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ABSTRACT
A general formalism is presented which shows how to rewrite a given Hamiltonian involving classical spin motion in an action-angle variable representation. The canonical transformation is made using the Lie transformation technique.

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## 1. Introduction

When we study orbital motion in circular accelerators, we often employ action-angle variables. For example, if we know the Twiss parameters of a linear betatron oscillation, the general solution can be written using two integration constants, the amplitude and the initial phase of the oscillation. The action variable is the square of the amplitude. The Hamiltonian becomes a simple function, the product of the betatron frequency and the action variable.

One may ask, however, "Why do we need such a form for a problem already solved?" In fact, it is pointless if one is only interested in the linear betatron motion. Nevertheless, we often use action-angle variables because they play an important role if we have perturbations such as nonlinear terms and synchrotron radiation. For the study of nonlinear effects, the action-angle variables are an indispensable tool in using classical perturbation theory. As for the radiation, we know that the equilibrium distribution is a function of the action variables only.

On the other hand, for the study of spin motion, we normally begin with the equation of motion, i.e., with the so-called BMT equation (ref.3), but not with the Hamiltonian or action-angle variables. However, when we have a perturbation, the equation of motion is not suited for a systematic investigation, although it is often easier to visualize.

The most important perturbation[1] on the spin motion in electron storage rings is the (quantized) radiation field, which causes spontaneous polarization through the Sokolov-Ternov mechanism. In this case action-angle variables provide a very effective way to describe the unperturbed system (classical spin motion). If the perturbation is semi-classical, or in other words, if we are only interested in calculating up to the first order in Planck's constant ($\hbar$), we can quantize the system approximately simply by dividing the action variables by $\hbar$ to get the quantum numbers. In fact, Derbenev and Kondratenko (ref.2. We call this DKII below because we shall refer it so frequently) used this approach to study the quantum perturbation and derived their famous formula for the equilibrium polarization. This is the classic paper that we have to refer to whenever we talk about spin resonances of higher than the lowest order.

Unfortunately, however, it seems to the present author that the Soviet papers are often

[1]The influence of orbit oscillation on the spin is not a perturbation in this context. It is a part of the unperturbed system.

regarded by physicists in the "West" as being difficult to understand and that this is probably due to the very compact presentation and to the large geographical separation between the Soviet and "Western" accelerator physicists. One of the aims of the present paper is a better understanding of DKII, especially chapter 3, where they discuss classical spin motion in an external field and introduce the famous vector $\vec{n}$ as the basis vector for the spin action-angle representation.

The transformation from the initial spin variables to the final action-angle variables is a canonical transformation. It is merely a rotation. Nevertheless, it is difficult or, at least, extremely tedious to describe this transformation by a conventional generating function (e.g. a function of the old angle variable and the new action variable). It is desirable to use another formalism. In this paper we shall employ a Lie transformation. The author does not know what method Derbenev and Kondratenko employed. The results are given in DKII, but we shall show that they are incomplete and contain some mistakes although the final Hamiltonian has the correct form.

Derbenev and Kondratenko also give another definition of $\vec{n}$ in ref.1, which we shall call DKI, but they do not prove the equivalence of these two definitions. The definition in DKI is more familiar to accelerator physicists because it uses the generalized machine azimuth $\theta$ as the independent variable. In DKII the time $t$ is chosen as the independent variable, since it is more convenient in problems related to quantum mechanics. However, these two definitions cannot be equivalent because the Hamiltonian is assumed to be independent of time in DKII, and this makes the application restrictive. In this paper we shall show that the two definitions are equivalent after generalizing DKII to a time-dependent Hamiltonian.

The plan of this paper is as follows. In the next section we shall explain the Lie transformation method of canonical transformations involving spin by a simple case with the spin degree of freedom only. In section 3, the method is then generalized for a system with both spin and orbital degrees of freedom. In section 4 we shall derive the action-angle representation of a time-independent Hamiltonian system with both spin and orbit using the Lie transformation given in section 3. In section 5, after generalizing the formalism to time-dependent systems, we rewrite the theory using machine azimuth as the independent variable, and discuss the relation between the two definitions of the vector $\vec{n}$.

## 2. Canonical Transformations with Spin Only

In this section we introduce the Lie transformation method of a canonical transformation for spin. In order to clarify the method, we discuss here a simple Hamiltonian system with the spin only. It is generalized to systems with orbital motion in the next section.

Let us consider the following Hamiltonian for the classical spin motion :

$$H^0 = \vec{W}(t)\cdot\vec{s}. \tag{2,1}$$

This is equivalent to the equation of motion

$$\frac{d\vec{s}}{dt} = \vec{W}(t)\times\vec{s}. \tag{2,2}$$

As is well known, if we observe the spin in a coordinate system rotating with instantaneous angular velocity $\vec{U}(t)$ (i.e., with angular velocity $|\vec{U}|$ around the axis $\vec{U}$), the equation of motion for the new spin components is given by

$$\frac{d\vec{s}}{dt} = (\vec{W}(t) - \vec{U}(t))\times\vec{s}, \tag{2,3}$$

which corresponds to the Hamiltonian

$$H = (\vec{W}(t) - \vec{U}(t))\cdot\vec{s}. \tag{2,4}$$

The aim of this chapter is quite simple; it is to express the change of the Hamiltonian from (2,1) to (2,4) in terms of a canonical transformation. These equations are, however, misleading and ambiguous, in particular when vector variables are differentiated, and it looks as if the same vector $\vec{s}$ satisfies different equations of motion. The two Hamiltonians (2,1) and (2,4) use different basis vectors to describe the spin $\vec{s}$. This fact must be expressed explicitly to avoid confusion. The spin $\vec{s}$ in the Hamiltonian (2,1) is represented in a time-independent right-handed orthonormal basis $\{\vec{e}_\alpha, \alpha = 1,2,3\}$, by

$$\vec{s} = \sum_\alpha \vec{e}_\alpha s^0_\alpha. \tag{2,5}$$

Instead of (2,1), let us write

$$H^0(s^0_\alpha, t) = \sum_\alpha \vec{W}(t)\cdot\vec{e}_\alpha s^0_\alpha. \tag{2,6}$$

On the other hand, in eq(2,4), $\vec{s}$ is represented in a rotating orthonormal basis $\{\vec{u}_\alpha(t)\}$, by

$$\vec{s} = \sum_\alpha \vec{u}_\alpha(t) s_\alpha \tag{2,7}$$

and the Hamiltonian (2,4) should be written as

$$H(s_\alpha, t) = \sum_\alpha (\vec{W}(t) - \vec{U}(t)) \cdot \vec{u}_\alpha(t) s_\alpha. \tag{2,8}$$

We shall still often use vector notation in order to avoid too many subscripts but then it must be understood that the vector components are referred to the time-independent basis $\{\vec{e}_\alpha\}$.

The fact that $\vec{u}_\alpha$ is rotating with the angular velocity $\vec{U}(t)$ is described by

$$\frac{d\vec{u}_\alpha}{dt} = \vec{U}(t) \times \vec{u}_\alpha. \tag{2,9}$$

If $\vec{u}_\alpha$ is given first, then $\vec{U}(t)$ can be obtained via

$$\vec{U}(t) = \frac{1}{2} \sum_\alpha \vec{u}_\alpha(t) \times \frac{d\vec{u}_\alpha}{dt}. \tag{2,10}$$

One can easily show the equivalence of these two equations by summing the vector product of $\vec{u}_\alpha$ and eq(2,9) over $\alpha$. Relations like this always hold for the derivatives of orthonormal basis vectors.

The vector $\vec{s}$ is a physical object, whereas $s^0_\alpha$ and $s_\alpha$ are dynamical variables representing it. Hamiltonians are written in terms of $s^0_\alpha$ or $s_\alpha$ but not $\vec{s}$. However, $s^0_\alpha$ and $s_\alpha$ consist of three numbers and, therefore, cannot be mutually conjugate variables. The canonical variables $J^0$ and $\Psi^0$ corresponding to $s^0_\alpha$ can be defined, for example, by

$$\begin{cases} s^0_1 = \sqrt{s^2 - J^{02}} \cos \Psi^0 \\ s^0_2 = \sqrt{s^2 - J^{02}} \sin \Psi^0 \\ s^0_3 = J^0, \end{cases} \tag{2,11}$$

where $s$ is a constant, the magnitude of the spin. We can similarly define $J$ and $\Psi$ corresponding to $s_\alpha$. The relation (2,11) is not unique, but we always assume in this paper that other quantities such as the time and the orbit variables do not enter the relation between $J, \Psi$ and $s_\alpha$.

At this stage, $J$ and $\Psi$ need not be the action and angle variables but must be canonical; i.e.,

$$\{\Psi, J\} = 1, \qquad \{\Psi, \Psi\} = \{J, J\} = 0. \tag{2,12}$$

Here, the Poisson brackets are denoted by { }, defined in general by

$$\{f, g\} = \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial q} \tag{2,13}$$

where $f$ and $g$ are functions of generalized momentum $p$ and coordinate $q$, including the spin. Note that our sign convention for the Poisson brackets is different from that of DKII. Now, using eqs(2,11) and (2,12), we have the Poisson brackets between the $s_\alpha$'s, viz.

$$\{s_\alpha, s_\beta\} = \sum_\gamma \epsilon_{\alpha\beta\gamma} s_\gamma \tag{2,14}$$

and similarly for $s^0_\alpha$. Here $\epsilon_{\alpha\beta\gamma}$ is the three dimensional completely antisymmetric tensor. The equation of motion corresponding to the Hamiltonian (2,6) is then

$$\frac{ds^0_\alpha}{dt} = \{s^0_\alpha, H^0\} = \sum_\beta \{s^0_\alpha, s^0_\beta\} \vec{W} \cdot \vec{e}_\beta = \sum_{\beta\gamma} \epsilon_{\alpha\beta\gamma} \vec{W} \cdot \vec{e}_\beta s^0_\gamma. \tag{2,15}$$

This is the rigorous statement of eq(2,2). Similarly, we have

$$\frac{ds_\alpha}{dt} = \{s_\alpha, H\} = \sum_{\beta\gamma} \epsilon_{\alpha\beta\gamma} (\vec{W} - \vec{U}) \cdot \vec{u}_\beta s_\gamma. \tag{2,16}$$

Now, since the transformation $s^0_\alpha \to s_\alpha$ , or, equivalently, $(J^0, \Psi^0) \to (J, \Psi)$, is expected to be canonical, it may be described by a generating function $S(J, \Psi^0, t)$, via

$$J^0 = \frac{\partial S}{\partial \Psi^0}, \qquad \Psi = \frac{\partial S}{\partial J} \tag{2,17}$$

or by other combinations of $J^0, \Psi^0, J$ and $\Psi$. However, it is extremely tedious to find a suitable generating function, even if possible. In this paper we shall do so with the help of a Lie transformation.

