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C. Wetterich

Deutsches Elektronen-Synchrotron DESY, Hamburg

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FERMION MASSES AND MIXINGS FROM HIGHER DIMENSIONS

C.Wetterich

Deutsches Elektronen-Synchrotron DESY
Notkestr. 85, 2000 Hamburg

Abstract:

A systematic discussion of the structure of fermion mass matrices in terms of quantum numbers is presented. Small ratios between fermion masses and small mixing angles are related to a fine structure of scales around the unification scale. We argue that in higher dimensional models all small fermion masses should be explained from symmetry considerations since no free small Yukawa couplings are available. This leads to a scanning procedure selecting higher dimensional models consistent with realistic fermion mass patterns.

We present a six dimensional model admitting "compactifications" with only $SU(3)_C \times SU(2)_L \times U(1)_Y$ gauge symmetry, a vanishing cosmological constant and three generations of quarks and leptons. The field equations have solutions with a gauge hierarchy for weak symmetry breaking for a large range of model parameters without the need of fine tuning. The weak scale M_W is a free integration constant and the mechanism determining its order of magnitude is not yet identified. These solutions have a good chance to be classically stable. For one particular solution the largest fermion mass is the top quark mass which is of the same order as M_W. At the next level the fermion masses m_b, m_τ and m_c are supressed by a small ratio of symmetry breaking scales γ. For the mixing between the second and third generation one finds $\theta_{23} \approx m_b/m_t \approx \gamma$. The relation $m_b(M) = m_\tau(M)(1+O(\gamma))$ is

predicted. Corrections of order γ^2 induce masses for the strange quark and the muon with the relation $m_s(M) = 1/3\ m_\mu(M)$. This reproduces the qualitative order of magnitude $m_s/m_b \approx m_c/m_t$. Unfortunately this particular solution fails by predicting maximal Cabibbo mixing and $\theta_{13} \approx \gamma$. The model can be interpreted as a subgroup analysis for $E_8 \times E_8$ superstrings.

We also give a systematic discussion of higher dimensional scalar fields in non trivial representations of the gauge group. We describe the higher dimensional Higgs effect which can lead to a stabilization of the ground state.

1. Introduction

Models in more than four space-time dimensions[1] have attracted much interest as candidates for a unified description of nature. However, a realistic model is still missing so far. To gain confidence that higher dimensional theories really work, we would like to have at least one model which satisfactorily reproduces the observed low energy physics. We need the existence proof for some prototype models comparable to the SU(5) or SO(10) models for the idea of grand unification[2]. In recent years, higher dimensional models have made several steps towards such a realistic model: We have understood the appearance of non abelian gauge symmetries[3]. Higher dimensional solutions with "spontaneous compactification" (solutions with small characteristic length scale of internal space and four dimensional Poincaré symmetry P_4) have first been discovered in higher dimensional gauge theories[4] and later in pure gravity using higher derivative terms[5] or non-compact internal space[6]. Classical stability was established[7] for some solutions with spontaneous compactification. This opened the way for a realistic Kaluza-Klein cosmology with a Friedmann universe for the late history of the universe[8]. The inflationary univers may originate from a higher dimensional world[9]. Another crucial development adressed the problem of chiral fermions[10]. The number of chiral generations in four dimensions was related to an index[11,12] of internal space. Massless fermions have first been found[13,7] in higher dimensional gauge theories and the first models[14,12,15] describing quarks and leptons arouse in this context. Ten dimensional superstring theories[16] have become candidates for a unification of all forces.

What are the next problems we have to solve for realistic model building? We have to reproduce the hierarchies of masses and mixing angles for quarks and leptons. This is a very restrictive requirement for higher dimensional models since they do not have free small Yukawa couplings and are therefore rather predictive. First attempts in this direction have already been made[17,18,19]. The second problem concerns the small scale of weak symmetry breaking. (In the context of an existence

proof we only need to establish the existence of solutions with a small scale and may postpone the naturalness question of the gauge hierarchy problem. This is similar to the cosmological constant.) So far this problem has mainly been addressed in the context of supersymmetric solutions[20]. Both problems are related to an understanding of the origin and couplings of the low energy weak Higgs doublet. Finally we are still missing compactifications where the low energy gauge group is $SU(3)_C \times SU(2)_L \times U(1)_Y$ without further gauge symmetries. Although additional U(1) gauge symmetries are not excluded experimentally, their symmetry breaking at low energies leads to several theoretical problems. (Supersymmetric compactifications have the additional problem to explain low energy baryon number conservation, absence of strangeness violating neutral currents etc.). There are other problems to be solved for a realistic model, but we think that the next most crucial step concerns the understanding of the Higgs doublet.

In this paper we describe a model which admits solutions with only $SU(3)_C \times SU(2)_L \times U(1)_Y$ symmetry, vanishing four dimensional cosmological constant, an arbitrarily small scale of weak symmetry breaking and good chances to be classically stable. This is the anomaly free[21] six dimensional SO(12) model[15,17,18]. We determine the structure of the mass matrices for quarks and leptons for various solutions. Although no completely realistic pattern is found so far, we discuss one particular solution which reproduces many characteristic features of the observed fermion mass matrices. It explains the hierarchy of masses $m_t >> m_b, m_\tau, m_c >> m_s, m_\mu >> m_d, m_u, m_e$ and predicts a mixing angle between the second and third generation in the right order of magnitude as well as the relation $m_b(M_c) = m_\tau(M_c)$ and the order of magnitude $m_s/m_b \approx m_c/m_t$. Unfortunately, the mixing angles for the first generation come out too large for this particular solution.

The main topic of this paper is a systematic discussion of the structure of fermion mass matrices in higher dimensional theories. We want to understand the observed small ratios of fermion masses and the smallness of mixing angles between different generations. (We do not discuss CP violating phases in this paper.) We develop criteria uniquely based on symmetry properties which determine if a given solution can reproduce the observed pattern or not.

In four dimensions Yukawa couplings are free parameters. The fermion mass matrices can easily be reproduced but there is very little predictivity. Higher dimensional theories predict the Yukawa couplings after dimensional reduction. They are typically of the order of the gauge coupling unless they vanish because of some symmetry or topological reason. This makes these theories much more predictive and it is not easy to reproduce the observed hierarchies of fermion masses and mixings. We propose that the structure of the fermion mass matrices is entirely determined by their quantum numbers with respect to symmetries at the unification scale. We assume that the symmetry G left unbroken at the unification scale M (the largest mass scale of the model) is larger than $SU(3)_C \times SU(2)_L \times U(1)_Y \times gen_4$. There should be additional continuous or discrete symmetries. These symmetries should be spontaneously broken at scales M_1, M_2.. somewhat below M so that the low energy gauge group is only $SU(3)_C \times SU(2)_L \times U(1)_Y$. (Ratios $M_1/M \approx 1/4$ may sometimes be sufficient.) We call this a fine structure of scales at the unification scale since mass levels which are degenerate in the limit of unbroken G split by scales M_i, small compared to the characteristic level splitting M. We propose that this fine structure at the unification scale is responsible for the observed structure in fermion mass matrices. Small ratios of quark and lepton masses are induced by various powers of M_i/M. The appearance of a fine structure may either be directly related to small quantities of D dimensional internal space like 1/D, the ratio of "radius" to volume L^D/V, inverse "monopole numbers" 1/N etc. or it may result from geometric properties ("the almost round sphere") of particular solutions.

How is a fine structure reflected in the fermion mass matrices? The different quarks and leptons in general have different quantum numbers with respect to G. The various fermion bilinears appearing as entries in mass matrices must therefore couple to colour singlet and electrically neutral components of scalars in $SU(2)_L$ doublets which have different quantum numbers with respect to G. The appearance of many scalar doublets is very natural in higher dimensional theories. Usually all scalars which can couple to chiral quarks and leptons are contained in the harmonic expansion of bosonic fields unless there is some topological restriction[17,18,19]. Their Yukawa couplings are typically all of the order of the gauge coupling g. We assume that there is only one low energy Higgs doublet which must be some linear combination of those various doublets. The typical mass for the other doublets is the unification scale M. In the limit of unbroken G, doublets with different G quantum numbers cannot mix. Spontaneous symmetry breaking induces mixings proportional to various powers of M_i/M. If the low energy Higgs doublet ϕ_L has only a small admixture γ_i of a given doublet d_i, the vacuum expectation value of d_i

$$\langle d_i \rangle \equiv \gamma_i \langle \phi_L \rangle \tag{1.1}$$

will be small compared to $\langle \phi_L \rangle$ and this reflects itself in a small entry to the fermion mass matrices.

As an illustration we give a possible realistic scenario for three generations (this is not unique): The low energy Higgs scalar should mainly consist of a leading doublet H_1 which couples to the top quark but is forbidden by G symmetry to couple to other quarks or charged leptons. There should be another doublet which only couples to bottom, tau and charm. Its admixture to H_1 should be supressed by one or two powers of M_1/M. The admixture of the doublet coupling to strange quark and muon should be further supressed by higher powers of M_1/M or by a still smaller scale ratio M_2/M. Finally, the admixture for doublets coupling to the first generation should only be around 10^{-4}. Corresponding supressions should hold for the off diagonal matrix elements leading to mixing. What appears as a small Yukawa coupling in the effective low energy theory corresponds to a small admixture of the corresponding doublet to the "leading doublet". This is in turn dictated by the fine structure M_i/M! We note that our scenario could also be implemented in a four dimensional framework, but this is not necessary. In contrast, it seems almost unavoidable in higher dimensions where no Yukawa couplings of order 10^{-5} is available for the electron. If the doublet coupling to the top quark is not forbidden by symmetries or topology to couple to the electron one ends with the prediction $m_t \approx m_e$!

In this paper we discuss two aspects of the fermion mass problem in parallel. In sections 3 and 7 we give a general discription of the above mechanism and propose a systematic procedure to select models with a realistic structure of fermion mass matrices. In sections 2 and 6 we develop a given model and try to push it to its limits. This demonstrates the predictivity of our approach. It also serves as an illustration that our mechanism has good prospects to produce realistic fermion masses using only few and relatively modest scale ratios M_i/M.

In section 2 we discuss two particular solutions of the six dimensional SO(12) model with a scalar in the fifth rank antisymmetric tensor representation. They lead to three and four generations of quarks and leptons, respectively. We establish the quantum numbers for chiral quarks and leptons and for the various doublets in the harmonic expansion with respect to symmetries beyond $SU(3)_C \times SU(2)_L \times U(1)_Y$. Such symmetries are subgroups of SO(12) like the abelian symmetries $U(1)_R$, $U(1)_{B-L}$ or $U(1)_q$ or isometries on two dimensional internal space like $U(1)_G$. The fermion mass matrices M_U, M_D and M_L are then determined as functions of vacuum expectation values of the various

doublets $<d_i>$. Treating for a first approach the $<d_i>$ as free parameters, we show for the three generation solution that there exists a possible order of scales for $<d_i>$ which leads to realistic patterns for M_U, M_D and M_L! This order of scales is almost unique (except some ambiguity for the smallest lepton masses). It is rather remarkable that this simple model allows for realistic patterns at this stage since for many other models and solutions no realistic order of scales for d_i exists at all.

In section 3 we give a systematic discussion which quantum numbers can lead to realistic mass matrices. The quantum numbers of chiral fermions determine the quantum numbers of the various bilinears in entries of M_U, M_D and M_L. If two entries have the same quantum numbers with respect to the symmetry G at the unification scale, these entries will also have the same order of magnitude. As our first necessary criterium for realistic mass patterns we require that there must exist at least one assignment of scales to entries with different G quantum numbers so that realistic mass matrices are reproduced. A systematic procedure starts with the heaviest fermions - top quark and possible fermions of a fourth or higher generation. The quantum numbers of the corresponding entries should not appear anywhere else in M_U, M_D or M_L. Then one assigns the next scale to bilinears coupling to bottom, tau and charm and so on. An each step one has to check if the doublets needed to generate masses do not couple to some other entries in the fermion mass matrices violating the observational upper bounds on various diagonal and off-diagonal entries. Also the mixing angles have to be generated in this procedure. We generalise the procedure to the case where mirror fermions are present. A systematic search for viable candidates can be done on a computer. Only quantum numbers are needed as an input. If a certain solution proves consistent with this first necessary criterium, we end after the scanning with an assignement of generation quantum numbers (Le, Lμ etc.) to the various chiral fermions. A necessary order of scales is established for the doublets generating the various fermion masses and mixings and upper bounds exist for the vacuum expectation values of all other doublets coupling to quarks and leptons.

In section 6 we turn back to the particular three generation solution discussed in section 2. We study in detail the mixing between the various doublets of the model. This requires a calculation of the mass matrix for all scalar doublets. We discuss how the problem of mixing for infinitely many four dimensional scalars with quantum numbers of the weak doublet can be reduced to the diagonalization problem of a mass matrix for a finite number of doublets. In appendix C we give explicit expressions for the various contributions to the doublet mass matrix for a general class of solutions with $SU(3)_C \times SU(2)_L \times U(1)_Y$ symmetry. Our main concern, however, are the symmetry properties of this mass matrix and the appearance of various powers M_1/M describing the mixing. We show how expectation values of $SU(3)_C \times SU(2)_L) \times U(1)_Y$ singlets S_i contained in the six dimensional scalar indeed lead to a mixing between the doublet H_1 coupling to the top quark and the doublet H_2 coupling to bottom, tau and charm. For suitable (rather generic) model parameters this mixing is indeed supressed by a small parameter $\gamma \approx M_1^2/M^2$. In our model the mixing angle between the second and third generation turns out to be of order γ as well, leading to the successful qualitative relations

$$\theta_{23} \approx \frac{m_b}{m_t} \approx \frac{m_\tau}{m_t} \approx \frac{m_c}{m_t} \approx \gamma \qquad (1.2)$$

There are solutions with a particular direction in group space for $<S_i>$ for which no other terms of order γ appear in M_U, M_D or M_L. There exist, however, additional contributions of order γ^2 in M_D and M_L. They give mass to muon and strange quark with the relation

$$m_s(M_c) = \frac{1}{3} m_\mu(M_c) \qquad (1.3)$$

Since m_s and m_μ are only of order γ^2 one predices the qualitative relation

$$\frac{m_s}{m_b} \approx \frac{m_c}{m_t} \approx \gamma \qquad (1.4)$$

Also, the mixing with superheavy fermions due to scalar expectation values gives small corrections to the relation

$$m_b(M_c) = m_\tau(M_c) \qquad (1.5)$$

Unfortunately the Cabibbo mixing comes out near unity and the mixing angle between the first and third generation is of order γ. Our simple model fails at this point.

Section 7 gives a systematic description which "chains" of supression factors $(M_i/M)^P$ appear for various entries in the fermion mass matrices. Again these chains can be established only in terms of quantum numbers with respect to G. Suppose that the symmetry G is spontaneously broken by an operator O_1 with given G quantum numbers and an associated scale M_1. Assume further that weak symmetry breaking is given by a leading doublet H_1 in a given representation of G. If the fermion mass matrix element corresponding to the bilinear $\psi_1\psi_2$ gets a nonvanishing contribution from G symmetry breaking, there must exist G-invariants

$$I \sim \psi_1 \psi_2 H_1 O_1 O_1 \ldots O_1 \qquad (1.6)$$

involving a certain number of operators O_1. If P is the minimal number of operators needed to produce an invariant, the supression factor for the corresponding mass matrix entry is at least $(M_1/M)^P$ (compared to the top quark mass). Determination of P is a group theoretical problem involving an analysis of subgroups of G. We call the sequence of fermion mass matrix entries supressed by M_1/M, $(M_1/M)^2$ etc. the "chain of the operator O_1".

The symmetry breaking effects of O_1 can appear in various chanels which can be represented graphically. We discuss explicitely doublet mixings and mixings involving the non-chiral superheavy fermions from harmonic expansion. This leads us to a second necessary criterium for realistic fermion mass patterns: For models where an order of scales for fermion mass entries with different G quantum numbers has been found consistent with our first necessary criterium, there must exist a choice of symmetry breaking operators O_1, O_2... whose "chains" can reproduce the required order of scales without violating the observational upper bounds on other entries. We again describe a systematic scanning procedure (which in principle could be done by computer) using only G quantum numbers as input. For a three generation model with heaviest top mass the leading symmetry breaking operator O_1 must induce a non-vanishing bottom mass. One establishes which other fermion mass entries are contained in the chain of O_1 and checks if no upper bounds are violated. If not all masses and mixings are generated by the chain of O_1 one needs a second operator O_2 to produce the largest entry which still needs to be generated and so on. All operators O_i should correspond to vacuum expectation values of fields contained in the model. In general, there will be only a small number of O_i since G should break to $SU(3)_C \times SU(2)_L \times U(1)_Y \times gen_4$ in a few steps. We note that models consistent with both our necessary criteria need only to establish the required scales M_i for the operators O_i and the identification of the leading doublet H_1 as the main component of the low energy Higgs doublet. This ensures that all entries in fermion mass matrices have an order of magnitude compatible with observation. Quantitative fermion mass relations may also follow partly from group theoretical considerations. We expect, however, that a complete quantitative prediction of all fermion masses will involve details of the model beyond the quantum number analysis discussed in this paper.

Besides the discussion of fermion mass matrices and the (not completely successful) attempt to give an existence proof for a realistic higher dimensional model, we are concerned in this paper with an investigation of effects of higher dimensional scalar fields. So far, most of the attention in discussions of spontaneous compactification was drawn to the graviton and gauge fields (plus certain antisymmetric tensors in supersymmetric theories). Many higher dimensional models also contain scalar fields. This may be dictated by supersymmetry. Scalars also arise in a field theory expansion of string theories. Scalar fields necessarily appear if a higher dimensional theory is not believed to be the "final" unified theory. In this case they reflect effects of a unification in still higher dimensions. For example, an embedding of the six dimensional SO(12) model into the ten dimensional $E_8 \times E_8$ superstring leads to six dimensional scalars in various representations of SO(12).

There is no reason why scalars in non-trivial representations of the gauge group should not acquire vacuum expectation values. In the six dimensional SO(12) model scalar vacuum expectation values are needed for several purposes: They accomplish the symmetry breaking of the gauge group to $SU(3)_C \times SU(2)_L \times U(1)_Y$ and can thereby stabilize the ground state. They give superheavy masses to the right handed neutrinos which guarantees that left handed neutrino masses are small enough[22-25]. Scalars also provide the necessary freedom in the dynamics of this model to allow for a gauge hierarchy in weak symmetry breaking. Finally they are responsible for the mixing of doublets with different G quantum numbers and therefore needed for realistic fermion mass patterns. To simplify the discussion we concentrate on only one six dimensional scalar field in the 792 dimensional fifth rank antisymmetric tensor representation of SU(12). Introduction of additional scalars would not lead to important qualitative changes at the level of our discussion.

In section 4 we derive the field equations in the presence of the 792 scalar for an ansatz with $SU(3)_C \times SU(2)_L \times U(1)_Y \times U(1)_{\tilde{q}}$ symmetry. (Algebraic properties needed for the discussion of antisymmetric tensor representations of SO(12) are collected in appendix A.) We establish the existence of a ten parameter family of local solutions (with topology R^6). Depending on the choice of the ten integration constants these solutions will correspond to compact internal geometry or a non-compact internal space once they are extended to the whole range of validity of the coordinate system choosen. For non-compact internal space the four dimensional cosmological constant Λ_0 is a free integration constant and no fine tuning of model parameters is needed to obtain $\Lambda_0 = 0$ [6,26]. Similarly the scale of weak symmetry breaking can be considered as a free integration constant[26] and can be arbitrarily small for a large range of model parameters without the need of fine tuning. These solutions can be interpreted as spontaneous symmetry breaking through a higher dimensional Higgs effect. In contrast to the algebraic problem of finding symmetry breaking minima of the scalar potential in four dimensions, the higher dimensional Higgs effect requires to solve a coupled system of nonlinear differential equations. This corresponds to the existence of infinitely many coupled four dimensional scalar modes. We also show that symmetry breaking scales M_i somewhat smaller than the highest scale M of the model can be obtained rather naturally.

In section 5 we discuss Yukawa couplings of the chiral quarks and leptons to the various doublets contained in the six dimensional scalar. These doublets will mix with the doublets from the gauge bosons which play the role of the "leading" doublet. In presence of singlet vacuum expectation values from the six dimensional scalars the wave functions for the chiral fermions are modified. This corresponds to mixing

with the infinitely many superheavy fermions in the harmonic expansion. We give explicit expressions for the various Yukawa couplings in terms of integrals over wave functions in appendix B. For the particular three generation solution of section 2 some of the wave functions are related by G symmetry. This leads to group theoretical mass relations $m_\tau(M) = m_b(M)$ and between m_μ and m_s which hold up to corrections of order M_i/M. For our particular solution we also have relations between m_c and m_b as well as between m_t and M_W. These relations depend, however, on the particular form of the wave function for a given solution.

We demonstrate in section 8 how scalar vacuum expectation values can stabilize the ground state. The mechanism is understood qualitatively in terms of the six dimensional Higgs effect. For vanishing scalar fields the harmonic expansion of the six dimensional gauge bosons contains several tachyons[17,27]. As in four dimensions, scalar vacuum expectation values induce positive mass terms for the gauge fields. For large enough scalar expectation values these contributions are dominant and the corresponding mode is stabilized. In our model, a gauge hierarchy for weak symmetry breaking can be realized in the transition region between stability and instability. Since the low energy Higgs doublet is a mixture between doublets in various representations of G, none of these representations is massless by itself. We argue that our mechanism inducing the structure of fermion mass matrices through doublet mixing is incompatible with the idea of a scalar doublet in a given representation of G remaining massless due to some topological reasons.

We conclude this paper in section 9 with a discussion of a possible embedding of the six dimensional SO(12) model into the ten dimensional $E_8 \times E_8$ model obtained from superstrings. Indeed, the six dimensional SO(12) model can be considered as a subgroup analysis for the ten dimensional $E_8 \times E_8$ model. The ten dimensional model can be formulated as some version of a six dimensional SO(12) model with infinitely many modes. Our particular quantum number analysis for the structure of fermion mass matrices in the six dimensional SO(12) model will be relevant for the $E_8 x E_8$ superstring provided the ground state after spontaneous compactification is in an appropriate SO(12) x gen_6-deformation class which will be specified in more detail. (This does not require a topology $M^6 \times K^4$ or $M^4 \times K^2 \times K^4$ or a compactification in steps $10 \to 6 \to 4$ with different scales.)

2. Fermion Mass Matrices from a Six Dimensional SO(12) Model

To gain some intuition about typical problems in attempts to construct realistic mass matrices from higher dimensions, we will first discuss two simple examples. Both are related to compactifications of a six dimensional SO(12) theory[17,18]. In addition to six dimensional Einstein gravity and SO(12) gauge fields this model contains a scalar in the 792 dimensional fifth rank antisymmetric tensor representation of SO(12) and two Majorana-Weyl spinors with opposite six dimensional helicity belonging to the inequivalent SO(12) spinor representations 32_1 and 32_2, respectively. This model is anomaly free[12,21]. The chiral fermion content after dimensional reduction is characterized by three "monopole numbers" n, m and p [17] (integers with n+p even). The spectrum of chiral fermions has been calculated[17] for solutions with geometry $M^4 x S^2$ and internal gauge fields in a monopole configuration. Due to stability properties of the chirality index[11] the content of chiral fermions is the same for a large class of "neighbouring" solutions, including vacuum expectation values of the six dimensional scalar field.

Our first example has three chiral generations and is characterized by monopole numbers $n = 3$, $m = p = 1$. We classify[17] different quarks and leptons by the charge q of the abelian subgroup in SO(12) commuting with SO(10) and by the $SU(2)_G$ "angular momentum" for the spherically symmetric monopole solution as $(2\ell+1)_q$:

$$\begin{aligned} &u_L,\ d_L,\ u_L^c,\ e_L^c &&:\ \underline{2}_{1/2} + \underline{1}_{-1/2} \\ &d_L^c,\ e_L &&:\ \underline{3}_{-1/2} \end{aligned} \qquad (2.1)$$

There are additional chiral neutrinos which are not discussed in this paper. A tentative labeling for generations according to q and the third component I of $SU(2)_G$-spin is given in table 1. Since we are concerned with mass eigenvalues and mixings we require the fields t', b' etc. to be weak eigenstates. Quarks within the same doublet must therefore have the same values of q and I[F1]. The labeling for leptons and antiquarks u^c and d^c is arbitrary.

In our model there are several colourless $SU(2)_L$ doublet scalars with electrically neutral components. The low energy Higgs doublet is a mixture of these fields. The SO(12) gauge fields contain a 10 plet of the SO(10) subgroup with charge q = ±1. We denote the two doublets in the 10 with q = 1 by H_1 and H_2. Harmonic expansion of the internal components of these gauge fields leads to four series of scalar doublets H_1^+, H_2^+, H_1^-, H_2^-. Only H_1^+ and H_2^+ can have Yukawa couplings to the chiral quarks and leptons. (These fields and their couplings have been extensively discussed in ref. 18.) The six dimensional scalar in the fifth rank antisymmetric tensor representation of SO(12) contains the SO(10) representations 126, $\overline{126}$ and 120 with q = 0. The 126 and 120 contain four fields d_1, d_2, d_3, d_4 with the quantum numbers of weak doublets (compare table 2). We note that the doublets d_i and H_j are not only distinguished by their different SO(10) transformation properties, but also by a different abelian charge q. Finally, the six dimensional scalar also contains a 210 with q = ±1. There are doublets in the $SU(4)_C \times SU(2)_L \times SU(2)_R$ representations (10,2,2) + ($\overline{10}$,2,2) with weak hypercharge -3, -1 and +3, +1, respectively. Even though the Y = ±1 doublets among these fields have the appropriate $SU(3)_C \times SU(2)_L \times U(1)_Y$ quantum numbers to contribute to the Higgs doublet, these fields cannot have Yukawa couplings to the chiral quarks and leptons as a consequence of SO(10) symmetry and $U(1)_q$ symmetry. We will omit them for the discussion of this section.

For the spherically symmetric monopole solutions the $SU(2)_G$ representations appearing in the harmonic expansion of the doublets have "angular momentum"

$$\ell = |\lambda_i|, \; |\lambda_i| + 1 \; \ldots \tag{2.2}$$

with

$$\begin{aligned} \lambda_{H_{1+}} &= 1 - \frac{n}{2} + \frac{m}{2} \\ \lambda_{H_{2+}} &= 1 - \frac{n}{2} - \frac{m}{2} \\ \lambda_{d_i} &= \frac{m}{2} \end{aligned} \tag{2.3}$$

In this paper we go beyond spherical symmetry in internal space and we discuss solutions with only internal abelian rotation symmetry in addition to $SU(3)_C \times SU(2)_L \times U(1)_Y$ gauge symmetries. We, therefore, classify the harmonic expansion according to the third component I of $SU(2)_G$ spin. For n = 3, m = p = 1 the harmonic expansion of H_1 and H_2 contains only integer I, while the expansion of d_i leads only to half integer I.

It is straigthforward quantum number analysis to write down the allowed couplings between the scalar doublets H_j and d_i and the chiral fermions. Omitting the explicit Yukawa couplings for a moment, we can schematically write the mass matrices M_U, M_D and M_L (for charge 2/3 quarks, charge -1/3 quarks and charged leptons, respectively) as functions of vacuum expectation values H_j and d_i:

$$M_U = \begin{array}{c c} & \begin{array}{ccc} t' & c' & u' \end{array} \\ \begin{array}{c} t^{c\prime} \\ c^{c\prime} \\ u^{c\prime} \end{array} & \begin{pmatrix} H_1 , & d_U^{1/2} , & d_U^{-1/2} \\ d_U^{1/2} , & (H_2^{-1})^* , & (H_2^{0})^* \\ d_U^{-1/2} , & (H_2^{0})^* , & (H_2^{+1})^* \end{pmatrix} \end{array} \tag{2.4}$$

$$M_D = \begin{array}{c} \\ b^{c\prime} \\ s^{c\prime} \\ d^{c\prime} \end{array} \begin{array}{c} \begin{array}{ccc} \quad b' \quad & \quad s' \quad & \quad d' \quad \end{array} \\ \begin{pmatrix} H_2^{-1}, & (d_D^{1/2})^*, & (d_D^{3/2})^* \\ H_2^{0}, & (d_D^{-1/2})^*, & (d_D^{1/2})^* \\ H_2^{+1}, & (d_D^{-3/2})^*, & (d_D^{-1/2})^* \end{pmatrix} \end{array} \qquad (2.5)$$

$$M_L = \begin{array}{c} \\ \tau^{c\prime} \\ \mu^{c\prime} \\ e^{c\prime} \end{array} \begin{array}{c} \begin{array}{ccc} \quad \tau' \quad & \quad \mu' \quad & \quad e' \quad \end{array} \\ \begin{pmatrix} H_2^{-1}, & H_2^{0}, & H_2^{+1} \\ (d_L^{1/2})^*, & (d_L^{-1/2})^*, & (d_L^{-3/2})^* \\ (d_L^{3/2})^*, & (d_L^{1/2})^*, & (d_L^{-1/2})^* \end{pmatrix} \end{array} \qquad (2.6)$$

Here we have indicated I by an upper index and we denote by $d_U(d_D, d_L)$ the linear combination of d_2, d_3 and d_4(d_1, d_3 and d_4) coupling to M_U (M_D,M_L). To derive this mass pattern, we use the fact that the states H_1, H_2, d_1, d_2, d_3, d_4 have $Y = 1$ and can only couple to M_U whereas the antiparticles with $Y = -1$ couple to M_D and M_L. The rest follows from I and q conservation plus the observation that the 5 plet in $\overline{126}$ couples only to M_U whereas the $\overline{45}$ in $\overline{126}$ couples only to M_D. We note that the choice of a basis d_1, d_2, d_3 and d_4 for the doublets contained in the six dimensional scalar is somewhat arbitrary. We will come back to this point in section 5.

The non vanishing Yukawa couplings of the doublets H_j are all of the same order as the four dimensional gauge coupling g[18]. All Yukawa couplings of doublets d_i are proportional to the six dimensional Yukawa coupling $\hat{f}$. This is the only free parameter appearing in the mass matrices once the ground state solution is known. In contrast to four dimensional unification, $\hat{f}$ is not a whole matrix of couplings, but just one real parameter. The model is, therefore, highly predictive! A realistic

mass and mixing pattern with the observed hierarchy of masses and small mixing angles is only possible if the vacuum expectation values for different doublets have different scales. This means that the low energy Higgs doublet must mainly consist of only one of the H_j or d_i with small admixtures of the others. (We do not consider the case of several light Higgs doublets with mass of the order of the Fermi scale.)

It is a necessary condition for a realistic fermion mass matrix that some order of scales for doublet vacuum expectation values exists which reproduces the qualitative features of all mass matrices. In our example the leading vacuum expectation value (VEV) must be H_1 which only couples to the top quark. The next to leading VEV must induce the bottom mass and we take it to be H_2^-. (H_2^+ would be equivalent up to a redefinition of fields and quantum numbers, whereas the other candidate H_2^0 is excluded since it would lead to a relation $m_u \approx m_c \approx m_b$.) The VEV H_2^- also induces a mass of the tau lepton and the charm quark. Its contribution should be a few GeV. Next we need mixing between the second and third generation of the order of a few percent. This either requires $d_U^{1/2}$ of the same order as H_2^- or $(d_D^{1/2})^*$ of the order of a few hundred MeV. The second alternative is excluded since $(d_D^{1/2})$ also appears in M_D in the column for d' and would lead to a down quark mass of a few hundred MeV. The first choice can only work if the scale for $d_D^{1/2}$ is much smaller than the VEV $d_U^{1/2} \sim$ a few GeV (for example if only d_2 has a VEV at this high scale). A VEV of a few hundred MeV is now required to induce m_s and m_μ. The only candidate for m_s which does not lead to unacceptable values for m_d is $d_D^{-3/2}$ and we also try $d_L^{-3/2}$ to generate m_μ. We therefore interchange the definition of $s^{c\prime}$ and $d^{c\prime}$ (μ' and e'). The next step must induce the relatively large Cabibbo angle by an off diagonal element of about 30 MeV in M_D. The only candidate is $d_D^{-1/2}$. This VEV can also account for m_d with the relatively successful relation $m_d/m_s \approx \sin^2\theta_c$. (A Cabibbo angle induced by $(H_2^0)^*$ in M_U would lead to a very high mass for m_u.) The electron mass would be induced by the off diagonal contribution of $d_L^{-1/2}$ in M_L and $m_e/m_d \approx O(10^{-1})$ requires for the entries in M_L and M_D

$$\frac{d_L^{-1/2}}{d_L^{-3/2}} \lesssim \frac{1}{4} \frac{d_D^{-1/2}}{d_D^{-3/2}} \tag{2.7}$$

(All our mass relations are predicted at the unification scale and we have corrected lepton masses for their different renormalization compared to the quark masses.) Finally we need a mass for the up quark either by H_2^{+1} or by off diagonal elements H_2^0 (or both).

One sees that the order of scales for different VEVs is rather constrained. In our example there is an almost unique possibility

$$M_U = \begin{pmatrix} A & B & 0 \\ B & B & 0 \\ 0 & 0 & E \end{pmatrix} \quad M_D = \begin{pmatrix} B & 0 & 0 \\ 0 & C & D \\ 0 & 0 & 0 \end{pmatrix} \quad M_L = \begin{pmatrix} B & 0 & 0 \\ 0 & C & D \\ 0 & D & 0 \end{pmatrix} \tag{2.8}$$

with central values

$$\begin{array}{llll} A & \gtrsim & 20\ \text{GeV} & (H_1) \\ B & \approx & 3\ \text{GeV} & (H_2^{-1};\ d_U^{1/2}) \\ C & \approx & 200\ \text{MeV} & (d_D^{-3/2};\ d_L^{-3/2}) \\ D & \approx & 30\ \text{MeV} & (d_D^{-1/2}) \\ E & \approx & 5\ \text{MeV} & (H_2^{+1};\ d_L^{-1/2}) \end{array} \tag{2.9}$$

(For possible modifications for the lower lepton masses compare section 6.) The different scales differ by roughly an order of magnitude. Instead of zeros there may be non-vanishing entries bounded from above to avoid unacceptably large contributions to masses or mixings:

$$\begin{array}{lll} (d_D^{1/2};\ d_L^{1/2}) & \lesssim & E \\ d_U^{-1/2} & \lesssim & C \\ (d_D^{3/2};\ d_L^{3/2}) & \lesssim & D \\ H_2^0 & \lesssim & D\ (C) \end{array} \tag{2.10}$$

(Particularly severe constraints come from the observed smallness of the mixing between the first and third generation and the small first generation masses.) We conclude that in our example there exists a possible order of VEVs with different quantum numbers which could produce the mass and mixing pattern of leptons and quarks!

In the limit of spherically symmetric monopole solutions (and also for the generalized solutions of ref. 26) the different scalar doublets do not mix. Is it possible that vacuum expectation values of scalar singlet fields induce such mixings? In our model there are four $SU(3)_C \times SU(2)_L \times U(1)_Y$ singlets contained in the six dimensional scalar (compare table 2): The three singlets S_2, S_3 and S_4 in 210_{+1} have $SU(4)_C \times SU(2)_L \times SU(2)_R$ transformation properties (1,1,1), (15,1,1) and (15,1,3), respectively, and the singlet S_1 in 126_0 transforms as $(\overline{10},1,3)$. On the spherically symmetric monopole solution the harmonic expansion for the S_i contains "angular momenta" $\ell = |\lambda_i|, |\lambda_i| + 1 \ldots$ with

$$\begin{aligned} \lambda_{S_1} &= -m - \tfrac{3}{2} p \\ \lambda_{S_2,S_3,S_4} &= -\tfrac{1}{2} n \end{aligned} \tag{2.11}$$

In our example with $n = 3$, $m = p = 1$ the harmonic expansion for all S_i contains only states with half integer I. Let us now assume that some linear combination of S_2, S_3 and S_4 with a given value $I = I_2$ and also S_1 with a fixed value $I = I_1$ acquire vacuum expectation values. Since S_2, S_3 and S_4 have $q = 1$, $Y_{B-L} = 0$, $I_{3R} = 0$ whereas S_1 has quantum numbers

$q = 0$, $Y_{B-L} = 2$, $I_{3R} = -1$, a linear combination of the generators I, q, Y_{B-L} and I_{3R} corresponds to an unbroken abelian symmetry $U(1)_{\tilde{q}}$ (in addition to weak hypercharge). The symmetry $SU(5) \times U(1) \times U(1)_q \times SU(2)_G$ of the monopole solutions is broken to $SU(3)_C \times SU(2)_L \times U(1)_Y \times U(1)_{\tilde{q}}$ (unless the linear combination of S_2, S_3 and S_4 is an SU(5) singlet). We will consider solutions with $U(1)_{\tilde{q}}$ symmetry as a first step of spontaneous symmetry breaking and assume that $U(1)_{\tilde{q}}$ gets broken at a somewhat lower scale.

