(kl - mD_x)_n$ we get:

$$\begin{aligned} < A_z >= \frac{1}{16\sin^4\pi Q} \\ [< \sum_{n=1}^{N}\sum_{m=1}^{N}\sum_{i=n}^{N}\sum_{j=m}^{N} \xi_n\xi_m (kl\sqrt{\beta})_i (kl\sqrt{\beta})_j \Delta z_i \cdot \Delta z_j \\ \cos(\Phi_n - \Phi_m)\cdot\cos(\Phi_i - \Phi_n - \pi Q)\cdot\cos(\Phi_j - \Phi_m - \pi Q) > \\ + < \sum_{n=1}^{N}\sum_{m=1}^{N}\sum_{i=1}^{n-1}\sum_{j=m}^{N} \xi_n\xi_m (kl\sqrt{\beta})_i (kl\sqrt{\beta})_j \Delta z_i \cdot \Delta z_j \\ \cos(\Phi_n - \Phi_m)\cdot\cos(\Phi_i - \Phi_n + \pi Q)\cdot\cos(\Phi_j - \Phi_m - \pi Q) > \\ + < \sum_{n=1}^{N}\sum_{m=1}^{N}\sum_{i=n}^{N}\sum_{j=1}^{m-1} \xi_n\xi_m (kl\sqrt{\beta})_i (kl\sqrt{\beta})_j \Delta z_i \cdot \Delta z_j \\ \cos(\Phi_n - \Phi_m)\cdot\cos(\Phi_i - \Phi_n - \pi Q)\cdot\cos(\Phi_j - \Phi_m + \pi Q) > \\ + < \sum_{n=1}^{N}\sum_{m=1}^{N}\sum_{i=1}^{n-1}\sum_{j=1}^{m-1} \xi_n\xi_m (kl\sqrt{\beta})_i (kl\sqrt{\beta})_j \Delta z_i \cdot \Delta z_j \\ \cos(\Phi_n - \Phi_m)\cdot\cos(\Phi_i - \Phi_n + \pi Q)\cdot\cos(\Phi_j - \Phi_m + \pi Q) >] \end{aligned} \tag{22}$$

Now we use that the quadrupole displacements Δz_i are randomly distributed and displacements at different positions are not correlated. Thus

$$< \Delta z_i \cdot \Delta z_j >= \delta_{ij}\Delta < z_i^2 >$$

and the sum over j gives only contributions for $j = i$. However the summation regions for i and j overlap only partly. Thus we cannot find for each index i an index j in each of the four terms and for any combinations of n and m. For the second term in expression (22) we don't have a contribution at all for all $n \geq m$. The same is the case for the third term if $n \leq m$. Furthermore the contributions from the summation over j are different for $n < m$ and vice versa. This leaves us with 6 different contributions to $< A_z >$. Again we make use of the fact that the lattice propertiesβ, kl are invariant under ensemble averaging and that the ensemble average of Δz_i^2 is the same for all positions i. The product of the last two cosine functions in eq (22) can be expressed respectively by the sum:

$$\frac{1}{2}\cos(2\Phi_i - \Phi_n - \Phi_m - 2\pi Q) + \frac{1}{2}\cos(\Phi_n - \Phi_m) \tag{23}$$

The term with the first of the two summands is expected to contribute much less than the one with the second one because it oscillates with increasing argument Φ_i whereas the other term is positive or negative definite if i is varied. Therefore we will neglect this term. Then we can carry out the sums over the index i using the approximation:

$$\sum_{i=1}^{n} k_i^2 l_i^2 \beta_i = n\overline{k^2 l^2 \beta}$$

Thus individual β, l and k values are replaced by the mean value $\overline{k^2 l^2 \beta}$ and we obtain:[1]