When $S$ is close to the identity, it can be written as

$$S(J, \Psi^0, t) = J\Psi^0 + \Delta S(J, \Psi^0, t) \tag{2,18}$$

and eq(2,17) becomes

$$J^0 = J + \frac{\partial \Delta S}{\partial \Psi^0}, \qquad \Psi = \Psi^0 + \frac{\partial \Delta S}{\partial J} \tag{2,19}$$

We may ignore the distinction between $(J^0, \Psi^0)$ and $(J, \Psi)$ in $\Delta S$ up to the first order in $\Delta S$, whence

$$\begin{aligned} J - J^0 &= -\frac{\partial \Delta S(J, \Psi, t)}{\partial \Psi} = \{J, \Delta S\} \\ \Psi - \Psi^0 &= +\frac{\partial \Delta S(J, \Psi, t)}{\partial J} = \{\Psi, \Delta S\} \end{aligned} \tag{2,20}$$

The point of the Lie transformation is to perform this infinitesimal canonical transformation successively to get a finite one. We parametrize this sequence by a scalar parameter $\kappa$ and write eq(2,20) as

$$\frac{dJ}{d\kappa} = \{J, L\}, \qquad \frac{d\Psi}{d\kappa} = \{\Psi, L\} \tag{2,21}$$

where $L = L(J, \Psi, t, \kappa)$ is called the Lie generating function, and a function of $\kappa$ in general. One has to note here that $J$ and $\Psi$ are independent (canonical) variables at each stage of the transformation (at each $\kappa$) and, therefore,

$$\frac{\partial J}{\partial \kappa} = \frac{\partial \Psi}{\partial \kappa} = 0. \tag{2,22}$$

The equations (2,21) can be generalized to an arbitrary function $f(J, \Psi)$ as

$$\frac{df}{d\kappa} = \{f, L\}, \tag{2,23}$$

under the assumption that $f$ does not contain $\kappa$ explicitly. Owing to this fact we can easily treat non-canonical variables such as $s_\alpha$. Moreover, we need not mix new and old variables in the generating function.

In order to describe the transformation $s_\alpha^0 \to s_\alpha$ corresponding to the transformation of the basis $\vec{e}_\alpha \to \vec{u}_\alpha$ by a Lie generating function, we need an intermediate orthonormal basis $\{\vec{v}_\alpha(t, \kappa)\}$ linking $\vec{e}_\alpha$ and $\vec{u}_\alpha$. Let $\kappa = 0$ and 1 denote the beginning and the end of the transformation, respectively; i.e.,

$$\vec{v}_\alpha(t, 0) = \vec{e}_\alpha, \qquad \vec{v}_\alpha(t, 1) = \vec{u}_\alpha(t). \tag{2,24}$$

We need not explicitly specify the route from $\vec{e}_\alpha$ to $\vec{u}_\alpha$. Since $\{\vec{v}_\alpha(t, \kappa)\}$ is an orthonormal basis, we have

$$\frac{d\vec{v}_\alpha}{d\kappa} = \vec{V}(t, \kappa) \times \vec{v}_\alpha(t, \kappa) \tag{2,25}$$

with

$$\vec{V}(t, \kappa) = \frac{1}{2} \sum_\alpha \vec{v}_\alpha(t, \kappa) \times \frac{d\vec{v}_\alpha}{d\kappa}. \tag{2,26}$$

We denote the intermediate spin variable by $s_\alpha^\kappa$, not by $s_\alpha(\kappa)$ since the $\kappa$-dependence of $s_\alpha$ is implicit. The physical spin vector $\vec{s}$ is

$$\vec{s} = \sum_\alpha s_\alpha^\kappa \vec{v}_\alpha(t,\kappa), \tag{2,27}$$

which does not depend on the choice of the basis vector; i.e.,

$$\frac{d}{d\kappa}(s_\alpha^\kappa \vec{v}_\alpha(t,\kappa)) = 0. \tag{2,28}$$

We can easily verify

$$\frac{ds_\alpha^\kappa}{d\kappa} = -\sum_{\beta\gamma} \epsilon_{\alpha\beta\gamma}(\vec{V}\cdot\vec{v}_\beta)s_\gamma^\kappa \tag{2,29}$$

from (2,25) and (2,28). The Lie generating function which leads to the transformation (2,29) is given by

$$L(s_\alpha^\kappa, t, \kappa) = -\sum_\alpha s_\alpha^\kappa \vec{v}_\alpha(t,\kappa)\cdot\vec{V}(t,\kappa). \tag{2,30}$$

In fact, we can easily get eq(2,29) by the reduction of

$$\frac{ds_\alpha^\kappa}{d\kappa} = \{s_\alpha^\kappa, L\} \tag{2,31}$$

which is a result of eq(2,23).

Logically speaking, we should start with the generating function (2,30), calculate (2,31) to get eq(2,29) and construct the orthonormal basis by using eq(2,28). All of them must be integrated over $\kappa$ to get the variables at $\kappa = 1$. We have described the story in an inverse manner.

Next, let us focus our attention on the change of the Hamiltonian under this transformation. The conventional generating function $S(J, \Psi^0, t)$ gives the new Hamiltonian via

$$H = H^0 + \frac{\partial S}{\partial t}$$

and the infinitesimal transformation (2,18) yields

$$H - H^0 = \frac{\partial \Delta S}{\partial t}.$$

The corresponding expression in the Lie transformation method is

$$\frac{dH^\kappa}{d\kappa} = \frac{\partial L}{\partial t}, \tag{2,32}$$

where the intermediate Hamiltonian $H^\kappa$ can depend on $\kappa$ both implicitly and explicitly. The generating function (2,30) leads to

$$\frac{dH^\kappa}{d\kappa} = -\sum_\alpha \frac{\partial}{\partial t}(s^\kappa_\alpha \vec{v}_\alpha\cdot\vec{V}) = -\frac{1}{2}\sum_{\alpha\beta}\frac{\partial}{\partial t}\Big[s^\kappa_\alpha \vec{v}_\alpha\cdot(\vec{v}_\beta\times\frac{d\vec{v}_\beta}{d\kappa})\Big], \tag{2,33}$$

where we have used the relation (2,26).

We now show that the partial derivative $\partial/\partial t$ and the total derivative $d/d\kappa$ on the right hand side of eq(2,33) can be exchanged; i.e.,

$$\sum_{\alpha\beta}\frac{\partial}{\partial t}\Big[s^\kappa_\alpha \vec{v}_\alpha\cdot(\vec{v}_\beta\times\frac{d\vec{v}_\beta}{d\kappa})\Big] = \sum_{\alpha\beta}\frac{d}{d\kappa}\Big[s^\kappa_\alpha \vec{v}_\alpha\cdot(\vec{v}_\beta\times\frac{\partial\vec{v}_\beta}{\partial t})\Big]. \tag{2,34}$$

Here, we have to note that $\vec{v}_\alpha$ depends on $\kappa$ only explicitly whereas $s^\kappa_\alpha$ does so only implicitly, that is to say

$$\frac{d\vec{v}_\alpha}{d\kappa} = \frac{\partial\vec{v}_\alpha}{\partial\kappa} \qquad \textit{and} \qquad \frac{ds^\kappa_\alpha}{d\kappa} \neq \frac{\partial s^\kappa_\alpha}{\partial\kappa} = 0. \tag{2,35}$$

Therefore, $d/d\kappa$ and $\partial/\partial t$ commute when operating on $\vec{v}_\alpha$ (but not on $s^\kappa_\alpha$).

With the help of eq(2,28) we can rewrite the difference between the l.h.s. and r.h.s. of eq(2,34) as

$$\begin{aligned}
&\sum_{\alpha\beta} s^\kappa_\alpha \frac{\partial}{\partial t}\vec{v}_\alpha\cdot(\vec{v}_\beta\times\frac{d\vec{v}_\beta}{d\kappa}) - \sum_{\alpha\beta} s^\kappa_\alpha \vec{v}_\alpha\cdot\frac{d}{d\kappa}(\vec{v}_\beta\times\frac{\partial\vec{v}_\beta}{\partial t})\\
&= \sum_{\alpha\beta} s^\kappa_\alpha \Big[\frac{\partial(\vec{v}_\alpha\times\vec{v}_\beta)}{\partial t}\cdot\frac{d\vec{v}_\beta}{d\kappa} - \vec{v}_\alpha\cdot(\frac{d\vec{v}_\beta}{d\kappa}\times\frac{\partial\vec{v}_\beta}{\partial t})\Big]\\
&= \sum_{\alpha\beta} s^\kappa_\alpha (\frac{\partial(\vec{v}_\alpha\times\vec{v}_\beta)}{\partial t} + \vec{v}_\alpha\times\frac{\partial\vec{v}_\beta}{\partial t})\cdot\frac{d\vec{v}_\beta}{d\kappa}\\
&= \sum_{\alpha\beta} s^\kappa_\alpha (2\vec{v}_\alpha\times\frac{\partial\vec{v}_\beta}{\partial t} + \frac{\partial\vec{v}_\alpha}{\partial t}\times\vec{v}_\beta)\cdot(\vec{V}\times\vec{v}_\beta)\\
&= \sum_{\alpha\beta} s^\kappa_\alpha \Big[\vec{v}_\beta\times(2\vec{v}_\alpha\times\frac{\partial\vec{v}_\beta}{\partial t} + \frac{\partial\vec{v}_\alpha}{\partial t}\times\vec{v}_\beta)\Big]\cdot\vec{V}\\
&= \sum_{\alpha\beta} s^\kappa_\alpha \Big[-2(\vec{v}_\alpha\cdot\vec{v}_\beta)\frac{\partial\vec{v}_\beta}{\partial t} + (\vec{v}_\beta\cdot\vec{v}_\beta)\frac{\partial\vec{v}_\alpha}{\partial t} - (\vec{v}_\beta\cdot\frac{\partial\vec{v}_\alpha}{\partial t})\vec{v}_\beta\Big]\cdot\vec{V}\\
&= \sum_{\alpha} s^\kappa_\alpha (-2\vec{v}_\alpha + 3\vec{v}_\alpha - \vec{v}_\alpha)\cdot\vec{V} \qquad = 0,
\end{aligned}$$