We want to induce a mixing between H_1 and $(H_2^{-1})^*$ by this first step of symmetry breaking. What are possible values for I_1 and I_2? We write the abelian charge

$$\tilde{q} = I + a(I_{3R} - \frac{1}{2} Y_{B-L}) + b\, q \qquad (2.12)$$

The vacuum expectation values of S_1 and $S_{2,3,4}$ are neutral under $U(1)_{\tilde{q}}$ implying

$$\begin{aligned} I_2 + b &= 0 \\ I_1 - 2a &= 0 \end{aligned} \qquad (2.13)$$

As a consequence b is half integer and $a = (2\bar{n}+1)/4$. To allow mixing between H_1 and $(H_2^{-1})^*$ these fields must have the same charge $\tilde{q}$. (Note that mixings between H_1 and H_2^{-1} are not allowed by hypercharge conservation.) This requires

$$\tilde{q}_{H_1} = b + \frac{1}{2} a = \tilde{q}_{(H_2^{-1})^*} = 1 - b + \frac{1}{2} a \qquad (2.14)$$

and determines

$$I_2 = -\frac{1}{2} \qquad (2.15)$$

whereas I_1 remains undetermined. Which other doublets can mix with H_1 and $(H_2^{-1})^*$? The charge $\tilde{q}$ for the doublets d_i depends on

the quantum number $I = I_d$ of the harmonic expansion:

$$\tilde{q}_{d_i} = I_d + \frac{1}{2} a \qquad (2.16)$$

Independent of a, the only fields which are allowed to mix with H_1 have $I_d = \frac{1}{2}$. Doublets d_i with other values of I_d as well as (H_2^0) and $H_2^{+1})^*$ can only mix with H_1 once $U(1)_{\tilde{q}}$ gets spontaneously broken. This has the encouraging feature that a value $d_U^{1/2}$ of the same order as H_2^{-1} as required for sufficient mixing of the third generation may indeed be generated at the first step! On the other hand, a large value for $d_D^{1/2}$ and $d_L^{1/2}$ is not excluded by $\tilde{q}$ conservation. If there is no other mechanism to suppress these VEVs to the order of a few MeV (2.10) the solution with $n = 3$, $m = p = 1$ would be ruled out! We will come back to this question in section 6.

For more than three generations or for additional mirror fermions (which are chiral with respect to symmetries outside $SU(3)_C \times SU(2)_L \times U(1)_Y$) the discussion becomes somewhat more complicated. We give a four generation example with $n = 4$, $m = p = 2$. The spherically symmetric monopole solutions are again SU(5) symmetric and the $SU(2)_G \times U(1)_q$ quantum numbers for chiral fermions are

$$\begin{aligned} u_L,\ d_L,\ u_L^c,\ e_L^c &: \ \underline{3}_{1/2} + \underline{1}_{-1/2} \\ d_L^c,\ e_L &: \ \underline{5}_{-1/2} \\ \bar{\bar{d}}_L^c,\ \bar{\bar{e}}_L &: \ \underline{1}_{-1/2} \end{aligned} \qquad (2.17)$$

We note the appearance of mirror particles $\bar{\bar{d}}_L^c$ and $\bar{\bar{e}}_L$ in this case. Similarly, the lowest harmonics for doublets are

$$\begin{aligned} H_{1+} &: \ \underline{1}_1 \\ H_{2+} &: \ \underline{5}_1 \\ d_i &: \ \underline{3}_0 \end{aligned} \qquad (2.18)$$

so that both H_j and d_i have integer I for this case. We give a tentative assignement for the quantum numbers I and q to the different generations in table 1. The fourth generation lepton and quarks are denoted by σ, a and z and the labeling for the quarks of type d^c and the leptons e remains to be specified. As before, we write the schematic form for the mass matrices.

$$M_U = \begin{array}{c} \\ a^{c'} \\ t^{c'} \\ c^{c'} \\ u^{c'} \end{array} \begin{array}{c} \begin{array}{cccc} a' & t' & c' & u' \end{array} \\ \begin{pmatrix} H_1 , & d_U^1 , & d_U^0 , & d_U^{-1} \\ d_U^1 , & (H_2^{-2})^* , & (H_2^{-1})^* , & (H_2^0)^* \\ d_U^0 , & (H_2^{-1})^* , & (H_2^0)^* , & (H_2^{+1})^* \\ d_U^{-1} , & (H_2^0)^* , & (H_2^{+1})^* , & (H_2^{+2})^* \end{pmatrix} \end{array} \tag{2.19}$$

$$M_D = \begin{array}{c} \\ D_1^c \\ D_2^c \\ D_3^c \\ D_4^c \\ D_5^c \end{array} \begin{array}{c} \begin{array}{ccccc} \bar{\bar{d}}^c & z' & b' & s' & d' \end{array} \\ \begin{pmatrix} s^{-2} , & H_2^{-2} , & (d_D^1)^* , & (d_D^2)^* , & (d_D^3)^* \\ s^{-1} , & H_2^{-1} , & (d_D^0)^* , & (d_D^1)^* , & (d_D^2)^* \\ s^0 , & H_2^0 , & (d_D^{-1})^* , & (d_D^0)^* , & (d_D^1)^* \\ s^1 , & H_2^1 , & (d_D^{-2})^* , & (d_D^{-1})^* , & (d_D^0)^* \\ s^2 , & H_2^2 , & (d_D^{-3})^* , & (d_D^{-2})^* , & (d_D^{-1})^* \end{pmatrix} \end{array} \tag{2.20}$$

$$M_L = \begin{array}{c} \\ \bar{\bar{e}} \\ \sigma^{c'} \\ \tau^{c'} \\ \mu^{c'} \\ e^{c'} \end{array} \begin{array}{c} \begin{array}{ccccc} E_1 & E_2 & E_3 & E_4 & E_5 \end{array} \\ \begin{pmatrix} s^{-2} , & s^{-1} , & s^0 , & s^1 , & s^2 \\ H_2^{-2} , & H_2^{-1} , & H_2^0 , & H_2^1 , & H_2^2 \\ (d_L^1)^* , & (d_L^0)^* , & (d_L^{-1})^* , & (d_L^{-2})^* , & (d_L^{-3})^* \\ (d_L^2)^* , & (d_L^1)^* , & (d_L^0)^* , & (d_L^{-1})^* , & (d_L^{-2})^* \\ (d_L^3)^* , & (d_L^2)^* , & (d_L^1)^* , & (d_L^0)^* , & (d_L^{-1})^* \end{pmatrix} \end{array} \tag{2.21}$$

Although we have a four generation example, the matrices M_D and M_L are 5x5 matrices because of the mirror fermions $\bar{d}^c$ and $\bar{\bar{e}}$. In consequence, singlet VEVs s appear in the mass matrices in addition to the doublets. These singlets will couple the mirror fermions to a linear combination of the chiral fermions and both will be eliminated from the low energy spectrum. This process eliminates from the mass matrices the mirror columns (or rows) with singlet entries and also a corresponding number of rows (or columns) for those standard fermions which have the leading coupling to the mirrors. In a first look on the model we may not know which standard fermions should be eliminated from the low energy spectrum. Nevertheless, we have to require that at least one choice of the remaining low energy fermions and of scales of VEVs for the Higgs doublets leads to a realistic mass and mixing pattern!

Let us try an example where E_4 and D_4^c are eliminated by coupling to $\bar{\bar{e}}$ and $\bar{d}^c$. The leading doublet VEVs should give mass to two up-type quarks, one down-type quark and one charged lepton. There are three different possible combinations: i) H_1 and H_2^{-2} with a possible addition of d_U^1; ii) d_U^1 and d_D^{-3} and either d_L^{-3} or d_L^{-2} or d_L^3 with a possible addition of H_1; iii) d_U^0 and d_D^{-2} and either d_L^{-3} or d_L^{-2} or d_L^3 with a possible addition of H_1. No other fields are allowed to acquire a VEV at this scale. We emphasize that there is no need that the mass pattern of the first three generations is repeated for the fourth generation. The quarks t and a could have degenerate mass, there could be maximal mixing between the third and fourth generation or the heaviest quark could have charge - 1/3!

We will not pursue a systematic discussion of possible mass patterns for this four generation example, but only briefly discuss a scenario similar to our first example, where in a first step of symmetry breaking some scalar $S_{2,3,4}$ with a definite value I = 1 gets a VEV. We check from (2.20) and (2.21) that E_4 and D_4^C are indeed elmininated from the low energy spectrum. We assume the leading doublet to be H_1. There remains again an unbroken symmetry $U(1)_{\tilde{q}}$ forbidding a mixing with H_1 for all other doublets except H_2^{-2}, d_U^1, d_L^1 and d_D^1. The VEVs of H_1, H_2^{-2} and d_U^1 would give masses to a, z, σ and t and induce mixing between the third and fourth generation. Large values for d_D^1 and d_L^1, however, would induce unacceptable terms in M_D and M_L and this scenario is ruled out unless other mechanisms forbid the mixing between d_L^1, d_D^1 and H_1 at this stage. The masses m_b and m_τ could be induced at a next step from d_D^{-3} whereas the charm quark could get its mass from H_2^{-1}, d_U^0, d_U^{-1} or H_2^{+2}. It becomes apparent that a systematic procedure to check all the various possibilities for mass patterns would be useful.

Our first two examples may be somewhat misleading since they may suggest that it is generically possible to find some order of scales for different VEVs which lead to a realistic mass pattern. Many models, however, lead to unacceptable mass patterns for all arbitrary assignments of scales for the different doublets!

3. Generation and Mixing Pattern of Fermion Mass Matrices

It is the aim of this section to give conditions on the quantum numbers of chiral fermions necessary to produce a realistic mass spectrum. Our approach is quite general, based only on symmetries and not on detailed dynamics. It applies especially to higher dimensional models, but also to a wide class of four dimensional unifications.

Our basic assumption states that there are no fundamental small coupling constants responsible for the smallness of certain fermion masses like the electron mass. The hierarchy of entries in the fermion mass matrices M_U, M_D and M_L is uniquely explained by symmetries and the scales of their spontaneous breaking. We will assume that the detailed dynamics may be responsible for factors of 3 or 5, but all mass ratios smaller than 1/10 have to be explained by symmetries. (Of course, this border line is somewhat arbitrary.) Our assumption is almost unavoidable in higher dimensional models with only one or two (or no!) free parameters in the coupling of fermions to bosons. It does not hold in standard grand unification which allows small Yukawa couplings and therefore has neither restrictions nor predictivity for the small fermion masses.

This approach requires that at some unification scale M the symmetries acting on quarks and leptons are larger than $SU(3)_C \times SU(2)_L \times U(1)_Y$. (In higher dimensional models M could be the scale of spontaneous compactification given by the inverse characteristic length of internal space.) These symmetries may be an abelian or non-abelian local generation group, embedding of $SU(3)_C \times SU(2)_L \times U(1)_Y$ into larger groups like $SU(5)$, $SO(10)$, E_6 or $SU(4)_C \times SU(2)_L \times SU(2)_R$, additional discrete symmetries, global Peccei-Quinn symmetries or some other remnant of higher dimensional symmetries. Our basic assumption implies that entries to the fermion mass matrices which are not distinguished by quantum numbers of these additional symmetries will have the same order of magnitude.

At this level we can distinguish between models which could possibly lead to realistic mass patterns and models for which all possible mass patterns are in contradiction to observation by the following strategy:

1) Determine the quantum numbers of all chiral fermions with respect to all additional symmetries.

2) Calculate the quantum numbers of the bilinears appearing in the mass matrices M_U, M_D and M_L. For abelian symmetries the charges of fermion bilinears are uniquely determined, whereas for non-abelian symmetries the bilinears typically contain several representations.

3) Investigate if the model provides scalar fields with the same quantum numbers as the fermion bilinears. If there is no scalar field with the quantum numbers of a given bilinear, set the corresponding entry to the mass matrices zero. (There may, however, be radiative corrections.) In higher dimensional models possible quantum numbers of scalars may be determined by direct inspection of the harmonic expansion. Alternatively, one may use topological criteria: Under some conditions the chirality index implies that certain fermions cannot get a mass or forbids certain mixings[18].

4) Label bilinear operators with different quantum numbers by X_1, X_2... X_i... Assign the same label X_i to two operators which have identical quantum numbers or are complex conjugate to each other. (If a doublet d_i can couple to a bilinear in M_U, its complex conjugate can couple to M_D or M_L if quantum numbers of the additional symmetries match. Note that by this procedure we are not sensitive to phases so that CP violation has to be checked separately.)

5) Look for arbitrary assignments of scales to X_i so that realistic mass patterns can be produced. If this fails for all possible assignments the model is inconsistent with our assumption and should be excluded. If this search is successful we are of course not guaranteed that the model really produces realistic mass patterns. However, it fulfills our first necessary criterion and can then be analyzed by the next set of conditions on quantum numbers explained in section 7.

An ambiguity may arise in this systematic search if a given fermion bilinear couples to several X_i. (This is possible for non-abelian symmetries.) If those X_i belong to the same higher dimensional field the pattern of spontaneous symmetry breaking may choose a direction in field space where the contributions of the different X_i to a given entry in a mass matrix cancel. In this case the cancellation has a group theoretical origin - some generalized Clebsch Gordon coefficient vanishes. No fine tuning of parameters, which would be in contradiction to our assumption, is needed for this cancellation. There is no problem if the corresponding X_i appear only in one entry in the mass matrices - we just may use a collective label X. If the same X_i appear in other entries of the mass matrices with different linear combinations, we have to use different colletive labels like X_{Uj}, X_{Dj}, X_{Lj} for those linear combinations which couple to different bilinears. The corresponding fields, however, are in general not linearily independent. If we find a viable scale assignment with X_{Uj}, X_{Dj} and X_{Lj} treated as independent labels, we still have to check if this assignment is not in contradiction with the fact that X_{Uj}, X_{Dj} and X_{Lj} are formed as different linear combinations of the same X_i. (For our example we discuss this question in section 5.)

What are our criteria for realistic mass patterns? Let us first discuss the three generation case: We note that there are upper bounds on the order of magnitude for the different entries in M_U, M_D and M_L:

$$M_U : \begin{pmatrix} A & B & C \\ A & B & C \\ A & B & D \end{pmatrix} \qquad (3.1)$$

$$M_D : \begin{pmatrix} B & C & D \\ B & C & D \\ B & C & D \end{pmatrix} \qquad (3.2)$$

$$M_L : \begin{pmatrix} B & C & C \\ B & C & C \\ B & C & D \end{pmatrix} \qquad (3.3)$$

with scales

$$\begin{aligned} A &\approx \text{a few hundred GeV} \\ B &\approx \text{a few GeV} \\ C &\approx \text{a few hundred MeV} \\ D &\approx \text{a few ten MeV} \end{aligned} \qquad (3.4)$$

If these bounds are violated, masses or mixings come out too large[F2]. We note that the observed smallness of the mixing between the first and third generations puts severe bounds on the entries $(M_U)_{13}$ and $(M_D)_{13}$. We have assumed that all mixing angles, including the Cabibbo-angle, are to be explained by symmetries. In addition, the smallness of m_u and m_e requires

$$\begin{aligned} (M_U)_{13}\,(M_U)_{31} &< DA \\ (M_U)_{23}\,(M_U)_{32} &\leq DB \\ (M_L)_{13}\,(M_L)_{31} &\leq DC \\ (M_L)_{23}\,(M_L)_{32} &\leq D^2 \end{aligned} \qquad (3.5)$$

We have only given a rough upper bound for the first generation and one may be more severe restricting M_{33}-entries by an upper bound $E \approx$ a few MeV. Our bounds are rather conservative and one could advocate more stringent bounds.

For the matrices M_U, M_D and M_L we have choosen a basis of weak eigenstates already adapted to the generation pattern (the first column refers to t', b', τ', the second to c', s', μ', the third to u', d', e' and similar for the rows and $t^{c'}$, $b^{c'}$... etc. We may have started our investigation of bilinears in some other basis of weak eigenstates. To connect with the basis in (3.1), (3.2) and (3.3) we may arbitrarily exchange rows in all matrices. The exchange of columns is arbitrary for M_L, whereas for M_U and M_D the same columns have to be exchanged in order to preserve weak eigenstates. Nothing is known about mixing in M_L and there could be another pattern where (3.3) is transposed[F3].

The search for a realistic mass pattern must first find some $X_i^{(1)}$ which appears only in one column in M_U and neither in M_D nor in M_L. This will be a candidate for the top mass and sets the scale A. The second step at the scale B must provide one (or several) $X_i^{(2)}$ appearing in M_D in the same column as $X^{(1)}$ and possibly in at most one column in M_L and at most one column different from $X^{(1)}$ in M_U, but nowhere else. This entry provides m_b. If it did not generate m_τ and m_c, other entries $X_j^{(2)}$ with scale B and the same criteria must be identified. The next step at the scale C has to provide masses for m_μ, m_s and guarantee enough mixing between the second and third generation. This mixing either requires non diagonal entries to M_D of order C or entries to M_U or order B. Finally, the Cabibbo angle and masses of the first generation have to be generated. Note that all masses may either be generated by diagonal or paired off diagonal entries[28] (or both).

For the case of four generations the upper bounds on entries are

$$M_U : \begin{pmatrix} A & A & A' & A' \\ A & A & A' & C \\ A & A & B & C \\ A & A & B & D \end{pmatrix} \qquad (3.6)$$

$$M_D \quad : \begin{pmatrix} A & A & A' & A' \\ A & B & C & D \\ A & B & C & D \\ A & B & C & D \end{pmatrix} \qquad (3.7)$$

$$M_L \text{ or } M_L^T \quad : \begin{pmatrix} A & A & A' & A' \\ A & B & B & B \\ A & B & C & C \\ A & B & C & D \end{pmatrix} \qquad (3.8)$$

Very little is known about mixings of a possible fourth generation: Only the relative decoupling of the first two generations requires A' to be roughly a factor 10 smaller than the largest fermion masses. In addition there will be several constraints of the type (3.5) guaranteeing that the contributions from paired off diagonal elements do not induce too large masses for the lower generations. The first step of the search for a realistic mass pattern must now provide masses for two up-type quarks, one down-type quark and one lepton (instead of only producing m_t for $N_G = 3$). The remainder of the analysis remains the same.

Some unification models may contain mirror quarks and leptons in addition to the standard quarks and leptons. These are left handed fermions in representations of $SU(3)_C \times SU(2)_L \times U(1)_Y$ which are complex conjugate to the standard left handed quarks and leptons. A mass term coupling mirror fermions to standard fermions is allowed by $SU(3)_C \times SU(2)_L \times U(1)_Y$ symmetry, but it may be forbidden by additional symmetries. Once these symmetries get broken by vacuum expectation values (VEV) of $SU(3)_C \times SU(2)_L \times U(1)_Y$ singlets S_i at the scale M_C or below, those mass terms will be induced. The mirror fermion generations and the same number of standard generations disappear from the

low energy spectrum, whereas exceeding standard fermions are protected from getting a mass by $SU(3)_C \times SU(2)_L \times U(1)_Y$ chirality. How do mirror fermions influence our systematic discussion of fermion mass and mixing patterns?

To illustrate the problem we take a three generation example with four standard quarks and one mirror quark. The quark mass matrix contains singlet and doublet entries (S_i, d_i):

$$M = \begin{array}{c} \\ q_1^C \\ q_2^C \\ q_3^C \\ \bar{\bar{q}} \end{array} \begin{array}{c} \begin{array}{cccc} q_1 & q_2 & q_3 & q_4 \end{array} \\ \begin{pmatrix} d_{11} & d_{12} & d_{13} & d_{14} \\ d_{21} & d_{22} & d_{23} & d_{24} \\ d_{31} & d_{32} & d_{33} & d_{34} \\ S_1 & S_2 & S_3 & S_4 \end{pmatrix} \end{array} \qquad (3.9)$$

If only the singlet S_1 acquires a VEV the mirror quark and the quark q_1 disappear from the low energy spectrum and we can discuss the remaining matrix for q_1^C, q_2^C, q_3^C, q_2, q_3 and q_4 as above. Assume that also S_2 acquires a VEV somewhat smaller than S_1. The superheavy quark q_1' is now mainly q_1 with an admixture of q_2 of order S_2/S_1. Correspondingly, the light quark q_2' has an admixture of q_1 of order S_2/S_1. This induces additional entries in the low energy mass matrix: The column for q_2' has in addition to the entries d_{i2} other entries of order $(S_2/S_1)d_{i1}$.[F4] This may influence the analysis for the low energy mass patterns.

We generalize our strategy for the case of mirrors: First one chooses some order of scales for the singlet bilinears S_i (which are distinguished by quantum numbers of the additional symmetries). As before, all arbitrary assignements of scales to these operators are allowed at this stage. One eliminates consecutively those standard fermions which are coupled by the largest operators S_i to mirrors until only the number of $SU(3)_C \times SU(2)_L \times U(1)_Y$- chiral fermions remains. The mass matrices for these low energy fermions are corrected by additional entries as described above. Then the analysis of the corrected mass matrices proceeds as described before by assigning arbitrary scales to doublet bilinears. We note that very often it may be possible to assign scales to S_i so that the problem of additional contributions to the low energy mass matrices is avoided. As a simplification for this case we just can try all possibilities for low energy mass matrices after eliminating arbitrary standard fermions by coupling to mirrors. However, new possibilities may be created by generating needed entries from mixing with superheavy fermions. We will see in section 7 that the order of scales for S_i is related to the order of scales for doublet bilinears X_i. This will impose much more severe restrictions.

Obviously, the chances for a model to pass the necessary criteria of this section are best if the bilinear operators are maximally differentiated by using the maximum amount of symmetry available. (If all bilinears have different quantum numbers, the choices for possible scale assignments become trivial.) It is, therefore, important for the search of realistic mass patterns to use all symmetries of a model. In some cases, however, where the maximal symmetry is broken at the scale M to a subgroup K, all scalars with the same K transformation properties get maximally mixed. (This is not always the case - compare section 7). If this happens, only the quantum numbers with respect to K can be used to differentiate between possible states. For our example with $n = 3$, $m = p = 1$, a restriction of the discussion to the symmetry $SU(5) \times U(1) \times U(1)_q \times SU(2)_G$ or subgroups of it would strengthen our criteria: In fact, this solution would be ruled out, since enough mixing between the second and third generation could not be generated without inducing much to high values for m_u or m_d.

At this point, a comment on the use of higher dimensional symmetries is in order. At the compactification scale M_c the higher dimensional coordinate, Lorentz and gauge transformations are broken to some four dimensional symmetry group. There is no

small scale ratio due to this symmetry breaking unless M_c is smaller than the overall characteristic mass scale M of the theory. In some respects it does not make sense in our approach to treat the higher dimensional symmetries as approximate symmetries acting on the chiral fermions. The chiral fermions can only be classified with respect to the four dimensional symmetry group. In our example, only some part of a given SO(12) representation contains chiral fermions, whereas the other part leads to superheavy particles. Nevertheless, the mixing of operators with the same transformation properties under the four dimensional symmetry is not always maximal. We will see that quantum numbers of the higher dimensional symmetry can indeed play a role in mass and mixing patterns. Another possible effect of higher dimensional symmetries concerns compactification in steps. For example, some fundamental (string) theory could compactify at a scale M to an intermediate higher dimensional model as the six dimensional SO(12) model. In this case small Yukawa couplings could appear in the intermediate model if the first step involved some small scale ratio. In our example, $\hat{f}/\overline{g}$ could be smaller than order one. The systematic search for viable mass patterns can easily be generalized to such a case.

To resume this section, we propose a systematic scanning procedure deciding if a given fermion content fulfills the necessary criteria for a realistic mass pattern consistent with our assumption of no small coupling parameters and no accidental cancellations. The only input are the quantum numbers of the chiral fermions with respect to symmetries beyond $SU(3)_C \times SU(2)_L \times U(1)_Y$. This concerns only rough properties of the theory and is independent of many dynamical details of a unification model. It seems to us that this is the next phenomenological step (after establishing the content of chiral fermions by index considerations) by which the compatibility with observation should be tested for higher dimensional models. A description of the logical steps needed to program this scanning on a computer will be presented elsewhere[29]. If the scanning is successful, one ends with one or several solutions where

i) the assignment of generation quantum numbers (like L_e, L_μ etc.) is fixed for the chiral fermions,

ii) an order of scales is established for the set of scalar vacuum expectation values responsible for masses and mixings,

iii) upper bounds are established for scales of other scalar vacuum expectation values coupling to fermion bilinears.

In section 7 we will give criteria to decide if this order of scales can be realized.

4. Spontaneous Symmetry Breaking with Higher Dimensional Scalar Fields

To substantiate the discussion of section 2 we will show in this section that classical solutions with $SU(3)_C \times SU(2)_L \times U(1)_Y \times U(1)_{\tilde{q}}$ symmetry indeed exist if a six dimensional scalar in the fifth rank antisymmetric tensor representation of SO(12) is present. One of these solutions should approximate the ground state up to effects breaking $U(1)_{\tilde{q}}$ and $SU(2)_L \times U(1)_Y$. We will derive the coupled field equations and discuss the general existence of solutions as well as some of their properties.

The action of our six dimensional model is

$$S = -\int d^6\hat{x}\, \hat{g}_6^{1/2} \{ \delta \hat{R} + \frac{1}{64} \mathrm{Tr}\, \hat{G}_{\hat{\mu}\hat{\nu}} \hat{G}^{\hat{\mu}\hat{\nu}} - \frac{1}{4} \mathrm{Tr}\,(D^{\hat{\mu}}\phi)(D_{\hat{\mu}}\phi) + V(\phi) - i\bar{\psi}\gamma^{\hat{\mu}} D_{\hat{\mu}}\psi - \hat{f}\,\bar{\psi}\phi\psi \} \tag{4.1}$$

Here $\hat{G}_{\hat{\mu}\hat{\nu}}$ is the field strength of the SO(12) gauge boson $\hat{A}_{\hat{\mu}}$ and ϕ is a scalar in the fifth rank antisymmetric tensor representation of SO(12). Both $\hat{G}_{\hat{\mu}\hat{\nu}}$ and ϕ are represented as 64x64 matrices. (Compare appendix A.) Otherwise we use conventions of refs. 15,17,18. The covariant derivative of ϕ is

$$D_{\hat{\mu}}\phi = \partial_{\hat{\mu}}\phi - i\bar{g}[\hat{A}_{\hat{\mu}}, \phi] \tag{4.2}$$

Neglecting fermionic excitations we obtain the bosonic field equations

$$\hat{R}_{\hat{\mu}\hat{\nu}} - \frac{1}{2}\hat{R}\,\hat{g}_{\hat{\mu}\hat{\nu}} = \frac{1}{2\delta}\hat{T}_{\hat{\mu}\hat{\nu}} \tag{4.3}$$

$$\hat{T}_{\hat{\mu}\hat{\nu}} = \frac{1}{64}\mathrm{Tr}\,\hat{G}_{\hat{\rho}\hat{\sigma}}\hat{G}^{\hat{\rho}\hat{\sigma}}\hat{g}_{\hat{\mu}\hat{\nu}} - \frac{1}{16}\mathrm{Tr}\,\hat{G}_{\hat{\mu}\hat{\rho}}\hat{G}_{\hat{\nu}}{}^{\hat{\rho}} - \frac{1}{4}\mathrm{Tr}\,(D^{\hat{\rho}}\phi)(D_{\hat{\rho}}\phi)\hat{g}_{\hat{\mu}\hat{\nu}} + \frac{1}{2}\mathrm{Tr}\,(D_{\hat{\mu}}\phi)(D_{\hat{\nu}}\phi) + V(\phi)\,\hat{g}_{\hat{\mu}\hat{\nu}} \tag{4.4}$$

$$\mathrm{Tr}\,\{\, T_{AB}\,(\, D_{\hat\mu}\,\hat G^{\hat\mu\hat\nu} - 8i\bar g\,[\phi, D^{\hat\nu}\phi\,])\} = 0 \qquad (4.5)$$

$$D_{\hat\mu} D^{\hat\mu}\phi + 2\,\frac{\partial V}{\partial\phi} = 0 \qquad (4.6)$$

Here T_{AB} are SO(12) generators in the Dirac basis. Normalization and derivative of V are defined so that a mass term reads

$$V = \frac{1}{4} M^2\, \mathrm{Tr}\,\phi^2 + \dots. \qquad (4.7)$$

$$\frac{\partial V}{\partial\phi} = \frac{1}{2} M^2 \phi + \dots. \qquad (4.8)$$

For independent complex scalar fields φ_i (compare (A49), (A84)) the scalar field equation has the standard form

$$D_{\hat\mu} D^{\hat\mu}\varphi_i + \frac{\partial V}{\partial\varphi_i^*} = 0 \qquad (4.9)$$

We are interested in solutions with $SU(3)_C \times SU(2)_L \times U(1)_Y \times U(1)_{\tilde q} \times \tilde P_4$ symmetry. The symmetry $U(1)_{\tilde q}$ is a combination of isometry- and gauge transformations. Its orbits in internal space form circles or fixed points. We parametrize the orbit (except for the fixed points) by a coordinate φ, whereas the other internal coordinate is denoted by χ. The orbits of the maximal four dimensional symmetry $\tilde P_4$ are the four coordinates x^μ of observable spacetime. The most general local form of the metric consistent with these symmetries is

$$\hat g_{\hat\mu\hat\nu} = \begin{pmatrix} \sigma(\chi)\, g^{\Lambda_0}_{\mu\nu}(x) & & \\ & -1 & \\ & & -\rho(\chi) \end{pmatrix} \qquad (4.10)$$

with $g^{\Lambda_0}_{\mu\nu}(x)$ the maximally symmetric four dimensional metric with cosmological constant Λ_0. We denote by $R_{\mu\nu\sigma\lambda}$ (without a hat)

the curvature tensor formed from $g^{\Lambda_0}_{\mu\nu}$ and one has

$$R_{\mu\nu} - \frac{1}{2} R\, g^{\Lambda_0}_{\mu\nu} = \Lambda_0\, g^{\Lambda_0}_{\mu\nu} \qquad (4.11)$$

The most general gauge field configuration consistent with our symmetries is

$$\begin{aligned} \hat A_\varphi &= H_6\, n(\chi) + (H_1 + H_2)\, m(\chi) + (H_3 + H_4 + H_5)\, p(\chi) \\ &= q\, n(\chi) - 2 I_{3R}\, m(\chi) + \frac{3}{2} Y_{B-L}\, p(\chi) \end{aligned} \qquad (4.12)$$

$$\hat A_\chi = 0$$
$$\hat A_\mu = 0 \qquad (4.13)$$

with generators H_i specified in appendix A. Finally, we need the $SU(3)_C \times SU(2)_L \times U(1)_Y$ singlet scalar fields in ϕ. There are four complex singlets S_1, S_2, S_3, S_4 whose properties are given in appendix A and table 2. The symmetry $U(1)_{\tilde q}$ requires

$$S_1 = u_1(\chi)\, \exp(i\alpha_1(\chi))\, \exp(i\bar m_1 \varphi) \qquad (4.14)$$

$$S_2 = u_2(\chi)\, \exp(i\alpha_2(\chi))\, \exp(i\bar m_2 \varphi) \qquad (4.15)$$

$$S_3 = u_3(\chi)\, \exp(i\alpha_3(\chi))\, \exp(i\bar m_2 \varphi) \qquad (4.16)$$

$$S_4 = u_4(\chi)\, \exp(i\alpha_4(\chi))\, \exp(i\bar m_2 \varphi) \qquad (4.17)$$

The functions u_i are real and α_i are phases. The integers $\bar m_1$ and $\bar m_2$ determine the unbroken symmetry $U(1)_{\tilde q}$ and will be discussed in section 6. The ansatz (4.10) - (4.17) has the most general local form consistent with the symmetry $SU(3)_C \times SU(2)_L \times U(1)_Y \times U(1)_{\tilde q} \times \tilde P_4$. In addition it is left form-invariant under

- constant translations in χ,
- rescaling of x^μ by a constant factor,
- global $U(1)_q$ and $U(1)_{B-L}$ transformations.

We will use this freedom later to fix some of our integration constants. (We remind that these transformations are symmetries of the action which are broken by any given solution of the above type provided S_1 and S_2, S_3 or S_4 do not vanish.)

We have to find solutions for the 13 unknown functions σ, ρ, n, m, p, u_1, u_2, u_3, u_4, α_1, α_2, α_3 and α_4. In terms of these functions, the scalar field equation (4.9) gives

$$u_1'' + (\tfrac{1}{2}\rho^{-1}\rho' + 2\sigma^{-1}\sigma')u_1' - \alpha_1'^2 u_1 - \rho^{-1}(\bar{m}_1 - \bar{g}(2m+3p))^2 u_1 - \tfrac{1}{2}\frac{\partial V}{\partial u_1} = 0 \quad (4.18)$$

$$u_2'' + (\tfrac{1}{2}\rho^{-1}\rho' + 2\sigma^{-1}\sigma')u_2' - \alpha_2'^2 u_2 - \rho^{-1}(\bar{m}_2 - \bar{g}n)^2 u_2 - \tfrac{1}{2}\frac{\partial V}{\partial u_2} = 0 \quad (4.19)$$

$$u_3'' + (\tfrac{1}{2}\rho^{-1}\rho' + 2\sigma^{-1}\sigma')u_3' - \alpha_3'^2 u_3 - \rho^{-1}(\bar{m}_2 - \bar{g}n)^2 u_3 - \tfrac{1}{2}\frac{\partial V}{\partial u_3} = 0 \quad (4.20)$$

$$u_4'' + (\tfrac{1}{2}\rho^{-1}\rho' + 2\sigma^{-1}\sigma')u_4' - \alpha_4'^2 u_4 - \rho^{-1}(\bar{m}_2 - \bar{g}n)^2 u_4 - \tfrac{1}{2}\frac{\partial V}{\partial u_4} = 0 \quad (4.21)$$

$$\alpha_1'' + (\tfrac{1}{2}\rho^{-1}\rho' + 2\sigma^{-1}\sigma' + 2u_1^{-1}u_1')\alpha_1' - \tfrac{1}{2}u_1^{-2}\frac{\partial V}{\partial \alpha_1} = 0 \quad (4.22)$$

$$\alpha_2'' + (\tfrac{1}{2}\rho^{-1}\rho' + 2\sigma^{-1}\sigma' + 2u_2^{-1}u_2')\alpha_2' - \tfrac{1}{2}u_2^{-2}\frac{\partial V}{\partial \alpha_2} = 0 \quad (4.23)$$

$$\alpha_3'' + (\tfrac{1}{2}\rho^{-1}\rho' + 2\sigma^{-1}\sigma' + 2u_3^{-1}u_3')\alpha_3' - \tfrac{1}{2}u_3^{-2}\frac{\partial V}{\partial \alpha_3} = 0 \quad (4.24)$$

$$\alpha_4'' + (\tfrac{1}{2}\rho^{-1}\rho' + 2\sigma^{-1}\sigma' + 2u_4^{-1}u_4')\alpha_4' - \tfrac{1}{2}u_4^{-2}\frac{\partial V}{\partial \alpha_4} = 0 \quad (4.25)$$

A prime denotes a derivative with respect to χ. The algebraic relations necessary for a derivation of field equations can be found in appendix A. The φ component of the gauge field equation (4.5) implies

$$n'' + (-\tfrac{1}{2}\rho^{-1}\rho' + 2\sigma^{-1}\sigma')n' + 2\bar{g}(\bar{m}_2 - \bar{g}n)(u_2^2 + u_3^2 + u_4^2) = 0 \quad (4.26)$$

$$m'' + (-\tfrac{1}{2}\rho^{-1}\rho' + 2\sigma^{-1}\sigma')m' + 2\bar{g}(\bar{m}_1 - \bar{g}(2m+3p))u_1^2 = 0 \quad (4.27)$$

$$p'' + (-\tfrac{1}{2}\rho^{-1}\rho' + 2\sigma^{-1}\sigma')p' + 2\bar{g}(\bar{m}_1 - \bar{g}(2m+3p))u_1^2 = 0 \quad (4.28)$$

whereas $D_{\hat{\mu}} G^{\hat{\mu}\hat{\chi}} = 0$ requires

$$\alpha_1' = 0 \quad (4.29)$$

$$\alpha_2' u_2^2 + \alpha_3' u_3^2 + \alpha_4' u_4^2 = 0 \quad (4.30)$$

Finally, one obtains for the gravitational equations

$$\hat{R}_{\mu\nu} - \tfrac{1}{2}\hat{R}\hat{g}_{\mu\nu} = \hat{g}_{\mu\nu}\{\Lambda_0\sigma^{-1} - \tfrac{3}{2}\sigma^{-1}\sigma'' - \tfrac{3}{4}\sigma^{-1}\sigma'\rho^{-1}\rho' - \tfrac{1}{2}\rho^{-1}\rho'' + \tfrac{1}{4}\rho^{-2}\rho'^2\} = \tfrac{1}{2\delta}\hat{T}_{\mu\nu} \quad (4.31)$$

$$\hat{R}_{\chi\chi} - \tfrac{1}{2}\hat{R}\hat{g}_{\chi\chi} = \hat{g}_{\chi\chi}\{2\Lambda_0\sigma^{-1} - \tfrac{3}{2}\sigma^{-2}\sigma'^2 - \sigma^{-1}\sigma'\rho^{-1}\rho'\} = \tfrac{1}{2\delta}\hat{T}_{\chi\chi} \quad (4.32)$$

$$\hat{R}_{\varphi\varphi} - \tfrac{1}{2}\hat{R}\hat{g}_{\varphi\varphi} = \hat{g}_{\varphi\varphi}\{2\Lambda_0\sigma^{-1} - 2\sigma^{-1}\sigma'' - \tfrac{1}{2}\sigma^{-2}\sigma'^2\} = \tfrac{1}{2\delta}\hat{T}_{\varphi\varphi} \quad (4.33)$$

with energy momentum tensor

$$\hat{T}_{\mu\nu} = \hat{g}_{\mu\nu}\{\tfrac{1}{2}\rho^{-1}(n'^2 + 2m'^2 + 3p'^2) + V + \sum_{i=1}^{4}[u_i'^2 + u_i^2\alpha_i'^2 + \rho^{-1}a_i^2 u_i^2]\} \quad (4.34)$$

$$\hat{T}_{\chi\chi} = \hat{g}_{\chi\chi}\{-\tfrac{1}{2}\rho^{-1}(n'^2 + 2m'^2 + 3p'^2) + V - \sum_{i=1}^{4}[u_i'^2 + u_i^2\alpha_i'^2 - \rho^{-1}a_i^2 u_i^2]\} \quad (4.35)$$

$$\hat{T}_{\varphi\varphi} = \hat{g}_{\varphi\varphi}\{-\tfrac{1}{2}\rho^{-1}(n'^2 + 2m'^2 + 3p'^2) + V + \sum_{i=1}^{4}[u_i'^2 + u_i^2\alpha_i'^2 - \rho^{-1}a_i^2 u_i^2]\} \quad (4.36)$$

Here we have defined the shorthands

$$a_1 = \bar{m}_1 - \bar{g}\,(2m + 3p) \qquad (4.37)$$

$$a_2 = a_3 = a_4 = \bar{m}_2 - \bar{g}\,n \qquad (4.38)$$

The component $\hat{T}_{\chi\varphi}$ vanishes due to (4.29), (4.30) and all other nonspecified components of $\hat{R}_{\hat{\mu}\hat{\nu}}$ or $\hat{T}_{\hat{\mu}\hat{\nu}}$ vanish trivially.