$$\begin{aligned}
< A_z > &= \frac{\overline{k^2 l^2 \beta} < \Delta z^2 >}{64 \sin^4 \pi Q} \\
&[\sum_{n=1}^{N} \sum_{m=1}^{n} (N-n+1)\xi_n \xi_m (1+\cos 2(\Phi_n - \Phi_m)) \\
&+ \sum_{n=1}^{N} \sum_{m=n+1}^{N} (N-m+1)\xi_n \xi_m (1+\cos 2(\Phi_n - \Phi_m)) \\
&+ \sum_{n=1}^{N} \sum_{m=1}^{n} (m-1)\xi_n \xi_m (1+\cos 2(\Phi_n - \Phi_m)) \\
&+ \sum_{n=1}^{N} \sum_{m=n+1}^{N} (n-1)\xi_n \xi_m (1+\cos 2(\Phi_n - \Phi_m)) \\
&+ \sum_{n=1}^{N} \sum_{m=n+1}^{N} |m-n|\xi_n \xi_m (cos(2\pi Q) + \cos(2|\Phi_n - \Phi_m| - 2\pi Q)) \\
&+ \sum_{n=1}^{N} \sum_{m=1}^{n} |m-n|\xi_n \xi_m (cos(2\pi Q) + \cos(2|\Phi_n - \Phi_m| - 2\pi Q))]
\end{aligned} \tag{24}$$

[1] using $\cos^2 \Phi = \frac{1}{2}(1 + \cos 2\Phi)$

The subsums 1 through 4 and 5 and 6 in eq (24) can be combined to

$$\begin{aligned}
< A_z > &= \frac{N\overline{k^2 l^2 \beta} < \Delta z^2 >}{64 \sin^4 \pi Q} \\
&[\sum_{n=1}^{N} \sum_{m=1}^{N} \xi_n \xi_m (1 + \cos(2\Phi_n)\cos(2\Phi_m) + \sin(2\Phi_n)\sin(2\Phi_m)) \\
&- \frac{1}{N} \sum_{n=1}^{N} \sum_{m=1}^{N} |n-m|\xi_n \xi_m (1 - \cos 2\pi Q) \\
&+ \frac{1}{N} \sum_{n=1}^{N} \sum_{m=1}^{N} |n-m|\xi_n \xi_m (\cos 2(\Phi_n - \Phi_m) - \cos(2|\Phi_n - \Phi_m| - 2\pi Q))]
\end{aligned} \tag{25}$$

We now recognize that the first summand of the first of the three double sums in eq (25) as composed of the chromaticity of the machine:

$$\xi = \frac{1}{4\pi} \sum_{n-1}^{N} \xi_n \tag{26}$$

The second part of the sum contains the width of the off energy stopband per $\Delta p/p$

$$\begin{aligned}
\Delta/(\Delta p/p) &= \sqrt{\Delta_s^2 + \Delta_c^2} \\
\Delta_c &= \frac{1}{4\pi} \sum_{n=1}^{N} \xi_n \cos(2\Phi_n)
\end{aligned} \tag{27}$$

The second and third double sum do not contain such well known machine properties. The third one has an oscillating factor of cosine functions

$$-2\sin(\pi Q)\sin(2|\Phi_n - \Phi_m| - \pi Q)$$

and the second one has the constant factor:

$$\cos(2\pi Q) - 1 = -2\sin^2(\pi Q)$$

If we neglect the term with the oscillating factor we finally obtain:

$$\begin{aligned}
< A_z > &= \frac{N\overline{k^2 l^2 \beta} < \Delta z^2 >}{64 \sin^4 \pi Q} \\
&[(4\pi\xi)^2 + [\frac{4\pi\Delta}{\Delta p/p}]^2 - 2\sin^2 \pi Q \sum_{n=1}^{N} \sum_{m=1}^{N} |m-n|\xi_n \xi_m]
\end{aligned} \tag{28}$$

The last term cannot be neglected as compared with the two others even in the case where the machine chromaticity is not compensated with sextupoles. In order to obtain the order of magnitude of that term, we estimate its value for a regular FODO structure. If we neglect summands to 1 in the the order of $\frac{1}{N}$ we obtain:

$$< A_z >= \frac{N\overline{k^2 l^2 \beta} < \Delta z^2 >}{64 \sin^4 \pi Q}$$
$$[\frac{2}{3}(4\pi\xi)^2(1+\frac{1}{2}\cos 2\pi Q) + [\frac{4\pi\Delta}{\Delta p/p}]^2 + \sin^2 \pi Q \frac{1}{N}\sum_{n=1}^{N} \xi_n^2] \quad (29)$$

The same procedure can be carried out for the expectation value of the dispersion amplitude $< D_z^2 >$ which differs from the dispersion emittance $< A_z >$ by a factor of 1/2 (if terms which oscillate with increasing phase Φ_k are neglected when D_{rms} is averaged around the machine). We recognize the similarity of this result with that given by ref /PIW76/ for the vertical dispersion generated by a single vertical kick.