which proves eq(2,34). (This is the most tedious manipulation in our formalism. Perhaps there is a better way.) Hence, the transformation of the Hamiltonian (2,33) is

$$\frac{dH^\kappa}{d\kappa} = -\frac{1}{2}\sum_{\alpha\beta}\frac{d}{d\kappa}\Big[s^\kappa_\alpha \vec{v}_\alpha\cdot(\vec{v}_\beta\times\frac{\partial\vec{v}_\beta}{\partial t})\Big], \tag{2,36}$$

which can easily be integrated with the result

$$H^\kappa = H^0 - \frac{1}{2}\sum_{\alpha\beta} s^\kappa_\alpha \vec{v}_\alpha \cdot (\vec{v}_\beta \times \frac{\partial \vec{v}_\beta}{\partial t}). \tag{2,37}$$

Here, the second term vanishes at $\kappa = 0$, since $\partial \vec{v}_\beta(t,0)/\partial t = \partial \vec{e}_\beta/\partial t = 0$. The first term $H^0$ must be expressed in terms of the intermediate variables

$$H^0 = \vec{W} \cdot \sum \vec{e}_\alpha s^0_\alpha = \vec{W} \cdot \sum \vec{v}_\alpha s^\kappa_\alpha. \tag{2,38}$$

whence

$$H^\kappa = \sum_\alpha (\vec{W} - \frac{1}{2}\sum_\beta \vec{v}_\beta \times \frac{\partial \vec{v}_\beta}{\partial t}) \cdot \vec{v}_\alpha s^\kappa_\alpha. \tag{2,39}$$

By putting $\kappa = 1$, we obtain the new Hamiltonian

$$H(s_\alpha, t) = \sum_\alpha (\vec{W} - \frac{1}{2}\sum_\beta \vec{u}_\beta \times \frac{d\vec{u}_\beta}{dt}) \cdot \vec{u}_\alpha s_\alpha. \tag{2,40}$$

Owing to eq (2,10) this is equivalent to the expected result (2,8). Thus, we have been able to contruct a canonical transformation (2,30) which leads to the required Hamiltonian (2,8).

## 3. Canonical Transformations with Spin and Orbital Motion

It is not difficult to generalize the method in the previous section to a system with orbital degrees of freedom. Let us start with the Hamiltonian

$$H^0(p^0, q^0, s^0_\alpha, t) = H_{orb}(p^0, q^0, t) + \sum_\alpha \vec{W}(p^0, q^0, t) \cdot \vec{e}_\alpha s^0_\alpha. \tag{3,1}$$

Here, $p^0$ and $q^0$ symbolically denote all the canonical variables of the orbital motion, $p^0_\lambda$ and $q^0_\lambda$ ($\lambda = 1, 2, 3$). The orbital Hamiltonian, in the absence of spin, is denoted by $H_{orb}$. The corresponding BMT equation is

$$\frac{ds^0_\alpha}{dt} = \sum_{\beta\gamma} \epsilon_{\alpha\beta\gamma} \vec{W}(p^0, q^0, t) \cdot \vec{e}_\beta s^0_\gamma. \tag{3,2}$$

Let us consider a new orthonormal basis $\{\vec{u}_\alpha(p^0, q^0, t)\}$ which rotates with the instantaneous angular velocity $\vec{U}(p^0, q^0, t)$;

$$\frac{d\vec{u}_\alpha}{dt} = \vec{U}(p^0(t), q^0(t), t) \times \vec{u}_\alpha \tag{3,3}$$

$$\vec{U} = \frac{1}{2}\sum_{\alpha} \vec{u}_{\alpha} \times \frac{d\vec{u}_{\alpha}}{dt}. \tag{3,4}$$

Here, $p^0(t)$ and $q^0(t)$ are solutions of the unperturbed system;

$$\frac{dp^0}{dt} = -\frac{\partial H_{orb}}{\partial q^0}, \qquad \frac{dq^0}{dt} = +\frac{\partial H_{orb}}{\partial p^0}. \tag{3,5}$$

In this paper we only consider terms up to the first order in spin. The spin motion is, of course, affected by the orbital motion but the orbit is also perturbed by the spin as in Stern and Gerlach's experiment. Since our formalism is canonical, the latter effect is automatically taken into account. We shall, however, ignore the second order perturbation of the perturbed orbit on the spin. Therefore, we used the unperturbed equation of motion in (3,5). The change of a spin variable by a Lie generating function linear in the spin is of the same order as the spin itself but the change of the orbital variables is small compared with their unperturbed values. Hence, when we work to the first order of spin, we need not distinguish between $p^0, q^0$ and $p, q$ (orbital variables after transformation) in the spin term. Therefore, we can also write $p$ and $q$ in eq(3,3) instead of $p^0$ and $q^0$.

Next, let us denote by $s_\alpha$ the spin component seen in the basis $\{\vec{u}_\alpha\}$ and express the canonical transformation $s_\alpha^0 \to s_\alpha$ using a Lie transformation. The intermediate orthonormal basis $\vec{v}_\alpha(p^\kappa, q^\kappa, t, \kappa)$ satisfies

$$\frac{d\vec{v}_\alpha}{d\kappa} = \vec{V} \times \vec{v}_\alpha \tag{3,6}$$

with

$$\vec{V}(p^\kappa, q^\kappa, t, \kappa) = \frac{1}{2}\sum_{\alpha} \vec{v}_\alpha \times \frac{d\vec{v}_\alpha}{d\kappa} \tag{3,7}$$

The corresponding Lie generating function is given by

$$L = L(p^\kappa, q^\kappa, s_\alpha^\kappa, t, \kappa) = -\sum_{\alpha} s_\alpha^\kappa \vec{v}_\alpha \cdot \vec{V} \tag{3,8}$$

and the transformation of $s_\alpha^\kappa$ is

$$\frac{ds_\alpha^\kappa}{d\kappa} = \{s_\alpha^\kappa, L\} = -\sum_{\beta\gamma} \epsilon_{\alpha\beta\gamma}(\vec{v}_\beta \cdot \vec{V}) s_\gamma^\kappa \tag{3,9}$$

However, in contrast to the previous section, the Lie generating function (3,8) also generates the transformation of the orbital variables;

$$\frac{dp_\lambda^\kappa}{d\kappa} = \{p_\lambda^\kappa, L\} = \frac{1}{2}\sum_{\alpha\beta} \frac{\partial}{\partial q_\lambda^\kappa} s_\alpha^\kappa \vec{v}_\alpha \cdot (\vec{v}_\beta \times \frac{d\vec{v}_\beta}{d\kappa}) \tag{3,10}$$

Without this change of orbital variables, our transformation $s_\alpha^0 \to s_\alpha$ would not be canonical. Let us solve this equation. We note that the following equation holds for any independent variable $x$ except the spin; i.e., $x = p_\lambda^\kappa, q_\lambda^\kappa$ or $t$;

$$\sum_{\alpha\beta} \frac{\partial}{\partial x} s_\alpha^\kappa \vec{v}_\alpha \cdot (\vec{v}_\beta \times \frac{d\vec{v}_\beta}{d\kappa}) = \sum_{\alpha\beta} \frac{d}{d\kappa} s_\alpha^\kappa \vec{v}_\alpha \cdot (\vec{v}_\beta \times \frac{\partial \vec{v}_\beta}{\partial x}) + \quad O((spin)^2), \tag{3,11}$$

which is a generalization of eq(2,34) and can be proved in the same manner. This time, however, it holds only up to the first order of spin. Since

$$\frac{d\vec{v}_\beta}{d\kappa} = \frac{\partial \vec{v}_\beta}{\partial \kappa} + \sum_\lambda (\frac{dp_\lambda^\kappa}{d\kappa} \frac{\partial \vec{v}_\beta}{\partial p_\lambda^\kappa} + \frac{dq_\lambda^\kappa}{d\kappa} \frac{\partial \vec{v}_\beta}{\partial q_\lambda^\kappa}) = \frac{\partial \vec{v}_\beta}{\partial \kappa} + \quad O(spin), \tag{3,12}$$

$d/d\kappa$ and $\partial/\partial x$ do not commute exactly but give

$$\frac{d}{d\kappa}(\frac{\partial \vec{v}_\beta}{\partial x}) - \frac{\partial}{\partial x}(\frac{d\vec{v}_\beta}{d\kappa}) = O(spin). \tag{3,13}$$

In the proof of eq(2,34) we commuted $d/d\kappa$ and $\partial/\partial t$. Hence, eq(3,11) holds only up to the first order in spin, but this is sufficient for our purposes. Then, with the help of (3,11), we can rewrite eq(3,10) up to the first order in spin as

$$\frac{dp_\lambda^\kappa}{d\kappa} = \frac{1}{2} \sum_{\alpha\beta} \frac{d}{d\kappa} s_\alpha^\kappa \vec{v}_\alpha \cdot (\vec{v}_\beta \times \frac{\partial \vec{v}_\beta}{\partial q_\lambda^\kappa}), \tag{3,14}$$

which can easily be integrated to yield

$$p_\lambda^\kappa = p_\lambda^0 + \frac{1}{2} \sum_{\alpha\beta} s_\alpha^\kappa \vec{v}_\alpha \cdot (\vec{v}_\beta \times \frac{\partial \vec{v}_\beta}{\partial q_\lambda^\kappa}). \tag{3,15}$$