Equations (4.18) - (4.33) form a complicated system of coupled nonlinear second order differential equations. The scalar field equations respond to the gauge field configurations n, m and p and other nonlinearities are induced by the potential. The gauge fields feel χ-dependent mass terms and source terms and the gravitational equations are complicated due to the structure of the energy momentum tensor. In addition, it may seem that one has more equations then free functions and that the system may be overdetermined and has no (or only a few special) solutions. This is, however, not the case since not all of the equations are independent.

Let us first look at the two equations (4.22) and (4.29) for α_1. Since the $U(1)_{B-L}$ subgroup of SO(12) acts as a translation in α_1 and the potential is SO(12) invariant one has

$$\frac{\partial V}{\partial \alpha_1} = 0 \qquad (4.39)$$

Equ. (4.29) implies conservation of the B-L current

$$\partial_{\hat{\mu}}\, J^{\hat{\mu}}_{B-L} = \partial_{\hat{\mu}} \left\{ i\hat{g}_6^{1/2} \left[s_1^* D^{\hat{\mu}} s_1 - s_1 D^{\hat{\mu}} s_1^* \right] \right\} = 0 \qquad (4.40)$$

and using (4.39) one finds that (4.22) is identically fulfilled. Similarily, $U(1)_q$ invariance implies that the potential can only depend on two independent phase differences for which we may take $\alpha_3 - \alpha_2$ and $\alpha_4 - \alpha_2$. The potential must obey

$$\frac{\partial V}{\partial \alpha_2} + \frac{\partial V}{\partial \alpha_3} + \frac{\partial V}{\partial \alpha_4} = 0 \qquad (4.41)$$

From (4.30) and (4.41) one finds that only two of the equations (4.23), (4.24) and (4.25) are linearily independent. Two integrals for the phases are trivially obtained:

$$\alpha_1 = \alpha_{10} \qquad (4.42)$$

$$\alpha_2 = \alpha_{20} - \int_0^{\chi} d\chi \; \frac{\alpha_3' \mu_3^2 + \alpha_4' \mu_4^2}{\mu_2^2} \qquad (4.43)$$

We may use the form invariance under global $U(1)_{B-L}$ and $U(1)_q$ transformations to put

$$\alpha_{10} = \alpha_{20} = 0 \qquad (4.44)$$

What remains are the two equations (4.23) and (4.24) for the two phases α_3 and α_4. We note that $\alpha_3 - \alpha_2 \neq 0$ or $\alpha_4 - \alpha_2 \neq 0$ indicates CP violation for the corresponding solution.

We also have three gravitational equations (4.31), (4.32) and (4.33) for two functions σ and ρ. Again, only two of them are independent. This can be seen most easily from the Bianchi identity

$$\left(\hat{R}^{\hat{\mu}\hat{\nu}} - \tfrac{1}{2} \hat{R}\, \hat{g}^{\hat{\mu}\hat{\nu}} \right)_{;\hat{\mu}} = 0 \qquad (4.45)$$

which implies

$$\partial_\chi \left(\hat{R}^{\chi\chi} - \tfrac{1}{2}\hat{R}\hat{g}^{\chi\chi} \right) + \left(\tfrac{1}{2}\rho^{-1}\rho' + 2\sigma^{-1}\sigma' \right)\left(\hat{R}^{\chi\chi} - \tfrac{1}{2}\hat{R}\hat{g}^{\chi\chi} \right)$$
$$+ \tfrac{1}{2}\rho^{-1}\rho'\, \hat{g}_{\varphi\varphi} \left(\hat{R}^{\varphi\varphi} - \tfrac{1}{2}\hat{R}\hat{g}^{\varphi\varphi} \right) + \tfrac{1}{2}\sigma^{-1}\sigma'\, \hat{g}_{\mu\nu} \left(\hat{R}^{\mu\nu} - \tfrac{1}{2}\hat{R}\hat{g}^{\mu\nu} \right) \qquad (4.46)$$
$$= 0$$

Since the energy momentum is conserved as well

$$\hat{T}^{\hat{\mu}\hat{\nu}}{}_{;\hat{\mu}} = 0 \qquad (4.47)$$

we conclude that we have only two independent equations for σ and ρ. The four dimensional cosmological constant Λ_0 is a free integration constant! As a check of our equations we have verified that the nontrivial component of energy momentum conservation

$$
\begin{aligned}
\hat{T}^{\hat{\mu}\chi}{}_{;\hat{\mu}} &= \partial_\chi \Big\{ \tfrac{1}{2}\rho^{-1}(n'^2+2m'^2+3p'^2) - V + \sum_i [u_i'^2 + u_i^2\alpha_i'^2 - \rho^{-1}a_i^2 u_i^2] \Big\} \\
&+ \rho^{-1}\rho' \sum_i [u_i'^2 + u_i^2\alpha_i'^2 - \rho^{-1}a_i^2 u_i^2] \\
&+ 2\sigma^{-1}\sigma' \Big\{ \rho^{-1}(n'^2+2m'^2+3p'^2) + 2\sum_i [u_i'^2 + u_i^2\alpha_i'^2] \Big\} = 0
\end{aligned} \tag{4.48}
$$

indeed follows from the field equations (4.18) - (4.28).

We are thus left with 13 independent equations for 13 functions. This system will always admit local solutions, depending on initial values, i. e. the values of the functions and its first derivative at some given $\chi = \chi_0$. In fact, due to (4.39) and (4.41), only $\alpha_3'(\chi_0)$ and $\alpha_4'(\chi_0)$ can be given independently. The most general local solution will, therefore, have 24 independent integration constants. One of them, for example $\rho'(\chi^0)$, can be replaced by Λ_0. The form invariances of our ansatz tell us that not all of them correspond to different physical situations since certain solutions are related by six dimensional coordinate and gauge transformations. We can eliminate this freedom by the choice (4.44) and fixing arbitrarily χ_0 and $\sigma(\chi_0)$, for example

$$
\begin{aligned}
\chi_0 &= 0 \\
\sigma(0) &= 1
\end{aligned} \tag{4.49}
$$

(care has to be taken if a function becomes singular at χ_0 or if ρ, σ or u_i vanish at χ_0.)

One more integral can be easily obtained by noting that the $U(1)_Y$ gauge field does not couple to the scalar fields S_i. We write

$$ m(H_1+H_2) + p(H_3+H_4+H_5) = yY + zZ \tag{4.50} $$

with

$$
\begin{aligned}
y &= \tfrac{3}{5}(p-m) \\
z &= \tfrac{1}{5}(3p+2m)
\end{aligned} \tag{4.51}
$$

$$
\begin{aligned}
Y &= -H_1 - H_2 + \tfrac{2}{3}(H_3+H_4+H_5) \\
Z &= H_1 + H_2 + H_3 + H_4 + H_5
\end{aligned} \tag{4.52}
$$

Equations (4.27) and (4.28) decouple

$$ \partial_\chi(\sigma^2\rho^{-1/2}y') = 0 \tag{4.53} $$

$$ \partial_\chi(\sigma^2\rho^{-1/2}z') + 2\bar{g}\,\sigma^2\rho^{-1/2}a_1u_1^2 = 0 \tag{4.54} $$

and (4.53) is integrated

$$ y = y_0 + C_y \int_0^\chi d\chi\, \sigma^{-2}\rho^{1/2} \tag{4.55} $$

We remain with 10 equations for σ, ρ, a_1, a_2, u_1, u_2, u_3, u_4, α_3 and α_4 which, after some rearrangements, read

$$ 2\sigma^{-1}\sigma'' - \sigma^{-2}\sigma'^2 - \sigma^{-1}\sigma'\rho^{-1}\rho' + \tfrac{1}{\delta}\sum_i(u_i'^2 + u_i^2\alpha_i'^2) = 0 \tag{4.56} $$

$$
\begin{aligned}
&2\sigma^{-1}\sigma'' + \tfrac{1}{2}\sigma^{-2}\sigma'^2 - 2\Lambda_0\sigma^{-1} - \tfrac{5}{6}\tfrac{C_y^2}{\delta}\sigma^{-4} - \tfrac{1}{48\bar{g}^2}\rho^{-1}\left(\tfrac{1}{5}a_1'^2 + a_2'^2\right) - \\
&- \tfrac{1}{2\delta}\rho^{-1}[a_1^2u_1^2 + a_2^2(u_2^2+u_3^2+u_4^2)] + \tfrac{V}{2\delta} + \tfrac{1}{2\delta}\sum_i(u_i'^2 + u_i^2\alpha_i'^2) = 0
\end{aligned} \tag{4.57}
$$

$$ a_1'' - \tfrac{1}{2}\rho^{-1}\rho'a_1' + 2\sigma^{-1}\sigma'a_1' - 10\bar{g}^2u_1^2a_1 = 0 \tag{4.58} $$

$$ a_2'' - \tfrac{1}{2}\rho^{-1}\rho'a_2' + 2\sigma^{-1}\sigma'a_2' - 2\bar{g}^2(u_2^2+u_3^2+u_4^2)a_2 = 0 \tag{4.59} $$

$$u_1'' + \left(\tfrac{1}{2}\rho^{-1}\rho' + 2\sigma^{-1}\sigma'\right)u_1' - \rho^{-1}a_1^2 u_1 - \frac{1}{2}\frac{\partial V}{\partial u_1} = 0 \tag{4.60}$$

$$u_2'' + \left(\tfrac{1}{2}\rho^{-1}\rho' + 2\sigma^{-1}\sigma'\right)u_2' - \left(\alpha_3' u_3^2 + \alpha_4' u_4^2\right)^2 u_2^{-3} - \rho^{-1}a_2^2 u_2 - \frac{1}{2}\frac{\partial V}{\partial u_2} = 0 \tag{4.61}$$

$$u_3'' + \left(\tfrac{1}{2}\rho^{-1}\rho' + 2\sigma^{-1}\sigma'\right)u_3' - \alpha_3'^2 u_3 - \rho^{-1}a_2^2 u_3 - \frac{1}{2}\frac{\partial V}{\partial u_3} = 0 \tag{4.62}$$

$$u_4'' + \left(\tfrac{1}{2}\rho^{-1}\rho' + 2\sigma^{-1}\sigma'\right)u_4' - \alpha_4'^2 u_4 - \rho^{-1}a_2^2 u_4 - \frac{1}{2}\frac{\partial V}{\partial u_4} = 0 \tag{4.63}$$

$$\alpha_3'' + \left(\tfrac{1}{2}\rho^{-1}\rho' + 2\sigma^{-1}\sigma' + 2u_3^{-1}u_3'\right)\alpha_3' - \frac{1}{2} u_3^{-2}\frac{\partial V}{\partial \alpha_3} = 0 \tag{4.64}$$

$$\alpha_4'' + \left(\tfrac{1}{2}\rho^{-1}\rho' + 2\sigma^{-1}\sigma' + 2u_4^{-1}u_4'\right)\alpha_4' - \frac{1}{2} u_4^{-2}\frac{\partial V}{\partial \alpha_4} = 0 \tag{4.65}$$

These equations simplify for special cases. For example, if SU(5) symmetry remains unbroken, one has $m(\chi) = p(\chi)$ and, therefore, $y_0 = C_y = 0$. There is only one SU(5) singlet with q = 1 and all phases can be set to zero ($\alpha_3 = \alpha_4 = 0$). The equations for u_2, u_3 and u_4 reduce to the corresponding field equation for the SU(5) singlet field (compare (A92)) $v = \frac{1}{\sqrt{10}}(u_2 + \sqrt{3}u_3 + \sqrt{6}u_4)$.

In equ. (4.56), (4.57) and (4.59) one replaces $u_2^2 + u_3^2 + u_4^2$ by v^2 and $u_2'^2 + u_3'^2 + u_4'^2$ by v'^2.

Let us discuss a class of solutions of (4.56) - (4.65) where the $U(1)_{\tilde{q}}$ isometry has a fixpoint at $\chi = 0$ which is included into the manifold. Without further restrictions this corresponds to a topology R^6. Regular behaviour of all functions requires that $\rho(\chi)$ vanishes like χ^2 and $m(\chi)$, $n(\chi)$ and $p(\chi)$ vanish like $c_{n,m,p}\chi^2$ for $\chi \to 0$. As well, the complex scalar functions S_i must be even (odd) in for $\bar{m}_i$ even (odd) and must vanish at $\chi = 0$ except for $\bar{m}_i = 0$. Let us discuss solutions with $\bar{m}_1 = \bar{m}_2 = 0$ (compare section 6). Near the fixpoint the different functions are then approximated

$$\rho(\chi) = \chi^2$$

$$\sigma(\chi) = 1 + C_\sigma \chi^2$$

$$a_i(\chi) = C_{ai}\,\chi^2 \tag{4.66}$$

$$u_i(\chi) = u_{io} + C_{ui}\,\chi^2$$

$$\alpha_i(\chi) = \alpha_{io} + C_{\alpha i}\,\chi^2$$

and the field equations imply the relations

$$\begin{aligned} C_\sigma &= \frac{1}{2}\Lambda_0 + \frac{1}{45\bar{g}^2}\left(\frac{1}{5}C_{a_1}^2 + C_{a_2}^2\right) + \frac{5}{24}\frac{C_y^2}{\delta} - \frac{V}{8\delta} \\ C_{ui} &= \frac{1}{8}\frac{\partial V}{\partial u_{io}} \\ C_{\alpha i} &= \frac{1}{8} u_{io}^{-2}\frac{\partial V}{\partial \alpha_{io}} \end{aligned} \tag{4.67}$$

Here the scalar potential V is treated as a function of u_{io} and α_{io}.

The requirement of a fixpoint at $\chi = 0$ determines half of the "initial values" for our system of differential equations. We remain with ten free integration constants Λ_0, C_y, C_{ai}, C_{ui}, $C_{\alpha i}$. We thus have found a ten parameter family of solutions with R^6 topology in the neighbourhood of $\chi = 0$. (For u_{io}, α_{io}, C_{ui}, $C_{\alpha i} = 0$ one recovers the solutions of ref. 26). This solution can be continued for growing χ either for all χ or until a singularity occurs at $\chi = \bar{\chi}$. The fate of a given solution depends on the choice of initial values. Not all choices may correspond to physically acceptable solutions and one may impose restrictions by boundary conditions[30]. If one requires compact internal space there should be another fixpoint at $\chi = \bar{\chi}$ (which in this case corresponds to a coordinate singularity). This would imply ten more constraints on the integration constants and fix them completely in terms of the model parameters. We expect that solutions with compact internal space and nonvanishing scalar fields indeed exist. In this case a fine tuning of parameters is needed to obtain $\Lambda_0 = 0$. There is, however, no need to require a priori compactness of internal space. Solutions with a genuine singularity at $\chi = \bar{\chi}$

where Λ_0 and other integration constants remain free may offer more interesting physics!

We can interprete our solutions as spontaneous symmetry breaking due to a higher dimensional Higgs mechanism. Whereas the spherically symmetric monopole solutions with $U(1)_q$ and $U(1)_{B-L}$ symmetry may correspond to $\Lambda_0 > 0$ the vacuum expectation values of scalars can reduce Λ_0. (For the final ground state one needs $\Lambda_0 = 0$, whereas the $U(1)_{\tilde{q}}$ symmetric approximate ground state may still have a cosmological constant$\sim M_{\tilde{q}}^4$.) Also the monopole solutions are classically unstable[17,27] and satisfactory spontaneous symmetry breaking requires a stable ground state. We will discuss in section 8 that scalar vacuum expectation values can indeed stabilize the ground state.

The detailed form of spontaneous symmetry breaking depends on the scalar potential $V(\phi)$. We will assume that the configuration space for the scalar singlets S_i is affine in the range where they take expectation values. (This is not necessarily the case if the six dimensional model is obtained from a fundamental higher dimensional theory[17].) In this case the chirality index is the same as for the corresponding monopole solution for topologies $M^4 \times$ compact internal space. We expect the same chirality index also for a wide class of solutions with noncompact internal space. Even for affine configuration space there are important differences between standard four dimensional symmetry breaking and the higher dimensional Higgs effect. In four dimensions, the search for symmetry breaking minima of the scalar potential is an algebraic problem. In higher dimensions one has to solve instead a coupled system of differential equations. This corresponds to the fact that there are infinitely many four dimensional scalars in a given representation of G. For any given χ the value of $S_i(\chi)$ will in general not correspond to an extremum of the six dimensional potential V. Indeed,

the effective four dimensional potential is not only determined by V but also by the four dimensional scalars contained in the higher dimensional graviton or gauge fields (σ, ρ, a_i). In particular, a positive six dimensional scalar mass term $1/4\ M^2\ \mathrm{Tr}\ \phi^2$ is not a sign for vanishing scalar expectation values. The instability of the spherically symmetric solutions in other modes can require $S_i \neq 0$ even for $M^2 > 0$.

What about scales of spontaneous symmetry breaking? We may denote by $M_{B-L} = g\langle\varphi_{B-L}\rangle$ and $M_{210} = g\langle\varphi_{210}\rangle$ the scales of spontaneous symmetry breaking of $U(1)_{B-L}$ and $U(1)_q$. Here g is the four dimensional gauge coupling and $\langle\varphi_{B-L}\rangle$, $\langle\varphi_{210}\rangle$ are the leading vacuum expectation values (in standard normalization) in a four dimensional language. The most natural order of magnitude for M_{B-L} and M_{210} is the compactification scale M_c. Larger values are possible if the six dimensional scalar potential has a deep minimum with a higher characteristic scale. Smaller values require a tuning of parameters or a special choice of "initial values". The characteristic mass scale of the six dimensional theory is the six dimensional gravitational constant and one may assume a typical scalar mass $M^2 \sim \delta^{1/2}$. We note, however, that there are two sorts of natural small parameters in the model. One is the ratio of internal characteristic length scale $L_0 = M_c^{-1}$ to the two dimensional volume V_2.

$$\frac{1}{\bar{V}_2} = \frac{L_0^2}{V_2} \tag{4.68}$$

(For spherical symmetry one has $\bar{V}_2 = 4\pi$.) The other is the inverse of the generalized monopole number

$$\frac{1}{\tilde{N}^2} = \frac{1}{(3p^2+2m^2+n^2)} \tag{4.69}$$

In terms of these quantities the Planck mass M_p and the compactification scale M_c are

$$M_p^2 \sim 16\pi \, \bar{V}_2 \, \tilde{N}^2 \, \delta^{1/2} \tag{4.70}$$

$$M_c^2 \sim \frac{1}{\tilde{N}^2} \, \delta^{1/2} \tag{4.71}$$

One should not take these relations to seriously since proportionality factors may change an order of magnitude. A scenario with M_c^2 smaller than $M^2 \sim \delta^{1/2}$ by an order of magnitude and between two and four orders of magnitude below M_p seems, however, not unrealistic.

5. Yukawa Couplings and Fermion Mass Relations

How do fermions propagate in the field configuration discussed in the last section? Fermions are coupled to gauge fields and gravity through the covariant kinetic term and to the scalars by a six dimensional Yukawa coupling $\hat{f}$ (compare appendix B):

$$\mathcal{L}_\psi = \hat{g}_6^{1/2} \, \bar{\psi} \{ i \hat{\gamma}^{\hat{\mu}} D_{\hat{\mu}} + \hat{f} \phi \} \psi \tag{5.1}$$

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} 32_1 \\ 32_2 \end{pmatrix} \tag{5.2}$$

Neglecting all excitations except the "ground state" configuration the field equations have the form

$$i\gamma^\mu \partial_\mu \psi + M \psi = 0 \tag{5.3}$$

with mass operator[31)]

$$M = \sigma^{1/2} \{ i\gamma^5 [\mathring{\Gamma}^\alpha \mathring{D}_\alpha + \mathring{\Gamma}^\alpha \sigma^{-1} \partial_\alpha \sigma] + \hat{f} \phi \} \tag{5.4}$$

Left handed (right handed) massless modes obey

$$(\mathring{\Gamma}^\alpha \mathring{D}_\alpha \underset{(+)}{-} i \hat{f} \phi)(\sigma \psi_0) = 0 \tag{5.5}$$

(Here $\mathring{\Gamma}^\alpha$, $\mathring{D}_\alpha$ and ϕ are formed with the ground state configuration of section 4.)

In general we do not expect a vanishing mass M except for those modes protected by chirality. The chiral fermions for spherically symmetric monopole solutions are listed in ref. 17 for arbitrary m, n and p and their wave functions $\psi_0(\chi, \varphi)$ can be found in ref. 18. For a large class of "neighbouring configurations" without spherical symmetry (including expectation values for scalars S_i) the spectrum of chiral fermions will remain the same. This is due to stability properties of the chirality index[11]. The wave functions $\psi_0(\chi,\varphi)$ for the chiral fermions, however, will get modified. Deviation from spherical symmetry due to $\sigma(\chi)$, $\rho(\chi)$, $m(\chi)$, $n(\chi)$ and $p(\chi)$ amounts to a mixing of the massless lowest angular momentum state with superheavy higher angular momentum states in the tower of the harmonic expansion for functions with given $SU(3)_C \times SU(2)_L \times U(1)_Y \times U(1)_R \times U(1)_q$ transformation properties and given I. (In addition to these symmetries the wave functions form SU(5) representations for $m(\chi) = p(\chi)$ and ϕ an SU(5) singlet.) A vacuum expectation value for ϕ induces mixings between fermions in ψ_1 and ψ_2. Including the scalars, the wave functions belong to representations of the unbroken symmetry ($SU(3)_C \times SU(2)_L \times U(1)_Y \times U(1)_{\tilde{q}}$ for the generic case above). Despite of that mixing, massless modes will be found after diagonalization of the mass operator in accordance to the index.

Mass terms for chiral quarks and leptons appear only once $SU(2)_L \times U(1)_Y$ symmetry is spontaneously broken. They will be proportional to the scale φ_L of this symmetry breaking. One way to study the quark and lepton masses would investigate solutions of the field equations (4.3) - (4.6) where $SU(2)_L \times U(1)_Y$ doublet fields have nonvanishing values and to study the mass operator (5.4) in this background. For small values of the scale φ_L we can instead calculate the Yukawa couplings of the chiral fermions to the various weak doublet scalar fields in the effective four dimensional theory obtained from dimensional reduction on the $SU(3)_C \times SU(2)_L \times U(1)_Y$ symmetric approximate ground state solution. We then have to determine how the low energy Higgs scalar is composed from these various doublets. This approach deviates from the correct result by terms of order φ_L/M_C with M_C^{-1} the characteristic length scale of spontaneous compactification. In a realistic theory they are negligible. If instead of dimensional reduction on the $SU(3)_C \times SU(2)_L \times U(1)_Y$ symmetric approximate ground state solution

we use the wave fundtions from an expansion on our solution with $SU(3)_C \times SU(2)_L \times U(1)_Y \times U(1)_{\tilde{q}}$ symmetry, there will be corrections of order $M_{\tilde{q}}/M_C$ to the Yukawa couplings. ($M_{\tilde{q}}$ is the scale where $U(1)_{\tilde{q}}$ is spontaneously broken.) We also should remember that we calculate the fermion masses from classical solutions derived from an <u>effective</u> action assumed to be valid at the compactification scale. Our fermion masses therefore correspond to a scale M_C and they have to be rescaled by the standard renormalization group procedure before comparison with experiment.

We denote by u_L^k, u_L^{ck} the left handed quarks and antiquarks with charge 2/3 and -2/3, respectively, and similar for down-type quarks and charged leptons. Four dimensional mass terms are written

$$\mathcal{L}_M^{(U)} = \tilde{u}_L^{cj} (M_U)_{jk}\, u_L^k + h.c. \tag{5.6}$$

The mass matrices are proportional to vacuum expectation values of the various doublet fields in our model:

$$\begin{aligned}(M_U)_{jk} = {} & 2i\, h^{(1)}_{u_j^c u_k \bar{H}_2} \langle H_2\rangle^* + 2i\, h^{(2)}_{u_j^c u_k H_1} \langle H_1 \rangle \\ & + h_{u_j^c u_k d_2} \langle d_2 \rangle + h_{u_j^c u_k d_3} \langle d_3 \rangle + h_{u_j^c u_k d_4} \langle d_4 \rangle \\ & + h_{u_k u_j^c d_2} \langle d_2 \rangle + h_{u_k u_j^c d_3} \langle d_3 \rangle + h_{u_k u_j^c d_4} \langle d_4 \rangle \end{aligned} \tag{5.7}$$

Similar expressions hold for M_D and M_L. The factor i in the first two terms comes from $i\gamma^5$ in the mass operator (5.4). It can be rotated away by a chiral phase rotation, but the relative phase between the different contributions may play a role for CP violation.

The Yukawa couplings $h^{(1)}$ and $h^{(2)}$ have been calculated in ref. 18. All nonvanishing couplings are found in the same order of magnitude as the four dimensional gauge coupling g. We have calculated in appendix B the Yukawa couplings for the doublets d_i contained in the six dimensional scalar field. All nonvanishing couplings are of order $(\hat{f}/\bar{g})\, g$ with $\hat{f}$ and $\bar{g}$ the six dimensional Yukawa and gauge couplings. The Yukawa couplings are given in terms of two dimensional integrals over wave functions of chiral fermions and scalar doublets. The definition of the wave functions for the scalar doublets will be discussed in the next section. Here we note that for our $U(1)_{\tilde{q}}$ symmetric solutions the φ integration is performed trivially and accounts for conservation of $\tilde{q}$ (or I) in the Yukawa couplings.

The choice of a basis d_1, d_2, d_3, d_4 for the weak doublets with q = 0 within the 792 scalar of SO(12) is not necessarily the most appropriate for a discussion of fermion masses. Indeed, both d_3 and d_4 give a nonvanishing contribution to M_U, but the mechanism which determines how the low energy Higgs scalar is composed from various doublets may single out a linear combination of d_2, d_3 and d_4 which does not couple to up quarks. To apply the general analysis of section 3 we have to avoid such possible cancellations since they could be caused naturally by symmetry reasons. We define linear combinations

$$\begin{aligned}
d_{U1} &= \frac{1}{\sqrt{3}} d_2 + \frac{1}{\sqrt{6}} d_3 + \frac{1}{\sqrt{2}} d_4 \\
d_{U2} &= \frac{1}{\sqrt{3}} d_2 - \frac{1}{\sqrt{6}} d_3 - \frac{1}{\sqrt{2}} d_4 \\
d_{D1} &= \frac{1}{\sqrt{3}} d_1 - \frac{1}{\sqrt{6}} d_3 + \frac{1}{\sqrt{2}} d_4 \\
d_{D2} &= \frac{1}{\sqrt{3}} d_1 + \frac{1}{\sqrt{6}} d_3 - \frac{1}{\sqrt{2}} d_4 \\
d_{L1} &= \sqrt{\frac{3}{5}} d_1 - \sqrt{\frac{3}{10}} d_3 - \frac{1}{\sqrt{10}} d_4 \\
d_{L2} &= \sqrt{\frac{3}{5}} d_1 + \sqrt{\frac{3}{10}} d_3 + \frac{1}{\sqrt{10}} d_4
\end{aligned} \tag{5.8}$$

In terms of these fields the nonvanishing Yukawa couplings from the six dimensional scalar read

$$\begin{aligned}
h_{u_j^c u_k d_{U1}} &= \frac{1}{\sqrt{2}} \hat{f} \int d^2y\, g_2^{1/2} \sigma^2 u_j^{c+} u_k^- d_{U1} \\
h_{u_j u_k^c d_{U2}} &= \frac{1}{\sqrt{2}} \hat{f} \int d^2y\, g_2^{1/2} \sigma^2 u_j^+ u_k^{c-} d_{U2} \\
h_{d_j^c d_k d_{\bar{D}1}} &= \frac{1}{\sqrt{2}} \hat{f} \int d^2y\, g_2^{1/2} \sigma^2 d_j^{c+} d_k^- d_{D1}^* \\
h_{d_j d_k^c d_{\bar{D}2}} &= \frac{1}{\sqrt{2}} \hat{f} \int d^2y\, g_2^{1/2} \sigma^2 d_j^+ d_k^{c-} d_{D2}^* \\
h_{e_j^c e_k d_{\bar{L}1}} &= -\frac{\sqrt{10}}{2} \hat{f} \int d^2y\, g_2^{1/2} \sigma^2 e_j^{c+} e_k^- d_{L1}^* \\
h_{e_j e_k^c d_{\bar{L}2}} &= -\frac{\sqrt{10}}{2} \hat{f} \int d^2y\, g_2^{1/2} \sigma^2 e_j^+ e_k^- d_{L2}^*
\end{aligned} \tag{5.9}$$

We note that d_{L1} and d_{L2} depend linearly on d_{U1}, d_{U2}, d_{D1} and d_{D2}. These fields are properly normalized, but not orthogonal to each other.

In our example with n = 3, m = p = 1 only the fields d_{U1}, d_{U2}, d_{D2} and d_{L1} are allowed to couple to chiral quarks and leptons in a leading approximation. The fields d_L^c and e_L are obtained from the harmonic expansion of ψ_2 (compare (2.1)) and one, therefore, has $d_j^{c+} \equiv 0$ and $e_j^+ \equiv 0$. This situation may get modified due to scalar singlet vacuum expectation values S_i. The term $\sim \hat{f}\phi$ in the mass operator (5.4) can in principle mix fields with quantum numbers of d^c and e in ψ_1 and ψ_2. It may, therefore, induce non-vanishing Yukawa couplings $h_{d_j^c d_k d_{\bar{D}1}}$ of order $(\langle S_i\rangle/M_c) \cdot h_{d_j d_k^c d_{\bar{D}2}}$. The existence of such couplings depends on details of the mixing between ψ_1 and ψ_2 for the chiral modes. We will come back to this question later and assume for the moment $d_j^{c+} = e_j^+ = 0$. The discussion of section 2 shows in this case that a realistic mass pattern for M_D and M_L requires a very small admixture of d_{D2} and d_{L1} to the Higgs doublet. This will be discussed in the next section.

What can we learn about fermion mass relations at this stage of our investigation? First we note that m_b, m_τ and m_c are all generated by Yukawa couplings to H_2^{-1}, leading to the relations[18] at M_c:

$$\left| \frac{m_\tau}{m_b} \right| = \left| \frac{\int d^2y\, g_2^{1/2} \sigma^2 \tau^- \tau^{c-} H_2^{-1}}{\int d^2y\, g_2^{1/2} \sigma^2 b^- b^{c-} H_2^{-1}} \right| \tag{5.10}$$

$$\left| \frac{m_c}{m_b} \right| = \left| \frac{\int d^2y\, g_2^{1/2} \sigma^2 c^+ c^{c+} (H_2^{-1})^* + \tilde{m}^c_{mix}}{\int d^2y\, g_2^{1/2} \sigma^2 b^- b^{c-} H_2^{-1}} \right| \tag{5.11}$$

Here we denote by τ, c, d, H_2^{-1} the wave functions for the corresponding leptons, quarks and scalar doublet. The quantity $\tilde{m}^c_{mix}$ stands for contributions to m_c from mixing between c' and t'. For the first relation we observe that $\tau^- = b^{c-}$ and $\tau^{c-} = b^-$ in the limit of SU(5) symmetry. For m = p one therefore predicts

$$\frac{m_\tau(M_c)}{m_b(M_c)} = 1 + O\left(\frac{\langle s_i \rangle}{M_c}\right) \tag{5.12}$$

A small ratio S_i/M_c would be sufficient to ensure the successful relation $m_b = m_\tau$. In contrast, the wave functions for b and c are not obviously related. A prediction of m_c/m_b will depend on details of the ground state solution even if a mixing between the second and third generation can be neglected. For any given solution this ratio is calculable and may serve as a good test to distinguish realistic solutions.

What about relations similar to (5.12) between m_s and m_μ? Up to SU(5) violating corrections the wave functions for s and μ^c are again equal. For our example n = 3, m = p = 1 discussed in section 2 we can completely neglect mixing effects between b' and s' or τ' and μ'. The only difference between m_s and m_μ comes from the fact that s couples to $d_{D2}^{-3/2}$ whereas μ^c couples to $d_{L1}^{-3/2}$. However, the wave functions for $d_{D2}^{-3/2}$ and $d_{L1}^{-3/2}$ are equal up to corrections $\langle S_i \rangle / M_c$. As a consequence, the ratio m_μ/m_s is mainly given by the ratio how strong d_{D2} and d_{L1} contribute to the low energy Higgs doublet.

$$\left| \frac{m_\mu(M_c)}{m_s(M_c)} \right| = \sqrt{5} \left| \frac{\langle d_{L1}^{-3/2} \rangle}{\langle d_{D2}^{-3/2} \rangle} \right| \left(1 + O\left(\frac{\langle s_i \rangle}{M_c}\right)\right) \tag{5.13}$$

For a given solution, this ratio will be calculable from group theoretical factors relevant in the breaking of $U(1)_{\tilde{q}}$. We observe that there is no reason to expect $m_s = m_\mu$ and a realistic value $m_\mu(M_c) \approx (2 - 3)\, m_s(M_c)$ could well be obtained.

Under the assumption that H_1 gives the leading contribution to the Higgs doublet we can calculate the absolute scale for the fermion masses and determine the top quark mass. The mass of the W-bosons is

$$M_W^2(M_c) = \frac{1}{2} g^2(M_c) \langle H_1 \rangle^2 \tag{5.14}$$

and the four dimensional gauge coupling g is related to the six dimensional gauge coupling $\bar{g}$ by

$$g^2(M_c) = \frac{\bar{g}^2}{\int d^2y\, g_2^{1/2}} \tag{5.15}$$

One finds the relation

$$\frac{m_t(M_c)}{M_W(M_c)} = 4\left(\int d^2y\, g_2^{1/2} \sigma^2 t^- t^{c-} H_1\right)\left(\int d^2y\, g_2^{1/2}\right)^{1/2} \tag{5.16}$$

and, using the normalization conditions

$$\frac{m_t(M_c)}{M_W(M_c)} = 2\left(\int d^2y\, g_2^{1/2} \sigma^2 t^- t^{c-} H_1\right)\left(\int d^2y\, g_2^{1/2}\right)^{1/2} \cdot \left(\int d^2y\, g_2^{1/2} \sigma^{-3/2} t^{-*} t^-\right)^{-1/2} \left(\int d^2y\, g_2^{1/2} \sigma^{3/2} (t^{c-})^* t^{c-}\right)^{-1/2} \left(\int d^2y\, g_2^{1/2} \sigma H_1^* H_1\right)^{-1/2} \tag{5.17}$$

The model predicts a top quark mass in the same order of magnitude as M_W.

6. The Low Energy Higgs Doublet

Our model has many scalar states with the $SU(3)_C \times SU(2)_L \times U(1)_Y$ quantum numbers of the Higgs doublet: There are H_1^+, H_2^+, H_1^- and H_2^- from the SO(12) gauge fields and the doublets d_1, d_2, d_3, d_4, K_1 and K_2 from the six dimensional 792-scalar. (We denote by K_1 and K_2 the doublets in 210_{+1} within the $SU(4)_C \times SU(2)_L \times SU(2)_R$ representation (10,2,2) and $(\overline{10},2,2)$.) For every one of these states harmonic expansion leads to an infinite tower of four dimensional scalars. How to choose the physical Higgs doublet? Which linear combination of these infinitely many states could correspond to the low energy doublet responsible for spontaneous symmetry breaking at a scale $\varphi_L \sim 170$ GeV? This question splits into two separate parts.:

1) How get the different doublets mixed? How is the lightest mass eigenstate composed from the different doublets? This is the question relevant for the structure of the fermion mass matrices and will be treated in this section.

2) Why is the mass of the lightest doublet very small? What could be the reason for the tiny ratio φ_L/M_c? This is the well known gauge hierarchy problem and we comment on it in section 8.

For our solutions with $U(1)_{\tilde{q}}$ symmetry one part of the mixing problem is easy to treat: Conservation of $\tilde{q}$ forbids any mixing between states with different $\tilde{q}$. The doublet sector can be decomposed into sectors with given $\tilde{q}$ for which the dependence on the internal angular coordinate φ is fixed. However, each sector still contains infinitely many states due to the dependence of all functions on the other internal coordinate χ. States with different χ dependence are not distinguished by any quantum numbers. We still are left with a mixing problem between infinitely many states.

The direct method to cope with this problem would solve the field equations for $H_1^+(\chi,\varphi)$, $H_2^+(\chi,\varphi)$, ..., $d_i(\chi,\varphi)$ etc. coupled to all the $SU(3)_C \times SU(2)_L \times U(1)_Y$ singlet excitations. For a given solution with nonzero doublet functions the symmetry $SU(2)_L \times U(1)_Y$ is spontaneously broken and the scale φ_L could be read off directly from the solution. In our example the relative φ dependence of the different doublet functions would be dictated by $U(1)_{\tilde{q}}$ symmetry up to effects of order $M_{\tilde{q}}/M_c$. ($M_{\tilde{q}}$ is the scale where $U(1)_{\tilde{q}}$ is spontaneously broken.) The remaining equations for the χ dependence, however, would form a complex system of nonlinear (second order) differential equations. Without some more insight into the structure of the problem this would be very difficult to solve.