Obviously it is essential to compensate for the chromaticity with sextupoles in order to control the vertical dispersion generated by random vertical kicks.

If the chromaticity ξ_0 is zero or small, the term proportional to $\Delta/\Delta p/p$ may become a relatively important contribution to the beam height. Compensation of the off energy stopband by using more than two sextupole families will eliminate this problem. If we do so the only remaining contribution is proportional to the average of the square of the locally produced chromaticity $\overline{(4\pi\xi_n)^2}$ which is about three orders of magnitude smaller than the expression with uncompensated chromaticity.

We use the above formula to estimate the rms value of the dispersion in HERA with no chromaticity compensation. For the luminosity optics with 60 degree phase advance per FODO cell /RoB85/

$$D_{rms}^z = 0.65m \cdot \Delta z_{rms}/10^{-4}m \quad (30)$$

with the vertical machine tune $Q_z = 47.28$. In the following section we will compare these results with computer simulations.

Computer Calculations

The computer code PETROS has been used to calculate the vertical emittance generated by orbit distortions. PETROS allows one to define field errors caused by horizontal or vertical magnet displacement, rotation of the magnet around the longitudinal axis and deviations in the magnet current.

PETROS calculates a given optics in two steps:

1. Find the closed orbit by iterative tracking of a particle trajectory through the ring and correction of its initial coordinates. Nonlinear magnets and chromaticity are taken into account.

2. The linear optics around this closed orbit are calculated. Using 5 × 5-Matrices, the coupling of the transverse motions and the dispersion functions are computed. If the closed orbit goes off center through a magnet, the horizontal and vertical dipole fields and the quadrupole and skew quadrupole fields are taken into account.

The HERA optics have been investigated by calculating 100 different sets of random vertical quadrupole displacement with a rms-value $\Delta z_{rms} = 3 \cdot 10^{-5}$ m. On avarage the closed orbit deviation $\bar{z}_{rms}$ is then 1 mm. The vertically deflecting spin rotator magnets and the sextupoles are switched off.

Fig 2 shows the distribution of the vertical dispersion rms-value averaged around the ring. The mean value ist 180 mm. This is in good agreement with the value estimated from eqn. 30, which gives 195 mm.

The vertical emittance is correlated to A_z in eqn 1 by:

$$\epsilon_z \sim \oint \frac{A_z(s)}{\rho(s)^3} ds$$

The mean value is $\epsilon_z = 1.7 \cdot 10^{-8} \pi$rad·m. The designed horizontal emittance is $3.45 \cdot 10^{-8} \pi$rad·m. Without correction of chromaticity a vertical displacement $\Delta z_{rms} = 0.03$ mm will generate a vertical emittance, which is about 50% of the horizontal emittance.

As stated before the emittance of the corrected machine will depend strongly on the accuracy of the beam monitor system. In the storage ring PETRA the monitor system has an accuracy of 0.5 mm. The emittance scales quadratically with the closed orbit rms-value. The mean value of a distibution of the vertical emittances with the quadrupole displacements scaled to produce a closed orbit rms-value of 0.5 mm is $0.11 \cdot 10^{-8}\pi\text{rad} \cdot \text{m}$, and the ratio is $\frac{\epsilon_z}{\epsilon_x} = 28\%$.

However, it is not planned to run HERA without compensation of chromaticity. An other set of optics has been calculated with sextupoles switched on. The mean value of the vertical dispersion then goes down to 28 mm, as eqn. 27 predicts. The mean value of the emittance is $0.014 \cdot 10^{-8}\pi\text{rad} \cdot \text{m}$ and the ratio $\frac{\epsilon_z}{\epsilon_x} = 0.4\%$.