Hence, the momentum at $\kappa = 1$ is given by

$$\begin{aligned} p_\lambda &= p_\lambda^0 + \frac{1}{2} \sum_{\alpha\beta} s_\alpha \vec{u}_\alpha \cdot (\vec{u}_\beta \times \frac{\partial \vec{u}_\beta}{\partial q_\lambda}) \\ &= p_\lambda^0 + \frac{1}{2} \sum_{\alpha\beta} s_\alpha \vec{u}_\alpha \cdot (\vec{u}_\beta \times \{\vec{u}_\beta, p_\lambda\}). \end{aligned} \tag{3,16}$$

Similarly, for $q_\lambda$ we have

$$\begin{aligned} q_\lambda &= q_\lambda^0 - \frac{1}{2} \sum_{\alpha\beta} s_\alpha \vec{u}_\alpha \cdot (\vec{u}_\beta \times \frac{\partial \vec{u}_\beta}{\partial p_\lambda}) \\ &= q_\lambda^0 + \frac{1}{2} \sum_{\alpha\beta} s_\alpha \vec{u}_\alpha \cdot (\vec{u}_\beta \times \{\vec{u}_\beta, q_\lambda\}). \end{aligned} \tag{3,17}$$

These equations can be generalized to any function $f(p^0, q^0, t)$ of the orbital variables $p_\lambda^0$ and $q_\lambda^0$ as

$$f(p,q,t) = f(p^0, q^0, t) + \frac{1}{2}\sum_{\alpha\beta} s_\alpha \vec{u}_\alpha \cdot (\vec{u}_\beta \times \{\vec{u}_\beta, f\}) \tag{3,18}$$

Here, $f(p,q,t)$ is a function given simply by replacing $p^0$ and $q^0$ in $f(p^0,q^0,t)$ by the new variables. This relation does not hold if $f$ is a function of $s_\alpha$.

Finally, we derive the new Hamiltonian. From (2,32) we have

$$\frac{dH^\kappa}{d\kappa} = \frac{\partial L}{\partial t} = -\frac{1}{2}\sum_{\alpha\beta}\frac{\partial}{\partial t} s_\alpha^\kappa \vec{v}_\alpha \cdot (\vec{v}_\beta \times \frac{d\vec{v}_\beta}{d\kappa}) = -\frac{1}{2}\sum_{\alpha\beta}\frac{d}{d\kappa} s_\alpha^\kappa \vec{v}_\alpha \cdot (\vec{v}_\beta \times \frac{\partial \vec{v}_\beta}{\partial t}), \tag{3,19}$$

whence

$$H^\kappa = H^0 - \frac{1}{2}\sum_{\alpha\beta} s_\alpha^\kappa \vec{v}_\alpha \cdot (\vec{v}_\beta \times \frac{\partial \vec{v}_\beta}{\partial t}) \tag{3,20}$$

and at $\kappa = 1$,

$$H = H^0 - \frac{1}{2}\sum_{\alpha\beta} s_\alpha \vec{u}_\alpha \cdot (\vec{u}_\beta \times \frac{\partial \vec{u}_\beta}{\partial t}). \tag{3,21}$$

In these expressions $H^0$ is the initial Hamiltonian defined in (3,1) but it must be rewritten in terms of new (or intermediate) variables. It consists of two terms, namely $H_{orb}$ and the spin term. Since we are only calculating up to the first order in spin, the orbital variables $p_\lambda^0$ and $q_\lambda^0$ in the spin term can simply be replaced by $p_\lambda$ and $q_\lambda$. On the other hand, $H_{orb}(p^0, q^0, t)$ can be written with the help of (3,18) as

$$H_{orb}(p^0, q^0, t) = H_{orb}(p,q,t) - \frac{1}{2}\sum_{\alpha\beta} s_\alpha \vec{u}_\alpha \cdot (\vec{u}_\beta \times \{\vec{u}_\beta, H_{orb}\}). \tag{3,22}$$

Hence,

$$H^0 = H_{orb}(p,q,t) - \frac{1}{2}\sum_{\alpha\beta} s_\alpha \vec{u}_\alpha \cdot (\vec{u}_\beta \times \{\vec{u}_\beta, H_{orb}\}) + \sum_\alpha \vec{W}(p,q,t)\cdot \vec{u}_\alpha s_\alpha. \tag{3,23}$$

Then the new Hamiltonian is

$$\begin{aligned} &H(p,q,s_\alpha,t) \\ &= H_{orb}(p,q,t) + \sum_\alpha \Big[\vec{W}(p,q,t) - \frac{1}{2}\sum_\beta \vec{u}_\beta \times \{\vec{u}_\beta, H_{orb}\} - \frac{1}{2}\sum_\beta \vec{u}_\beta \times \frac{\partial \vec{u}_\beta}{\partial t}\Big]\cdot \vec{u}_\alpha s_\alpha \\ &= H_{orb}(p,q,t) + \sum_\alpha (\vec{W}(p,q,t) - \frac{1}{2}\sum_\beta \vec{u}_\beta \times \frac{d\vec{u}_\beta}{dt})\cdot \vec{u}_\alpha s_\alpha \\ &= H_{orb}(p,q,t) + \sum_\alpha (\vec{W}(p,q,t) - \vec{U}(p,q,t))\cdot \vec{u}_\alpha s_\alpha. \end{aligned} \tag{3,24}$$

Here, $H_{orb}(p, q, t)$ is given simply by replacing $p^0$ and $q^0$ in $H_{orb}(p^0, q^0, t)$ by $p$ and $q$.

## 4. Action-Angle Variables of Spin Motion

In this section we derive the action-angle representation of a system with spin and orbital degrees of freedom using the canonical transformation introduced in the preceding sections. This is the central part of this paper. We assume that the Hamiltonian of the orbital motion, in the absence of spin, is integrable, and can therefore be written in terms of action-angle variables $I_\lambda^0$ and $\Phi_\lambda^0$. As in the previous section, we restrict attention to the first order of spin. In this section we shall discuss time-independent systems only. The initial Hamiltonian has the form

$$H^0(I^0, \Phi^0, s_\alpha^0) = H_{orb}(I^0) + \sum_\alpha \vec{W}(I^0, \Phi^0) \cdot \vec{e}_\alpha s_\alpha^0 \tag{4,1}$$

and the BMT equation is

$$\frac{ds_\alpha^0}{dt} = \sum_{\beta\gamma} \epsilon_{\alpha\beta\gamma} \vec{W} \cdot \vec{e}_\beta s_\gamma^0. \tag{4,2}$$

The instantaneous angular velocity $\vec{W}$ is a periodic function of the $\Phi_\lambda^0$'s.

The key issue in this section is to find a transformation, amongst the general canonical transformations given in the previous section, (or equivalently to find $\vec{U}$ and $\{\vec{u}_\alpha\}$) so that the final Hamiltonian (3,24) is a function of the action variables only.

$$H = H_{orb}(I) + \sum_\alpha (\vec{W}(I, \Phi) - \vec{U}) \cdot \vec{u}_\alpha s_\alpha \tag{4,3}$$

Let us define the action variable, $J$, for spin as $s_3$ according to eq(2,11). In order that the expression (4,3) be a function of $I_\lambda$ and $J$ only, the following conditions are required.

(a) The vector $\vec{\Omega} \equiv \vec{W} - \vec{U}$ is parallel to $\vec{u}_3$; i.e.,

$$(\vec{W} - \vec{U}) \cdot \sum \vec{u}_\alpha s_\alpha = \vec{\Omega} \cdot \vec{u}_3 s_3 = |\vec{\Omega}| J.$$

(b) Its magnitude $\Omega \equiv |\vec{\Omega}|$ is a function of $I_\lambda$ only. (Of course, it can depend on $J$ from the requirements of the action-angle representation, but in practice $J$ does not appear because we are calculating only to first order in spin.)

(c) The $\vec{u}_\alpha$'s ($\alpha = 1, 2, 3$), which are solutions of

$$\frac{d\vec{u}_\alpha}{dt} = \vec{U} \times \vec{u}_\alpha, \tag{4,4}$$

do not depend on the time explicitly. They are periodic in $\Phi_\lambda$.

If these conditions are satisfied, the new Hamiltonian (4,3) becomes

$$H(I,J) = H_{orb}(I) + \Omega(I)J. \tag{4,5}$$

It is evident that $\Omega(I)$ is the spin precession frequency;

$$\frac{\partial H}{\partial J} = \Omega(I). \tag{4,6}$$

Perhaps, the last condition (c) needs some explanation. If the new orthonormal basis $\{\vec{u}_\alpha\}$ depends on the time explicitly, the basis vectors vary with time even at a fixed space point. In this case the spin precession frequency is not defined. In fact, it can take any value in an arbitrarily rotating frame. We cannot define a spin angle variable in such a frame. The requirement that $\vec{u}_\alpha$ be periodic in $\Phi_\lambda$ is almost obvious. Without it, we lose the meaning of $\Phi_\lambda$ as an angle variable because a period in $\Phi_\lambda$ becomes different from that in $\Phi_\lambda^0$.

We can easily show that $\vec{\Omega}$ satisfies the BMT equation if the conditions (a) and (b) hold;

$$\text{(d)} \qquad \frac{d\vec{\Omega}}{dt} = |\vec{\Omega}|\frac{d\vec{u}_3}{dt} = |\vec{\Omega}|\vec{U} \times \vec{u}_3 = \vec{U} \times \vec{\Omega} = (\vec{U} + \vec{\Omega}) \times \vec{\Omega} = \vec{W} \times \vec{\Omega} \tag{4,7}$$

It is also evident that $\vec{\Omega}$ does not depend on the time explicitly if (a), (b) and (c) hold.