An important simplification occurs if the low energy Higgs doublet consists mainly of an excitation of one of the doublet states - for definiteness we may take H_1^+. Assume the wave function $H_1^+(\chi,\varphi)$ is known. Then the field equations for $H_2^+(\chi,\varphi)$, $d_i(\chi,\varphi)$ etc. can be linearized in the doublet fields, including the field $H_1^+(\chi,\varphi)$ which is treated in these equations as a given source term. This source term will be responsible for the admixture of H_2^+, d_i etc. to the low energy doublet. Its strength compared to the mass term will determine the amount of mixing.

The mixing problem can be studied by determining the doublet mass matrix in the effective four dimensional theory obtained from dimensional reduction on the $SU(3)_C \times SU(2)_L \times U(1)_Y$ symmetric approximate ground state. Expanding on this state, the correct mixing (as obtained from the full field equations) is reproduced up to corrections of order φ_L/M_c. If we use instead of the approximate ground state our solutions with $U(1)_{\tilde{q}}$ symmetry we can still calculate the mixing up to corrections of order $M_{\tilde{q}}/M_c$. We first can study the structure and order of magnitude of the doublet mixing by assuming that the final wave functions

$H_2^+(\chi,\varphi)$, $d_i(\chi,\varphi)$ etc. are known. Using these functions for a determination of the mass matrix, we have decoupled all the infinitely many modes which do not acquire a vacuum expectation value. In this basis, only a finite number of fields contributes to the low energy Higgs doublet. The other fields correspond to massive doublets which are not mixed to the modes $H_2^+(\chi,\varphi)$, $d_i(\chi,\varphi)$. (Any such mixing would distort the χ dependence of the low energy Higgs doublet in contradiction to our assumption that $H_2^+(\chi,\varphi)$ is already the solution of the field equations.) We have calculated in appendix C the doublet mass matrix for given functions $H_{1,2}^+(\chi,\varphi)$, $d_i(\chi,\varphi)$. At this stage we can already discuss the general structure. Orders of magnitude being known, we then could derive the wave functions a posteriori using an expansion in the small mixings to solve the field equations. One can proceed by steps and first solve the nonlinear field equation for the leading field H_1^+ in the approximation that all other doublets vanish. The wave function for the next to leading doublet can be calculated in an approximation linear in this field and H_1^+ (but not linear in the singlet fields), neglecting fields with even smaller admixture. This process can be repeated, and the structure of the mass matrix immediately determines which doublet fields have to be included at each step.

We will not attempt a calculation of scalar wave functions in this paper but rather concentrate on a qualitative analysis of the scalar mass matrix. How can the mass terms calculated in appencix C be understood in terms of symmetries? First we note that the doublets H_1, H_2, d_i have all $Y_{B-L} = 0$. The weak hypercharge Y is, therefore, given by $2I_{3R}$ and Y conservation implies that the only allowed mixings are between H_1, d_i and $(H_2)^*$ (not H_1 and H_2, for example). The conserved symmetry $U(1)_{\tilde q}$ is responsible for the Kronecker symbols for the m_i which characterize the φ dependence of the wave functions: The charge $\tilde q$ is

given in terms of the third component I of $SU(2)_G$ spin and the abelian charges q, I_{3R} and Y_{B-L} (see (2.12), (2.13)):

$$\tilde q_i = I_i - I_2 q - \frac{1}{2} I_1 \left(I_{3R} - \frac{1}{2} Y_{B-L}\right) \qquad (6.1)$$

Here I_i is composed from the third component m_i of angular momentum on internal space and helicity λ_i

$$I_i = m_i + \lambda_i \qquad (6.2)$$

and one has

$$\begin{aligned}\tilde q_i = m_i + \lambda_i + \tfrac{1}{2} m q + \left(\tfrac{1}{2} m + \tfrac{3}{4} p\right)\left(I_{3R} - \tfrac{1}{2} Y_{B-L}\right)\\ - \bar m_2 q - \tfrac{1}{2} \bar m_1 \left(I_{3R} - \tfrac{1}{2} Y_{B-L}\right)\end{aligned} \qquad (6.3)$$

In general, all doublets have nonzero $\tilde q$:

$$\begin{aligned}\tilde q(H_1) &= m_6 - \bar m_2 - \tfrac{1}{4}\bar m_1 + 1 + \tfrac{3}{4} m + \tfrac{3}{8} p\\ \tilde q(H_2^+) &= m_5 + \bar m_2 - \tfrac{1}{4}\bar m_1 - 1 + \tfrac{3}{4} m + \tfrac{3}{8} p\\ \tilde q(d_i) &= m_i - \tfrac{1}{4}\bar m_1 + \tfrac{3}{4} m + \tfrac{3}{8} p \qquad , i = 1 \dots 4\end{aligned} \qquad (6.4)$$

Mixings between H_1 and d_i or H_2^* and d_i are allowed by $\tilde q$ conservation if

$$\begin{aligned} m_6 - m_i - \bar m_2 + 1 &= 0 \qquad \text{for } H_1 - d_i\\ m_5 - m_i + \bar m_2 - 1 &= 0 \qquad \text{for } H_2^* - d_i\end{aligned} \qquad (6.5)$$

In our example for n = 3, m = p = 1, we need a mixing between H_1 and $(H_2^{-1})^*$ to obtain a realistic bottom mass. This requires[F5]

$$\bar m_2 = 1 \qquad (6.6)$$

Allowed are mixings between the following groups of doublets:

$$H_1^{\circ} - d^{1/2} - (H_2^{-1})^* \qquad (m_6 = m_5 = m_i = 0)$$
$$d^{-1/2} - (H_2^{\circ})^* \qquad (m_5 = m_i = -1) \qquad (6.7)$$
$$d^{-3/2} - (H_2^{+1})^* \qquad (m_5 = m_i = -2)$$

The fields $d^{-1/2}$, $d^{-3/2}$, $d^{3/2}$, $(H_2^0)^*$ and $(H_2^{+1})^*$ cannot mix with the leading doublet H_1 in the limit of $U(1)_{\tilde{q}}$ conservation.

No mixing occurs in the limit of vanishing expectation values for the six dimensional scalars S_i. For $S_i = 0$ the unbroken symmetry (for m = p) is $SU(5) \times U(1) \times U(1)_q \times U(1)_I$. The conserved charge q forbids mixing between H_1 and H_2^* or H_1 and d_i. SU(5) symmetry forbids mixing between doublets in the 5 and 45 representation of SU(5). However, a mixing between d_2 (d_1) and the 5 (45) contained in 120 is not forbidden by any of the continuous symmetries. The mass terms in L_2 break SO(10) symmetry due to nonvanishing gauge configurations $m(\chi)$, $p(\chi)$. Although SO(10) invariants can be constructed from 120, 126 and 45 or $(45)^2$, these invariants are not present in the six dimensional model. Six dimensional gauge invariance only allows the combination

$$(1,45 \times 120)_{120} \times (1,45 \times 126)_{126} \qquad (6.68)$$

which does not contain a singlet. Compared to generic four dimensional theories of spontaneous symmetry breaking, higher dimensional symmetries lead to restrictions[F6] on allowed invariants! At this level, mixing between d_1, d_2, d_3 and d_4 is forbidden by six dimensional symmetry properties.

Mixing between different doublets is induced by nonvanishing scalars $S_i \neq 0$. There are mass terms (L_3 and L_4) from the covariant derivative containing H_1 or H_2 applied on the six di-

mensional scalar field. They induce off diagonal elements in the doublet mass matrix between H_1 and d_i and between H_2^* and d_i of order

$$M_{Hd}^2 \sim M_{210} M_C \qquad (6.9)$$

with

$$M_{210} = g \langle S_{210} \rangle \qquad (6.10)$$

involving the four dimensional gauge coupling g and the leading VEV in 210 (calculated in the four dimensional theory by using the appropriate normalization for the scalar wave functions). Comparing with the diagonal mass terms

$$M_{HH}^2 \sim M_C^2 + M_{210}^2 + \ldots \qquad (6.11)$$

$$M_{dd}^2 \sim M^2 + M_{210}^2 + \ldots \qquad (6.12)$$

we find that any admixture to H_1 of doublets d_i is indeed small of $M_{210}^2 \ll M^2$. As we have discussed in section 4 this can easily be realized in our model!

We observe that no direct mixing appears between H_1 and H_2^*. SO(12) symmetry would allow a term $M_{H_1H_2^*}^2 \sim M_{210}^2$ and the mass term L_5 is a candidate to produce such a term. Again, a higher dimensional symmetry forbids the appearance of a term $M_{H_1H_2^*}^2$. In this case the relevant quantum number is two dimensional helicity on internal space. Although the internal two dimensional Lorentzgroup $SO(2) \doteq U(1)$ is not a symmetry of the four dimensional effective action after compactification (only a linear combination of this Lorentz group with several other U(1) groups leads to the unbroken group $U(1)_q$), its presence in the six dimensional action influences the pattern of mixings. Whereas six dimensional scalars have two dimensional helicity $\lambda_{\mathcal{L}} = 0$, the wave functions for H_1^+ and H_2^+ correspond to two dimensional gauge bosons with $\lambda_{\mathcal{L}} = 1$. (Note that one should distinguish between the pure Lorentz-helicity $\lambda_{\mathcal{L}}$ and the generalized helicity λ in presence of monopole configurations which determines the spectrum.) Since the symmetry breaking operator $S_i S_i^*$ in L_5 has $\lambda_{\mathcal{L}} = 0$ it cannot mix the doublets H_1^+ and $(H_2^+)^*$ which have opposite Lorentz-helicity. It only gives to the diagonal terms M_{HH}^2 a positive contribution of order M_{210}^2. On the other hand, the doublets H_1^-, H_2^- (which have no couplings to chiral fermions) have $\lambda_{\mathcal{L}} = -1$. It is easy to verify that mixings

$$M^2_{H_1^+(H_2^-)^*} \sim M^2_{210}$$
$$M^2_{H_1^-(H_2^+)^*} \sim M^2_{210} \tag{6.13}$$

indeed occur. Two dimensional derivates $\partial_\pm$ as well as the monopole fields $A_{S\pm}$ carry $\lambda_{\mathcal{L}} = \pm 1$. This explains why the mixings (6.9) as well as

$$M^2_{H_1^- d} \sim M_{210}\, M_C$$

$$M_{(H_2^-)^* d} \sim M_{210}\, M_C$$

can be induced. As another consequence, higher order terms involving $\lambda_{\mathcal{L}} = 2$ operators could in principle lead to direct $H_1^+ - (H_2^+)^*$ mixing, but those contributions should be suppressed by the large mass scales appearing as coefficients of higher order invariants.

So far all mixings are completely independent on the vacuum expectation value of S_1 in the 126 of SO(10). The VEV$\langle S_1\rangle$ is responsible for $U(1)_{B-L}$ breaking and we denote its scale

$$M_{B-L} = g\langle S_1\rangle \tag{6.15}$$

The field S_4 has $Y_{B-L} = 2$. Since all doublets H and d have $Y_{B-L} = 0$, B-L conservation implies that all contributions to the mass matrix for H and d must involve the operator $S_1 S_1^*$. If $\langle S_1\rangle$ has a definite value of I the operator $S_1 S_1^*$ has $I = 0$ and cannot induce mixings between doublets with different I. It only can contribute diagonal terms for H and induce some mixings between different d_i of order M^2_{B-L} through the scalar potential L_7. (Since S_1 is a SU(5) singlet it only can mix d_1 with $\hat{d}_{45}$ and d_2 with $\hat{d}_5$, but other mixings between d_i of order M^2_{210} can be induced by VEVs S_2, S_3, S_4 in the potential L_7.) However, the doublets K_1 and K_2 in 210 have $Y_{B-L} = \pm 2$. Mixing terms

$$M^2_{HK} \sim M_{B-L}\, M_C \tag{6.16}$$

are indeed induced by the mass terms L_3 and L_4.

We schematically summarize the order of magnitude for the different mass terms for doublets with a given charge $\tilde{q}$:

$$\mathcal{M}^2 = \begin{array}{c|ccccccc} & H_1^+ & (H_2^+)^* & d & H_1^- & (H_2^-)^* & K_1 & K_2^* \\ \hline H_1^+ & M_c^2+M_{210}^2+M_{B-L}^2, & 0, & M_c M_{210}, & 0, & M_{210}^2, & M_c M_{B-L}, & 0 \\ (H_2^+)^* & 0, & M_c^2+M_{210}^2+M_{B-L}^2, & M_c M_{210}, & M_{210}^2, & 0, & 0, & M_c M_{B-L} \\ d & M_c M_{210}, & M_c M_{210}, & M^2+M_c^2+M_{210}^2+M_{B-L}^2, & M_c M_{210}, & M_c M_{210}, & M_{210} M_{B-L}, & M_{210} M_{B-L} \\ H_1^- & 0, & M_{210}^2, & M_c M_{210}, & M_c^2+M_{210}^2+M_{B-L}^2, & 0, & M_c M_{B-L}, & 0 \\ (H_2^-)^* & M_{210}^2, & 0, & M_c M_{210}, & 0, & M_c^2+M_{210}^2+M_{B-L}^2, & 0, & M_c M_{B-L} \\ K_1 & M_c M_{B-L}, & 0, & M_{210} M_{B-L}, & M_c M_{B-L}, & 0, & M^2+M_c^2+M_{210}^2+M_{B-L}^2, & M_{210}^2 \\ K_2^* & 0, & M_c M_{B-L}, & M_{210} M_{B-L}, & 0, & M_c M_{B-L}, & M_{210}^2, & M^2+M_c^2+M_{210}^2+M_{B-L}^2 \end{array}$$

Here we have used a short hand "d" to denote the linear combination of d_1, d_2, d_3 and d_4 which finally acquires a vacuum expectation value. The doublets K induce effective mixings between H_1 and H_2 by two step processes where first H_1^+ mixes with K and this in turn mixes with $(H_2^+)^*$ etc. In the limit M^2_{B-L}, M^2_C << M^2 the two and three step mixings induce effective mixing terms in the mass matrix for H:

$$\begin{aligned}
\left(M^{(K)}_{H_1^+(H_2^+)^*}\right)^2 &\sim \left(M^{(K)}_{H_1^-(H_2^-)^*}\right)^2 \sim \frac{M_C^2 M_{B-L}^2 M_{210}^2}{M^4} \\
\left(M^{(K)}_{H_1^+(H_2^-)^*}\right)^2 &\sim \left(M^{(K)}_{H_1^-(H_2^+)^*}\right)^2 \sim \frac{M_C^2 M_{B-L}^2 M_{210}^2}{M^4} \qquad (6.18)\\
\left(M^{(K)}_{H_1^+ H_1^-}\right)^2 &\sim \left(M^{(K)}_{(H_2^+)^*(H_2^-)^*}\right)^2 \sim \frac{M_C^2 M_{B-L}^2}{M^2}
\end{aligned}$$

Indeed, we may integrate out the doublets K in tree approximation. Some graphs leading to (6.18) are shown in fig. 1.

The mixing between H_1^+ and $(H_2^-)^*$ depends on the ratios M^2_{210}/M^2_C or M^2_{210}/M^2_{B-L} and may be large if M_{210} is of the same order as M_C (see the discussion in section 8). For large mixing, the VEV of the leading doublet $\langle H_1^+\rangle$ coupling to quarks and leptons will be smaller than ~ 170 GeV thereby reducing the ratio m_t/M_W. Many step mixings including H^- induce an effective mixing between H_1^+ and $(H_2^-)^*$ or order

$$\left(M^{(K,H)}_{H_1^+(H_2^+)^*}\right)^2 \sim \frac{M_C^2 M_{210}^2 (M_{B-L}^2 + M_{210}^2)}{M^2(M_C^2 + M_{210}^2 + M_{B-L}^2)} \qquad (6.19)$$

To keep our discussion simple we will concentrate in this section on the case of small mixing between H_1^+ and $(H_2^-)^*$. In this case the many step contributions to H_1^+ - $(H_2^+)^*$ mixing from H^- and K are small compared to those involving an intermediate d and we can neglect them. (For large H_1^+ $(H_2^-)^*$ mixing both contributions may be of the same order, but the qualitative features remain the same.) Integrating out the fields K and H^- the effective mass matrix for H_1^+, $(H_2^+)^*$ and d reads

$$\mathcal{M}^2 = \begin{array}{c} \\ H_1 \\ H_2^* \\ d \end{array} \begin{array}{c} \begin{array}{ccc} H_1 & H_2^* & d \end{array} \\ \begin{pmatrix} M_1^2 & 0 & \alpha M_C M_{210} \\ 0 & M_2^2 & \beta M_C M_{210} \\ \alpha M_C M_{210} & \beta M_C M_{210} & M^2 \end{pmatrix} \end{array} \qquad (6.20)$$

Here the order of magnitude of M_1^2 and M_2^2 is bound by the maximum of M_C^2, M^2_{210} and M^2_{B-L}. The coefficients α and β are expected of order one and we have neglected corrections of order M_{210}/M, M_{B-L}/M, M_C/M. As we will discuss in section 8 a gauge hierarchy requires $M_1^2 \approx \alpha^2 M_C^2 M^2_{210}/M^2$ and we take both M_{210} and M_{B-L} either of the same order of magnitude as M_C or smaller. Diagonalization of (6.20) is straightforward, and the leading doublet H_1 induces admixtures

$$\begin{aligned}
\langle d \rangle &\approx \frac{\alpha M_C M_{210}}{M^2} \langle H_1 \rangle \\
\langle H_2^* \rangle &\approx \frac{\alpha\beta M_C^2 M_{210}^2}{M^2 M_2^2} \langle H_1 \rangle
\end{aligned} \qquad (6.21)$$

We therefore expect $\langle d\rangle$ and $\langle H_2^*\rangle$ of the same order of magnitude. Both are suppressed by a factor M_C M_{210}/M^2 compared to $\langle H_1\rangle$. Realistic masses for bottom, charm and tau require this factor to be around 1/10 (or smaller for very large m_t). This is well compatible with our discussion of scales in section 4!

We next address the question which linear combination of d_1, d_2, d_3 and d_4 acquires a vacuum expectation value. Remembering the discussion of sections 2 and 5, it is crucial for the viability of our example that d_{D2} has a very small VEV. Is this possible? The direction of $\langle d\rangle$ in the space of d_1, d_2, d_3

and d_4 will depend on the direction of $\langle S_{210}\rangle$ in the space of S_2, S_3 and S_4. We observe that to leading order all d_i have the same mass M^2. (Diagonal and off diagonal terms in the mass matrix for d_i induced by the scalar potential L_7 are suppressed by factors M^2_{210}/M^2 or M^2_{B-L}/M^2.) The mixing of the different d_i with H_1 is, therefore, determined by the structure of the mass terms M^2_{Hd} from L_3 and L_4. The dependence on wave functions in $M^{(3)}_{i_6 m_j m_6}$ in (C43) is the same for all i and the direction in d_i space is determined by SO(12)-Clebsch-Gordon coefficients. We observe that the same coefficients appear in L_3 and L_4 and we can collect the mixing terms (with S_i denoting here the group theoretical direction rather than wave functions):

$$\begin{aligned} M^{(3+4)}_{16} &= \mathcal{M}_1^2\left(S_3^* - \tfrac{1}{\sqrt{2}} S_4^*\right) \\ M^{(3+4)}_{26} &= \mathcal{M}_1^2\left(-S_3^* - \tfrac{1}{\sqrt{2}} S_4^*\right) \\ M^{(3+4)}_{36} &= \mathcal{M}_1^2\left(-S_4^*\right) \\ M^{(3+4)}_{46} &= \mathcal{M}_1^2\left(\sqrt{2}\, S_2^*\right) \end{aligned} \tag{6.22}$$

$$\begin{aligned} M^{(3+4)}_{15} &= \mathcal{M}_2^2\left(-S_3 + \tfrac{1}{\sqrt{2}} S_4\right) \\ M^{(3+4)}_{25} &= \mathcal{M}_2^2\left(S_3 + \tfrac{1}{\sqrt{2}} S_4\right) \\ M^{(3+4)}_{35} &= \mathcal{M}_2^2\left(-S_4\right) \\ M^{(3+4)}_{45} &= \mathcal{M}_2^2\left(\sqrt{2}\, S_2\right) \end{aligned} \tag{6.23}$$

(As in appendix C, 5 and 6 denote H_2^* and H_1, respectively.) Using the definitions (5.8) one finds the following mixings:

$$\begin{aligned} H_1 d_{U1} &\sim S_2^* - \tfrac{1}{\sqrt{3}} S_3^* - \tfrac{2}{\sqrt{6}} S_4^* \\ H_1 d_{U2} &\sim -S_2^* - \tfrac{1}{\sqrt{3}} S_3^* \end{aligned} \tag{6.24}$$

$$\begin{aligned} H_1 d_{D1} &\sim S_2^* + \tfrac{1}{\sqrt{3}} S_3^* \\ H_1 d_{D2} &\sim -S_2^* + \tfrac{1}{\sqrt{3}} S_3^* - \tfrac{2}{\sqrt{6}} S_4^* \\ H_1 d_{L1} &\sim \tfrac{1}{\sqrt{5}}\left(-S_2^* + \sqrt{3}\, S_3^*\right) \\ H_1 d_{L2} &\sim \tfrac{1}{\sqrt{5}}\left(S_2^* + \sqrt{3}\, S_3^* - \sqrt{6}\, S_4^*\right) \end{aligned} \tag{6.24}$$

$$\begin{aligned} H_2^* d_{U1} &\sim S_2 + \tfrac{1}{\sqrt{3}} S_3 \\ H_2^* d_{U2} &\sim -S_2 + \tfrac{1}{\sqrt{3}} S_3 + \tfrac{2}{\sqrt{6}} S_4 \\ H_2^* d_{D1} &\sim S_2 - \tfrac{1}{\sqrt{3}} S_3 + \tfrac{2}{\sqrt{6}} S_4 \\ H_2^* d_{D2} &\sim -S_2 - \tfrac{1}{\sqrt{3}} S_3 \\ H_2^* d_{L1} &\sim \tfrac{1}{\sqrt{5}}\left(-S_2 - \sqrt{3}\, S_3 + \sqrt{6}\, S_4\right) \\ H_2^* d_L^2 &\sim \tfrac{1}{\sqrt{5}}\left(S_2 - \sqrt{3}\, S_3\right) \end{aligned} \tag{6.25}$$

As a check of these SO(12)-Clebsch-Gordon coefficients we may use the outer automorphism I_{11} which changes the sign of the eleventh component of the fundamental 12 dimensional vector representation. This transforms

$$\begin{aligned} I_{11} : \quad d_{U1} &\leftrightarrow d_{U2} \\ d_{D1} &\leftrightarrow d_{D2} \\ d_{L1} &\leftrightarrow d_{L2} \\ H_1 &\leftrightarrow -H_2^* \\ S_i &\leftrightarrow S_i^* \end{aligned} \tag{6.26}$$

Neither the combination d_{D2} nor d_{L1} mixes with H_1 if

$$
\begin{aligned}
S_3 &= \frac{1}{\sqrt{3}} S_3 \\
S_4 &= -\frac{2}{\sqrt{6}} S_2
\end{aligned}
\qquad (6.27)
$$

However, a vanishing of d_{D2} and d_{L1} mixings with H_1 and H_2^* is impossible due to I_{11} (6.26). The combination (6.27) gives

$$
\begin{aligned}
H_2^* d_{D2} &\sim -\frac{4}{3} S_2 \\
H_2^* d_{L1} &\sim -\frac{4}{\sqrt{5}} S_2
\end{aligned}
\qquad (6.28)
$$

The vanishing of $H_1 d_{D2}$ and $H_1 d_{L1}$ for (6.27) has a group theoretical origin. The "generator" for the combination (6.27) (compare appendix A)

$$
\begin{aligned}
\tilde{S}_0 &= \frac{1}{\sqrt{2}} \tilde{S}_2 + \frac{1}{\sqrt{6}} \tilde{S}_3 - \frac{1}{\sqrt{3}} \tilde{S}_4 \\
&= \frac{i}{\sqrt{24}} \Big\{ 3\,\Gamma_{-1+1-2+2+6} + \Gamma_{-4+4-5+5+6} + \Gamma_{-3+3-4+4+6} + \Gamma_{-3+3-5+5+6} \\
&\quad - \Gamma_{-1+1-3+3+6} - \Gamma_{-1+1-4+4+6} - \Gamma_{-1+1-5+5+6} \\
&\quad - \Gamma_{-2+2-3+3+6} - \Gamma_{-2+2-4+4+6} - \Gamma_{-2+2-5+5+6} \Big\} \\
&= \frac{i}{2} \Gamma_{+6} \left(I_{3R}^2 - I_{3L}^2 + \frac{3}{8} Y_{B-L}^2 - \frac{1}{8} + Y_{B-L} I_{3R} \right)
\end{aligned}
\qquad (6.29)
$$

belongs to the representation $\overline{210}$ of the SU(6) subgroup of SO(12). The SO(12) spinors 32_1 and 32_2 decompose differently under SU(6). (This is the origin of the different Clebsch-Gordon coefficients for d_{D1} and d_{D2}.):

$$
32_1 \xrightarrow{SO(12)\to SU(6)} \begin{cases} 1 & \xrightarrow{SU(6)\to SU(5)} 1\,(\nu^c, \bar{\nu}^c) \\ +1 & \longrightarrow 1\,(\nu^c, \bar{\bar{\nu}}^c) \\ +15 & \longrightarrow 5(\bar{\bar{d}}^c, \bar{\bar{e}}, \bar{\bar{\nu}}) + 10(u, u^c, d, e^c) \\ +\overline{15} & \longrightarrow \bar{5}(d^c, e, \nu) + \overline{10}(\bar{\bar{u}}, \bar{\bar{u}}^c, \bar{\bar{d}}, \bar{\bar{e}}^c) \end{cases}
\qquad (6.30)
$$

$$
32_2 \longrightarrow \begin{cases} 6 & \longrightarrow 1(\nu^c, \bar{\bar{\nu}}^c) + 5(\bar{\bar{d}}^c, \bar{\bar{e}}, \bar{\bar{\nu}}) \\ +\bar{6} & \longrightarrow 1(\nu^c, \bar{\bar{\nu}}^c) + \bar{5}(d^c, e, \nu) \\ +20 & \longrightarrow 10(u, u^c, d, e^c) + \overline{10}(\bar{\bar{u}}, \bar{\bar{u}}^c, \bar{\bar{d}}, \bar{\bar{e}}^c) \end{cases}
$$

The six dimensional scalar decomposes under SU(6)

$$
792 \longrightarrow 6 + 6 + \bar{6} + \bar{6} + 20 + 20 + 70 + \overline{70} + 84 + \overline{84} + 210 + \overline{210} \qquad (6.31)
$$

and the combinations d_{U1}, d_{U2}, d_{D1} and d_{L2} belong to 6, 84 and 210 whereas d_{D2} and d_{L1} only belong to 6 and 84. Similarly, the doublets H_1 and H_2^* belong to the SU(6) representations 35 and 15, respectively. Since H_1 is in the adjoint of SU(6) the covariant derivative $D_\mu(H_1)d_{D2}$ in L_4 belongs to 6+84 and there can be no mixing with S^* in 210. Similarly $D_\mu(H_1)S$ in L_3 belongs to $\overline{210}$ forbidding a mixing term with d_{D2}. This is the same mechanism which forbids all doublet mixings for $\langle S \rangle = 0$ and is a consequence of six dimensional gauge invariance. The mixings $H_1 d_{D2}$ and $H_1 d_{L1}$ are not forbidden by global SU(6) symmetry alone since $84 \times \overline{210} \times 35$ contains singlets. However, there may be a subgroup of SU(6) or some other subgroup of SO(12) excluding these mixings. We note $S(\overline{210})$ belongs to a 75 of SU(5). Both H_1 and H_2^* can, therefore only mix with doublets d_i in the 45 representation of SU(5)!

As another consequence of SU(6) symmetry we can read from (6.30) the transformation properties for bilinears involving quarks and leptons and mirror fermions:

$$u_1\bar{\bar{u}}_2\,,\ d_1\bar{\bar{d}}_2\,,\ u_1^c\bar{\bar{u}}_2^c\,,\ e_1^c\bar{\bar{e}}_2^c \subset \bar{6}+\overline{84}+\overline{210}$$

$$u_2\bar{\bar{u}}_1\,,\ d_2\bar{\bar{d}}_1\,,\ u_2^c\bar{\bar{u}}_1^c\,,\ e_2^c\bar{\bar{e}}_1^c \subset 6+84+210 \qquad (6.32)$$

$$d_1^c\bar{\bar{d}}_2^c\,,\ e_1\bar{\bar{e}}_2 \subset \bar{6}+\overline{84}$$

$$d_c^2\bar{\bar{d}}_1^c\,,\ e_2,\bar{\bar{e}}_1 \subset 6+84$$

(The index indicates if a field is contained in 32_1 or 32_2.) This has consequences for the wave functions in (5.10) and (5.13): There are no corrections from $S(\overline{210})$ to the wave functions for b^c and τ in 32_2 (as well as s^c, μ, d^c and e). However, the fields b and τ^c in 32_2 are contaminated with higher modes in 32_1 and their relativ Clebsch-Gordon coefficients for the mixing through $S(\overline{210})$ differ by a factor (-3). The relation $m_b = m_\tau$ indeed gets corrections of order $(\hat{f}/\bar{g})(M_{210}/M_C)$ (see (5.12)).

Do we expect that VEVs for S_2, S_3 and S_4 in the linear combination corresponding to the $\overline{210}$ of SU(6) are exact solutions of the field equations (4.56) to (4.65)? The answer depends on the question if $S(\overline{210})$ induces terms linear in the orthogonal combinations $S(\overline{84})$ and $S(\bar{6})$. The only source is the scalar potential. For a generic potential we expect invariants $\bar{6}\times 210\times\overline{210}\times 210$ and $\overline{84}\times 210\times\overline{210}\times 210$ to appear. As a consequence, a generic solution has a contamination of $S(\bar{6})$ and $S(\overline{84})$ to a leading $S(\overline{210})$:

$$\langle S(\bar{6})\rangle, \langle S(\overline{84})\rangle \approx \frac{M_{210}^2}{M^2} \langle S(\overline{210})\rangle \qquad (6.33)$$

This gives a contribution to d_{D2} and d_{L1} of the same order as the mixing with H_2^* (6.28).

In summary, we have found a natural suppression factor M_{210}^2/M^2 for the ratios $d_{D2}/d_{U1,2}$, $d_{L1}/d_{U1,2}$. Unfortunately, this suppression seems not enough. Indeed, typical entries d_{D2} or d_{L1} are of order

$$\langle d_{D2}\rangle, \langle d_{L1}\rangle \approx \frac{m_{b,c}}{m_t} \langle d_{U1,2}\rangle \qquad (6.34)$$

Since $\langle d_{U2}\rangle$ must be responsible for the mixing between the second and third generation, there are entries to M_D and M_L of order ϑ_{23} x a few GeV $\approx$ 50 to 300 GeV. (ϑ_{23} is the corresponding mixing angle $\approx$ 4 to 5 %.) If m_s and m_μ are generated by $\langle d^{-3/2}\rangle$ these entries contribute directly to m_d and m_e (see section 2). Especially for the electron this mass is far outside the acceptable range even if we account for some suppression due to integrals over wave functions. We can consider the alternative that $\langle d_{L1}^{1/2}\rangle$ is responsible for the μ-mass. Taking renormalization effects into account, the muon mass indeed requires an entry of about 300 MeV. However, there is also an entry

$$\langle d_{D2}\rangle = -\frac{1}{3}\langle d_{L1}\rangle \qquad (6.35)$$

to M_D of about 100 MeV. (The factor -1/3 is due to the fact that $d_{D2,L1}$ belong to a 45 of SU(5).) Unfortunately, this entry is in the column for the down quark and cannot be used to generate m_s without inducing an unacceptably large Cabibbo angle. The wave functions for the entries to M_L and M_D are related by SU(5) symmetry (similar to (5.13)) and we remain with a real problem either for the down quark mass or the Cabibbo angle!

We could try another alternative where both mixings $H_1 d_{D2}$ and $H_2^* d_{D2}$ are forbidden. This is the case for

$$\begin{aligned} S_3 &= -\sqrt{3}\,S_2 \\ S_4 &= -\sqrt{6}\,S_2 \end{aligned} \qquad (6.36)$$

There is large mixing between H_1 and d_{L1} and $\langle d_{L1}\rangle$ could be responsible for the muon mass. Unfortunately, the mixing $H_1 d_{U2}$ (6.24) vanishes as well. The mixing angle between the second and the third generation comes out much too small! (The matrix element $(M_U)_{12}$ responsible for this mixing is proportional to $\langle d_{U2}\rangle$.) Once again, a realistic mixing pattern imposes severe constraints on model building!

To conclude this section, we have found that the mixing of doublets can indeed give a fermion mass pattern with a hierarchy of generations! Our example $n = 3$, $m = p = 1$ accounts for a large top quark mass of order M_W, masses m_b, m_τ and m_c of a few GeV, the successful relation $m_b(M_c) = m_\tau(M_c)$, a mixing angle between the second and third generation of a few percent and a muon mass in the right order of magnitude. It also could account for $m_s \approx 1/3\, m_\mu$, but unfortunately the Cabibbo angle comes out maximal for this case. (Small masses for the first generations could be induced by $U(1)_{\tilde{q}}$ breaking effects.) Although it is surprising how well this example agrees with the observed fermion mass pattern for the heavy generations, its problems will be difficult to cure. A more realistic model is required. Our analysis shows that it will not be easy to find a model obeying all the restrictions for realistic fermion mass hierarchies and mixings. To facilitate the search, we give in the next section a systematic procedure to calculate orders of magnitudes of scalar doublet mixings. We find it, nevertheless, encouraging that in this simple model a relatively modest scale ratio $M_{210}, M_c/M \approx 1/3$ to 1/4 not only could explain $m_{b,c,\tau}/m_t \approx 1/10$ to 1/20 and mixing between the second and third generation around five percent, but also $m_\mu/m_t \lesssim 10^{-2}$!

7. Scales in Fermion Mass Matrices

In this section we describe a general mechanism how small ratios of fermion masses can be induced. The main idea is that a small ratio of symmetry breaking scales at the unification scale reproduces itself in the fermion mass matrices. In higher dimensional models, the fine structure of scales for spontaneous compactification is responsible for the structure in the fermion mass matrices[18]. Since small scale ratios at the unification scale are reproduced with various powers in the fermion masses, relatively modest ratios ($M_1/M \sim 1/4$ say) may sometimes be sufficient. A small ratio M_1/M may correspond to an intrinsic small parameter of the theory. In higher dimensional models, it may alternatively be a property of a given compactification solution. Examples for small numbers are the inverse of the number of internal dimensions, the ratio of "radius" to volume of internal space, the inverse of "monopole numbers" or two different scales in internal geometry (the "almost round" sphere).

Suppose that at the unification scale M the symmetry group G acting on quarks and leptons is larger than $SU(3)_C \times SU(2)_L \times U(1)_Y$ and that the various fermion bilinears in the fermion mass terms have different quantum numbers with respect to G (compare section 3). Suppose further a vacuum expectation value (~170 GeV) for a "leading" scalar doublet H_1 (the main component of the low energy Higgs doublet) in a given representation of G. In the limit of unbroken symmetry G the leading doublet will not couple to all fermion bilinears and, therefore, induce masses only for a subset of quarks and leptons. (This should be the top quark in a realistic three generation example.) Next assume that G is spontaneously broken at a scale $M_1 < M$ by an operator O_1 (typically a VEV for a scalar field). This operator will mix doublets with other quantum numbers to H_1. The amount of mixing is suppressed by a factor $(M_1/M)^P$ where P counts the power of the operator O_1 needed to induce the doublet mixing. As a consequence, a chain of scales with various suppression factors $(M_1/M)^P$ appears in the fermion mass matrices. The order of this chain will be determined by the fermion quantum numbers.

This mechanism also appears in four dimensional unification. Indeed, it has first been discussed to explain why neutrino masses are naturally small[24,25] and why m_d differs from m_e in models predicting $m_\tau = m_b$ [32]. It has been used[33] in supersymmetric theories to produce a scale $M_W \sim M_I^2/M_P$. In the context of family unification it was applied[34] to generation splitting. However, in four dimensional theories it may only partially be responsible for the structure of fermion mass matrices, since other small parameters (Yukawa couplings) are available. In contrast, higher dimensional theories have typically no small Yukawa couplings in the effective four dimensional theory. In this case, all structure of the fermion mass matrices has to be described by this mechanism. As we have seen in section 6, this gives severe constraints on model building, but also may offer an understanding of the fermion mass puzzle!

For higher dimensional models we will first assume for simplicity that the compactification scale M_c (the inverse of the characteristic length of internal space which may be defined by the mass gap of harmonic expansion) equals the largest relevant scale M in the model. (See, however, below for a discussion of the case $M_c < M$.) One makes a harmonic expansion on a state with maximal symmetry G unbroken at M_c. This state should approximate the true ground state up to symmetry breaking effects with a scale M_1 below M_c. Different quarks and leptons as well as scalar doublets are classified in representations of G according to section 3. The appearance of small factors M_1/M in the fermion mass matrices is now mainly a group theoretical problem. We give a systematic procedure in several steps:

1) Determine the symmetry breaking operator O_1 with its associated scale M_1. Determine the subgroup K of G left unbroken by O_1. Classify the Higgs doublets in representations of K. Since O_1 cannot mix doublets in different K representations, this determines the space D_1 of doublets which can mix with the "leading" doublet H_1 through O_1. Doublets outside D_1 get no VEV at this stage. (In our example, K corresponds to $U(1)_{\tilde{q}}$.)