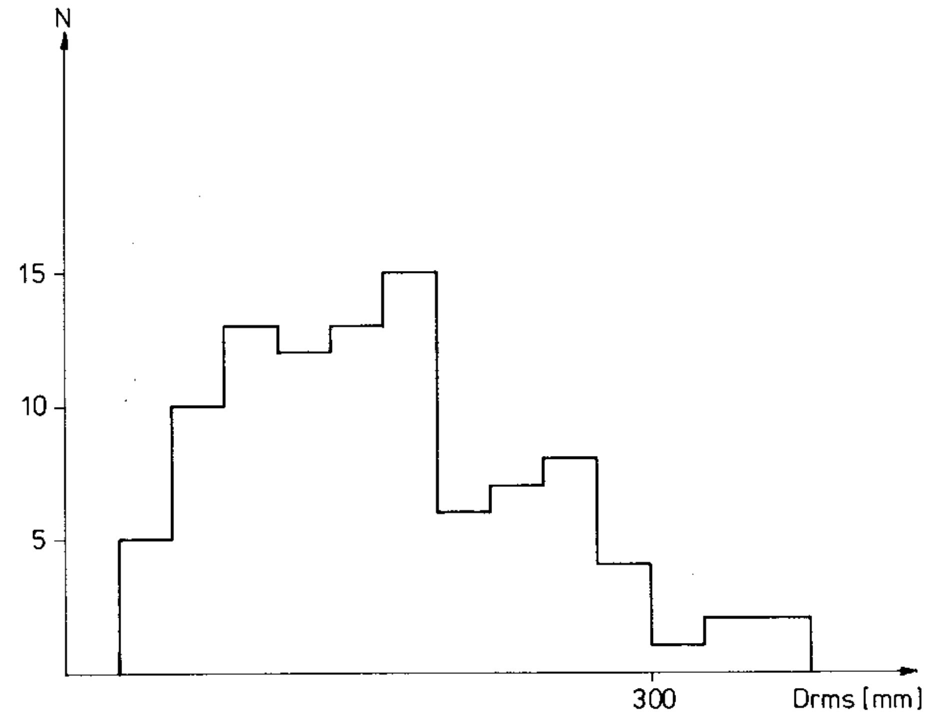

Fig 2 : Distribution of Averaged Dispersion Amplitude (PETROS results)

In the third set of calculations all important machine errors are included. The quadrupoles are displaced in the horizontal direction with a rms-value of 0.03 mm and the magnet current has an error of $\frac{\Delta I}{I} = 5 \cdot 10^{-4}$, which is assumed to be the accuracy of the HERA power supplies.

These focussing errors cause a mismatch of the beta- and dispersion functions: The peak value of β_z changes from 235 m to 242 m, the peak value of D_x from 2.23 m to 2.31 m. The rms-values of the remaining chromaticities are $\xi_x = 0.02$ and $\xi_z = 0.01$. The mean value of the vertical dispersion is again 28 mm. The mean value of the emittance is $0.013 \cdot 10^{-8}\pi\text{rad} \cdot \text{m}$.

The calculations show that once the chromaticity has been corrected by sextupoles HERA seems to be quite insensitive to machine errors.

(5) Conclusion

It has been shown that closed orbit deviations at different positions in a storage ring are strongly correlated even if the closed orbit is generated by many small random kicks as, for example, by displacements of quadrupole magnets.

As a consequence the rms value of the vertical dispersion generated in quadrupoles scales like $N^{3/2}$ if the chromaticity is not compensated, where N is the number of error kicks[1]. If the chromaticity is of the order of unity, the scale factor reduces to $\sqrt{N}$. The compensation of the off energy half integer stopband eliminates another source of spurious dispersion.

Another important aspect is that we now understand the efficiency of the "most effective correction kick" orbit correction procedure. Experience at the PETRA storage ring shows that it gives good results if the orbit is generated by a few sources of errors but that it also works for an orbit which is apparently generated by many random kicks. That is because even such an apparently random orbit contains a large "coherent" part. This result may have a large impact upon the design of orbit correction systems for any large accelerator.

[1] For constant phase advance per FODO cell and constant envelope functions, N is proportional to the radius of a storage ring.

(6) Acknowledgement

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