Conversely, we can prove that $|\vec{\Omega}|$ is a function of $I_\lambda$ only, if $\vec{\Omega}$ does not depend on the time explicitly and if it satisfies the BMT equation (4,7):

$$\frac{d}{dt}|\vec{\Omega}|^2 = 2\vec{\Omega}\cdot\frac{d\vec{\Omega}}{dt} = 2\vec{\Omega}\cdot(\vec{W} \times \vec{\Omega}) = 0$$

Therefore,

$$\sum_\lambda \omega_\lambda \frac{\partial|\vec{\Omega}|}{\partial\Phi_\lambda} = \frac{d|\vec{\Omega}|}{dt} - \frac{\partial|\vec{\Omega}|}{\partial t} = 0, \tag{4,8}$$

where $\omega_\lambda = \partial H/\partial I_\lambda$ is the orbit angular frequency. Since $\Omega$ is a periodic function of $\Phi_\lambda$, it can be expanded in a Fourier series in $\Phi_\lambda$. Then, the above equation states that $\Omega$ contains only the zero-harmonic in $\Phi_\lambda$ as long as

$$\sum_\lambda m_\lambda\omega_\lambda \neq 0 \tag{4,9}$$

holds for any set of integers $m_\lambda(\lambda = 1,2,3)$. Therefore, $\Omega$ is a function of $I_\lambda$ only.

We will assume that (4,9) holds. In other words, the orbital motion is not in resonance. Rigorously speaking, if the unperturbed system (without spin) is nonlinear (but integrable), $\omega_\lambda$ can depend on $I_\lambda$ and the condition (4,9) is not satisfied for some values of $I_\lambda$. A tiny perturbation by the spin can destroy the integrability of the system in the immediate vicinity of such resonances. This problem is beyond our scope and will not be discussed in this paper. We can say that it is not interesting in practice as a problem of spin.

From the above considerations we can replace the requirement (b) with the BMT equation (d).

Let us state some theorems here which will be used frequently without notice below.

(1) $d/dt$ and $\partial/\partial t$ commute if the Hamiltonian does not depend on the time explicitly.

(2) $d/dt$ and $\partial/\partial\Phi_\lambda$ commute if $\Phi_\lambda$ is the true angle variable. ($d/dt$ and $\partial/\partial\Phi_\lambda^0$ commute only approximately.)

(3) Let $\mathbf{B}(\vec{A},x)$ be the set of vectors $\vec{a}$ which satisfy $d\vec{a}/dx = \vec{A}\times\vec{a}$. Then if $\vec{a},\vec{b}\in\mathbf{B}(\vec{A},x)$, we have $c_1\vec{a}+c_2\vec{b}\in\mathbf{B}(\vec{A},x)$ and $\vec{a}\times\vec{b}\in\mathbf{B}(\vec{A},x)$, where $c_1$ and $c_2$ are constants, i.e., $dc_1/dx = dc_2/dx = 0$. Here, $d/dx$ can be a partial derivative if used correctly. In more sophisticated terms, $\mathbf{B}(\vec{A},x)$ forms a Lie algebra with respect to the vector product. (This has nothing to do with the Lie transformation in the previous section.)

All of these can be proved easily.

Before we discuss how to choose $\{\vec{u}_\alpha\}$, we have to show that the basis $\{\vec{u}_\alpha\}$ which satisfies the requirements given above is not in fact unique. This is not only interesting mathematically but also useful for elucidating the physical picture.

As will be shown later, $\vec{u}_3(\equiv\vec{n})$ is unique. The difference between $(\vec{u}_1,\vec{u}_2)$ and $\vec{u}_3$ is that the first two satisfy (4,4) only but the third vector satisfies the BMT equation (4,7), too. Let $\{\vec{u}'_\alpha\}$ be the new orthonormal basis which is given by rotating $\{\vec{u}_\alpha\}$ around $\vec{u}_3$ through the angle $\sum m_\lambda\Phi_\lambda$, where $\{m_\lambda\}$ is an arbitrary set of integers;

$$\begin{cases} \vec{u}'_1 + i\vec{u}'_2 = e^{-i\sum m_\lambda\Phi_\lambda}(\vec{u}_1 + i\vec{u}_2) \\ \vec{u}'_3 = \vec{u}_3. \end{cases} \tag{4,10}$$

As is easily seen, $\vec{u}'_\alpha$ satisfies

$$\frac{d\vec{u}'_\alpha}{dt} = \vec{U}' \times \vec{u}'_\alpha \tag{4,11}$$

where

$$\vec{U}' = \vec{U} + (\sum_\lambda m_\lambda \omega_\lambda)\vec{n}. \tag{4,12}$$

Therefore, if we define $\vec{\Omega}'$ by

$$\vec{\Omega}' = \vec{W} - \vec{U}' = \vec{\Omega} - (\sum m_\lambda \omega_\lambda)\vec{n} = (\Omega - \sum m_\lambda \omega_\lambda)\vec{n}, \tag{4,13}$$

all the conditions (a) to (d) are satisfied by using $\vec{U}', \vec{\Omega}'$ and $\vec{u}'_\alpha$ instead of $\vec{U}, \vec{\Omega}$ and $\vec{u}_\alpha$.

This fact shows that if $\Omega$ is a spin precession frequency, then $\Omega' \equiv \Omega - \sum m_\lambda \omega_\lambda$ can be so regarded. It is ambiguous up to a linear combination of orbital frequencies. Hence, if the spin resonant condition

$$\Omega = \sum m_\lambda \omega_\lambda \tag{4,14}$$

holds, then the spin precession frequency seen in some (time-independent) frame is zero. This resonance is more important in our problem than the orbit resonance (4,9).

As is well known, the spin tune (=precession frequency/revolution frequency) in circular accelerators can only be defined in general up to the fractional part. This may be seen by noting the fact stated above. As is shown in section 5, one of the $\omega_\lambda$ is the revolution frequency $\omega_0$. Hence, the integer part of $\Omega/\omega_0$ is ambiguous. However, we do not have this ambiguity when considering betatron oscillations. This comes from the fact that we observe the particle at a fixed location (machine azimuth) in the ring but at *any* transverse coordinate. If instead we observe it in a three-dimensionally limited (but finite) region, the particle will only appear in this region with a long time interval which is some "least common multiple" of orbital oscillation periods, and we shall have more ambiguity in defining the spin tune.

So far, things have been defined only heuristically. We shall now redefine all the quantities rigorously. Let us proceed with the problem of choosing $\{\vec{u}_\alpha\}$. First, we define $\vec{n}(=\vec{u}_3)$ as follows.

$$\vec{n}(I,\Phi) \equiv \text{explicitly time-independent solution to the BMT equation with unit length, periodic in } \Phi_\lambda. \tag{4,15}$$

or, equivalently, $\vec{n}(I,\Phi)$ is the solution to the coupled linear partial differential equation

$$\sum_\lambda \omega_\lambda \frac{\partial \vec{n}}{\partial \Phi_\lambda} = \vec{W}(I,\Phi) \times \vec{n} \tag{4,16}$$

with the periodic boundary conditions

$$\vec{n}(I,\Phi_1+2\pi,\Phi_2,...) = \vec{n}(I,\Phi_1,\Phi_2+2\pi,....) = .... = \vec{n}(I,\Phi_1,\Phi_2,...). \tag{4,17}$$

Since it is evident from the requirements (a) through (d) that the above property of $\vec{n}$ is necessary, the point is whether or not it determines $\vec{n}$ uniquely. Suppose there were another vector $\vec{n}'(I,\Phi)$ which satisfied (4,15). Define $\vec{u}_1'$ by

$$\vec{u}_1' = \frac{\vec{n} - \vec{n}'(\vec{n}\cdot\vec{n}')}{\sqrt{1-(\vec{n}\cdot\vec{n}')^2}}.$$

This is the vector given by the Schmidt orthogonalization from $\vec{n}$ and $\vec{n}'$. It is perpendicular to $\vec{n}'$. Since $\vec{n}\cdot\vec{n}'$ is conserved $(d(\vec{n}\cdot\vec{n}')/dt = 0)$, and since we have assumed $\vec{n} \neq \vec{n}'$, the denominator does not vanish. Also, $\vec{u}_1'$ satisfies the BMT equation. Define $\vec{u}_2' \equiv \vec{n}' \times \vec{u}_1'$ and $\vec{u}_3' \equiv \vec{n}'$. These obviously satisfy the BMT equation as is seen from the theorem (3). Now, all the $\vec{u}_\alpha'$ satisfy the BMT equation and none depends on $t$ explicitly. Therefore, by transforming to the $\{\vec{u}_\alpha'\}$ system, we have

$$\vec{W} = \frac{1}{2}\sum_\alpha \vec{u}_\alpha' \times \frac{d\vec{u}_\alpha'}{dt} = \vec{U}.$$

This means that $\vec{\Omega} = 0$, which is equivalent to the spin resonant condition (4,14). Therefore, the above definition of $\vec{n}$ is unique away from the spin resonance.

The reason we need a time-independent solution can be explained in a different way. If we have any solution $\vec{\xi}(I,\Phi,t)$ of the BMT equation, then the projection of the spin onto this vector $\vec{s}\cdot\vec{\xi}$ is a constant of the motion. What we want, however, is not a constant of the motion but an integral of the motion. The former is a conserved quantity consisting of the canonical variables and time, but the latter is a time-independent function. So, if we know $\vec{n}(I,\Phi)$, the projection onto $\vec{n}$ is an integral of the motion.