2) The next step involves the analysis of abelian quantum numbers. (We treat it separately from the non-abelian case (step 3) since it is easier and often sufficient to establish upper bounds on mass entries.) One determines all abelian quantum numbers $Q^{(i)}$ (in our example I_{3L}, I_{3R}, Y_{B-L}, q, I) for the operator O_1 and for all doublets in D_1, including H_1. This establishes an upper bound on the VEV of a doublet d in D_1

$$<d> \; \leq \left(\frac{M_1}{M}\right)^{P_0} <H_1> \qquad (7.1)$$

The number P_o is determined by

$$Q^{(i)}(d) \pm Q^{(i)}(H_1) \pm P_0\, Q^{(i)}(O_1) \;=\; 0 \qquad (7.2)$$

(The signs account for mixing with H_1 or H_1^* through P_0 powers of O_1 or $\overline{O}_1$. Equation (7.2) must hold with the same choice of signs for all abelian charges $Q^{(i)}$.) The bound (7.1) arises since the mixing involves at least a factor $M_1^{P_0}$. It is then suppressed by a factor M^{-P_0} since M is the only other mass scale and mixing angles are dimensionless.

3) More severe bounds can be obtained from a non-abelian analysis. One considers various non-abelian subgroups of G and determines the representations $R(O_1)$, $R(H_1)$, $R(d)$ to which O_1 and the various doublets belong. We generalize (7.2) to the non-abelian case: An upper bound on a doublet <d> is suppressed by only one power of M_1/M if the direct product of representations for H_1 and d contains the representation of O_1:

$$R(d) \times R(H_1) \qquad R(O_1) \quad \text{or} \quad \overline{R}(O_1) \qquad (7.3)$$

(Inclusion of the complex conjugate $\overline{R}(H_1)$ should be understood.) The suppression factor is $(M_1/M)^2$ if $R(d) \times R(H_1)$ contains a representation which also appears in $R(O_1) \times R(O_1)$, $\overline{R}(O_1) \times R(O_1)$ or $\overline{R}(O_1) \times \overline{R}(O_1)$ and so on for higher powers of M_1/M. It is obvious that the bounds from non-abelian symmetries may be stronger than those from their abelian subgroups. For an example where $R(d) \times R(H_1)$ does not contain $R(O_1)$, but only a representation contained in $R(O_1) \times R(O_1) \times \overline{R}(O_1)$, the non-abelian suppression factor is $(M_1/M)^3$ whereas abelian analysis only gives a bound with one power of M_1/M. In principle, all subgroups of G (including non-maximal subgroups) should be analyzed. In practice, most restrictions come from subgroups where O_1, H_1 or d belong only to one irreducible representation, especially if they are singlets. (Instead of a complete subgroup analysis one may establish the power of the suppression factor by a direct calculation of non vanishing Clebsch-Gordon coefficients for mixings through the expected power of O_1.)

4) If doublet mixing with a given power of O_1 is consistent with all subgroups G, this determines the group theoretical value $(M_1/M)^{P_G}$ for the suppression factor. In the generic case without unnatural cancellations, this will not only be an upper bound but give the actual order of magnitude of the mixing, at least if all required invariants appear in the action (without scales heavier than M). If there is only a restricted set of invariants - for example as a consequence of a higher dimensional gauge symmetry - it is still possible that the relevant Clebsch-Gordon coefficient for a doublet mixing vanishes, even if the pure group theoretical bound is fulfilled. In this case, an explicit calculation of Clebsch-Gordon coefficients for the existing invariants may be necessary. It may happen that certain mixings are not induced at all or only occur with $P > P_G$.

5) The doublet mixing described above can be represented graphically (see fig. 2). In addition, the symmetry breaking operator O_1 may induce other contributions to the fermion mass

matrix. In fig. 3 we have depicted the mixing with superheavy fermions which was discussed in the preceding section. Again, such contributions are suppressed by an appropriate power of M_1/M which can be calculated by similar group theoretical arguments as for the doublet mixing. An upper bound for <u>all</u> contributions to fermion mass matrix entries with given G quantum numbers is easily established: Let ψ_1 and ψ_2 be two left handed chiral fermions for which we want to calculate the mass matrix element M_{12}. The direct product of representations for ψ_1 and ψ_2 with respect to various subgroups of G will in general contain several irreducible representations:

$$R(\psi_1) \times R(\psi_2) = \sum_i R_i \tag{7.4}$$

The lowest poser $\overline{P}$ of O_1 for which

$$\big(R(H_1) \text{ or } \overline{R}(H_1)\big) \times \big(R(O_1) \text{ or } \overline{R}(O_1)\big) \times \big(R(O_1) \text{ or } \overline{R}(O_1)\big) \times \ldots \big(R(O_1) \text{ or } \overline{R}(O_1)\big) \tag{7.5}$$

$$(\ \overline{P} \ \text{times}\ )$$

contains a representation $\overline{R}_i$ (this must hold for all subgroups of G) determines the group theoretical suppression factor

$$M_{12} \leq g_y \langle H_1 \rangle \left(\frac{M_1}{M}\right)^{\overline{P}} \tag{7.6}$$

for all possible contributions from O_1. (Here g_y is the Yukawa coupling of H_1. In higher dimensional theories it is typically of the order of the gauge coupling.)

At this point we should comment on the case of a higher dimensional theory with compactification scale M_c somewhat smaller than the largest characteristic scale M of the theory. In this case we may consider spontaneous compactification itself as a symmetry breaking operator O_1 with $M_c = M_1$. The maximal unbroken symmetry G at the scale M is now a higher dimensional symmetry group. We can essentially proceed as before, except one important modification: The superheavy fermions and some of the superheavy scalar doublets have now a characteristic scale M_c rather than M and one has to account for this in the suppression factors.

We are now in a position to combine the results of this section with the systematic fermion mass matrix scanning of section 3. As a result a realistic model will be subject to more severe necessary criteria. After a successful scanning in section 3 we end with candidate mass matrices with a required order of scales for the different entries. We now check if this is consistent with our operator analysis for spontaneous symmetry breaking. For the case of three generations the doublet H_1 (with VEV $\sim A$) is uniquely determined. One now has to find a symmetry breaking operator O_1 which induces the necessary entry for m_b of order B. For a given operator O_1 one can now check if the suppression factors are strong enough for the various matrix entries to be consistent with the bounds of the scanning process (3.1) to (3.5). If this fails for all possible O_1, the model should be discarded. If successful, one records the other necessary entries (like m_τ, m_c, m_μ etc.) generated by O_1. If not all necessary entries are generated by O_1, one has to look for a second operator O_2 with scale $M_2 \leq M_1$. The analysis can now be repeated with the combined set of operators O_1 and O_2. Some care is needed in the discussion of suppression factors since ratios M_2/M_1 may appear instead of M_2/M for graphs mediated by particles with mass M_1. In this way one has to proceed until all necessary entries are generated without ever conflicting with the upper bounds (3.1) to (3.5). The analysis for four generations is similar. If there are mirror fermions in the model, there will be additional restrictions: The same $SU(3)_C \times SU(2)_L \times U(1)_Y$ singlet operator O_i can be responsible for the superheavy masses discarding mirrors plus associated quarks and leptons from the low energy spectrum

as well as for the mixing of various doublets to H_1 etc. For given quantum numbers of O_i the contribution to superheavy masses is easily established and the removal of the "mirror partners" from the low energy spectrum no longer arbitrary.

One may do again a systematic scanning for operator trees O_i consistent with the needed hierarchy of scales. Not too many models will pass this second test, especially if one allows only for a few (two or three) symmetry breaking operators. For successful models, however, the problem of structure of the three fermion mass matrices (mass hierarchies and small mixing angles) will be reduced to the problem of explaining its symmetry breaking scales M_i. Also remains the problem of quantitative predictions of fermion masses. Some of them will depend on dynamical details of the model. On the other hand, the group theoretical content of symmetry breaking of G will be determined to a large extent for any model successfully passing the scanning. Several fermion masses and mixings may then be predicted from the corresponding Clebsch Gordon coefficients and serve as a further test for such models.

8. Classical Stability and the Gauge Hierarchy Problem

Let us come back to our six dimensional SO(12) model and discuss some problems related to the observed smallness of weak symmetry breaking. Neglecting the scalar vacuum expectation values and expanding on a spherically symmetric monopole solution, the lowest modes in the harmonic expansion for H_1^+ and H_2^+ have negative mass squared[17]. These tachyons indicate classical instability of the corresponding monopole solution. Classical instability could be a sign of spontaneous symmetry breaking if a less symmetric stable ground state is found. In six dimensional gauge theories this requires[35] either geometries which are not a direct product of four dimensional space and internal space or additional fields (for example scalars). Both features are realized for our solutions in section 4 and we can investigate if they should indeed be interpreted as spontaneous symmetry breaking (for $n = 3$, $m = p = 1$)

$$\begin{aligned} &SU(5) \times U(1) \times U(1)_q \times SU(2)_G \\ &\rightarrow SU(3)_C \times SU(2)_L \times U(1)_Y \times U(1)_{\tilde{q}} \end{aligned} \tag{8.1}$$

For small symmetry breaking scales M_{210}, M_{B-L} we expect unstable solutions. Indeed, for a wide class of deformations from spherical symmetry the diagonal contributions $\sim M_C^2$ in the mass matrix for H_1^+ and H_2^+ (6.17) will be negative. (All mixings are small for M_{210}, $M_{B-L} << M_C$.) On the other hand, the diagonal contributions $\sim M_{210}^2$, M_{B-L}^2 in (6.17) are positive (compare appendix C). For M^2, $M_{B-L}^2 >> M_C^2$, M_{210}^2 all mixings are small and all eigenvalues of (6.17) are positive. There is no classical instability in the doublet sector anymore! We expect classical stability for the doublet sector for a wide range of solutions with M_{B-L} and/or M_{210} sufficiently large. A similar behaviour is expected for other modes which would be tachyonic[17] on spherically

symmetric monopole solutions with $M_{B-L} = M_{210} = 0$. The six dimensional Higgs mechanism can stabilize the "compactifying" solutions! Stability depends on the scales M_{B-L} and M_{210} which in turn depend on parameters in the six dimensional scalar potential and on "initial conditions" for the solutions in section 4. For large enough M_{B-L} and M_{210} we can indeed interprete these solutions as spontaneous symmetry breaking of the higher symmetric monopole solutions.

For some intermediate range of M_{B-L} and M_{210} there must be a transition from stability to instability. It is this transition region we are most interested in since $U(1)_{\tilde{q}}$ and $SU(2)_L \times U(1)_Y$ must be spontaneously broken for a realistic theory. There is a class of solutions in this transition region where $U(1)_{\tilde{q}}$ is broken at a scale $M_{\tilde{q}} << M_{210}$, M_{B-L}, M_C. This happens if we choose potential parameters and initial conditions so that the mass term for the lowest mode from six dimensional scalars with $\tilde{q} \neq 0$ is negative and small compared to M_C^2. In this case we can use the four dimensional effective theory for the corresponding scalar mode in a good approximation and do not need to discuss the complicated φ dependence of the corresponding higher dimensional solution explicitely. We now want to study the symmetry breaking of the weak interaction gauge group $SU(2)_L \times U(1)_Y$. The mass matrix for the doublets in H_1^+, H_2^+ and d (after integrating out the other doublet modes) will have the form (6.20), up to small corrections proportional to some power of $M_{\tilde{q}}$. Consider for a moment M_{B-L} and M_{210} as free parameters. Decreasing the overall scale for the scalar vacuum expectation values will induce a change from positive to negative M_1^2 or M_2^2. This corresponds to a phase transition where $SU(2)_L \times U(1)_Y$ is spontaneously broken. If the quartic coupling for the doublet is not too small (so that Coleman-Weinberg symmetry breaking[36] is a small effect) this transition is essentially second order. There is, therefore, a critical scale of singlet VEV's where the lowest doublet mass vanishes. For values sufficiently near this critical point the lowest doublet mass can be arbitrarily small and a gauge hierarchy is realized!

Of course, M_{B-L} and M_{210} are not free parameters. The exact location of the phase transition depends on the different parameters of the model (including the scalar potential) and the "initial values" for the different solutions. For a wide range of model parameters there will be a second order phase transition on a hypersurface in the space of "initial values". (This hypersurface has one dimension less than the total dimension of that space.) Solutions near this hypersurface realize a gauge hierarchy independent on any fine tuning of model parameters! This realizes the idea of a continuous spectrum of classical solutions where the weak symmetry breaking scale is a free integration constant[26)]. Solutions realizing a gauge hierarchy cover only a very small range within the continuous spectrum of solutions. It is a difficult open dynamical question to understand why such a particular solution should be preferred and what determines the scale of weak symmetry breaking. We only note here that a small doublet mass at the compactification scale M_c remains small in the whole energy range down to 100 GeV even if quantum fluctuations are included. This "naturalness" of a small quantity is due to the second order character of the phase transition[37)]. For generic model parameters the gauge hierarchy solutions correspond to non-compact internal space[26)]. Only if one insists on compactness of internal space a fine tuning of model parameters would be needed for a gauge hierarchy. In this respect, the status of the gauge hierarchy problem is now very similar to the problem of a vanishing four dimensional cosmological constant Λ_0. The cosmological constant is another free integration constant[6,26)] in a continuous spectrum of classical solutions with non-compact internal space.

Critical solutions with vanishing doublet mass are generically expected due to the second order character of the phase transition. A given model may predict which one of the candidate Higgs doublets becomes massless. One has to check if this doublet coincides with the leading doublet needed for realistic

fermion mass matrices. This places further restrictions on models which fulfill the necessary criteria of sections 3 and 7. In our expample ($n = 3$, $m = p = 1$) all doublets except the lowest modes in H_1^+ and H_2^+ have positive mass squared in the limit of spherical symmetry. The massless doublet will be either H_1^+ or H_2^+ depending on M_1^2 smaller or gereater M_2^2. In our example we need a range of solutions for which $M_1^2 < M_2^2$ so that H_1^+ is indeed the leading doublet.

We also can determine the quartic scalar coupling λ for the doublet H_1^+. Since H_1 is a component of the six dimensional gauge field its interactions are determined by six dimensional gauge symmetry. We can read from ref. 15 that λ is positive. For any given solution the quartic coupling is easily calculated and the physical Higgs mass, therefore, predicted. Details depend only on the specific form of the wave function $H_1^+(y)$. One finds λ of the order g^2 and the Higgs mass is, therefore, expected to be of the order of the W-boson mass and the top quark mass.

One more detail is important for the general setting of the gauge hierarchy problem: From the doublet mass matrix (6.20) we learn that the phase transition is not at $M_1^2 = 0$. (We assume $M_1^2 < M_2^2$ for definiteness.) Due to doublet mixing, a zero mass eigenvalue rather occurs for

$$M_1^2 \approx \alpha^2 M_c^2 M_{210}^2/M^2 \tag{8.2}$$

This situation is generic for all cases of mixing. Even though $M_{210}^2 M_c^2/M^2$ may be small compared to M_2^2 or M_c^2, it is still enormous compared to the weak scale M_W^2. There is no gauge hierarchy for $M_1^2 = 0$!

This shows a serious dilemma for ideas which want to obtain a massless scalar in a certain representation of G at the compactification scale, for example due to some Betti number[F7], and then keep it massless due to supersymmetry. (This idea is popular in some discussions on string "phenomenology".) Since all mixings with other doublets would destroy the gauge hierarchy they must be forbidden in this case. Preventing mixing due to symmetry breaking is by itself not easy, but in addition it also creates serious problems for an understanding of realistic fermion mass matrices. Without doublet mixing, the structure of fermion mass matrices must be explained by Yukawa couplings which are small without any symmetry reason and all quarks and leptons must get their masses from couplings to one massless doublet in a given G-representation.

One first might have thought that doublet mixing could be replaced by mixing through heavy fermions as in fig. 3. Fermion mixing alone, however, does not lead to realistic mass patterns: Assume that in the limit of unbroken G symmetry the massless doublet couples only to one up type quark

$$\mathcal{L}_M \sim H_1\, t'\, t^{c\prime} \qquad (8.3)$$

Symmetry breaking of G can induce mixing

$$\begin{aligned} t' &= a_{1i}\, u_i \\ t^{c\prime} &= b_{1i}\, u_i^c \end{aligned} \qquad (8.4)$$

Here the sum runs over the light quarks u_1, u_2, u_3 as well as infinitely superheavy many superheavy quarks. The mass matrix for the light quarks u_j, u_k^c has the form

$$M_{jk}^{(U)} \sim H_1\, a_{1j}\, b_{1k} \qquad (8.5)$$

This matrix has still two zero eigenvalues and nonzero masses for charm and up quark cannot be explained by fermion mixing.

Also the small ratios m_b/m_t and m_τ/m_t cannot result from fermion mixing. Doublet mixing is needed if small fermion masses and mixings are to be understood from G-symmetry breaking effects.

In conclusion one has to choose between our explanation of the structure in fermion mass matrices by symmetries and the approach with a massless doublet in a given G representation. In our opinion, not only the radiative stability of a small doublet mass between the scales M_c and M_W does not need supersymmetry[37], but also the mechanism generating a vanishing or small doublet mass at the compactification scale is probably not related to specific properties of supersymmetric potentials. The most important property for both questions seems to be the essential second order character of the weak phase transition.

9. Conclusions

In this paper an attempt was made to give a systematic discussion of the structure in fermion mass matrices. The hierachies of fermion masses and the small mixing angles are explained by a fine structure of scales near the unification scale. With the assumption that all small ratios of entries in fermion mass matrices are due to ratios of symmetry breaking scales, we have formulated several necessary criteria for possible quantum numbers of quarks and leptons with respect to a symmetry G at the unification scale (which is larger than $SU(3)_C \times SU(2)_L \times U(1)_Y$). We propose a systematic scanning procedure (which can be done on a computer) to select "viable" quantum numbers.

We have demonstrated these ideas for a specific class of solutions of the six dimensional SO(12) gauge theory. They can reproduce the hierarchy of fermion masses $m_t \gg m_b, m_\tau, m_c \gg m_s, m_\mu \gg m_d, m_u, m_e$, but tend to predict unacceptably large mixing angles for the first generation. The six dimensional SO(12) model is the simplest model to discuss the problem of fermion masses in a realistic setting. So far we have treated it mainly as an illustration. Since this model has proven to be relatively successful one may wonder if there could be some real physics in it.

Our discussion is only based on symmetries and their breaking scales and we observe that six dimensional general coordinate and Lorentz transformations (gen_6) plus SO(12) gauge transformations are a subgroup of the symmetry group for various interesting unification models, as $gen_{10} \times E_8 \times E_8$ for superstrings or gen_{18} for the simplest pure gravitational model[15]. The SO(12) spinor representations for fermions discussed in this paper appear in the decomposition of spinors for such unified models. One may ask to what extent our discussion can be considered as a subgroup analysis for unified models?

Subgroup analysis in higher dimensional unification is much more involved than in usual four dimensional "grand unification". This is due to the appearance of infinitely many representations of the subgroups if they relate to a reduced number of dimensions. Suppose a theory with a symmetry which has $gen_6 \times SO(12)_6$ as a subgroup, as for example the ten dimensional $E_8 \times E_8$ superstring or its field theory limit. We always can expand such a theory on a state with global six dimensional Poincaré symmetry and SO(12) symmetry (plus possible additional symmetries). (For the $E_8 \times E_8$ example one may think of some space $\mathcal{M}^6 \times K$ with an appropriate gauge field configuration.) This expansion leads to an action with local $gen_6 \times SO(12)_6$ symmetry and the corresponding massless graviton and gauge fields. In addition, there will be an infinite tower of other fields which are in general massive unless protected by some symmetry or topological reason. We note that our expansion state is not necessarily a solution of the higher dimensional field equations. If it is not, the effective action obtained from the expansion will contain terms linear in six dimensional scalars which are singlets of SO(12).

We claim that any arbitrary classical configuration of the higher dimensional theory can be represented by appropriate values for the infinitely many fields of the six dimensional theory. In particular, the (classical) ground state can be expressed by vacuum expectation values of the six dimensional bosonic fields. This is the generalization of subgroup analysis to the higher dimensional case. These statements may at first sight look somewhat surprising. How can a ground state like $\mathcal{M}^4 \times S^6$ be expressed in terms of expansion on the topologically inequivalent state $\mathcal{M}^6 \times S^4$? Locally, it is obvious that we can express the metric of $\mathcal{M}^4 \times S^6$ by expectation values of bosonic fields in the six dimensional theory obtained from harmonic expansion on $\mathcal{M}^6 \times S^4$:

$$\begin{pmatrix} \eta_{\mu\nu} & 0 \\ 0 & g_{\alpha\beta}(S^6) \end{pmatrix} = \begin{pmatrix} \eta_{\mu\nu} & 0 & 0 \\ 0 & \langle\text{metric}\rangle, & \langle\text{vectors}\rangle \\ 0 & \langle\text{vectors}\rangle, & \langle\text{scalars}\rangle \end{pmatrix} \qquad (9.1)$$

(In this schematic notation the six dimensional metric has the form of spontaneous compactification $g^{(6)} = \begin{pmatrix} \eta_{\mu\nu} & 0 \\ 0 & \langle metric \rangle \end{pmatrix}$.)

Such a local region with topology R^{10} can be extended everywhere except the south poles of S^6 and S^4, respectively. They appear as singularities in a cartesian coordinate system with the north poles as origin. The structure of the singularity being different for $\mathcal{M}^4 \times S^6$ and $\mathcal{M}^6 \times S^4$, we conclude that the state $\mathcal{M}^4 \times S^6$ appears as a singular configuration of bosonic fields in the six dimensional theory obtained from $\mathcal{M}^6 \times S^4$! (This is another reason why we considered in section 4 solutions of the six dimensional field equations corresponding to non-compact geometry.) In general, such a singular configuration may involve infinitely many harmonics on $\mathcal{M}^6 \times S^4$. Higher dimensional ground states with topology not admitting $gen_6 \times SO(12)_6$ will appear as singular configurations of some six dimensional SO(12) theory (if the unification group contains $gen_6 \times SO(12)_6$). We, therefore, can always formulate a ten dimensional $E_8 \times E_8$ theory as a six dimensional SO(12) theory with infinitely many modes. There is even an infinite number of such formulations corresponding to expansions on different states with $P_6 \times SO(12)$ symmetry. The question whether such a six dimensional formulation is useful depends on the criterium if the ground state can be well approximated by a finite number of six dimensional fields.

What are possible modifications of our six dimensional model if it is embedded into a higher dimensional unified theory? First of all we have all the massive modes, in particular infinitely many six dimensional scalars instead of only the fifth rank antisymmetric tensor representation. The scalar potential will be more complicated. More general, the six dimensional field equations will be modified, but we expect that qualitative features of the solutions and scale arguments remain unaffected. Only very few modes can couple to bilinears of chiral six dimensional fermions and, therefore, influence the structure of fermion mass matrices. In our example there

will be two more scalars in the vector and third rank antisymmetric tensor representation of SO(12) (compare appendices A, B). The structure of their contributions to fermion masses is essentially the same as for the 792 scalar.

More important, their may be traces of the higher symmetry of the unified theory (for example $E_8 \times E_8$) in form of relations between six dimensional couplings. This could explain cancellations between contributions to fermion masses which would look unnatural in a purely six dimensional context. In our example, a cancellation between contributions from scalars in 12, 210 and 792 (which may all belong to the same multiplet of the unified theory) could make the unwanted mixing between d_{D2} and H_2 vanish and, therefore, cure the problem of too large mixings of the first generation!

The content of four dimensional chiral fermions obtained from the chiral six dimensional spinors 32_1 and 32_2 could be more complicated than the one given by the monopole numbers m, n and p. This is possible if the non-compact solutions deviate sufficiently from the compact monopole solutions. Our analysis could easily cover this case as well. We note, however, that the possible content of four dimensional chiral fermions is strongly restricted due to anomaly cancellation (including $U(1)_q$, $U(1)_G$ and $U(1)_{B-L}$). There also could be other massless six dimensional chiral fermions. This possibility is restricted by the requirements that all six dimensional anomalies must cancel and the additional chiral spinors should contain fermions with quantum numbers of quarks and leptons.

There is, however, an even more drastic possible modification of the six dimensional theory: This is the case if the four dimensional chiral fermions are not obtained from the six dimensional chiral fermions, but rather involve the infinitely many massive fermions of the six dimensional theory. This can happen if the four dimensional symmetry group cannot be embedded into the unification group G via the chain

$$\breve{G} \supset SO(12)_6 \times gen_6 \supset SU(3)_C \times SU(2)_L \times U(1)_Y \times gen_4 \qquad (9.2)$$

or if the ground state does not belong to an appropriate $SO(12)_6 \times gen_6$-deformation class[17)]. For an appropriate deformation class it must be possible to "blow up" two dimensions so that in the corresponding limit $M_C \to 0$ the massless fermions form a set of chiral six dimensional spinors containing the two spinor representations of SO(12). Note that the concept of blowing up two (arbitrary) dimensions defines the notion of a $SO(12)_6 \times gen_6$-deformation class even for a ground state which has only four flat dimensions. One $SO(12)_6 \times gen_6$-deformation class contains many $SU(3)_C \times SU(2)_L \times U(1)_Y \times gen_4$-deformation classes. If the four dimensional chiral spinors are not contained in six dimensional chiral spinors a subgroup analysis with respect to $SO(12)_6 \times gen_6$ is meaningless for the chiral fermions and the six dimensional SO(12) model is irrelevant for the structure of fermion mass matrices. (This is for example the situation for the subgroup $SO(12)_4 \times gen_4$. For m, n or p different from zero the four dimensional deformation class of the ground state does not admit SO(12) symmetry.) The relative success of our simple model may indicate that the four dimensional gauge symmetries are indeed contained in $SO(12)_6 \times gen_6$ and the deformation class of the ground state admits a six dimensional formulation for the chiral spinors.

We conclude that for a large class of compactifications a suitable version of a six dimensional SO(12) theory reflects many relevant features of a ten dimensional $E_8 \times E_8$ theory. For our discussion of fermion masses we only have used the symmetries SO(12) and $U(1)_G$ (isometry of φ-rotations on internal space). It is possible that $U(1)_G$ could be replaced by a discrete subgroup. It is also possible and perhaps necessary that

other continuous or discrete symmetries play a role for the understanding of fermion mass matrices. We find it encouraging for higher dimensional theories (in particular for those containing $SO(12)_6 \times gen_6$ as a subgroup as the $E_8 \times E_8$ superstring) that for a first time we have a model with predictive power about the structure of fermion mass matrices. Even a relatively simple solution can explain the hierarchies of fermion masses. Unfortunately not all mixings come out satisfactory so far. It is our hope that a systematic analysis of possible quantum numbers of quarks and leptons with respect to symmetries at the unification scale will lead to a complete understanding of all small quantities in the fermion mass matrices in terms of ratios of symmetry breaking scales.

A P P E N D I X   A

Totally Antisymmetric Tensor Representations of SO(12)

a) Scalars in Odd Rank Antisymmetric Tensor Representations

In this appendix we perform the analysis of the SO(12) algebra for the totally antisymmetric tensors appearing in this paper. We give a systematic treatment which is easily generalized to orthogonal groups in arbitrary dimensions. Totally antisymmetric tensors will be represented in terms of the Clifford algebra. We start by defining the usual Dirac matrices in 12 dimensions by

$$\{\Gamma_A, \Gamma_B\} = 2\eta_{AB} = -2\delta_{AB} \quad , \quad A,B = 1\ldots 12 \tag{A1}$$

and their totally antisymmetric products of rank k

$$\Gamma^{(k)}_{A_1\ldots A_k} = \frac{1}{k!}\Gamma_{[A_1}\Gamma_{A_2}\ldots\Gamma_{A_k]} \tag{A2}$$

The bracket $[\;]$ on indices means antisymmetrisation and we will often omit the label (k). We are interested in scalars in the representations 12, 220 and 792 of SO(12) corresponding to totally antisymmetric tensors of rank one, three and five. We represent them by 64x64 matrices

$$\phi^{(k)} = \frac{i^k}{k!\sqrt{32}}\varphi^{A_1\ldots A_k}\Gamma^{(k)}_{A_1\ldots A_k} \tag{A3}$$

with

$$\varphi^{B_1\ldots B_k} \equiv (-1)^P\varphi^{A_1\ldots A_k} \tag{A4}$$

and $(-1)^P$ the degree of permutation of the indices $(B_1\ldots B_k)$ compared to $(A_1\ldots A_k)$.

This representation is adapted to the fact that the direct product of two Dirac spinors contains all totally antisymmetric tensors. A SO(12)-Dirac spinor has 64 components and transforms

$$\psi \to U\psi$$
$$U = \exp\left(\frac{i}{2}\varepsilon^{AB}T_{AB}\right) \tag{A5}$$

with T_{AB} hermitean generators of SO(12) in the spinor representations:

$$T_{AB} = \frac{i}{4}[\Gamma_A, \Gamma_B] = \frac{i}{2}\Gamma_{AB} \tag{A6}$$

Correspondingly, totally antisymmetric tensors transform

$$\phi^{(k)} \to U\phi^{(k)}U^{-1} \tag{A7}$$

Using

$$U\Gamma_A U^{-1} = \Gamma_B O^B{}_A$$
$$O^B{}_A = \left[\exp\left(\frac{i}{2}\varepsilon^{CD}T^{(V)}_{CD}\right)\right]^B{}_A \tag{A8}$$

with $T^{(V)}_{CD}$ the SO(12) generators in the vector representation given as 12x12 matrices

$$(T^{(V)}_{CD})_{ST} = -i\,\delta_{CS}\delta_{DT} + i\,\delta_{CT}\delta_{DS} \tag{A9}$$

one obtains

$$\varphi^{A_1A_2\ldots A_k} \to O^{A_1}{}_{A_1'}O^{A_2}{}_{A_2'}\ldots O^{A_k}{}_{A_k'}\varphi^{A_1'A_2'\ldots A_k'} \tag{A10}$$

We have chosen our normalization (A3) so that the standard kinetic term for scalars in antisymmetric tensor representations reads

$$S_{kin} = \int d^6\hat{x}\, \hat{g}_6^{1/2}\, \frac{1}{4}\, Tr\, (D^{\hat{\mu}}\phi)^{\dagger} D_{\hat{\mu}}\phi$$

$$\phi = \sum_{k} \phi^{(k)} \tag{A11}$$

This is easily checked by using the following relations for traces of Γ matrices in even dimensions:

$$Tr\, \Gamma_{A_1}\Gamma_{A_2}\cdots\Gamma_{A_\ell} = 0 \quad \text{for } \ell \text{ odd or at least one index appearing an odd number of times.} \tag{A12}$$

$$Tr\, \Gamma^{(k_1)}_{A_1\cdots A_{k_1}}\Gamma^{(k_2)}_{B_1\cdots B_{k_2}} = 0 \quad \text{for } k_1 \neq k_2 \tag{A13}$$

$$Tr\, \Gamma^{(k)}_{A_1\cdots A_k}\Gamma^{(k)}_{B_1\cdots B_k} = (-1)^{\frac{k(k+1)}{2}}\, 64\, P\begin{pmatrix} A_1\cdots A_k \\ B_1\cdots B_k \end{pmatrix} \tag{A14}$$

$$P\begin{pmatrix} A_1\cdots A_k \\ B_1\cdots B_k \end{pmatrix} = \begin{cases} +1 & \text{for } B_1\cdots B_k \text{ even permutation of } A_1\cdots A_k \\ -1 & \text{for } B_1\cdots B_k \text{ odd permutation of } A_1\cdots A_k \\ 0 & \text{otherwise} \end{cases} \tag{A15}$$

For real $\phi^{A_1\cdots A_k}$ we note that $\phi^{(1)}$ and $\phi^{(5)}$ are hermitean whereas $\phi^{(3)}$ is antihermitean

$$(\Gamma^{(k)}_{A_1\cdots A_k})^{\dagger} = (-1)^{\frac{k(k+1)}{2}}\Gamma^{(k)}_{A_1\cdots A_k}$$

$$\begin{aligned} \phi^{(1)\dagger} &= \phi^{(1)} \\ \phi^{(3)\dagger} &= -\phi^{(3)} \\ \phi^{(5)\dagger} &= \phi^{(5)} \end{aligned} \tag{A16}$$

We also define the SO(12) charge conjugation matrix B_{12} by

$$\begin{aligned} \Gamma_A^{*} &= -B_{12}\Gamma_A B_{12}^{-1}, \quad B_{12}^{*}B_{12} = -1 \\ \Gamma_A^{T} &= B_{12}\Gamma_A B_{12}^{-1} \end{aligned} \tag{A17}$$

The matrices

$$\tilde{\phi}^{(k)} = B_{12}\,\phi^{(k)} \tag{A18}$$

obey

$$\begin{aligned} (\tilde{\phi}^{(1)})^T &= -\tilde{\phi}^{(1)} \\ (\tilde{\phi}^{(3)})^T &= \tilde{\phi}^{(3)} \\ (\tilde{\phi}^{(5)})^T &= -\tilde{\phi}^{(5)} \end{aligned} \tag{A19}$$

$$\begin{aligned} (\tilde{\phi}^{(k)})^{*} &= B_{12}\,\tilde{\phi}^{(k)}B_{12}^{-1} = \phi^{(k)}B_{12} \\ (\phi^{(k)})^{*} &= B_{12}\,\phi^{(k)}B_{12}^{-1} \end{aligned} \tag{A20}$$

b) <u>SO(12) Gauge Bosons</u>

The gauge bosons of SO(12) belong to the totally antisymmetric tensor representation of rank two. It will be useful in our context to represent them in the Dirac representation as 64x64 matrices

$$\begin{aligned} \hat{A}_{\hat{\mu}} &= \frac{i}{4}\hat{A}_{\hat{\mu}}^{AB}\Gamma_{AB} = \frac{1}{2}\hat{A}_{\hat{\mu}}^{AB}T_{AB} \\ \hat{A}_{\hat{\mu}}^{AB} &= -\hat{A}_{\hat{\mu}}^{BA} \end{aligned} \tag{A21}$$

Their transformation properties are modified by an inhomogenous term

$$\hat{A}_{\hat{\mu}} \rightarrow U\hat{A}_{\hat{\mu}}U^{-1} + \frac{i}{g}U\partial_{\hat{\mu}}U^{-1} \tag{A22}$$

The field strength is defined as usual

$$\hat{G}_{\hat{\mu}\hat{\nu}} = \partial_{\hat{\mu}}\hat{A}_{\hat{\nu}} - \partial_{\hat{\nu}}\hat{A}_{\hat{\mu}} - i\hat{g}[\hat{A}_{\hat{\mu}},\hat{A}_{\hat{\nu}}] = \frac{1}{2}\hat{G}_{\hat{\mu}\hat{\nu}}^{AB}T_{AB} \tag{A23}$$

The normalization (A21) implies that the gauge invariant kinetic term in the standard normalization reads

$$S_6 = -\int d^6\hat{x}\, \hat{g}_6^{1/2} \frac{1}{64} \mathrm{Tr}\, \hat{G}^{\hat{\mu}\hat{\nu}} \hat{G}_{\hat{\mu}\hat{\nu}} = -\frac{1}{8} \int d^6\hat{x}\, \hat{g}_6^{1/2}\, \hat{G}^{\hat{\mu}\hat{\nu}}_{AB}\, \hat{G}^{AB}_{\hat{\mu}\hat{\nu}} \qquad (A24)$$

c) Weight Basis for Dirac Matrices

We want to identify physical fields of interest like $SU(3)_C \times SU(2)_L \times U(1)_Y$ singlet scalar fields responsible for spontaneous symmetry breaking of the unification group or Higgs doublets inducing fermion masses. For this program a change of basis for the matrices $\Gamma^{(\Sigma)}$ is appropriate. It will also be useful to study various subgroups of SO(12) and the projection operators for their representations.