Next, we define two other vectors $\vec{u}_1$ and $\vec{u}_2$. In order to do so, we introduce two other orthonormal solutions, $\vec{\eta}_1$ and $\vec{\eta}_2$, to the BMT equation, perpendicular to $\vec{n}$. For notational convenience let $\vec{\eta}_3 = \vec{n}$. Then, we can write

$$\frac{d\vec{\eta}_\alpha}{dt} = \vec{W} \times \vec{\eta}_\alpha \tag{4,18}$$

and

$$\vec{W}(I,\Phi) = \frac{1}{2}\sum_\alpha \vec{\eta}_\alpha(I,\Phi,t) \times \frac{d\vec{\eta}_\alpha(I,\Phi,t)}{dt} \qquad (4,19)$$

There is still arbitrariness in the initial values of $\vec{\eta}_1$ and $\vec{\eta}_2$. These will be specified later. One sees from the discussion above that $\vec{\eta}_1$ and $\vec{\eta}_2$ do depend on $t$ explicitly. Otherwise, we have the spin resonant condition (4,14)

The orthonormal basis $\{\vec{\eta}_\alpha\}$ is not the one we want, but the difference between it and $\{\vec{u}_\alpha\}$ is merely a rotation around the axis $\vec{\eta}_3 = \vec{u}_3 = \vec{n}$. This rotation contains a component uniformly increasing with time as well as periodically oscillating terms. In order to simplify the formulation, let us subtract this oscillatory part from $\vec{\eta}_1$ and $\vec{\eta}_2$. Consider the vector

$$\vec{a} = \frac{1}{2}\sum_\alpha \vec{\eta}_\alpha \times \frac{\partial\vec{\eta}_\alpha}{\partial t} = \frac{1}{2}(\vec{\eta}_1 \times \frac{\partial\vec{\eta}_1}{\partial t} + \vec{\eta}_2 \times \frac{\partial\vec{\eta}_2}{\partial t}). \qquad (4,20)$$

The term $\alpha = 3$ vanishes because $\vec{\eta}_3$ is independent of time. As can be seen by differentiating $\vec{\eta}_1\cdot\vec{\eta}_3 = \vec{\eta}_2\cdot\vec{\eta}_3 = 0$, $\partial\vec{\eta}_1/\partial t$ is parallel to $\vec{\eta}_2$ and $\partial\vec{\eta}_2/\partial t$ to $\vec{\eta}_1$. Hence $\vec{a}$ is parallel to $\vec{\eta}_3$, say, $\vec{a} = F(I,\Phi,t)\vec{n}$. One can see by partial differentiation of the BMT equation with respect to $t$ that $\partial\vec{\eta}_\alpha/\partial t$ also satisfies the BMT equation, thus so does $\vec{a}$. On the other hand, $\vec{\eta}_3 = \vec{n}$ is already a solution to the BMT equation, hence, $dF/dt$ must vanish; i.e., $F$ is a function of $I_\lambda$ and $\Phi_\lambda - \omega_\lambda t$ only. Next, we impose the constraint on $\vec{\eta}_\alpha$ that $\vec{a}$ does not depend on $t$ explicitly;

$$\frac{\partial\vec{a}}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}\sum_\alpha \vec{\eta}_\alpha \times \frac{\partial\vec{\eta}_\alpha}{\partial t} = 0. \qquad (4,21)$$

This is always possible because if we replace $\vec{\eta}_1 + i\vec{\eta}_2$ by $e^{-if(I,\Phi-\omega t)}(\vec{\eta}_1 + i\vec{\eta}_2)$, the new $\vec{\eta}_1$ and $\vec{\eta}_2$ satisfy the BMT equation and the new $\vec{a}$ is

$$(F(I,\Phi-\omega t) - \frac{\partial}{\partial t}f(I,\Phi-\omega t))\vec{n}.$$

Therefore, if we choose $f$ to satisfy

$$\frac{\partial}{\partial t}f(I,\Phi-\omega t) = F(I,\Phi-\omega t) - \langle F \rangle, \qquad (4,22)$$

then $\vec{a}$ does not depend on $t$. Here, $\langle F \rangle$ is the zero-harmonic in the Fourier expansion of $F$ with respect to $\Phi_\lambda - \omega_\lambda t$. Unless we subtract $\langle F \rangle$ from $F$, $f$ cannot be a function of $\Phi - \omega t$ only but depends on $t$ explicitly. From now on, we denote by $\vec{\eta}_1$ and $\vec{\eta}_2$ only those that satisfy the requirement (4,21). Let us define $\vec{\Omega}(I,\Phi)$ and $\Omega(I)$ by

$$\vec{\Omega}(I,\Phi) = \frac{1}{2}\sum_\alpha \vec{\eta}_\alpha \times \frac{\partial\vec{\eta}_\alpha}{\partial t} \equiv \Omega(I)\vec{n}(I,\Phi). \qquad (4,23)$$

In reality, $\Omega$ is essentially equal to $< F >$. As can be verified easily,

$$\frac{\partial \vec{\eta}_\alpha}{\partial t} = \vec{\Omega}(I, \Phi) \times \vec{\eta}_\alpha. \tag{4,24}$$

Now, we are ready to define $\vec{u}_1$ and $\vec{u}_2$. Since eq(4,24) states that the explicit time-dependence of $\vec{\eta}_\alpha$ is a rotation around $\vec{\Omega}$ with constant angular velocity, we get time independent vectors $\vec{u}_1$ and $\vec{u}_2$ by rotating $\vec{\eta}_1$ and $\vec{\eta}_2$ backwards by the steady angular velocity $-\Omega$; thus

$$\vec{u}_1(I,\Phi) + i\vec{u}_2(I,\Phi) = e^{i\Omega(I)t}(\vec{\eta}_1(I,\Phi,t) + i\vec{\eta}_2(I,\Phi,t)). \tag{4,25}$$

It is evident from eqs(4,23) and (4,24) that $\partial\vec{u}_\alpha/\partial t$ vanishes.

The procedure stated above is exactly the one employed in standard canonical perturbation theory, where we attribute the average of the perturbation ($\vec{\Omega}$ in our case) to the Hamiltonian of the next order and the rest to the generating function. Our generating function will be written in terms of $\vec{u}_\alpha$. Since the rotation angle $\Omega(I)t$ in the expression (4,25) has non-vanishing total time derivative, the new vectors $\vec{u}_1$ and $\vec{u}_2$ no longer satisfy the BMT equation. Let us find the equation they do satisfy. From (4,24) and (4,25) we have

$$\frac{d}{dt}(\vec{u}_1 + i\vec{u}_2) = i\Omega(I)(\vec{u}_1 + i\vec{u}_2) + e^{i\Omega t}\vec{W} \times (\vec{\eta}_1 + i\vec{\eta}_2) = (\vec{W} - \vec{\Omega}) \times (\vec{u}_1 + i\vec{u}_2),$$

because $\vec{\Omega} \parallel \vec{u}_3$. Since $\vec{u}_3$ obviously satisfies the same equation, we deduce

$$\frac{d\vec{u}_\alpha(I,\Phi)}{dt} = \vec{U}(I,\Phi) \times \vec{u}_\alpha(I,\Phi) \tag{4,26}$$

with

$$\vec{U}(I,\Phi) = \vec{W}(I,\Phi) - \vec{\Omega}(I,\Phi). \tag{4,27}$$

One can easily verify that

$$\vec{U} = \frac{1}{2}\sum_\alpha \vec{u}_\alpha \times \frac{d\vec{u}_\alpha}{dt} = \frac{1}{2}\sum_{\alpha\lambda} \omega_\lambda \vec{u}_\alpha \times \frac{\partial \vec{u}_\alpha}{\partial \Phi_\lambda} = \frac{1}{2}\sum_{\alpha\lambda} \omega_\lambda \vec{\eta}_\alpha \times \frac{\partial \vec{\eta}_\alpha}{\partial \Phi_\lambda}. \tag{4,28}$$

The last equality comes from the fact that the relation between $\{\vec{u}_\alpha\}$ and $\{\vec{\eta}_\alpha\}$ (eq(4,25)) does not contain $\Phi_\lambda$.

Thus, we have finished the preparation for the canonical transformation to the action-angle representation. It is now easy to perform the final step by applying the methods of the previous section. We have only to replace $p_\lambda$ and $q_\lambda$ with $I_\lambda$ and $\Phi_\lambda$. It is even easier because the transformation in this section is time-independent. It is sufficient to quote the results only.

The Lie generating function is given by eq(3,8) and the relation between the old and the new spin variables is

$$\vec{s} = \sum_\alpha s_\alpha^0 \vec{e}_\alpha = \sum_\alpha s_\alpha \vec{u}_\alpha. \tag{4,29}$$

The transformations of the orbital variables are

$$I_\lambda = I_\lambda^0 + \frac{1}{2}\sum_{\alpha\beta} s_\alpha \vec{u}_\alpha \cdot (\vec{u}_\beta \times \{\vec{u}_\beta, I_\lambda\}) \tag{4,30}$$

and

$$\Phi_\lambda = \Phi_\lambda^0 + \frac{1}{2}\sum_{\alpha\beta} s_\alpha \vec{u}_\alpha \cdot (\vec{u}_\beta \times \{\vec{u}_\beta, \Phi_\lambda\}). \tag{4,31}$$

The new Hamiltonian is

$$H(I,J) = H_{orb}(I) + \sum_\alpha \vec{\Omega}(I,\Phi)\cdot \vec{u}_\alpha s_\alpha = H_{orb}(I) + J\Omega(I). \tag{4,32}$$

Let us note the following point, though trivial. The spin precession frequency $\partial H/\partial J = \Omega(I)$ is a function of the orbital action variables only. It does not depend on $J$ or, equivalently, on the precession amplitude. This means that the spin motion is linear. This is not the case, however, if we take into account the fact that the influence of the spin on the orbit can affect the spin itself in the second order of perturbation, albeit an extremely small effect.

Before closing this section we mention the relation between our formalism and the one in DKII. Derbenev and Kondratenko's argument can be summarized as follows. First, they denote by $\vec{\xi}_\alpha$ "some(какие-либо)" orthonormal basis vectors satisfying the BMT equation and they define $\vec{\Omega}$ and $\vec{n}$ by

$$\vec{\Omega} = \frac{1}{2}\sum_\alpha \vec{\xi}_\alpha(I,\Phi,t) \times \frac{\partial \vec{\xi}_\alpha(I,\Phi,t)}{\partial t}, \qquad \vec{n} = \vec{\Omega}/|\vec{\Omega}|. \tag{4,33}$$

They claim (1) $\vec{\Omega}$ does not depend on the time explicitly, and (2) $\vec{\Omega}$ satisfies the BMT equation, and (3) $|\vec{\Omega}|$ is a function of $I_\lambda$ only, and (4) $\vec{n}$ does not depend on the choice of $\{\vec{\xi}_\alpha\}$.

Of these, (2) is obvious because $\partial\vec{\xi}_\alpha/\partial t \in \mathbf{B}(\vec{W},t)$, and (3) follows from (1) and (2) as we proved above. They proved (4) using (1). The statement (1) is essential in the action angle formalism. This is a very beautiful argument in that it gives a closed expression for $\vec{\Omega}$ and $\vec{n}$. Only the BMT equation is needed for the definition of $\vec{\xi}_\alpha$. Our definitions (4,15),(4,21) and (4,23) look clumsy by comparison. Unfortunately, however, the statement (1) does not hold in general. The following simple example will illustrate the problem.

Consider a precession around $\vec{e}_3$ with the angular velocity modulated by the orbital motion which has one degree of freedom. The Hamiltonian is

$$H^0(I^0,\Phi^0,s_\alpha^0) = H_{orb}(I^0) + \vec{W}(I^0,\Phi^0)\cdot\vec{e}_\alpha s_\alpha^0 \tag{4,34}$$

with

$$\vec{W}(I^0,\Phi^0) = (W_0(I^0) + a(I^0)\cos\Phi^0)\vec{e}_3. \tag{4,35}$$

Obviously, $\vec{\xi}_3 = \vec{e}_3$ is a solution of the BMT equation. The other two solutions are linear combinations of $\vec{e}_1$ and $\vec{e}_2$. Let us write

$$\begin{cases} \vec{\xi}_1 = +\vec{e}_1\cos\vartheta + \vec{e}_2\sin\vartheta \\ \vec{\xi}_2 = -\vec{e}_1\sin\vartheta + \vec{e}_2\cos\vartheta \\ \vec{\xi}_3 = \vec{e}_3. \end{cases} \tag{4,36}$$

The BMT equation yields

$$\frac{d\vartheta}{dt} = W_0(I^0) + a(I^0)\cos\Phi^0, \tag{4,37}$$

and its general solution is given by

$$\vartheta = W_0(I^0)t + \frac{a(I^0)}{\omega(I^0)}(\sin\Phi^0 - f(I^0,\Phi^0-\omega t)) \tag{4,38}$$

with

$$\omega(I^0) = \frac{\partial H_{orb}(I^0)}{\partial I^0}. \tag{4,39}$$

Here, $f$ is an arbitrary function of $I^0$ and $\Phi^0-\omega t$, since $df/dt=0$. Then the definition (4,33) of $\vec{\Omega}$ gives

$$\vec{\Omega} = \frac{\partial\vartheta}{\partial t}\vec{e}_3 = \left[\vec{W}_0 - \frac{a(I^0)}{\omega(I^0)}\frac{\partial}{\partial t}f(I^0,\Phi^0-\omega t)\right]\vec{e}_3. \tag{4,40}$$

Now, if we choose the initial condition $\vec{\xi}_\alpha = \vec{e}_\alpha$, we have $f = \sin(\Phi^0 - \omega t)$ which leads to $\partial\vec{\Omega}/\partial t \neq 0$. The correct choice for obtaining a time-independent $\vec{\Omega}$ is $f=0$. This is obvious in this simple example, but it is not easy to express the correct choice in a general form.

There is a cure. Since they did not give any condition on $\vec{\xi}_\alpha$ except the BMT equation and the orthonormality, we have to interpret "какие-либо" as "arbitrary". We can, however, demand a constraint

$$(\frac{\partial \vec{\Omega}}{\partial t})_{t=0} = 0. \tag{4,41}$$

i.e., on the initial value of $\vec{\xi}_\alpha$ (its dependence on $\Phi_\lambda$ at $t = 0$). Since $\partial\vec{\Omega}/\partial t$ satisfies the BMT equation, which is homogeneous, $\partial\vec{\Omega}/\partial t$ vanishes at any time $t$ if eq(4,41) is satisfied.

In practice, however, the constraint (4,41) is a partial differential equation which is not easy to solve. Even the existence of a solution is not trivial. Moreover, it spoils the beautiful definition of $\vec{n}$ because $\vec{\xi}_\alpha$ cannot be given explicitly. Although not elegant, this extra constraint can save Derbenev and Kondratenko's formalism. Then, it is evident that their definition of $\vec{n}$ is equivalent to ours, since the new constraint (4,41) guarantees the validity of the statement (1).

Next, they define $\vec{\eta}_1$ and $\vec{\eta}_2$ as solutions of the BMT equation perpendicular to $\vec{n}$ and rotate them around $\vec{n}$ to define $\vec{l}_1$ and $\vec{l}_2$. They claim $\vec{l}_1$ and $\vec{l}_2$ are time independent. It is evident that in general this does not hold either, if one considers why we imposed the constraint (4,21) on $\vec{\eta}_\alpha$ to make $\vec{u}_\alpha$ explicitly time independent. If one imposes the condition (4,21) on their vectors $\vec{\eta}_1$ and $\vec{\eta}_2$, then their $\vec{\eta}_1$ and $\vec{\eta}_2$ are equivalent to ours and their $\vec{l}_1$ and $\vec{l}_2$ to our $\vec{u}_1$ and $\vec{u}_2$.

They give the transformations of $I_\lambda$ and $\Phi_\lambda$ (eqs(3,3) and (3,4) in DKII), which correspond to our (4,30) and (4,31), but $\vec{\xi}_\alpha$ in their formulae should be replaced by our $\{\vec{u}_\alpha\}$ or their $\{\vec{n}, \vec{l}_1, \vec{l}_2\}$. (The transformation for $I_\lambda$, eq(3,3) in DKII, is still correct owing to the relation (4,28) but that for $\Phi_\lambda$ is not correct in general.) If we observe the spin in the frame $\{\vec{\xi}_\alpha\}$, the spin term totally disappears from the Hamiltonian, contrary to their final Hamiltonian.[1] We shall see the spin at rest in this frame and thus cannot find its precession frequency.

Thus, by demanding the two additional conditions (4,21) and (4,41), their formalism becomes basically correct and the transformed Hamiltonian eventually becomes the same as ours. However, the basis $\{\vec{\xi}_\alpha\}$, with a new, complicated, constraint, is no longer attractive.

[1]There might be an objection to this statement from people who claim to have verified the final Hamiltonian (3,5) in DKII by substituting the new orbital variables, (3,3) and (3,4), into the initial Hamiltonian (3,2). Note that the transformation to the basis $\{\vec{\xi}_\alpha\}$ is time-dependent. We need an extra term (partial time-derivative of the generating function) and it eliminates the spin term entirely.

I think that in reality the only possible form for $\vec{\xi}_\alpha$ is

$$\vec{\xi}_\alpha(I, \Phi, t) = \sum_\alpha R_{\alpha\beta}(I)\vec{\eta}_\beta(I, \Phi, t),$$

where $R_{\alpha\beta}$ is an arbitrary $3\times 3$ rotation matrix whose elements are functions of $I_\lambda$ only. I have not yet proved this, but, provided it holds, there is no reason to complicate the formalism by adding another orthonormal basis $\{\vec{\xi}_\alpha\}$, which merely makes a constant angle with $\{\vec{\eta}_\alpha\}$.

## 5. Time-Dependent Systems and the Accelerator Coordinate

So far, we have restricted attention to a time-independent Hamiltonian system, but this is not sufficient for accelerator physics because the field in the RF cavities depends on the time explicitly. We cannot take into account the synchrotron oscillation without explicit time-dependence.[1] There are three degrees of freedom in the orbital motion in the Hamiltonian in section 4. One may think that they are the horizontal and vertical betatron oscillations and the synchrotron oscillation, but, in reality, the third degree of freedom is the revolution around the ring. The particle energy is assumed to be constant there. Therefore, one of the three oscillation frequencies $\omega_\lambda$ is the revolution frequency.

It is very easy, however, to generalize formally the method in the previous section to time-dependent Hamiltonian systems, since, as is well known, a time-dependent system with $n$ degrees of freedom is equivalent to a time-independent system with $(n+1)$ degrees of freedom.

For a time dependent Hamiltonian $H_{orb}(p_\lambda, q_\lambda, t)$ $(\lambda = 1,2,3)$, we introduce a new Hamiltonian with the independent variable $\tau$ as

$$\bar{H}_{orb}(p_1, q_1, p_2, q_2, p_3, q_3, p_4, q_4) = H_{orb}(p_1, q_1, p_2, q_2, p_3, q_3, t) + (-E), \qquad (5,1)$$

where $q_4 = t$ and $p_4 = -E$. The equations of motion for $t$ and $E$ are

$$\frac{d(-E)}{d\tau} = -\frac{\partial \bar{H}_{orb}}{\partial t} = -\frac{\partial H_{orb}}{\partial t} \qquad (5,2)$$

and

$$\frac{dt}{d\tau} = +\frac{\partial \bar{H}_{orb}}{\partial(-E)} = 1. \qquad (5,3)$$

[1]The footnote (3) in DKII states that the explicit time dependence due to the RF field can be neglected. In practice, however, we know that the synchrotron oscillation often has significant effects on the spin motion.

Eq(5,3) shows that $t$ and $\tau$ are equivalent as the solution to the equation of motion

$$t = \tau + const. \tag{5,4}$$

Eq(5,2) coincides with the rate of change of the Hamiltonian in the original system $dH_{orb}/dt = \partial H_{orb}/\partial t$.