We define the generators of the Cartan subalgebra of SO(12)

$$H_W = T_{2W-1,2W} = \frac{i}{2} \Gamma_{2W-1} \Gamma_{2W} \quad , \quad W = 1\ldots 6 \qquad (A25)$$

They are related to the usual quantum numbers[18]

$$\begin{aligned} H_1 &= -(I_{3L} + I_{3R}) \\ H_2 &= I_{3L} - I_{3R} \\ H_3 + H_4 + H_5 &= \tfrac{3}{2} Y_{B-L} \\ H_6 &= q \\ Q_{em} &= -H_1 + \tfrac{1}{3}(H_3 + H_4 + H_5) \end{aligned} \qquad (A26)$$

The weight basis for the Dirac matrices consists of eigenstates of H_W:

$$\Gamma_{\varepsilon W} = \frac{1}{\sqrt{2}} (\Gamma_{2W-1} + i\varepsilon \Gamma_{2W}) \qquad \varepsilon - -1, +1 \qquad (A27)$$

$$[H_{W'}, \Gamma_{\varepsilon W}] = \varepsilon\, \delta_{W'W}\, \Gamma_{\varepsilon W} \qquad (A28)$$

$$(\Gamma_{\varepsilon W})^{\dagger} = -\Gamma_{-\varepsilon, W} \qquad (A29)$$

In a metric formulation, we can define the operations of raising and lowering indices (εW): Writing

$$\begin{aligned} \Gamma_{\varepsilon W} &= e^A_{\varepsilon W} \Gamma_A \quad ; \quad e^A_{\varepsilon W} = \frac{1}{\sqrt{2}} (\delta^A_{2W-1} + i\varepsilon\, \delta^A_{2W}) \\ \Gamma_A &= e^{\varepsilon W}_A \Gamma_{\varepsilon W} \quad ; \quad e^{\varepsilon W}_A = \frac{1}{\sqrt{2}} (\delta^{2W-1}_A - i\varepsilon\, \delta^{2W}_A) \end{aligned} \qquad (A30)$$

we note that $e^A_{\varepsilon W}$ and $e^{\varepsilon W}_A$ play the role of a vielbein and its inverse

$$\begin{aligned} e^A_{\varepsilon W}\, e^{\varepsilon' W'}_A &= \delta^{\varepsilon'}_{\varepsilon}\, \delta^{W'}_W \\ e^{\varepsilon W}_A\, e^{A'}_{\varepsilon W} &= \delta^{A'}_A \end{aligned} \qquad (A31)$$

The metric which lowers and raises (εW) indices is

$$\begin{aligned} \eta_{\varepsilon_1 W_1 \varepsilon_2 W_2} &= e^{A_1}_{\varepsilon_1 W_1}\, e^{A_2}_{\varepsilon_2 W_2}\, \eta_{A_1 A_2} = -\delta_{W_1 W_2}\, \delta_{\varepsilon_1, -\varepsilon_2} \\ \eta^{\varepsilon_1 W_1 \varepsilon_2 W_2} &= e^{\varepsilon_1 W_1}_{A_1}\, e^{\varepsilon_2 W_2}_{A_2}\, \eta^{A_1 A_2} = \eta_{\varepsilon_1 W_1 \varepsilon_2 W_2} \\ \eta_{\varepsilon_1 W_1 \varepsilon_2 W_2}\, \eta^{\varepsilon_2 W_2 \varepsilon_3 W_3} &= \delta^{\varepsilon_3}_{\varepsilon_1}\, \delta^{W_3}_{W_1} \end{aligned} \qquad (A32)$$

Lowering and raising indices is related to hermitean conjugation

$$\begin{aligned} e^{\varepsilon W}_A &= \eta^{\varepsilon W \tilde{\varepsilon} \tilde{W}}\, e^{\tilde{A}}_{\tilde{\varepsilon} \tilde{W}}\, \eta_{\tilde{A} A} \equiv (e^A_{\varepsilon W})^* \\ e^A_{\varepsilon W} &= \eta_{\varepsilon W \tilde{\varepsilon} \tilde{W}}\, e^{\tilde{\varepsilon} \tilde{W}}_{\tilde{A}}\, \eta^{\tilde{A} A} = (e^{\varepsilon W}_A)^* \end{aligned} \qquad (A33)$$

$$\Gamma^{\varepsilon W} = \eta^{\varepsilon W \tilde{\varepsilon} \tilde{W}}\, \Gamma_{\tilde{\varepsilon} \tilde{W}} = -\Gamma_{-\varepsilon W} = (\Gamma_{\varepsilon W})^{\dagger} \qquad (A34)$$

In the weight basis the anticommutator is

$$\{ \Gamma_{\varepsilon_1 w_1}, \Gamma_{\varepsilon_2 w_2} \} = 2\, \eta_{\varepsilon_1 w_1 \varepsilon_2 w_2} = -2\, \delta_{w_1 w_2}\, \delta_{\varepsilon_1, -\varepsilon_2} \quad (A35)$$

implying

$$(\Gamma_{\varepsilon w})^2 = 0 \quad , \quad \{ \Gamma^+_{\varepsilon w}, \Gamma_{\varepsilon w} \} = 2 \quad (A36)$$

The adjoint representation is again given by the commutator of two Γ matrices

$$\Gamma_{\varepsilon_1 w_1 \varepsilon_2 w_2} = \frac{1}{2} [\Gamma_{\varepsilon_1 w_1}, \Gamma_{\varepsilon_2 w_2}] = \Gamma_{\varepsilon_1 w_1} \Gamma_{\varepsilon_2 w_2} + \delta_{w_1 w_2} \delta_{\varepsilon_1, -\varepsilon_2} \quad (A37)$$

The generators of the Cartan subalgebra read

$$H_w = \frac{1}{2} \Gamma_{-w+w} = \frac{1}{4} [\Gamma_{-w}, \Gamma_{+w}] \quad (A38)$$

The ladder operators are obtained for $w_1 \neq w_2$

$$E_{\varepsilon_1 w_1 \varepsilon_2 w_2} = \frac{i}{2} \Gamma_{\varepsilon_1 w_1 \varepsilon_2 w_2} \quad ; \quad w_1 \neq w_2 \quad (A39)$$

They fulfill simple commutation relations

$$[E_{\varepsilon_1 w_1 \varepsilon_2 w_2}, \Gamma_{\varepsilon_3 w_3}] = -i\, \delta_{w_3 w_2} \delta_{\varepsilon_3, -\varepsilon_2} \Gamma_{\varepsilon_1 w_1} + i\, \delta_{w_3 w_1} \delta_{\varepsilon_3, -\varepsilon_1} \Gamma_{\varepsilon_2 w_2} \quad (A40)$$

$$[H_{w_3}, E_{\varepsilon_1 w_1 \varepsilon_2 w_2}] = (\varepsilon_1 \delta_{w_3 w_1} + \varepsilon_2 \delta_{w_3 w_2}) E_{\varepsilon_1 w_1 \varepsilon_2 w_2} \quad (A41)$$

$$[E_{\varepsilon_1 w_1 \varepsilon_2 w_2}, E_{-\varepsilon_1 w_1, -\varepsilon_2 w_2}] = \varepsilon_1 H_{w_1} + \varepsilon_2 H_{w_2} \quad (A42)$$

Under hermitean conjugation one has

$$H_w^+ = H_w$$
$$E^+_{\varepsilon_1 w_1 \varepsilon_2 w_2} = E_{-\varepsilon_1 w_1, -\varepsilon_2 w_2} \quad (A43)$$

Using the weight basis, all fields are eigenstates of the abelian charges in the Cartan subalgebra. The gauge fields

$$\hat{A}_{\hat{\mu}} = \frac{i}{4} \hat{A}_{\hat{\mu}}^{AB} \Gamma_{AB} = \frac{i}{4} \hat{A}_{\hat{\mu}}^{\varepsilon_1 w_1 \varepsilon_2 w_2} \Gamma_{\varepsilon_1 w_1 \varepsilon_2 w_2}$$
$$\hat{A}_{\hat{\mu}}^{\varepsilon_1 w_1 \varepsilon_2 w_2} = e^{\varepsilon_1 w_1}_{A_1} e^{\varepsilon_2 w_2}_{A_2} \hat{A}_{\hat{\mu}}^{A_1 A_2} \quad (A44)$$

can be divided into neutral fields in the Cartan subalgebra

$$\hat{U}_{\hat{\mu}} = \hat{U}_{\hat{\mu}}^{w} H_w = \sum_w \frac{i}{2} \hat{A}_{\hat{\mu}}^{-w+w} \Gamma_{-w+w}$$
$$\hat{U}_{\hat{\mu}}^{w} = i\, \hat{A}_{\hat{\mu}}^{-w+w} = \hat{A}_{\hat{\mu}}^{2w-1, 2w} \quad (A45)$$

and into charged fields

$$\hat{E}_{\hat{\mu}} = \sum_{w_1 < w_2} \sum_{\varepsilon_1, \varepsilon_2} \hat{A}_{\hat{\mu}}^{\varepsilon_1 w_1 \varepsilon_2 w_2} E_{\varepsilon_1 w_1 \varepsilon_2 w_2} \quad (A46)$$

$$\hat{A}_{\hat{\mu}} = \hat{U}_{\hat{\mu}} + \hat{E}_{\hat{\mu}} \quad (A47)$$

For example, the field $\hat{A}_{\hat{\mu}}^{-1+2}$ has charges $H_1 = -1$, $H_2 = 1$, $H_3 = H_4 = H_5 = H_6 = 0$ and, therefore, the quantum numbers of the W_L^+-boson. For the scalars we define

$$\Gamma^{(k)}_{\varepsilon_1 w_1 \ldots \varepsilon_k w_k} = e^{A_1}_{\varepsilon_1 w_1} \cdots e^{A_k}_{\varepsilon_k w_k} \Gamma^{(k)}_{A_1 \ldots A_k} = \frac{1}{k!} \Gamma_{[\varepsilon_1 w_1} \cdots \Gamma_{\varepsilon_k w_k]} \quad (A48)$$

$$\varphi^{\varepsilon_1 w_1 \ldots \varepsilon_k w_k} = e^{\varepsilon_1 w_1}_{A_1} \cdots e^{\varepsilon_k w_k}_{A_k} \varphi^{A_1 \ldots A_k} \quad (A49)$$

$$\varphi^{\tilde{\varepsilon}_1 \tilde{w}_1 \ldots \tilde{\varepsilon}_k \tilde{w}_k} = (-1)^P \varphi^{\varepsilon_1 w_1 \ldots \varepsilon_k w_k} \quad (A50)$$

with $(-1)^P$ the degree of permutation of $(\tilde{\varepsilon}_1 \tilde{w}_1, \ldots, \tilde{\varepsilon}_k \tilde{w}_k)$ compared to $(\varepsilon_1 w_1, \ldots, \varepsilon_k w_k)$. The fields $\varphi^{\varepsilon_1 w_1 \ldots \varepsilon_k w_k}$ have standard normalization and

$$\phi^{(k)} = \frac{i^k}{k! \sqrt{32}} \varphi^{\varepsilon_1 w_1 \ldots \varepsilon_k w_k} \Gamma^{(k)}_{\varepsilon_1 w_1 \ldots \varepsilon_k w_k} \quad (A51)$$

Properties under hermitean conjugation are given by

$$\left(\Gamma^{(\ell)}_{\varepsilon_1 w_1 \cdots \varepsilon_\ell w_\ell}\right)^{+} = (-1)^{\frac{\ell(\ell+1)}{2}} \Gamma^{(\ell)}_{-\varepsilon_1 w_1 \cdots -\varepsilon_\ell w_\ell} \tag{A52}$$

$$\left(\varphi^{\varepsilon_1 w_1 \cdots \varepsilon_\ell w_\ell}\right)^{*} = \varphi^{-\varepsilon_1 w_1 \cdots -\varepsilon_\ell w_\ell} \tag{A53}$$

d) Projections with $\bar{\Gamma}_N$

In even dimensions N the totally antisymmetric tensors of rank N/2 are reducible. Irreducible representations are obtained by using the projection operator $\frac{1}{2}(1 \pm \bar{\Gamma}_N)$. In our context antisymmetric tensors of rank N/2 appear for various subgroups SO(N) of SO(12) and we present the relevant properties for arbitrary N even. The operator

$$\bar{\Gamma}_N = \eta_N \Gamma_1 \Gamma_2 \cdots \Gamma_N = \eta_N \Gamma^{(N)}_{12\ldots N} \tag{A54}$$

$$= \frac{\eta_N}{N!} \varepsilon^{A_1 \cdots A_N} \Gamma_{A_1} \cdots \Gamma_{A_N} = \frac{\eta_N}{N!} \varepsilon^{A_1 \cdots A_N} \Gamma^{(N)}_{A_1 \cdots A_N}$$

$$\eta_N = i^{N/2} \tag{A55}$$

Commutes with all SO(N) generators T_{AB} . It induces a duality between totally antisymmetric tensors of rank ℓ and $N-\ell$:

$$\bar{\Gamma}_N \Gamma^{(\ell)}_{A_1 \cdots A_\ell} = \frac{\eta_N}{(N-\ell)!} (-1)^{\frac{\ell(\ell+1)}{2}} \varepsilon_{A_1 \cdots A_\ell}{}^{A_{\ell+1} \cdots A_N} \Gamma^{(N-\ell)}_{A_{\ell+1} \cdots A_N} \tag{A56}$$

The totally antisymmetric tensor $\varepsilon_{A_1 \cdots A_N}$ is an SO(N) invariant with $\varepsilon_{12\ldots N} = 1$ and indices raised and lowered with $\eta_{AB} = -\delta_{AB}$. Thus $\phi^{(\ell)}$ and $\phi^{(N-\ell)}$ belong to the same representation

$$\bar{\Gamma}_N \varphi^{A_1 \cdots A_\ell} \Gamma^{(\ell)}_{A_1 \cdots A_\ell} = \tilde{\varphi}^{A_{\ell+1} \cdots A_N} \Gamma^{(N-\ell)}_{A_{\ell+1} \cdots A_N}$$

$$\tilde{\varphi}^{A_{\ell+1} \cdots A_N} = \frac{\eta_N}{(N-\ell)!} (-1)^{\frac{\ell(\ell-1)}{2}} \varepsilon^{A_{\ell+1} \cdots A_N}{}_{A_1 \cdots A_\ell} \varphi^{A_1 \cdots A_\ell} \tag{A57}$$

For $\ell = N/2$ the operators $\frac{1}{2}(1 \pm \bar{\Gamma}_N)$ project out irreducible representations

$$\Gamma^{(\frac{N}{2})\pm}_{A_1 \cdots A_{N/2}} = \frac{1}{2}(1 \pm \bar{\Gamma}_N) \Gamma^{(\frac{N}{2})}_{A_1 \cdots A_{N/2}} \tag{A58}$$

$$\varphi_{\pm}^{A_1 \cdots A_{N/2}} = \frac{1}{2}\left(\varphi^{A_1 \cdots A_{N/2}} \pm \tilde{\varphi}^{A_1 \cdots A_{N/2}}\right) \tag{A59}$$

In the weight basis, relation (A56) reads

$$\bar{\Gamma}_N \Gamma^{(\ell)}_{\varepsilon_1 w_1 \cdots \varepsilon_\ell w_\ell} = \frac{\eta_N}{(N-\ell)!} (-1)^{\frac{\ell(\ell+1)}{2}} \varepsilon_{\varepsilon_1 w_1 \cdots \varepsilon_\ell w_\ell}{}^{\varepsilon_{\ell+1} w_{\ell+1} \cdots \varepsilon_N w_N} \Gamma^{(N-\ell)}_{\varepsilon_{\ell+1} w_{\ell+1} \cdots \varepsilon_N w_N} \tag{A60}$$

with

$$\varepsilon_{\varepsilon_1 w_1 \cdots \varepsilon_\ell w_\ell}{}^{\varepsilon_{\ell+1} w_{\ell+1} \cdots \varepsilon_N w_N} = e^{A_1}_{\varepsilon_1 w_1} \cdots e^{A_\ell}_{\varepsilon_\ell w_\ell} e^{\varepsilon_{\ell+1} w_{\ell+1}}_{A_{\ell+1}} \cdots e^{\varepsilon_N w_N}_{A_N} \varepsilon_{A_1 \cdots A_\ell}{}^{A_{\ell+1} \cdots A_N}$$

$$= \eta^{\varepsilon_{\ell+1} w_{\ell+1} \tilde{\varepsilon}_{\ell+1} \tilde{w}_{\ell+1}} \cdots \eta^{\varepsilon_N w_N \tilde{\varepsilon}_N \tilde{w}_N} \varepsilon_{\varepsilon_1 w_1 \cdots \varepsilon_\ell w_\ell \tilde{\varepsilon}_{\ell+1} \tilde{w}_{\ell+1} \cdots \tilde{\varepsilon}_N \tilde{w}_N} \tag{A61}$$

and $\varepsilon_{\varepsilon_1 w_1 \cdots \varepsilon_N w_N}$ the totally antisymmetric tensor of rank N

$$\varepsilon_{-1+1-2+2\cdots-\frac{N}{2}+\frac{N}{2}} = (i)^{N/2} = \eta_N$$

$$\varepsilon^{-1+1-2+2\cdots-\frac{N}{2}+\frac{N}{2}} = (-i)^{N/2} = 1/\eta_N = \eta_N^{*} \tag{A62}$$

The matrix $\bar{\Gamma}_N$ is

$$\bar{\Gamma}_N = \Gamma^{(N)}_{-1+1-2+2\cdots-\frac{N}{2}+\frac{N}{2}} \qquad \text{(A63)}$$

and for antisymmetric tensors of rank N/2 one has

$$\bar{\Gamma}_N \Gamma^{(\frac{N}{2})}_{\varepsilon_1 w_1 \cdots \varepsilon_{\frac{N}{2}} w_{N/2}} = (-1)^{\frac{N}{4}(\frac{N}{2}+1)} (-1)^{\tilde{P}} \Gamma^{(\frac{N}{2})}_{-\tilde{\varepsilon}_1 \tilde{w}_1 \cdots -\tilde{\varepsilon}_{N/2} \tilde{w}_{N/2}} \qquad \text{(A64)}$$

Here $(\varepsilon_1 w_1, \ldots, \varepsilon_{N/2} w_{N/2})$ and $(\tilde{\varepsilon}_1 \tilde{w}_1, \ldots, \tilde{\varepsilon}_{N/2} \tilde{w}_{N/2})$ are two disjoint lists of N/2 double indices $\varepsilon_i w_i$ (no double index appearing in both lists) and $(-1)^{\tilde{P}}$ is the sign of permutation of the set $(\varepsilon_1 w_1, \ldots, \varepsilon_{N/2} w_{N/2}, \tilde{\varepsilon}_1 \tilde{w}_1, \ldots \tilde{\varepsilon}_{N/2} \tilde{w}_{N/2})$ compared to $(-1, +1, \ldots, -\frac{N}{2}, +\frac{N}{2})$.

e) Subgroups of SO(12)

For a classification of states we use various subgroups of SO(12). Let us first consider the maximal subgroup SO(10) x $U(1)_q$. The SO(10) generators are given by T_{MN}; $M,N = 1 \ldots 10$, and the $U(1)_q$ generator is $T_{11,12}$. In the weight basis SO(10) labels correspond to $w = 1 \ldots 5$ and $U(1)_q$ labels to $w = 6$. The antisymmetric tensors considered in this paper decompose as follows

$$\begin{aligned} 12 &\rightarrow 10_0 + 1_{\pm 1} \\ 66 &\rightarrow 45_0 + 10_{\pm 1} + 1_0 \\ 220 &\rightarrow 120_0 + 45_{\pm 1} + 10_0 \\ 792 &\rightarrow 126_0 + \overline{126}_0 + 210_{\pm 1} + 120_0 \end{aligned} \qquad \text{(A65)}$$

The fifth rank antisymmetric tensor of SO(10) consists of two irreducible representations complex conjugate to each other

$$\begin{pmatrix} 126 \\ \overline{126} \end{pmatrix} = \tfrac{1}{2}(1 \pm \bar{\Gamma}_{10}) \Phi^{(5)} = \Phi^{(5)} \tfrac{1}{2}(1 \mp \bar{\Gamma}_{10}) \qquad \text{(A66)}$$

The irreducible complex spinor representations 16 and $\overline{16}$ obey

$$\bar{\Gamma}_{10} \begin{pmatrix} 16 \\ \overline{16} \end{pmatrix} = \begin{pmatrix} +16 \\ -\overline{16} \end{pmatrix} \qquad \text{(A67)}$$

With respect to $U(1)_q$, tensor fields with charge q = 1 (q = -1) have one label $\Gamma_{\ldots +6}$ ($\Gamma_{\ldots -6}$) whereas fields with no label -6 or +6 or both labels -6 and +6 have q = 0. Spinors with charge q = 1/2 (-1/2) are eigenstates of $\bar{\Gamma}_2$ with eigenvalue +1 (-1).

The group SO(10) can be further reduced to various subgroups. (For useful tables on SO(10) representations and their decomposition see ref. 38, although our conventions differ from this author. For a systematic treatment and additional material compare ref. 39.) The subgroup SO(4) = $SU(2)_L$ x $SU(2)_R$ is spanned by the labels w = 1,2 (M,N = 1...4). The spinor representations of $SU(2)_L$ x $SU(2)_R$ obey

$$\begin{aligned} \bar{\Gamma}_4 \, (2,1) &= -(2,1) \\ \bar{\Gamma}_4 \, (1,2) &= +(1,2) \end{aligned} \qquad \text{(A68)}$$

The singlets (1,1) have either no label w = 1 or w = 2 or labels $|-1+1-2+2\rangle$. The SO(4) vector (2,2) has states with labels

$$|-1\rangle;\ |+1\rangle;\ |-2\rangle;\ |+2\rangle \qquad \text{(A69)}$$

or

$$-|+1,-2+2\rangle;\ |-1,-2,+2\rangle;\ -|-1,+1,+2\rangle;\ |-1,+1,-2\rangle \qquad \text{(A70)}$$

The antisymmetric tensor of rank two gives the adjoint representations of $SU(2)_L$ and $SU(2)_R$:

$$\begin{aligned} (1,3):&\quad |-1,-2\rangle;\ \tfrac{1}{\sqrt{2}}\{|-1+1\rangle + |-2+2\rangle\};\ |+1+2\rangle \\ (3,1):&\quad |-1,+2\rangle;\ \tfrac{1}{\sqrt{2}}\{|-1+1\rangle - |-2+2\rangle\};\ |+1,-2\rangle \end{aligned} \qquad \text{(A71)}$$

and fulfills

$$\begin{pmatrix} (3,1) \\ (1,3) \end{pmatrix} = \frac{1}{2}(1 \mp \bar{\Gamma}_4)\phi^{(2)} = \phi^{(2)}\frac{1}{2}(1 \mp \bar{\Gamma}_4) \qquad \text{(A72)}$$

The subgroup SO(6) = $SU(4)_c$ is spanned by the labels w = 3,4,5 (M,N=5,6...10). The 4 component complex spinor representations obey

$$\begin{aligned} \bar{\Gamma}_6 \ \underline{4} &= -\underline{4} \\ \bar{\Gamma}_6 \ \bar{\underline{4}} &= +\bar{\underline{4}} \end{aligned} \qquad \text{(A73)}$$

Singlets of SO(6) have no label w = 3,4 or 5 or are labelled $|-3,+3,-4,+4,-5,+5\rangle$. The vector ($\underline{6}$) has states

$$|-3\rangle;\ |+3\rangle;\ |-4\rangle;\ |+4\rangle;\ |-5\rangle;\ |+5\rangle \qquad \text{(A74)}$$

or the corresponding dual states with five SO(6) labels obtained by applying $\bar{\Gamma}_6$. The adjoint ($\underline{15}$) has two (or four) labels with w = 3,4,5. The antisymmetric tensor of rank 3 decomposes into two complex representations $\underline{10}$ and $\underline{\overline{10}}$ with

$$\begin{pmatrix} 10 \\ \overline{10} \end{pmatrix} = \frac{1}{2}(1 \mp \bar{\Gamma}_6)\phi^{(3)} = \phi^{(3)}\frac{1}{2}(1 \pm \bar{\Gamma}_6) \qquad \text{(A75)}$$

The colour interactions correspond to the subgroup $SU(3)_c$ of $SU(4)_c$. The Octet within the 15 of $SU(4)_c$ is spanned by the generators

$$\begin{aligned} &E_{-3+4}\ ,\quad E_{-3+5}\ ,\quad E_{-4+5}\ , \\ &E_{+3-4}\ ,\quad E_{+3-5}\ ,\quad E_{+4-5}\ , \\ &-\tfrac{1}{\sqrt{2}}(H_3 - H_4)\ ,\quad -\tfrac{1}{\sqrt{6}}(H_3 + H_4 - 2H_5) \end{aligned} \qquad \text{(A76)}$$

(The generator $\frac{1}{\sqrt{3}}(H_3+H_4+H_5)$ corresponds to the abelian group

$U(1)_{B-L}$ commuting with $SU(3)_c$.) Singlets of $SU(3)_c$ occur in various $SU(4)_c$ representations. The trivial one in the singlet of $SU(4)_c$ has no labels w = 3,4,5 and Y_{B-L} = 0. Other singlets with Y_{B-L} = 0 are in the 15 of $SU(4)_c$ and correspond to the states

$$\begin{aligned} &\tfrac{1}{\sqrt{3}}\{|-3+3\rangle + |-4+4\rangle + |-5+5\rangle\} \\ &\tfrac{1}{\sqrt{3}}\{|-3+3-4+4\rangle + |-3+3-5+5\rangle + |-4+4-5+5\rangle\} \end{aligned} \qquad \text{(A77)}$$

Finally, there are singlets with Y_{B-L} = -2 (+2) in the representations 10 ($\overline{10}$):

$$\begin{aligned} &|-3-4-5\rangle \qquad (Y_{B-L} = -2) \\ &|+3+4+5\rangle \qquad (Y_{B-L} = 2) \end{aligned} \qquad \text{(A78)}$$

Triplets are contained in the 6, 10, $\overline{10}$ and 15 of $SU(4)_c$ as for example

$$\begin{aligned} &3: \quad |-3\rangle;\ |-4\rangle;\ |-5\rangle \qquad (Y_{B-L} = -\tfrac{2}{3}) \\ &\bar{3}: \quad |+3\rangle;\ |+4\rangle;\ |+5\rangle \qquad (Y_{B-L} = \tfrac{2}{3}) \\ &3: \quad |+4+5\rangle;\ |+5+3\rangle;\ |+3+4\rangle \qquad (Y_{B-L} = 4/3) \\ &\bar{3}: \quad |-4-5\rangle;\ |-5-3\rangle;\ |-3-4\rangle \qquad (Y_{B-L} = -4/3) \end{aligned} \qquad \text{(A79)}$$

We finally note that the SU(5) embedding in SO(10) is completely parallel to the $SU(3)_c$ embedding in SO(6). SU(5) labels run from w = 1 to w = 5 instead of w = 3,4,5 for $SU(3)_c$. Among the SU(5) singlets are the following states

no label w = 1,2,3,4,5

$|+1+2+3+4+5\rangle$

$|-1-2-3-4-5\rangle$

$$\frac{1}{\sqrt{5}}\left\{|-1+1\rangle+|-2+2\rangle+|-3+3\rangle+|-4+4\rangle+|-5+5\rangle\right\} \tag{A80}$$

$$\frac{1}{\sqrt{10}}\left\{|-1+1-2+2\rangle+|-1+1-3+3\rangle+|-1+1-4+4\rangle+|-1+1-5+5\rangle+|-2+2-3+3\rangle\right.$$
$$\left.+|-2+2-4+4\rangle+|-2+2-5+5\rangle+|-3+3-4+4\rangle+|-3+3-5+5\rangle+|-4+4-5+5\rangle\right\}$$

and some 5-plets are given by

$$|-1\rangle,\ |-2\rangle,\ |-3\rangle,\ |-4\rangle,\ |-5\rangle;$$

$$|+2+3+4+5\rangle,\ |+1+3+4+5\rangle,\ |+1+2+4+5\rangle,\ |+1+2+3+5\rangle,\ |+1+2+3+4\rangle;$$

$$\frac{1}{2}\left\{|-1-2+2\rangle+|-1-3+3\rangle+|-1-4+4\rangle+|-1-5+5\rangle\right\},$$
$$\frac{1}{2}\left\{|-2-1+1\rangle+|-2-3+3\rangle+|-2-4+4\rangle+|-2-5+5\rangle\right\},\ \ldots; \tag{A81}$$

$$\frac{1}{\sqrt{6}}\left\{|-1-2+2-3+3\rangle+|-1-2+2-4+4\rangle+|-1-2+2-5+5\rangle\right.$$
$$\left.+|-1-3+3-4+4\rangle+|-1-3+3-5+5\rangle+|-1-4+4-5+5\rangle\right\},$$

$$\frac{1}{\sqrt{6}}\left\{|-2-1+1-3+3\rangle+|-2-1+1-4+4\rangle+|-2-1+1-5+5\rangle\right.$$
$$\left.+|-2-3+3-4+4\rangle+|-2-3+3-5+5\rangle+|-2-4+4-5+5\rangle\right\},\ \ldots$$

f) Physical States

We are now at a point where the classification of "physical states" in the various antisymmetric tensor representations becomes an easy task. Let us first study the adjoint of SO(12). In table 3 we give a list of the abelian quantum numbers H_w and the labelling for all states in the 66. A similar analysis can be done for the other totally antisymmetric tensors. As an example, we discuss the electrically neutral colour singlets in the fifth rank tensor 792. We have given in table 2 a list of these fields with their quantum numbers. For the weak doublets, we have omitted the neutral doublet components in the 210 of SO(10) (with $SU(4)_c \times SU(2)_L \times SU(2)_R$ transformation properties

$(10,2,2) + (\overline{10},2,2)$) since they do not admit Yukawa couplings to the chiral quarks and leptons. To specify the labelling, we write this neutral part of $\phi^{(5)}$ explicitely

$$\begin{aligned}
\phi_o^{(5)} = \frac{i}{\sqrt{32}}\Big\{ & -s_1\,\Gamma_{+1+2+3+4+5} + s_2\,\Gamma_{-1+1-2+2+6} \\
& +\tfrac{1}{\sqrt{3}}s_3\left(\Gamma_{-4+4-5+5+6}+\Gamma_{-3+3-5+5+6}+\Gamma_{-3+3-4+4+6}\right) \\
& +\tfrac{1}{\sqrt{6}}s_4\left(\Gamma_{-1+1-3+3+6}+\Gamma_{-1+1-4+4+6}+\Gamma_{-1+1-5+5+6}\right. \\
& \qquad \left.+\Gamma_{-2+2-3+3+6}+\Gamma_{-2+2-4+4+6}+\Gamma_{-2+2-5+5+6}\right) \\
& +\tfrac{1}{\sqrt{6}}d_1\left(-\Gamma_{-2-4+4-5+5}-\Gamma_{-2-3+3-5+5}-\Gamma_{-2-3+3-4+4}\right. \\
& \qquad \left.+\Gamma_{-1+1-2-3+3}+\Gamma_{-1+1-2-4+4}+\Gamma_{-1+1-2-5+5}\right) \\
& +\tfrac{1}{\sqrt{6}}d_2^{*}\left(\Gamma_{+2-4+4-5+5}+\Gamma_{+2-3+3-5+5}+\Gamma_{+2-3+3-4+4}\right. \\
& \qquad \left.+\Gamma_{-1+1+2-3+3}+\Gamma_{-1+1+2-4+4}+\Gamma_{-1+1+2-5+5}\right) \\
& +\tfrac{1}{\sqrt{3}}d_3\left(\Gamma_{-2-3+3-6+6}+\Gamma_{-2-4+4-6+6}+\Gamma_{-2-5+5-6+6}\right) \\
& -d_4\,\Gamma_{-1+1-2-6+6} \\
& -t_1\,\Gamma_{-1+2-3-4-5} \\
& +\tfrac{1}{\sqrt{6}}t_2\left(\Gamma_{-1+1-3+3+6}+\Gamma_{-1+1-4+4+6}+\Gamma_{-1+1-5+5+6}\right. \\
& \qquad \left.-\Gamma_{-2+2-3+3+6}-\Gamma_{-2+2-4+4+6}-\Gamma_{-2+2-5+5+6}\right)\Big\} + h.c.
\end{aligned} \tag{A82}$$

To proceed further, we note the following identities

$$\Gamma^{(k)}_{A_1\cdots A_m,A_{m+1}\cdots A_k} = \Gamma^{(m)}_{A_1\cdots A_m}\,\Gamma^{(k-m)}_{A_{m+1}\cdots A_k} \quad \text{if } A_1 \neq \cdots A_m \cdots \neq A_k \tag{A83}$$

$$\Gamma^{(k)}_{\varepsilon_1 w_1,\ldots,-i,+i,\ldots\varepsilon_k w_k} = \Gamma^{(2)}_{-i+i}\,\Gamma^{(k-2)}_{\varepsilon_1 w_1\cdots\varepsilon_k w_k} \quad \text{if } w_1 \ldots w_k \neq i \tag{A84}$$

$$\Gamma^{(2)}_{-i+i} = 2H_i$$

$$\Gamma^{(k)}_{\varepsilon_1 w_1, \varepsilon_2 w_2, \dots \varepsilon_k w_k} = \Gamma_{\varepsilon_1 w_1}\Gamma_{\varepsilon_2 w_2}\cdots\Gamma_{\varepsilon_k w_k} \quad \text{if} \quad w_1 \neq w_2 \neq \cdots w_k \tag{A85}$$

In equ. (A82) the list of indices of $\Gamma^{(k-2)}$ is the same as the one of $\Gamma^{(k)}$ with indices -i, +i taken away. Using (A26) one finds

$$\begin{aligned}\phi_0^{(5)} = \frac{i}{\sqrt{2}}\Gamma_{-2}\{ & S_1^+ s_1^* + D_1 d_1 + D_2 d_2 + D_3 d_3 + D_4 d_4 + T_1^+ t_1^* \} \\ + \frac{i}{\sqrt{2}}\Gamma_{+2}\{ & S_1 s_1 + D_1 d_1^* + D_2 d_2^* + D_3 d_3^* + D_4 d_4^* + T_1 t_1 \} \\ + \frac{i}{\sqrt{2}}\Gamma_{+6}\{ & S_2 s_2 + S_3 s_3 + S_4 s_4 + T_2 t_2 \} \\ + \frac{i}{\sqrt{2}}\Gamma_{-6}\{ & S_2 s_2^* + S_3 s_3^* + S_4 s_4^* + T_2 t_2^* \}\end{aligned} \tag{A86}$$

with

$$\begin{aligned} S_1^+ &= T_1 = \tfrac{1}{4}\Gamma_{-1}\Gamma_{-3}\Gamma_{-4}\Gamma_{-5} \\ S_1 &= T_1^+ = \tfrac{1}{4}\Gamma_{+1}\Gamma_{+3}\Gamma_{+4}\Gamma_{+5} \\ D_1 &= \tfrac{1}{\sqrt{6}}\{-\tfrac{9}{8}Y_{B-L}^2 + \tfrac{3}{8} - \tfrac{3}{2}Y_{B-L}(I_{3L}+I_{3R})\} \\ D_2 &= \tfrac{1}{\sqrt{6}}\{\tfrac{9}{8}Y_{B-L}^2 - \tfrac{3}{8} - \tfrac{3}{2}Y_{B-L}(I_{3L}+I_{3R})\} \\ D_3 &= \tfrac{\sqrt{3}}{2}Y_{B-L}\,q \\ D_4 &= (I_{3L}+I_{3R})q \\ S_2 &= I_{3R}^2 - I_{3L}^2 \\ S_3 &= \tfrac{1}{\sqrt{3}}(\tfrac{9}{8}Y_{B-L}^2 - \tfrac{3}{8}) \\ S_4 &= -\tfrac{3}{\sqrt{6}}Y_{B-L}I_{3R} \\ T_2 &= -\tfrac{3}{\sqrt{6}}Y_{B-L}I_{3L} \end{aligned} \tag{A87}$$

The fields d_2 and d_1 belong to a 5 and a 45 of SU(5) and to $\overline{126}$ and 126 of SO(10), respectively. The neutral fields in the 5 and 45 within the 120 of SO(10) are linear combinations of d_3 and d_4. We write the corresponding contribution to $\phi^{(5)}$ as

$$d_3\tilde{D}_3 + d_4\tilde{D}_4 = \hat{d}_5\hat{D}_5 + \hat{d}_{45}\hat{D}_{45} \tag{A88}$$

with

$$\begin{aligned}\hat{D}_5 &= \tfrac{1}{2}\sqrt{3}\tilde{D}_3 - \tfrac{1}{2}\tilde{D}_4 \\ &= \tfrac{1}{2}\{\Gamma_{-2-3+3-6+6} + \Gamma_{-2-4+4-6+6} + \Gamma_{-2-5+5-6+6} + \Gamma_{-2-1+1-6+6}\} \\ \hat{D}_{45} &= \tfrac{1}{2}\sqrt{3}\tilde{D}_4 + \tfrac{1}{2}\tilde{D}_3\end{aligned} \tag{A89}$$

$$\begin{aligned}\hat{d}_5 &= \tfrac{1}{2}\sqrt{3}\,d_3 - \tfrac{1}{2}d_4 \\ \hat{d}_{45} &= \tfrac{1}{2}d_3 + \tfrac{1}{2}\sqrt{3}\,d_4\end{aligned} \tag{A90}$$

Similarly, the singlets s_2, s_3 and s_4 in 210 of SO(10) belong to SU(5) representations 1, 24 and 75

$$s_2\tilde{S}_2 + s_3\tilde{S}_3 + s_4\tilde{S}_4 = \hat{s}_1\hat{S}_1 + \hat{s}_{24}\hat{S}_{24} + \hat{s}_{75}\hat{S}_{75} \tag{A91}$$

with

$$\begin{aligned}\hat{S}_1 &= \tfrac{1}{\sqrt{10}}(\tilde{S}_2 + \sqrt{3}\tilde{S}_3 + \sqrt{6}\tilde{S}_4) \\ \hat{s}_1 &= \tfrac{1}{\sqrt{10}}(s_2 + \sqrt{3}s_3 + \sqrt{6}s_4)\end{aligned} \tag{A92}$$

We may use the explicit basis for the matrices Γ_A in ref.18 to work out the matrices $\Gamma_{\varepsilon w}$. One finds

$$\frac{i}{\sqrt{2}}\Gamma_{-1} = i\,\tau_0\otimes\tau_0\otimes\tau_+\otimes\tau_d\otimes\tau_1\otimes\tau_1 + i\,\tau_0\otimes\tau_0\otimes\tau_-\otimes\tau_u\otimes\tau_1\otimes\tau_1$$

$$\frac{i}{\sqrt{2}}\Gamma_{-2} = \tau_0\otimes\tau_0\otimes\tau_d\otimes\tau_0\otimes\tau_-\otimes\tau_1 - \tau_0\otimes\tau_0\otimes\tau_u\otimes\tau_0\otimes\tau_+\otimes\tau_1$$

$$\frac{i}{\sqrt{2}}\Gamma_{-3} = \tau_+\otimes\tau_-\otimes\tau_2\otimes\tau_+\otimes\tau_1\otimes\tau_1 + \tau_-\otimes\tau_-\otimes\tau_2\otimes\tau_-\otimes\tau_1\otimes\tau_1 \qquad \text{(A93)}$$
$$- \tau_+\otimes\tau_+\otimes\tau_2\otimes\tau_-\otimes\tau_1\otimes\tau_1 - \tau_-\otimes\tau_+\otimes\tau_2\otimes\tau_+\otimes\tau_1\otimes\tau_1$$

$$\frac{i}{\sqrt{2}}\Gamma_{-4} = \tau_d\otimes\tau_2\otimes\tau_2\otimes\tau_-\otimes\tau_1\otimes\tau_1 - \tau_u\otimes\tau_2\otimes\tau_2\otimes\tau_+\otimes\tau_1\otimes\tau_1$$

$$\frac{i}{\sqrt{2}}\Gamma_{-5} = -\tau_2\otimes\tau_u\otimes\tau_2\otimes\tau_+\otimes\tau_1\otimes\tau_1 - \tau_2\otimes\tau_d\otimes\tau_2\otimes\tau_-\otimes\tau_1\otimes\tau_1$$

$$\frac{i}{\sqrt{2}}\Gamma_{-6} = \tau_0\otimes\tau_0\otimes\tau_0\otimes\tau_0\otimes\tau_d\otimes\tau_- - \tau_0\otimes\tau_0\otimes\tau_0\otimes\tau_0\otimes\tau_u\otimes\tau_+$$

Here we used the following combinations of Pauli matrices ($\tau_0 = 1$):

$$\tau_+ = \tfrac{1}{2}(\tau_1 - i\tau_2) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\tau_- = \tfrac{1}{2}(\tau_1 + i\tau_2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$
$$\qquad\qquad\qquad \text{(A94)}$$
$$\tau_u = \tfrac{1}{2}(\tau_0 + \tau_3) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\tau_d = \tfrac{1}{2}(\tau_0 - \tau_3) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

In this basis one finds

$$S_1^+ = \tfrac{1}{4}\Gamma_{-1}\Gamma_{-3}\Gamma_{-4}\Gamma_{-5} = -\tau_d\otimes\tau_d\otimes\tau_u\otimes\tau_-\otimes\tau_0\otimes\tau_0 \qquad \text{(A95)}$$

To derive the field equations in section 4, we need a few trace relations for the $SU(3)_C \times SU(2)_L \times U(1)_Y$ singlet components of $\phi_0^{(5)}$. Writing

$$\phi_s = s_1\tilde{S}_1 + s_2\tilde{S}_2 + s_3\tilde{S}_3 + s_4\tilde{S}_4 + h.c. \qquad \text{(A96)}$$

one has the following relations for i,j = 2,3,4:

$$\tilde{S}_i\,\tilde{S}_j = 0$$
$$[\tilde{S}_i, \tilde{S}_j^+] = 2\,S_i S_j H_6 \qquad \text{(A97)}$$
$$\{\tilde{S}_i, \tilde{S}_j^+\} = S_i S_j$$

We use the identities

$$\mathrm{Tr}\; H_i = 0$$
$$\mathrm{Tr}\; H_i H_j = 16\,\delta_{ij} \qquad \text{(A98)}$$
$$\mathrm{Tr}\; H_{i_1} H_{i_2} \cdots H_{i_k} = 0 \qquad \text{for } i_1 \neq i_2 \neq \cdots i_k$$

to derive the trace relations

$$\mathrm{Tr}\; \tilde{S}_i\,\tilde{S}_j = 0$$
$$\mathrm{Tr}\; \tilde{S}_i\,\tilde{S}_j^+ = 2\,\delta_{ij} \qquad \text{(A99)}$$

$$\mathrm{Tr}\; E_{\varepsilon_1 w_1 \varepsilon_2 w_2}[\tilde{S}_i, \tilde{S}_j] = 0$$
$$\mathrm{Tr}\; E_{\varepsilon_1 w_1 \varepsilon_2 w_2}[\tilde{S}_i, \tilde{S}_j^+] = 0 \qquad \text{(A100)}$$

$$\mathrm{Tr}\; H_k[\tilde{S}_i, \tilde{S}_j] = 0$$
$$\mathrm{Tr}\; H_k[\tilde{S}_1, \tilde{S}_1^+] = \begin{cases} 2 & \text{for } k = 1,2,3,4,5 \\ 0 & \text{for } k = 6 \end{cases}$$

$$\mathrm{Tr}\; H_k[\tilde{S}_i, \tilde{S}_j^+] = \begin{cases} 2\delta_{ij} & \text{for } i,j = 2,3,4 \text{ and } k = 6 \\ 0 & \text{for } i,j = 2,3,4 \text{ and } k = 1,2,3,4,5 \end{cases} \qquad \text{(A101)}$$

A P P E N D I X   B

Yukawa Couplings of Scalars in Totally Antisymmetric Tensor Representations of SO(12)

In this appendix we perform the SO(1,5) and SO(12) algebra necessary for a calculation of Yukawa couplings in the six dimensional SO(12) model. The analysis is completely parallel to ref. 18. We represent the two irreducible Majorana-Weyl spinors ψ_1 and ψ_2 in the form of a Dirac spinor with 64x8 components

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \tag{B1}$$

obeying the Weyl and Majorana constraints

$$\bar{\Gamma}_{12}\, \bar{\gamma}_6\, \psi = \psi \tag{B2}$$

$$B_{12}^{-1} B_6^{-1} \psi^* = \psi \tag{B3}$$

In six dimensions, Yukawa couplings of this spinor are possible to scalars in totally antisymmetric tensor representations of rank one, three and five. The action involves at most three independent real Yukawa couplings $\hat{f}_1$, $\hat{f}_3$ and $\hat{f}_5$ and can be written in the form

$$S_{yuk.} = \int d^6\hat{x}\, \hat{g}_6^{1/2} \{ \hat{f}_1 \bar{\psi} \phi^{(1)} \psi + \hat{f}_3 \bar{\psi} \phi^{(3)} \bar{\Gamma}_{12} \psi + \hat{f}_5 \bar{\psi} \phi^{(5)} \psi \} \tag{B4}$$

All terms are hermitean and $\bar{\psi}$ is defined by

$$\bar{\psi} = \psi^{+} \gamma^{0} = \tilde{\psi} = \psi^{T} B_{12} B_6 \gamma^{0} = \psi^{T} C_6 B_{12} \tag{B5}$$

Terms like $\bar{\psi}\phi^{(1)}\bar{\Gamma}_{12}\psi$, $\bar{\psi}\phi^{(3)}\psi$ or $\bar{\psi}\phi^{(5)}\bar{\Gamma}_{12}\psi$ are excluded by Fermi statistics. The matrix C_6 fulfills

$$(\gamma^{\hat{m}})^{T} = - C_6 \gamma^{\hat{m}} C_6^{-1} \tag{B6}$$

and can be written

$$\begin{aligned} C_6 &= C_4 B_2 \\ (\gamma^{m})^{T} &= - C_4 \gamma^{m} C_4^{-1} \quad ; m = 0 \ldots 3 \\ (\Gamma^{a})^{T} &= - B_2 \Gamma^{a} B_2^{-1} = -(\Gamma_a)^{*} \quad ; a = 1,2 \end{aligned} \tag{B7}$$

We use index conventions of ref 18.

We are mainly interested in the Yukawa couplings of $SU(3)_C \times U(1)_{em}$ singlet scalar fields to chiral quarks and leptons. The harmonic expansion of the chiral fermions reads

$$\psi^{0}(y,x) = \psi_{V_j}(y)\, \psi^{V_j}(x) \tag{B8}$$

whereas the electrically neutral colour singlets in the scalars $\phi^{(k)}$ are expanded

$$\phi_0^{(k)}(y,x) = \varphi_{S_i}(y)\, \varphi^{S_i}(x) \tag{B9}$$

Here the indices V and S refer to the quantum numbers of SO(12) or its subgroups with V = u_1, e, ν^c... etc. and S = s_1, s_2, s_3, s_4, d_1, d_2, d_3, d_4, t_1, t_2 for $\phi_0^{(5)}$ (compare appendix A) and similar for $\phi_0^{(1)}$ and $\phi_0^{(3)}$. The index j counts the number of chiral u-quarks etc. and i labels infinitely many scalar fields with given SO(12) quantum numbers.

The action (B4) for any given (k) is then written

$$S_k = \int d^4x\, g_4^{1/2}(x)\, \tilde{\psi}^{V_j}(x)\, \varphi^{S_i}(x)\, \psi^{V'_{j'}}(x) \cdot \hat{f}_k \int d^2y\, \sigma^2(y)\, g_2^{1/2}(y)\, \psi_{V_j}^{T}(y)\, B_2 B_{12}\, \varphi_{S_i}(y)\, \bar{\Gamma}_{(k)}\, \psi_{V'_{j'}}(y) \tag{B10}$$

with

$$\tilde{\psi}^{\nu j}(x) = (\psi^{\nu j}(x))^T C_4 \tag{B11}$$

$$\bar{\Gamma}_{(\ell)} = \begin{cases} 1 & \text{for } \ell = 1,5 \\ \bar{\Gamma}_{12} & \text{for } \ell = 3 \end{cases} \tag{B12}$$

$$\hat{g}_6^{1/2} = g_4^{1/2}(x)\,\sigma^2(y)\, g_2^{1/2}(y) \tag{B13}$$

Using the Weyl-constraint (B2) and the following definitions (τ_3 acts on two dimensional spinor indices with $\bar{\gamma}_6 = \tau_3 \gamma^5$)

$$\begin{pmatrix} \psi_L(x) \\ \psi_R(x) \end{pmatrix} = \tfrac{1}{2}(1 \pm \gamma^5)\,\psi(x)$$

$$\begin{aligned} \psi_1^+(y) &= \tfrac{1}{4}(1+\bar{\Gamma}_{12})(1+\tau_3)\,\psi(y) \\ \psi_1^-(y) &= \tfrac{1}{4}(1+\bar{\Gamma}_{12})(1-\tau_3)\,\psi(y) \\ \psi_2^+(y) &= \tfrac{1}{4}(1-\bar{\Gamma}_{12})(1+\tau_3)\,\psi(y) \\ \psi_2^-(y) &= \tfrac{1}{4}(1-\bar{\Gamma}_{12})(1-\tau_3)\,\psi(y) \end{aligned} \tag{B14}$$

one has

$$\begin{aligned} S_\ell = {} & \int d^4x\, g_4^{1/2}\, \tilde{\psi}_L^{\nu j} \varphi^{si} \psi_L^{\nu' j'} \cdot \\ & \cdot \hat{f}_\ell \int d^2y\, \sigma^2 g_2^{1/2} (\psi_{1\nu j}^+ + \psi_{2\nu j}^-)^T B_2 B_{12} \varphi_{si} \bar{\Gamma}_{(\ell)} (\psi_{1\nu' j'}^+ + \psi_{2\nu' j'}^-) \\ & + \int d^4x\, g_4^{1/2}\, \tilde{\psi}_R^{\nu j} \varphi^{si} \psi_R^{\nu' j'} \cdot \\ & \cdot \hat{f}_\ell \int d^2y\, \sigma^2 g_2^{1/2} (\psi_{2\nu j}^+ + \psi_{1\nu j}^-)^T B_2 B_{12} \varphi_{si} \bar{\Gamma}_{(\ell)} (\psi_{2\nu' j'}^+ + \psi_{1\nu' j'}^-) \end{aligned} \tag{B15}$$

Using the Majorana constraint (B3) one can show that the second term is the hermitean conjugate of the first term (compare ref.18). Using the explicit form $B_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and noting that $B_{12} \varphi_{si} \bar{\Gamma}_{(\ell)}$ is an antisymmetric 64x64 matrix we write the Yukawa couplings

$$S_\ell = \int d^4x\, g_4^{1/2}\, \tilde{\psi}_L^{\nu j} h_{\nu j \nu' j' s i}\, \varphi^{si} \psi_L^{\nu' j'} + h.c. \tag{B16}$$

$$h_{\nu j \nu' j' s i} = 2 \hat{f}_\ell \int d^2y\, g_2^{1/2} \sigma^2 (\psi_{1\nu j}^+)^T B_{12} \varphi_{si} \bar{\Gamma}_{(\ell)}\, \psi_{2\nu' j'}^- \tag{B17}$$

For a calculation of h we have to specify the normalization of $\psi_{\nu j}$ and φ_{si}. The standard kinetic terms for the fermions is obtained for

$$\int d^2y\, g_2^{1/2} \sigma^{3/2} \{ (\psi_{1\nu j}^+)^\dagger \psi_{1\nu' j'}^+ + (\psi_{2\nu j}^-)^\dagger \psi_{2\nu' j'}^- \} = \tfrac{1}{2} \delta_{\nu\nu'} \delta_{jj'} \tag{B18}$$

Correspondingly, we obtain the standard kinetic term for scalar fields $\varphi^{si}(x)$ if

$$\int d^2y\, g_2^{1/2} \sigma\, \mathrm{Tr}\, \varphi_{si}^\dagger \varphi_{s'i'} = 2\, \delta_{ss'} \delta_{ii'} \tag{B19}$$

Note that in our formalism a complex scalar field and its complex conjugate are treated as different φ^{si}. The normalization (B19) takes care of the necessary identifications.

To perform the necessary SO(12) algebra we need the matrix elements of $\tilde{\Phi}_0^{(\ell)} = B_{12} \Phi_0^{(\ell)}$ in a physical basis. We decompose ψ with respect to SO(10) x $U(1)_q$

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} 16_{1/2} \\ \overline{16}_{1/2} \\ 16_{-1/2} \\ \overline{16}_{-1/2} \end{pmatrix} \tag{B20}$$

The scalars $\Phi^{(\ell)}$ give

$$12 \to 10_0 + 1_{\pm 1}$$

$$220 \to 120_0 + 45_{\pm 1} + 10_0 \qquad \text{(B21)}$$

$$792 \to 126_0 + \overline{126}_0 + 210_{\pm 1} + 120_0$$

In an obvious notation we can write $\tilde{\phi}^{(k)}$ as blocks of 16x16 matrices

$$\tilde{\phi}^{(1)} = \tilde{\phi}^{(1)}_{10} + \tilde{\phi}^{(1)}_{1} = \begin{pmatrix} 0 & 0 & \varphi^{(1)}_{10} & 0 \\ 0 & 0 & 0 & \chi^{(1)}_{10} \\ -\varphi^{(1)}_{10} & 0 & 0 & 0 \\ 0 & -\chi^{(1)}_{10} & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & \varphi^{-}_{1} \\ 0 & 0 & \varphi^{+}_{1} & 0 \\ 0 & -(\varphi^{+}_{1})^{T} & 0 & 0 \\ -(\varphi^{-}_{1})^{T} & 0 & 0 & 0 \end{pmatrix} \qquad \text{(B22)}$$

$$\tilde{\phi}^{(3)} = \tilde{\phi}^{(3)}_{120} + \tilde{\phi}^{(3)}_{45} + \tilde{\phi}^{(3)}_{10} = \begin{pmatrix} 0 & 0 & \varphi^{(3)}_{120} & 0 \\ 0 & 0 & 0 & \chi^{(3)}_{120} \\ -\varphi^{(3)}_{120} & 0 & 0 & 0 \\ 0 & -\chi^{(3)}_{120} & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & \varphi^{-}_{45} \\ 0 & 0 & \varphi^{+}_{45} & 0 \\ 0 & (\varphi^{+}_{45})^{T} & 0 & 0 \\ (\varphi^{-}_{45})^{T} & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & \varphi^{(3)}_{10} & 0 \\ 0 & 0 & 0 & \chi^{(3)}_{10} \\ \varphi^{(3)}_{10} & 0 & 0 & 0 \\ 0 & \chi^{(3)}_{10} & 0 & 0 \end{pmatrix} \qquad \text{(B23)}$$

$$\tilde{\phi}^{(5)} = \tilde{\phi}^{(5)}_{126} + \tilde{\phi}^{(5)}_{210} + \tilde{\phi}^{(5)}_{120} = \begin{pmatrix} 0 & 0 & \varphi_{\overline{126}} & 0 \\ 0 & 0 & 0 & \varphi_{126} \\ -\varphi_{\overline{126}} & 0 & 0 & 0 \\ 0 & -\varphi_{126} & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & \varphi^{-}_{210} \\ 0 & 0 & \varphi^{+}_{210} & 0 \\ 0 & -(\varphi^{+}_{210})^{T} & 0 & 0 \\ -(\varphi^{-}_{210})^{T} & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & \varphi^{(5)}_{120} & 0 \\ 0 & 0 & 0 & \chi^{(5)}_{120} \\ \varphi^{(5)}_{120} & 0 & 0 & 0 \\ 0 & \chi^{(5)}_{120} & 0 & 0 \end{pmatrix} \qquad \text{(B24)}$$

It is checked easily that $\varphi^{(k)}_{10}$, $\chi^{(k)}_{10}$, φ_{126} and $\varphi_{\overline{126}}$ are symmetric matrices whereas $\varphi^{(k)}_{120}$ and $\chi^{(k)}_{120}$ are antisymmetric.

Let us now concentrate on the electrically neutral colour singlets in the fifth rank antisymmetric tensor representation $\phi^{(5)}$. These fields are given explicitely in (A80) where we write now explicitely

$$d_1 = d_{1i}(y)\, d_1^{i}(x) \qquad \text{(B25)}$$

$$\int d^2y\, g_2^{1/2} \sigma\, d_{1i}^{*}\, d_{1i'} = \delta_{ii'} \qquad \text{(B26)}$$

and similar for the other fields. Using (A84), A(85), the explicit representation (A 93) for the matrices Γ_{EW} and

$$B_{12} = i\, \tau_0 \otimes \tau_0 \otimes \tau_0 \otimes \tau_1 \otimes \tau_2 \otimes \tau_3 \qquad \text{(B27)}$$

one has

$$\begin{aligned} \frac{i}{\sqrt{2}} B_{12} \Gamma_{-2} &= - \tau_0 \otimes \tau_0 \otimes \tau_u \otimes \tau_1 \otimes \tau_u \otimes i\tau_2 \\ &\quad - \tau_0 \otimes \tau_0 \otimes \tau_d \otimes \tau_1 \otimes \tau_d \otimes i\tau_2 \\ \frac{i}{\sqrt{2}} B_{12} \Gamma_{+2} &= \tau_0 \otimes \tau_0 \otimes \tau_u \otimes \tau_1 \otimes \tau_d \otimes i\tau_2 \\ &\quad + \tau_0 \otimes \tau_0 \otimes \tau_d \otimes \tau_1 \otimes \tau_u \otimes i\tau_2 \end{aligned} \qquad \text{(B28)}$$

We now can give explicitely the Yukawa couplings for the doublets d_1, d_2, d_3 and d_4. The non-vanishing couplings for the quarks, antiquarks, leptons and antileptons are (we omit couplings to mirror particles here):

$$\begin{aligned} h_{d_j^c d_k d_{1i}^*} &= \tfrac{1}{16} \hat{f} \int d^2y\, g_2^{1/2} \sigma^2 d_j^{c+} d_k^{-} d_{1i}^{*} \\ h_{d_j d_k^c d_{1i}^*} &= \tfrac{1}{16} \hat{f} \int d^2y\, g_2^{1/2} \sigma^2 d_j^{+} d_k^{c-} d_{1i}^{*} \\ h_{e_j^c e_k d_{1i}^*} &= -\tfrac{3}{16} \hat{f} \int d^2y\, g_2^{1/2} \sigma^2 e_j^{c+} e_k^{-} d_{1i}^{*} \\ h_{e_j e_k^c d_{1i}^*} &= -\tfrac{3}{16} \hat{f} \int d^2y\, g_2^{1/2} \sigma^2 e_j^{+} e_k^{c-} d_{1i}^{*} \end{aligned}$$

$$
\begin{aligned}
h_{u^c_j u_k d_{2i}} &= \frac{1}{\sqrt{6}}\hat{f}\int d^2y\, g_2^{1/2}\sigma^2\, u^{c+}_j u^-_k\, d_{2i} \\
h_{u_j u^c_k d_{2i}} &= \frac{1}{\sqrt{6}}\hat{f}\int d^2y\, g_2^{1/2}\sigma^2\, u^+_j u^{c-}_k\, d_{2i} \\
h_{\nu^c_j \nu_k d_{2i}} &= -\frac{3}{\sqrt{6}}\hat{f}\int d^2y\, g_2^{1/2}\sigma^2\, \nu^{c+}_j \nu^-_k\, d_{2i} \\
h_{\nu_j \nu^c_k d_{2i}} &= -\frac{3}{\sqrt{6}}\hat{f}\int d^2y\, g_2^{1/2}\sigma^2\, \nu^+_j \nu^{c-}_k\, d_{2i} \\
h_{u^c_j u_k d_{3i}} &= \frac{1}{2\sqrt{3}}\hat{f}\int d^2y\, g_2^{1/2}\sigma^2\, u^{c+}_j u^-_k\, d_{3i} \\
h_{u_j u^c_k d_{3i}} &= -\frac{1}{2\sqrt{3}}\hat{f}\int d^2y\, g_2^{1/2}\sigma^2\, u^+_j u^{c-}_k\, d_{3i} \\
h_{\nu^c_j \nu_k d_{3i}} &= -\frac{3}{2\sqrt{3}}\hat{f}\int d^2y\, g_2^{1/2}\sigma^2\, \nu^{c+}_j \nu^-_k\, d_{3i} \\
h_{\nu_j \nu^c_k d_{3i}} &= \frac{3}{2\sqrt{3}}\hat{f}\int d^2y\, g_2^{1/2}\sigma^2\, \nu^+_j \nu^{c-}_k\, d_{3i} \\
h_{d^c_j d_k d^*_{3i}} &= -\frac{1}{2\sqrt{3}}\hat{f}\int d^2y\, g_2^{1/2}\sigma^2\, d^{c+}_j d^-_k\, d^*_{3i} \\
h_{d_j d^c_k d^*_{3i}} &= \frac{1}{2\sqrt{3}}\hat{f}\int d^2y\, g_2^{1/2}\sigma^2\, d^+_j d^{c-}_k\, d^*_{3i} \\
h_{e^c_j e_k d^*_{3i}} &= \frac{3}{2\sqrt{3}}\hat{f}\int d^2y\, g_2^{1/2}\sigma^2\, e^{c+}_j e^-_k\, d^*_{3i} \\
h_{e_j e^c_k d^*_{3i}} &= -\frac{3}{2\sqrt{3}}\hat{f}\int d^2y\, g_2^{1/2}\sigma^2\, e^+_j e^{c-}_k\, d^*_{3i} \\
h_{u^c_j u_k d_{4i}} &= \frac{1}{2}\hat{f}\int d^2y\, g_2^{1/2}\sigma^2\, u^{c+}_j u^-_k\, d_{4i} \\
h_{u_j u^c_k d_{4i}} &= -\frac{1}{2}\hat{f}\int d^2y\, g_2^{1/2}\sigma^2\, u^+_j u^{c-}_k\, d_{4i} \\
h_{\nu^c_j \nu_k d_{4i}} &= \frac{1}{2}\hat{f}\int d^2y\, g_2^{1/2}\sigma^2\, \nu^{c+}_j \nu^-_k\, d_{4i} \\
h_{\nu_j \nu^c_k d_{4i}} &= -\frac{1}{2}\hat{f}\int d^2y\, g_2^{1/2}\sigma^2\, \nu^+_j \nu^{c-}_k\, d_{4i} \\
h_{d^c_j d_k d^*_{4i}} &= \frac{1}{2}\hat{f}\int d^2y\, g_2^{1/2}\sigma^2\, d^{c+}_j d^-_k\, d^*_{4i} \\
h_{d_j d^c_k d^*_{4i}} &= -\frac{1}{2}\hat{f}\int d^2y\, g_2^{1/2}\sigma^2\, d^+_j d^{c-}_k\, d^*_{4i} \\
h_{e^c_j e_k d^*_{4i}} &= \frac{1}{2}\hat{f}\int d^2y\, g_2^{1/2}\sigma^2\, e^{c+}_j e^-_k\, d^*_{4i} \\
h_{e_j e^c_k d^*_{4i}} &= -\frac{1}{2}\hat{f}\int d^2y\, g_2^{1/2}\sigma^2\, e^+_j e^{c-}_k\, d^*_{4i}
\end{aligned}
\tag{B29}
$$

Here u^+_k and u^-_k are shorthands for ψ^+_{1uk} and ψ^-_{2uk}, respectively. We note that the Yukawa couplings of the doublets in 120 give no contribution to diagonal terms in the mass matrix (j = k) only if the functions $u^{c\pm}_j$ and $u^\pm_j$ etc. are the same. (This is necessarily the case if SO(10) symmetry is unbroken.)

The singlets s_1 and the triplet t_1 contribute to Majorana-masses for the right handed and left handed neutrinos, respectively. Using (A84) and observing

$$
\begin{aligned}
\frac{i}{\sqrt{2}}B_{12}\Gamma_{-2}S_1^+ &= \tau_d\otimes\tau_d\otimes\tau_u\otimes\tau_d\otimes\tau_u\otimes i\tau_2 \\
\frac{i}{\sqrt{2}}B_{12}\Gamma_{+2}S_1 &= -\tau_d\otimes\tau_d\otimes\tau_u\otimes\tau_u\otimes\tau_d\otimes i\tau_2 \\
\frac{i}{\sqrt{2}}B_{12}\Gamma_{-2}S_1 &= \tau_d\otimes\tau_d\otimes\tau_u\otimes\tau_u\otimes\tau_u\otimes i\tau_2 \\
\frac{i}{\sqrt{2}}B_{12}\Gamma_{+2}S_1^+ &= -\tau_d\otimes\tau_d\otimes\tau_u\otimes\tau_d\otimes\tau_d\otimes i\tau_2
\end{aligned}
\tag{B30}
$$

one finds the following non-vanishing Yukawa couplings:

$$
\begin{aligned}
h_{\nu^c_j \nu^c_k s^*_1} &= 2\hat{f}\int d^2y\, g_2^{1/2}\sigma^2\, \nu^{c+}_j \nu^{c-}_k\, s^*_1 \\
h_{\nu_j \nu_k t^*_1} &= 2\hat{f}\int d^2y\, g_2^{1/2}\sigma^2\, \nu^+_j \nu^-_k\, t^*_1 \\
h_{\bar{\bar\nu}^c_j \bar{\bar\nu}^c_k s_1} &= -2\hat{f}\int d^2y\, g_2^{1/2}\sigma^2\, \bar{\bar\nu}^{c+}_j \bar{\bar\nu}^{c-}_k\, s_1 \\
h_{\bar{\bar\nu}_j \bar{\bar\nu}_k t_1} &= -2\hat{f}\int d^2y\, g_2^{1/2}\sigma^2\, \bar{\bar\nu}^+_j \bar{\bar\nu}^-_k\, t_1
\end{aligned}
\tag{B31}
$$

Finally the singlets s_2, s_3 and s_4 (and also the triplet t_2) have Yukawa couplings between fermions and mirror fermions. They are easily worked out using

$$
\begin{aligned}
\frac{i}{\sqrt{2}}B_{12}\Gamma_{-6} &= \tau_0\otimes\tau_0\otimes\tau_0\otimes\tau_1\otimes\begin{pmatrix} & & 1 \\ & 0 & \\ -1 & & \end{pmatrix} \\
\frac{i}{\sqrt{2}}B_{12}\Gamma_{+6} &= \tau_0\otimes\tau_0\otimes\tau_0\otimes\tau_1\otimes\begin{pmatrix} & & 0 \\ & -1 & \\ 0 & & \end{pmatrix}
\end{aligned}
\tag{B32}
$$

A P P E N D I X   C

The Mass Matrix for Scalar Doublets

In this appendix we calculate the mass terms for the various colour singlet weak doublets of our model. This will determine how these scalars mix to form the low energy Higgs doublet. We restrict the discussion to the mass terms for the electrically neutral components of weak doublets coupling to the chiral quarks and leptons. They are given by H_1^+, H_2^+, d_1, d_2, d_3 and d_4. (Compare tables 1 and 2.) The corresponding higher dimensional fields are contained in internal components of the six dimensional gauge fields[18])

$$A_{da} = \overset{\circ}{e}_a{}^{\alpha} A_{d\alpha} \tag{C1}$$

$$A_{d\pm} = \frac{1}{\sqrt{2}}\left(A_{d1} \mp i A_{d2}\right) \tag{C2}$$

$$A_{d+} = i H_1(y,x) E_{-2+6} + i H_2(y,x) E_{+2+6} \tag{C3}$$

$$A_{d-} = -i H_1(y,x)^* E_{+2-6} - i H_2(y,x)^* E_{-2-6} = (A_{d+})^\dagger \tag{C4}$$

and in the six dimensional scalar fields

$$\phi_d = d_1(y,x)\tilde{D}_1 + d_2(y,x)\tilde{D}_2 + d_3(y,x)\tilde{D}_3 + d_4(y,x)\tilde{D}_4 + h.c. \tag{C5}$$

where we read from (A80)

$$\tilde{D}_1 = \frac{i}{\sqrt{32}}\frac{1}{\sqrt{6}}\Big(\Gamma_{-1+1-2-3+3} + \Gamma_{-1+1-2-4+4} + \Gamma_{-1+1-2-5+5} - \Gamma_{-2-4+4-5+5} - \Gamma_{-2-3+3-5+5} - \Gamma_{-2-3+3-4+4}\Big) \tag{C6}$$

$$\tilde{D}_2 = \frac{i}{\sqrt{32}}\frac{1}{\sqrt{6}}\Big(\Gamma_{-1+1-2-3+3} + \Gamma_{-1+1-2-4+4} + \Gamma_{-1+1-2-5+5} + \Gamma_{-2-4+4-5+5} + \Gamma_{-2-3+3-5+5} + \Gamma_{-2-3+3-4+4}\Big)$$

$$\tilde{D}_3 = \frac{i}{\sqrt{32}}\frac{1}{\sqrt{3}}\Big(\Gamma_{-2-3+3-6+6} + \Gamma_{-2-4+4-6+6} + \Gamma_{-2-5+5-6+6}\Big) \tag{C6}$$

$$\tilde{D}_4 = -\frac{i}{\sqrt{32}}\Gamma_{-1+1-2-6+6}$$

We make the harmonic expansion according to the $U(1)_{\tilde{q}}$ symmetry discussed in the main text

$$\begin{aligned} H_j(y,x) &= H_{jm_jn_j}(\chi)\exp(im_j\varphi)\, H_j^{m_jn_j}(x) \\ d_i(y,x) &= d_{im_in_i}(\chi)\exp(im_i\varphi)\, d_i^{m_in_i}(x) \end{aligned} \tag{C7}$$

The index n labels different states with a given charge $\tilde{q}$. For the sphere, this quantum number would label total angular momentum, but for the more general χ dependence of our solutions the choice of eigenfunctions for the harmonic expansion is somewhat arbitrary. As discussed in section 6 we make a choice so that only the lowest (n = 1) field gets a vacuum expectation value and we drop the index n from now on. After dimensional reduction the mass terms for the four dimensional scalars $H_j^{m_j}$, $d_i^{m_i}$ read

$$S_H = -\int d^4x\, g_4^{1/2}\, \varphi_i^{m_i}(x)^*\, \varphi_j^{m_j}(x)\, M_{ijm_im_j} \tag{C8}$$

where we use

$$\begin{aligned} \varphi_i &= d_i \quad \text{for} \quad i = 1 \ldots 4 \\ \varphi_5 &= H_2^* \\ \varphi_6 &= H_1 \end{aligned} \tag{C9}$$

It is the purpose of this appendix to calculate the hermitean mass matrix $M_{ijm_im_j}$.

There are different sources for doublet mass terms in our model

$$S_M = -\int d^4x\, g_4^{1/2} \left(\sum_{R=1}^{7} L_R \right) \tag{C10}$$

The first contribution comes from the kinetic term for the gauge bosons and contributes to M_{55} and M_{66} :

$$L_1 = \frac{1}{64} \int d^2y\, \sigma^2 \varrho^{1/2}\, \mathrm{Tr}\left(G_{\varphi\beta} G^{\varphi\beta} \right)_d \tag{C11}$$

where $(G_{\varphi\beta} G^{\varphi\beta})_d$ is the contribution quadratic in the fields $A_{d\alpha}$. The next terms come from the scalar kinetic term

$$L_2 = -\frac{1}{4} \int d^2y\, \sigma^2 \varrho^{1/2} \mathring{g}^{\alpha\beta}\, \mathrm{Tr} \left\{ \partial_\alpha \phi_d \partial_\beta \phi_d - 2i\bar{g} [A_{s\alpha}, \phi_d] \partial_\beta \phi_d - \bar{g}^2 [A_{s\alpha}, \phi_d][A_{s\beta}, \phi_d] \right\} \tag{C12}$$

$$L_3 = -\frac{1}{4} \int d^2y\, \sigma^2 \varrho^{1/2} \mathring{g}^{\alpha\beta}\, \mathrm{Tr} \left\{ -2i\bar{g} [A_{d\alpha}, \phi_s] \partial_\beta \phi_d - 2\bar{g}^2 [A_{d\alpha}, \phi_s][A_{s\beta}, \phi_d] \right\} \tag{C13}$$

$$L_4 = -\frac{1}{4} \int d^2y\, \sigma^2 \varrho^{1/2} \mathring{g}^{\alpha\beta}\, \mathrm{Tr} \left\{ -2i\bar{g} [A_{d\alpha}, \phi_d] \partial_\beta \phi_s - 2\bar{g}^2 [A_{d\alpha}, \phi_d][A_{s\beta}, \phi_s] \right\} \tag{C14}$$

$$L_5 = \frac{1}{4} \bar{g}^2 \int d^2y\, \sigma^2 \varrho^{1/2} \mathring{g}^{\alpha\beta}\, \mathrm{Tr} \left\{ [A_{d\alpha}, \phi_s][A_{d\beta}, \phi_s] \right\} \tag{C15}$$

Here $A_{s\alpha}$ and ϕ_s denote the vacuum expectation values of $SU(3)_C \times SU(2)_L \times U(1)_Y$ singlets. In the limit of $U(1)_{\tilde{q}}$ invariance they correspond to the solutions of the field equations in section 4:

$$\begin{aligned} A_{s\varphi} &= m(\chi)(H_1 + H_2) + p(\chi)(H_3 + H_4 + H_5) + n(\chi) H_6 \\ A_{s\chi} &= 0 \\ \phi_s &= s_1(\chi) \exp(i\bar{m}_1 \varphi) \tilde{S}_1 + s_2(\chi) \exp(i\bar{m}_2 \varphi) \tilde{S}_2 \\ &\quad + s_3(\chi) \exp(i\bar{m}_2 \varphi) \tilde{S}_3 + s_4(\chi) \exp(i\bar{m}_2 \varphi) \tilde{S}_4 + \text{h.c.} \end{aligned} \tag{C16}$$

$$\begin{aligned} \tilde{S}_1 &= -\frac{i}{\sqrt{32}} \Gamma_{+1+2+3+4+5} \\ \tilde{S}_2 &= \frac{i}{\sqrt{32}} \Gamma_{-1+1-2+2+6} \\ \tilde{S}_3 &= \frac{i}{\sqrt{32}} \frac{1}{\sqrt{3}} \left(\Gamma_{-4+4-5+5+6} + \Gamma_{-3+3-5+5+6} + \Gamma_{-3+3-4+4+6} \right) \\ \tilde{S}_4 &= \frac{i}{\sqrt{32}} \frac{1}{\sqrt{6}} \left(\Gamma_{-1+1-3+3+6} + \Gamma_{-1+1-4+4+6} + \Gamma_{-1+1-5+5+6} \right. \\ &\quad \left. + \Gamma_{-2+2-3+3+6} + \Gamma_{-2+2-4+4+6} + \Gamma_{-2+2-5+5+6} \right) \end{aligned} \tag{C17}$$

Finally there are contributions from the scalar potential $V(\phi)$ which we split into the mass term and additional terms involving vacuum expectation values of S_i:

$$L_6 = \frac{1}{4} M^2 \int d^2y\, \sigma^2 \varrho^{1/2}\, \mathrm{Tr}\, \phi_d \phi_d \tag{C18}$$

$$L_7 = \int d^2y\, \sigma^2 \varrho^{1/2} \Delta V(\phi) \tag{C19}$$

The appropriate normalization to obtain the standard kinetic term for the four dimensional scalar doublets is (without summation over indices)

$$\begin{aligned} 2\pi \int d\chi\, \sigma \varrho^{1/2} H^*_{jm_j}(\chi) H_{jm_j}(\chi) &= 1 \\ 2\pi \int d\chi\, \sigma \varrho^{1/2} d^*_{im_i}(\chi) d_{im_i}(\chi) &= 1 \end{aligned} \tag{C20}$$

We first study the mass terms which are independent of the scalar singlet vacuum expectation values $S_i(\chi)$. Using the trace relations

$$Tr\ \tilde{D}_i \tilde{D}_j = 0$$
$$Tr\ \tilde{D}_i \tilde{D}_j^{+} = 2\,\delta_{ij} \qquad \text{(C21)}$$

the mass terms from (C18) are

$$M^{(6)}_{ij m_i m_j} = M^2 \delta_{ij} \delta_{m_i m_j} \frac{\int d\chi\, \sigma^2 \rho^{1/2} d^*_{i m_i} d_{j m_j}}{\int d\chi\, \sigma \rho^{1/2} d^*_{i m_i} d_{j m_j}} \quad \text{for } i,j=1..4 \qquad \text{(C22)}$$

The term (C11) reads

$$L_1 = \frac{1}{32}\int d^2y\, \sigma^2 \rho^{-1/2}\, Tr\{(\partial_\varphi A_{d\chi} - \partial_\chi A_{d\varphi} - i\bar{g}\,[A_{s\varphi}, A_{d\chi}])^2 + 2i\bar{g}\,\partial_\chi A_{s\varphi}[A_{d\varphi}, A_{d\chi}]\} \qquad \text{(C23)}$$

With the ground state values of the vielbein

$$\mathring{e}_\varphi^{\ 1} = \rho^{1/2}(\chi)\cos\varphi \quad , \quad \mathring{e}_\varphi^{\ 2} = \rho^{1/2}(\chi)\sin\varphi$$
$$\mathring{e}_\chi^{\ 1} = \sin\varphi \quad , \quad \mathring{e}_\chi^{\ 2} = -\cos\varphi \qquad \text{(C24)}$$

we have

$$A_{d\varphi} = \frac{1}{\sqrt{2}}(A_{d+}\exp i\varphi + A_{d-}\exp -i\varphi)\,\rho^{1/2}$$
$$A_{d\chi} = -\frac{i}{\sqrt{2}}(A_{d+}\exp i\varphi - A_{d-}\exp -i\varphi) \qquad \text{(C25)}$$

We insert the harmonic expansion (C7). Using the relations

$$[A_{s\varphi}, E_{-2+6}] = (-m(\chi)+n(\chi))\,E_{-2+6}$$
$$[A_{s\varphi}, E_{+2+6}] = (m(\chi)+n(\chi))\,E_{+2+6}$$
$$[A_{s\varphi}, E_{+2-6}] = (m(\chi)-n(\chi))\,E_{+2-6}$$
$$[A_{s\varphi}, E_{-2-6}] = -(m(\chi)+n(\chi))\,E_{-2-6} \qquad \text{(C26)}$$

$$Tr\ E_{\varepsilon_1 w_1 \varepsilon_2 w_2} E_{\varepsilon_3 w_3 \varepsilon_4 w_4} = 16(\delta_{w_1 w_3}\delta_{\varepsilon_1,-\varepsilon_3}\delta_{w_2 w_4}\delta_{\varepsilon_2,-\varepsilon_4} - \delta_{w_1 w_4}\delta_{\varepsilon_1,-\varepsilon_4}\delta_{w_2 w_3}\delta_{\varepsilon_2,-\varepsilon_3}) \qquad \text{(C27)}$$

$$Tr\{H_i[E_{\varepsilon_1 w_1 \varepsilon_2 w_2}, E_{\varepsilon_3 w_3 \varepsilon_4 w_4}]\} = 16(\varepsilon_1 \delta_{i w_1} + \varepsilon_2 \delta_{i w_2})\cdot(\delta_{w_1 w_3}\delta_{\varepsilon_1,-\varepsilon_3}\delta_{w_2 w_4}\delta_{\varepsilon_2,-\varepsilon_4} - \delta_{w_1 w_4}\delta_{\varepsilon_1,-\varepsilon_4}\delta_{w_2 w_3}\delta_{\varepsilon_2,-\varepsilon_3}) \qquad \text{(C28)}$$

we easily can perform the integration over φ and obtain the following mass terms

$$M^{(1)}_{55\,m m'} = \pi\,\delta_{m m'}\int d\chi\,\sigma^2\{H_{2m}H^*_{2m}\cdot[\rho^{-1/2}(m+1-\bar{g}(m(\chi)+n(\chi))-\tfrac{1}{2}\rho^{-1/2}\rho')^2 - 2\bar{g}(m'(\chi)+n'(\chi))] - (m+1-\bar{g}(m(\chi)+n(\chi))-\tfrac{1}{2}\rho^{-1/2}\rho')(H_{2m}H^{*\prime}_{2m} + H'_{2m}H^*_{2m}) + \rho^{1/2}H'_{2m}H^{*\prime}_{2m}\}$$

$$M^{(1)}_{66\,m m'} = \pi\,\delta_{m m'}\int d\chi\,\sigma^2\{H_{1m}H^*_{1m}\cdot[\rho^{-1/2}(m+1+\bar{g}(m(\chi)-n(\chi))-\tfrac{1}{2}\rho^{-1/2}\rho')^2 + 2\bar{g}(m'(\chi)-n'(\chi))] - (m+1+\bar{g}(m(\chi)-n(\chi))-\tfrac{1}{2}\rho^{-1/2}\rho')(H_{1m}H^{*\prime}_{1m} + H'_{1m}H^*_{1m}) + \rho^{1/2}H'_{1m}H^{*\prime}_{1m}\} \qquad \text{(C29)}$$

$$M^{(1)}_{ij m_i m_j} = 0 \qquad \text{otherwise}$$

We have checked this formula by inserting the monopole solution of ref. 17 with $\sigma = 1$, $\rho = L_0^2 \sin^2(\chi/L_0)$, $n(\chi) = 3\, m(\chi) = \frac{3}{2\bar g}(1-\cos\frac{\chi}{L_0})$ for the field $H_{2,1}(\chi) = \left(\frac{3}{8\pi L_0^2}\right)^{1/2} \sin \chi/L_0$. One recovers the tachyon with $M^{(1)}_{5511} = -2/L_0^2$. We expect negative values for the lowest $M^{(1)}_{55}$ and $M^{(1)}_{66}$ entries for a large variety of more general solutions. There is finally a contribution from L_2: Using

$$[A_{s\varphi}, \phi_d] = -m(\chi) d_i \tilde D_i + m(\chi) d_i^* \tilde D_i^+ \qquad (C30)$$

the contribution to the mass matrix for i, j = 1...4 is

$$M^{(2)}_{ijm_im_j} = 2\pi \delta_{ij} \delta_{m_im_j} \int d\chi\, \sigma^2 \left(\rho^{1/2} d'_{im_i} d^{*\prime}_{im_i} + \rho^{-1/2} (m_i + \bar g\, m(\chi))^2 d_{im_i} d^*_{im_i} \right) \qquad (C31)$$

This contribution gives positive diagonal entries. As a consequence, the mass matrix $M^{(1)} + M^{(2)} + M^{(6)}$ is diagonal. No mixing occurs for vanishing expectation values of the scalar singlets S_i. The masses for d_i may be positive, whereas the lowest H_j could be tachyons.

How do expectation values S_i change this situation? There are additional contributions to the mass matrix for H_j from

$$\begin{aligned} L_5 = -\tfrac{1}{2}\bar g^2 \int d^2y\, \sigma^2 \rho^{1/2} \{ & H_1 H_1^* \,\mathrm{Tr}[E_{-2+6}, \phi_s][E_{+2-6}, \phi_s] \\ & + H_2 H_2^* \,\mathrm{Tr}[E_{+2+6}, \phi_s][E_{-2-6}, \phi_s] \\ & + H_1 H_2^* \,\mathrm{Tr}[E_{-2+6}, \phi_s][E_{-2-6}, \phi_s] \\ & + H_2 H_1^* \,\mathrm{Tr}[E_{+2+6}, \phi_s][E_{+2-6}, \phi_s] \} \end{aligned} \qquad (C32)$$

We use the relation

$$\begin{aligned} [E_{\varepsilon_1 w_1 \varepsilon_2 w_2}, \Gamma_{\varepsilon_3 w_3 \varepsilon_4 w_4 \varepsilon_5 w_5 \ldots}] = \\ i\,\delta_{w_1 w_3} \delta_{\varepsilon_1,-\varepsilon_3} \Gamma_{\varepsilon_2 w_2 \varepsilon_4 w_4 \varepsilon_5 w_5 \ldots} - i\,\delta_{w_2 w_3} \delta_{\varepsilon_2,-\varepsilon_3} \Gamma_{\varepsilon_1 w_1 \varepsilon_4 w_4 \varepsilon_5 w_5 \ldots} \\ + i\,\delta_{w_1 w_4} \delta_{\varepsilon_1,-\varepsilon_4} \Gamma_{\varepsilon_3 w_3 \varepsilon_2 w_2 \varepsilon_5 w_5 \ldots} - i\,\delta_{w_2 w_4} \delta_{\varepsilon_2,-\varepsilon_4} \Gamma_{\varepsilon_3 w_3 \varepsilon_1 w_1 \varepsilon_5 w_5 \ldots} \\ + i\,\delta_{w_1 w_5} \delta_{\varepsilon_1,-\varepsilon_5} \Gamma_{\varepsilon_3 w_3 \varepsilon_4 w_4 \varepsilon_2 w_2} - i\,\delta_{w_2 w_5} \delta_{\varepsilon_2,-\varepsilon_5} \Gamma_{\varepsilon_3 w_3 \varepsilon_4 w_4 \varepsilon_1 w_1 \ldots} \\ + \ldots \end{aligned} \qquad (C33)$$

to calculate the commutators

$$\begin{aligned} [E_{-2+6}, \phi_s] = & -\frac{1}{\sqrt{32}} S_1 \Gamma_{+1+3+4+5+6} \\ & + \frac{1}{\sqrt{32}} S_2^* \Gamma_{-1+1-2-6+6} \\ & + \frac{1}{\sqrt{32}} S_3^* \frac{1}{\sqrt 3} (\Gamma_{-2-4+4-5+5} + \Gamma_{-2-3+3-5+5} + \Gamma_{-2-3+3-4+4}) \\ & + \frac{1}{\sqrt{32}} S_4^* \frac{1}{\sqrt 6} (\Gamma_{-1+1-2-3+3} + \Gamma_{-1+1-2-4+4} + \Gamma_{-1+1-2-5+5} \\ & \quad + \Gamma_{-2-3+3-6+6} + \Gamma_{-2-4+4-6+6} + \Gamma_{-2-5+5-6+6}) \\ = & -\frac{1}{\sqrt{32}} S_1 \Gamma_{+1+3+4+5+6} \\ & + i S_2^* \tilde D_4 + \frac{i}{\sqrt 2} S_3^* (\tilde D_1 - \tilde D_2) - \frac{i}{2} S_4^* (\tilde D_1 + \tilde D_2 + \sqrt 2\, \tilde D_3) \end{aligned} \qquad (C34)$$

$$\begin{aligned} [E_{+2+6}, \phi_s] = & -\frac{1}{\sqrt{32}} S_1^* \Gamma_{-1-3-4-5-6} \\ & - \frac{1}{\sqrt{32}} S_2^* \Gamma_{-1+1+2-6+6} \\ & + \frac{1}{\sqrt{32}} S_3^* \frac{1}{\sqrt 3} (\Gamma_{+2-4+4-5+5} + \Gamma_{+2-3+3-5+5} + \Gamma_{+2-3+3-4+4}) \\ & + \frac{1}{\sqrt{32}} S_4^* \frac{1}{\sqrt 6} (\Gamma_{-1+1+2-3+3} + \Gamma_{-1+1+2-4+4} + \Gamma_{-1+1+2-5+5} \\ & \quad - \Gamma_{+2-3+3-6+6} - \Gamma_{+2-4+4-6+6} - \Gamma_{+2-5+5-6+6}) \end{aligned} \qquad (C35)$$

$$= -\frac{1}{\sqrt{32}}\, s_1^* \,\Gamma_{-1-3-4-5+6}$$
$$-i\, s_2^* \,\tilde{D}_4^{\,+} + \frac{i}{\sqrt{2}} s_3^* (\tilde{D}_1^{\,+} - \tilde{D}_2^{\,+}) - \frac{i}{2} s_4^* (\tilde{D}_1^{\,+} + \tilde{D}_2^{\,+} - \sqrt{2}\,\tilde{D}_3^{\,+}) \tag{C35}$$

$$[\,E_{+2-6},\, \phi_s\,] = -[\,E_{-2+6},\, \phi_s\,]^+$$
$$[\,E_{-2-6},\, \phi_s\,] = -[\,E_{+2+6},\, \phi_s\,]^+ \tag{C36}$$

For the evaluation of the traces one uses (C21) and the fact that the trace is an SO(12) singlet and, therefore,

$$\mathrm{Tr}\; \Gamma^{(\ell_1)}_{\varepsilon_1 w_1 \varepsilon_2 w_2 \ldots}\; \Gamma^{(\ell_2)}_{\tilde{\varepsilon}_1 \tilde{w}_1 \tilde{\varepsilon}_2 \tilde{w}_2 \ldots}\; \Gamma^{(\ell_m)}_{\hat{\varepsilon}_1 \hat{w}_1 \hat{\varepsilon}_2 \hat{w}_2 \ldots} \neq 0$$

only if for each weight w_i the sum over all ε_i under the trace vanishes. One finds

$$\mathrm{Tr}\,[\,E_{-2+6}, \phi_s\,][\,E_{+2-6}, \phi_s\,] = \mathrm{Tr}\,[\,E_{+2+6}, \phi_s\,][\,E_{-2-6}, \phi_s\,]$$
$$= -2\,(\,s_1 s_1^* + s_2 s_2^* + s_3 s_3^* + s_4 s_4^*\,) \tag{C37}$$

$$\mathrm{Tr}\,[\,E_{-2+6}, \phi_s\,][\,E_{-2-6}, \phi_s\,] = \mathrm{Tr}\,[\,E_{+2+6}, \phi_s\,][\,E_{+2-6}, \phi_s\,] = 0 \tag{C38}$$

One obtains the following contributions to the mass matrices

$$M^{(5)}_{55mm'} = 2\pi \bar{g}^{-2} \delta_{mm'} \int d\chi\, \sigma^2 \rho^{1/2} (s_1 s_1^* + s_2 s_2^* + s_3 s_3^* + s_4 s_4^*) H_{2m} H_{2m}^*$$

$$M^{(5)}_{66mm'} = 2\pi \bar{g}^{-2} \delta_{mm'} \int d\chi\, \sigma^2 \rho^{1/2} (s_1 s_1^* + s_2 s_2^* + s_3 s_3^* + s_4 s_4^*) H_{1m} H_{1m}^* \tag{C39}$$

$$M^{(5)}_{ij m_i m_j} = 0 \qquad \text{otherwise}$$

In contrast to $M^{(1)}$ these contributions are positive and the tachyons become stable for large enough $S_i^* S_i$. At this point there is still no mixing.

Mixing between different doublets is induced by the term

$$L_3 = -\frac{1}{2\sqrt{2}}\, \bar{g} \int d^2y\, \sigma^2 \rho^{1/2} \cdot$$
$$\cdot \{ \exp i\varphi\; \mathrm{Tr}\,[A_{d+}, \phi_s](\phi_d' + i\rho^{-1/2}\partial_\varphi \phi_d + \bar{g} \rho^{-1/2}[A_{s\varphi}, \phi_d]) \tag{C40}$$
$$+ \exp(-i\varphi)\; \mathrm{Tr}\,[A_{d-}, \phi_s](-\phi_d' + i\rho^{-1/2}\partial_\varphi \phi_d + \bar{g} \rho^{-1/2}[A_{s\varphi}, \phi_d]\}$$

We integrate over φ and obtain for i = 1 ... 4

$$M^{(3)}_{i6m_im_6} = -\frac{i\pi}{\sqrt{2}}\bar{g}\, \delta_{-m_i+m_6+\bar{m}_k+1,0} \int d\chi\, \sigma^2 \cdot$$
$$\cdot \{ \rho^{1/2} d^{*\,\prime}_{im_i} + (m_i + \bar{g}\, m(\chi)) d^*_{im_i} \} H_{1m_6}\, \mathrm{Tr}\, \tilde{D}_i^{\,+} [E_{-2+6},\, s_k \tilde{S}_k + s_k^* \tilde{S}_k^{\,+}]$$

$$M^{(3)}_{i5m_im_5} = -\frac{i\pi}{\sqrt{2}}\,\bar{g}\, \delta_{-m_i+m_5+\bar{m}_k-1,0} \int d\chi\, \sigma^2 \cdot$$
$$\cdot \{ \rho^{1/2} d^{*\,\prime}_{im_i} - (m_i + \bar{g}\, m(\chi)) d^*_{im_i} \} H^*_{2,-m_5} \cdot \tag{C41}$$
$$\cdot\, \mathrm{Tr}\, \tilde{D}_i^{\,+} [\,E_{-2-6},\, s_k \tilde{S}_k + s_k^* \tilde{S}_k^{\,+}]$$

$$M^{(3)}_{6im_6m_i} = (M^{(3)}_{i6m_im_6})^*$$
$$M^{(3)}_{5im_5m_i} = (M^{(3)}_{i5m_im_5})^*$$

Here we use $\bar{m}_k = \bar{m}_1$, $-\bar{m}_1$, $\bar{m}_2$, $-\bar{m}_2$ for S_1, S_1^*, (S_2, S_3, S_4) or (S_2^*, S_3^*, S_4^*), respectively. Employing the commutators (C34) to (C36) and

$$\mathrm{Tr}\, \tilde{D}_i\, [\,E_{\pm 2 \pm 6},\, \tilde{S}_1\,] = \mathrm{Tr}\, \tilde{D}_i\, [\,E_{\pm 2 \pm 6},\, \tilde{S}_1^{\,+}] = 0 \tag{C42}$$

one finds the following mixing terms in the mass matrix:

$$M^{(3)}_{16m_1m_6} = \pi\bar{g}\,\delta_{-m_1+m_6-\bar{m}_2+1,0}\int d\chi\,\sigma^2\cdot$$
$$\cdot[\rho^{1/2}d^{*\,\prime}_{1m_1} + (m_1+\bar{g}m(\chi))d^*_{1m_1}]H_{1m_6}(s_3^* - \tfrac{1}{\sqrt{2}}s_4^*)$$

$$M^{(3)}_{26m_2m_6} = -\pi\bar{g}\,\delta_{-m_2+m_6-\bar{m}_2+1,0}\int d\chi\,\sigma^2\cdot$$
$$\cdot[\rho^{1/2}d^{*\,\prime}_{2m_2} + (m_2+\bar{g}m(\chi))d^*_{2m_2}]H_{1m_6}(s_3^* + \tfrac{1}{\sqrt{2}}s_4^*)$$

$$M^{(3)}_{36m_3m_6} = -\pi\bar{g}\,\delta_{-m_3+m_6-\bar{m}_2+1,0}\int d\chi\,\sigma^2$$
$$\cdot[\rho^{1/2}d^{*\,\prime}_{3m_3} + (m_3+\bar{g}m(\chi))d^*_{3m_3}]H_{1m_6}\,s_4^*$$

$$M^{(3)}_{46m_4m_6} = \sqrt{2}\,\pi\bar{g}\,\delta_{-m_4+m_6-\bar{m}_2+1,0}\int d\chi\,\sigma^2\cdot$$
$$\cdot[\rho^{1/2}d^{*\,\prime}_{4m_4} + (m_4+\bar{g}m(\chi))d^*_{4m_4}]H_{1m_6}\,s_2^*$$

$$M^{(3)}_{15m_1m_5} = -\pi\bar{g}\,\delta_{-m_1+m_5+\bar{m}_2-1,0}\int d\chi\,\sigma^2\cdot \qquad (C43)$$
$$\cdot[\rho^{1/2}d^{*\,\prime}_{1m_1} - (m_1+\bar{g}m(\chi))d^*_{1m_1}]H^*_{2,-m_5}(s_3 - \tfrac{1}{\sqrt{2}}s_4)$$

$$M^{(3)}_{25m_2m_5} = \pi\bar{g}\,\delta_{-m_2+m_5+\bar{m}_2-1,0}\int d\chi\,\sigma^2\cdot$$
$$\cdot[\rho^{1/2}d^{*\,\prime}_{2m_2} - (m_2+\bar{g}m(\chi))d^*_{2m_2}]H^*_{2,-m_5}(s_3 + \tfrac{1}{\sqrt{2}}s_4)$$

$$M^{(3)}_{35m_3m_5} = -\pi\bar{g}\,\delta_{-m_3+m_5+\bar{m}_2-1,0}\int d\chi\,\sigma^2\cdot$$
$$\cdot[\rho^{1/2}d^{*\,\prime}_{3m_3} - (m_3+\bar{g}m(\chi))d^*_{3m_3}]H^*_{2,-m_5}\,s_4$$

$$M^{(3)}_{45m_4m_5} = \sqrt{2}\,\pi\bar{g}\,\delta_{-m_4+m_5+\bar{m}_2-1,0}\int d\chi\,\sigma^2\cdot$$
$$\cdot[\rho^{1/2}d^{*\,\prime}_{4m_4} - (m_4+\bar{g}m(\chi))d^*_{4m_4}]H^*_{2,-m_5}\,s_2$$

The next term is

$$L_4 = -\frac{1}{2\sqrt{2}}\bar{g}\int d^2y\,\sigma^2\cdot$$
$$\{\exp i\varphi\,\mathrm{Tr}\,[A_{d+},\phi_d](\rho^{1/2}\partial_\chi\phi_s + i\partial_\varphi\phi_s + \bar{g}[A_{s\varphi},\phi_s]) \qquad (C44)$$
$$+\exp{-i\varphi}\,\mathrm{Tr}\,[A_{d-},\phi_d](-\rho^{1/2}\partial_\chi\phi_s + i\partial_\varphi\phi_s + \bar{g}[A_{s\varphi},\phi_s])\}$$

One obtains for i = 1 ... 4, k = 2, 3, 4:

$$M^{(4)}_{i6m_im_6} = -\frac{i\pi}{\sqrt{2}}\bar{g}\,\delta_{-m_i+m_6-\bar{m}_2+1,0}\int d\chi\,\sigma^2 H_{1m_6}\cdot$$
$$\cdot d^*_{im_i}\,\mathrm{Tr}\,[E_{-2+6},\tilde{D}_i^+](\rho^{1/2}s_k^{*\,\prime}\,\tilde{S}_k^+ + (\bar{m}_2 - \bar{g}n(\chi))s_k^*\,\tilde{S}_k^+)$$

$$M^{(4)}_{i5m_im_5} = -\frac{i\pi}{\sqrt{2}}\bar{g}\,\delta_{-m_i+m_5+\bar{m}_2-1,0}\int d\chi\,\sigma^2 H^*_{2,-m_5}\cdot \qquad (C45)$$
$$\cdot d^*_{im_i}\,\mathrm{Tr}\,[E_{2-6},\tilde{D}_i^+](\rho^{1/2}s_k'\,\tilde{S}_k + (\bar{m}_2 - \bar{g}n(\chi))s_k\,\tilde{S}_k)$$

$$M^{(4)}_{6im_6m_i} = M^{(4)\,*}_{i6m_im_6}$$

$$M_{5im_5m_i} = M^{(4)\,*}_{i5m_im_5}$$

We evaluate the commutators

$$[E_{-2+6},\tilde{D}_1^+] = -\frac{i}{\sqrt{2}}\tilde{S}_3 + \frac{i}{2}(\tilde{S}_4 + \tilde{T}_2)$$
$$[E_{-2+6},\tilde{D}_2^+] = \frac{i}{\sqrt{2}}\tilde{S}_3 + \frac{i}{2}(\tilde{S}_4 + \tilde{T}_2) \qquad (C46)$$
$$[E_{-2+6},\tilde{D}_3^+] = \frac{i}{\sqrt{2}}(\tilde{S}_4 - \tilde{T}_2)$$
$$[E_{-2+6},\tilde{D}_4^+] = -i\,\tilde{S}_2$$

$$[E_{-2-6}, \tilde{D}_1^+] = -\frac{i}{\sqrt{2}}\tilde{S}_3^+ + \frac{i}{2}(\tilde{S}_4^+ + \tilde{T}_2^+)$$

$$[E_{-2-6}, \tilde{D}_2^+] = \frac{i}{\sqrt{2}}\tilde{S}_3^+ + \frac{i}{2}(\tilde{S}_4^+ + \tilde{T}_2^+)$$

$$[E_{-2-6}, \tilde{D}_3^+] = -\frac{i}{\sqrt{2}}(\tilde{S}_4^+ - \tilde{T}_2^+) \qquad \text{(C46)}$$

$$[E_{-2-6}, \tilde{D}_4^+] = i\,\tilde{S}_2^+$$

From this we read

$$M^{(4)}_{16 m_1 m_6} = -\pi\bar{g}\,\delta_{-m_1+m_6-\bar{m}_2+1,0}\int d\chi\,\sigma^2 H_{1m_6}\,d^*_{1m_1}\cdot$$
$$\cdot\left[\rho^{1/2}(s_3^{*\prime} - \tfrac{1}{\sqrt{2}}s_4^{*\prime}) + (\bar{m}_2 - \bar{g}n(\chi))(s_3^* - \tfrac{1}{\sqrt{2}}s_4^*)\right]$$

$$M^{(4)}_{26 m_2 m_6} = \pi\bar{g}\,\delta_{-m_2+m_6-\bar{m}_2+1,0}\int d\chi\,\sigma^2 H_{1m_6}\,d^*_{2m_2}\cdot$$
$$\cdot\left[\rho^{1/2}(s_3^{*\prime} + \tfrac{1}{\sqrt{2}}s_4^{*\prime}) + (\bar{m}_2 - \bar{g}n(\chi))(s_3^* + \tfrac{1}{\sqrt{2}}s_4^*)\right]$$

$$M^{(4)}_{36 m_3 m_6} = \pi\bar{g}\,\delta_{-m_3+m_6-\bar{m}_2+1,0}\int d\chi\,\sigma^2 H_{1m_6}\,d^*_{3m_3}\cdot$$
$$\cdot\left[\rho^{1/2}s_4^{*\prime} + (\bar{m}_2 - \bar{g}n(\chi))\,s_4^*\right]$$

$$M^{(4)}_{46 m_4 m_6} = -\sqrt{2}\,\pi\bar{g}\,\delta_{-m_4+m_6-\bar{m}_2+1,0}\int d\chi\,\sigma^2 H_{1m_6}\,d^*_{4m_4}\cdot \qquad \text{(C47)}$$
$$\cdot\left[\rho^{1/2}s_2^{*\prime} + (\bar{m}_2 - \bar{g}n(\chi))\,s_2^*\right]$$

$$M^{(4)}_{15 m_1 m_5} = -\pi\bar{g}\,\delta_{-m_1+m_5+\bar{m}_2-1,0}\int d\chi\,\sigma^2 H^*_{2,-m_5}\,d^*_{1m_1}\cdot$$
$$\cdot\left[\rho^{1/2}(s_3' - \tfrac{1}{\sqrt{2}}s_4') + (\bar{m}_2 - \bar{g}n(\chi))(s_3 - \tfrac{1}{\sqrt{2}}s_4)\right]$$

$$M^{(4)}_{25 m_2 m_5} = \pi\bar{g}\,\delta_{-m_2+m_5+\bar{m}_2-1,0}\int d\chi\,\sigma^2 H^*_{2,-m_5}\,d^*_{2m_2}\cdot$$
$$\cdot\left[\rho^{1/2}(s_3' + \tfrac{1}{\sqrt{2}}s_4') + (\bar{m}_2 - \bar{g}n(\chi))(s_3 + \tfrac{1}{\sqrt{2}}s_4)\right]$$

$$M^{(4)}_{35 m_3 m_5} = -\pi\bar{g}\,\delta_{-m_3+m_5+\bar{m}_2-1,0}\int d\chi\,\sigma^2 H^*_{2,-m_5}\,d^*_{3m_3}\cdot$$
$$\cdot\left[\rho^{1/2}s_4' + (\bar{m}_2 - \bar{g}n(\chi))\,s_4\right] \qquad \text{(C47)}$$

$$M^{(4)}_{45 m_4 m_5} = \sqrt{2}\,\pi\bar{g}\,\delta_{-m_4+m_5+\bar{m}_2-1,0}\int d\chi\,\sigma^2 H^*_{2,-m_5}\,d^*_{4m_4}\cdot$$
$$\cdot\left[\rho^{1/2}s_2' + (\bar{m}_2 - \bar{g}n(\chi))\,s_2\right]$$

We observe that the structure of the mixing terms from $M^{(4)}$ is similar to $M^{(3)}$ except for a different χ dependence of the integrands.

Finally, the contribution L_7 from the scalar potential generates mass terms for the doublets d_i which are proportional to an even power of s_k. Mixings can only be induced if the quantum numbers of the operator $d_i^* d_j$ appear in some even polynomial of s_k. The exact form of these mixings, which depends on details of $V(\phi)$, is not important in our context.

Table 2 Electrically neutral colour singlets in 792

	H_1	H_2	H_3	H_4	H_5	H_6	I_{3L}	I_{3R}	Y_{B-L}	$SO(10)\times U(1)_q$	$SU(4)_C \times SU(2)_L \times SU(2)_R$	SU(5)
s_1	1	1	1	1	1	0	0	-1	2	126_0	$(\overline{10},1,3)$	1
s_2	0	0	0	0	0	1	0	0	0	210_1	(1,1,1)	1,24,75
s_3	0	0	0	0	0	1	0	0	0	210_1	(15,1,1)	1,24,75
s_4	0	0	0	0	0	1	0	0	0	210_1	(15,1,3)	1,24,75
d_1	0	-1	0	0	0	0	-1/2	1/2	0	126_0	(15,2,2)	45
d_2	0	-1	0	0	0	0	-1/2	1/2	0	$\overline{126}_0$	(15,2,2)	5
d_3	0	-1	0	0	0	0	-1/2	1/2	0	120_0	(15,2,2)	5,45
d_4	0	-1	0	0	0	0	-1/2	1/2	0	120_0	(1,2,2)	5,45
t_1	-1	1	-1	-1	-1	0	1	0	-2	126_0	(10,3,1)	$\overline{15}$
t_2	0	0	0	0	0	1	0	0	0	210_1	(15,3,1)	24

Table 1 Abelian quantum numbers q and I for chiral fermions

a) $n = 3,\ m = p = 1$

	t'	$t^{c\prime}$	c'	$c^{c\prime}$	u'	$u^{c\prime}$	b'	$b^{c\prime}$	s'	$s^{c\prime}$	d'	$d^{c\prime}$	τ'	$\tau^{c\prime}$	μ'	$\mu^{c\prime}$	e'	$e^{c\prime}$
I	0	0	-1/2	-1/2	1/2	1/2	0	1	-1/2	0	1/2	-1	1	0	0	-1/2	-1	1/2
q	-1/2	-1/2	1/2	1/2	1/2	1/2	-1/2	-1/2	1/2	-1/2	1/2	-1/2	-1/2	-1/2	-1/2	1/2	-1/2	1/2

b) $n = 4,\ m = p = 2$

	a'	$a^{c\prime}$	t'	$t^{c\prime}$	c'	$c^{c\prime}$	u'	$u^{c\prime}$	z'	b'	s'	d'	$\sigma^{c\prime}$	$\tau^{c\prime}$	$\mu^{c\prime}$	$e^{c\prime}$
I	0	0	-1	-1	0	0	1	1	0	-1	0	1	0	-1	0	1
q	-1/2	-1/2	1/2	1/2	1/2	1/2	1/2	1/2	-1/2	1/2	1/2	1/2	-1/2	1/2	1/2	1/2

	D_1^c	D_2^c	D_3^c	D_4^c	D_5^c	E_1	E_2	E_3	E_4	E_5	$\bar{\bar{d}}_c$	$\bar{\bar{e}}$
I	2	1	0	-1	-2	2	1	0	-1	-2	0	0
q	-1/2	-1/2	-1/2	-1/2	-1/2	-1/2	-1/2	-1/2	-1/2	-1/2	-1/2	-1/2

Table 3 continued

	H_1	H_2	H_3	H_4	H_5	H_6	I_{3L}	I_{3R}	Y_{B-L}	Q	state	$SU(3)_c \times SU(2)_L$ rep
X_1	-1	0	-1	0	0	0	1/2	1/2	-2/3	2/3	$E_{-1,-3}$	(3,2)
	-1	0	0	-1	0	0	1/2	1/2	-2/3	2/3	$E_{-1,-4}$	
	-1	0	0	0	-1	0	1/2	1/2	-2/3	2/3	$E_{-1,-5}$	
	0	-1	-1	0	0	0	-1/2	1/2	-2/3	-1/3	$E_{-2,-3}$	
	0	-1	0	-1	0	0	-1/2	1/2	-2/3	-1/3	$E_{-2,-4}$	
	0	-1	0	0	-1	0	-1/2	1/2	-2/3	-1/3	$E_{-2,-5}$	
X_2	0	1	-1	0	0	0	1/2	-1/2	-2/3	-1/3	$E_{+2,-3}$	(3,2)
	0	1	0	-1	0	0	1/2	-1/2	-2/3	-1/3	$E_{+2,-4}$	
	0	1	0	0	-1	0	1/2	-1/2	-2/3	-1/3	$E_{+2,-5}$	
	1	0	-1	0	0	0	-1/2	-1/2	-2/3	-4/3	$E_{+1,-3}$	
	1	0	0	-1	0	0	-1/2	-1/2	-2/3	-4/3	$E_{+1,-4}$	
	1	0	0	0	-1	0	-1/2	-1/2	-2/3	-4/3	$E_{+1,-5}$	
H_1^+	-1	0	0	0	0	1	1/2	1/2	0	1	$E_{-1,+6}$	(1,2)
H_1^o	0	-1	0	0	0	1	-1/2	1/2	0	0	$E_{-2,+6}$	
H_2^o	0	1	0	0	0	1	1/2	-1/2	0	0	$E_{+2,+6}$	(1,2)
H_2^-	1	0	0	0	0	1	-1/2	-1/2	0	-1	$E_{+1,+6}$	

Table 3 Quantum numbers for the adjoint representation 66 of SO(12).
States labelled with $E_{\varepsilon_1 w_1 \varepsilon_2 w_2}$ are complex. Their complex conjugate states have opposite quantum numbers and all labels εw replaced by $-\varepsilon w$. They are not listed separately.

	H_1	H_2	H_3	H_4	H_5	H_6	I_{3L}	I_{3R}	Y_{B-L}	Q	state	$SU(3)_c \times SU(2)_L$ rep
U_q	0	0	0	0	0	0	0	0	0	0	H_6	(1,1)
W_L^+	-1	1	0	0	0	0	1	0	0	1	$E_{-1,+2}$	(1,3)
W_{3L}	0	0	0	0	0	0	0	0	0	0	$1/\sqrt{2}(H_1-H_2)$	
W_R^+	-1	-1	0	0	0	0	0	1	0	1	$E_{-1,-2}$	(1,1)
W_{3R}	0	0	0	0	0	0	0	0	0	0	$1/\sqrt{2}(H_1+H_2)$	(1,1)
U_{B-L}	0	0	0	0	0	0	0	0	0	0	$1/\sqrt{3}(H_3+H_4+H_5)$	(1,1)
G	0	0	-1	1	0	0	0	0	0	0	$E_{-3,+4}$	(8,1)
	0	0	-1	0	1	0	0	0	0	0	$E_{-3,+5}$	
	0	0	0	-1	1	0	0	0	0	0	$E_{-4,+5}$	
	0	0	0	0	0	0	0	0	0	0	$-1/\sqrt{2}(H_3-H_4)$	
	0	0	0	0	0	0	0	0	0	0	$-1/\sqrt{6}(H_3+H_4-2H_5)$	
C	0	0	0	1	1	0	0	0	4/3	2/3	$E_{+4,+5}$	(3,1)
	0	0	1	0	1	0	0	0	4/3	2/3	$E_{+5,+3}$	
	0	0	1	1	0	0	0	0	4/3	2/3	$E_{+3,+4}$	

Table 3 continued

	H_1	H_2	H_3	H_4	H_5	H_6	I_{3L}	I_{3R}	Y_{B-L}	Q	state	$SU(3)_c xSU(2)_L$ rep
T_1	0	0	-1	0	0	1	0	0	-2/3	-1/3	$E_{-3,+6}$	(3,1)
	0	0	0	-1	0	1	0	0	-2/3	-1/3	$E_{-4,+6}$	
	0	0	0	0	-1	1	0	0	-2/3	-1/3	$E_{-5,+6}$	
T_2	0	0	-1	0	0	-1	0	0	-2/3	-1/3	$E_{-3,-6}$	(3,1)
	0	0	0	-1	0	-1	0	0	-2/3	-1/3	$E_{-4,-6}$	
	0	0	0	0	-1	-1	0	0	-2/3	-1/3	$E_{-5,-6}$	

Footnote

F1) This requirement was not made in ref. 18 where we have only been interested in mass eigenvalues. For a discussion of weak eigenstates the role of s' and d' should be interchanged in table 3 and equs. (71) and (72) of this earlier paper.

F2) For a very large top quark mass we may release the bound on $(M_U)_{12}$ and introduce a higher bound A' which is about an order of magnitude smaller than m_t.

F3) The author thanks H. Bijnens for pointing out this possibility.

F4) The assumption $s_2 < s_1$ is not necessary. For equal order of magnitude for s_2 and s_1 both entries d_{i2} and d_{i1} appear in the low energy mass matrices with equal weight.

F5) There is another possibility where H_1 mixes with $(H_2^{\pm 1})^*$ for $\overline{m}_2 = 2$.

F6) It may be possible that these restrictions reflect themselves in terms of conserved four dimensional discrete symmetries or additional effective global symmetries in the scalar sector.

F7) In addition, massless scalars due to Betti numbers are only possible if the full scalar mass operator only involves the Laplacian[17]. In generic theories, including string theories, we do not expect this to be the case.

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Fig. 1:

Effective mixings induced by doublets K

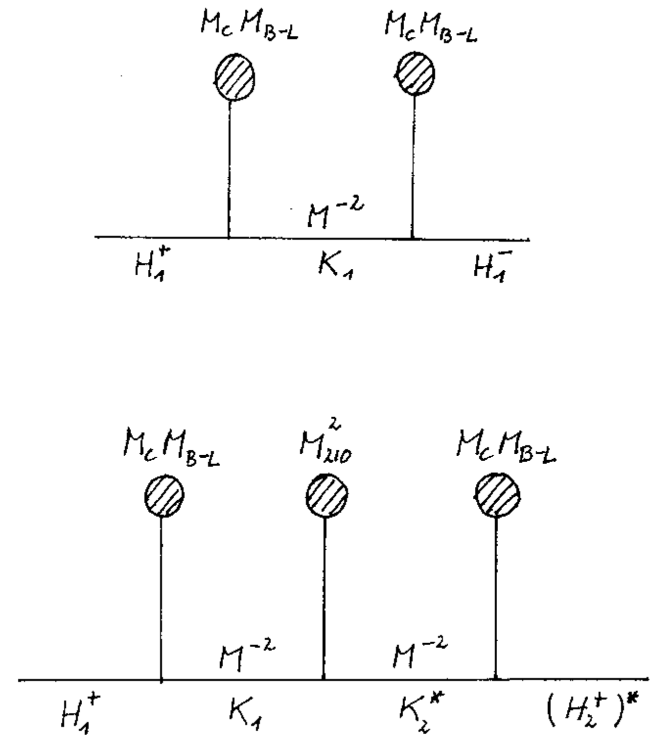

Fig. 2:

Fermion mass contribution from doublet mixing

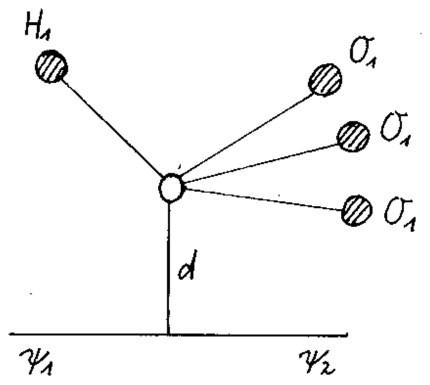

Fig. 3:

Fermion mass contribution from mixing with superheavy fermions (ψ4)