We express the extended Hamiltonian (5,1) using the action-angle variables, assuming integrability, and write

$$\bar{H}(I_\lambda^0, \Phi_\lambda^0, s_\alpha^0) = \bar{H}_{orb}(I_\lambda^0) + \sum_\alpha \vec{W}(I_\lambda^0, \Phi_\lambda^0) \cdot \vec{e}_\alpha s_\alpha^0 \qquad (\lambda = 1,2,3,4) \tag{5,5}$$

This has the same form as the Hamiltonian in the previuos section (4,1). All the discussions in the previous section are valid if $t$ is replaced by $\tau$ and the sum over $\lambda$ ranges from 1 to 4. The number of orthonormal spin vectors is of course three. Then we can define a $\tau$-independent vector $\vec{n}(I, \Phi)$ uniquely.

We have used the time $t$, or its substitute $\tau$, as the independent variable so far. This is a convenient choice for problems related to quantum theory. In the theories of modern circular accelerators, however, we normally use the arc-length $s$ along the design orbit (or the closed orbit) or the so-called generalized machine azimuth $\theta$ which is proportional to $s$ and normalized to $2\pi$ for one revolution over the ring. Let us rewrite our formalism using $\theta$ as the independent variable.

The Hamiltonian (5,5) can be written in the action angle representation as

$$K(I_\lambda, J) = H_{orb}(I_\lambda) + J\Omega(I_\lambda), \tag{5,6}$$

where we have omitted the bar which was used to denote four degrees of freedom of orbital motion. The four modes $\lambda = 1,2,3,4$ describe oscillations in the horizontal transverse, vertical transverse and the longitudinal displacements from the closed orbit (including possible coupling between them), and the revolution around the ring. (Otherwise, it is not an accelerator.) We denote the latter by $\lambda = 4$ and write $\Phi_4 = \theta$.

Let the inverse of $H_{orb}(I_1, I_2, I_3, I_4) = a$ with respect to $I_4$ be $I_4 = F(I_1, I_2, I_3, a)$. Then, solving (5,6) with respect to $I_4$ up to first order in spin, we obtain

$$I_4 = F - J\Omega(I_1, I_2, I_3, F)\frac{\partial F}{\partial K} \qquad F = F(I_1, I_2, I_3, K). \tag{5,7}$$

Therefore, the new Hamiltonian with $\theta$ as the independent variable is given by

$$H(I_1, I_2, I_3, -K, J) = -I_4 = -F + J\Omega \frac{\partial F}{\partial K}. \tag{5,8}$$

Here, the canonical variables are $(I_1, \Phi_1), (I_2, \Phi_2), (I_3, \Phi_3)$ and $(-K, \tau)$. Note that $\partial F/\partial K$ is the reciprocal of the revolution angular frequency $\omega_0$. We can identify $\Omega \partial F/\partial K$ as the spin tune. Since we are not interested in the value of $K$, which was introduced formally to make the Hamiltonian time-independent, and since $K$ is conserved, it can be eliminated from the Hamiltonian. Then, we have

$$H(I_1, I_2, I_3, J) = -F(I_1, I_2, I_3) + J\nu(I_1, I_2, I_3) \tag{5,9}$$

where $\nu(I_1, I_2, I_3)$ is the spin tune. This is the form usual in accelerator physics.

How do we visualize our orthonormal vectors $\{\vec{u}_\alpha\}$, when we choose the above variables? Firstly, let us recall the definition of $\vec{n}$, (4,15) and (4,16). Since $\Phi_4 = \theta$, eq(4,15) can be written as

$$\sum_{\lambda=1}^{3} \nu_\lambda \frac{\partial \vec{n}}{\partial \Phi_\lambda} + \frac{\partial \vec{n}}{\partial \theta} = \vec{w}(I, \Phi, \theta) \times \vec{n}, \tag{5,10}$$

where $\nu_\lambda = \nu_\lambda(I_1, I_2, I_3) = \omega_\lambda/\omega_0$ $(\lambda = 1, 2, 3)$ are the tunes of the orbital motion and $\vec{w}(I, \Phi, \theta) = \vec{W}/\omega_0$. We have one more argument $I_4$. It is related to the operating beam energy. ($I_4$ is now the Hamiltonian itself.) But since we usually regard it merely as a parameter, we have omitted this argument. Note that, when we first introduced eq(4,15), the orbital motion had three degrees of freedom. In eq(5,10) we also have three but the meaning is different.

Since $d\Phi_\lambda/d\theta = \nu_\lambda$, we can rewrite (5,10) as

$$\frac{d\vec{n}}{d\theta} = \vec{w}(I, \Phi, \theta) \times \vec{n}. \tag{5,11}$$

The periodicity of $\vec{n}$, eq(4,17), can now be written as

$$\begin{aligned} \vec{n}(I, \Phi_1 + 2\pi, \Phi_2, \Phi_3, \theta) &= \vec{n}(I, \Phi_1, \Phi_2 + 2\pi, \Phi_3, \theta) \\ &= \vec{n}(I, \Phi_1, \Phi_2, \Phi_3 + 2\pi, \theta) = \vec{n}(I, \Phi_1, \Phi_2, \Phi_3, \theta + 2\pi) = \vec{n}(I, \Phi_1, \Phi_2, \Phi_3, \theta). \end{aligned} \tag{5,12}$$

Derbenev and Kondratenko gave their first definition of $\vec{n}$ in DKI using the BMT equation (5,11) and the periodicity (5,12). It is now evident that our definition (4,14) is the same as their first definition. In the previous section we showed that our definition leads to that in DKII after adding some constraints. Thus we see that all the three definitions are equivalent.

The properties of the other two vectors $\vec{u}_1$ and $\vec{u}_2$ are obvious, since they are also $\tau$-independent. Their periodicity in $\Phi_\lambda$ ($\lambda = 1,2,3,4$) is now the periodicity in $\Phi_\lambda$ ($\lambda = 1,2,3$) and in $\theta$.

There is a subtle problem as to the other orthonormal basis $\{\vec{\eta}_\alpha\}$ because, except for $\vec{\eta}_3 = \vec{n}$, it depends on $\tau$ explicitly. The original BMT equation is

$$\frac{d\vec{\eta}_\alpha}{d\tau} = \frac{\partial\vec{\eta}_\alpha}{\partial\tau} + \sum_{\lambda=1}^{4} \omega_\lambda \frac{\partial\vec{\eta}_\alpha}{\partial\Phi_\lambda} = \vec{W}(I,\Phi) \times \vec{\eta}_\alpha. \tag{5,13}$$

Replacing $\Phi_4$ with $\theta$ and dividing by $\omega_0 (= \omega_4)$, we have

$$\left(\frac{1}{\omega_0}\frac{\partial}{\partial\tau} + \frac{\partial}{\partial\theta}\right)\vec{\eta}_\alpha + \sum_{\lambda=1}^{3} \nu_\lambda \frac{\partial\vec{\eta}_\alpha}{\partial\Phi_\lambda} = \vec{w}(I,\Phi,\theta) \times \vec{\eta}_\alpha. \tag{5,14}$$

Therefore, in general, $\vec{\eta}_\alpha$ is a function of five arguments, i.e.,

$$(\Phi_1, \Phi_2, \Phi_3, \Phi_4, \tau) = (\Phi_1, \Phi_2, \Phi_3, \theta_0 + \omega_0\tau, \tau) = (\Phi_1, \Phi_2, \Phi_3, \theta, (\theta - \theta_0)/\omega_0).$$

However, we need not regard $\theta_0$ as a variable because it is merely the origin of the independent variable. (We cannot discard the initial values of the $\Phi_\lambda$'s, viz. $\Phi_\lambda - \nu_\lambda\theta$.) Therefore, we can absorb the last argument into $\theta$. But by this process $\theta = \Phi_4$ loses its character as an angle variable and the periodicity of $\vec{\eta}_\alpha$ in $\theta$ is lost. In this sense we can replace the differential operator in the parenthesis in eq(5,14) with $\partial/\partial\theta$ and write

$$\frac{\partial\vec{\eta}_\alpha}{\partial\theta} + \sum_{\lambda=1}^{3} \nu_\lambda \frac{\partial\vec{\eta}_\alpha}{\partial\Phi_\lambda} = \frac{d\vec{\eta}_\alpha}{d\theta} = \vec{w} \times \vec{\eta}_\alpha, \tag{5,15}$$

which is the familiar form of the BMT equation. The point is that the $\tau$-independence (dependence) is replaced by the periodicity (non-periodicity) in $\theta$ in a formalism where $\theta$ is the independent variable.

Finally, let us make some comments on the nature of $\vec{n}(I,\Phi,\theta)$. Note that $\vec{n}$ is a function of $\Phi_\lambda$, not only of $\theta$. The BMT equation (5,11) is usually considered as an ordinary differential equation

$$\frac{d\vec{n}(\theta)}{d\theta} = \vec{w}(I,\Phi(\theta),\theta) \times \vec{n}(\theta) \tag{5,16}$$

where $\Phi(\theta)$ is a solution to the orbital equation of motion;

$$\Phi_\lambda(\theta) = \Phi_{\lambda 0} + \nu_\lambda \theta \tag{5,17}$$

If we regard (5,16) as an initial value problem, then the solution is a function of the initial values: $\vec{n}(\theta) = \vec{f}(I, \Phi_{\lambda 0}, \theta, \vec{n}(0))$. By rewriting $\Phi_{\lambda 0}$ using (5,17), we have

$$\vec{n}(\theta) = \vec{f}(I, \Phi_\lambda - \nu_\lambda \theta, \theta, \vec{n}(0)) = \vec{n}(I, \Phi_\lambda, \theta, \vec{n}(0)).$$

If we choose a proper initial value $\vec{n}(0)$, which is not easy in practice, then the periodicity in $\theta$ is restored in this process and we get the correct $\vec{n}$.

It may be of help to condider that $\vec{n}$ is a solution to the partial differential equation (5,10) under the periodic boundary condition (5,12). Then, we can conceive of $\vec{n}$ without tracing the orbit of a particle. Eqs(5,10) and (5,11) are related by the characteristic equation of the partial differential equation;

$$d\theta = \frac{d\Phi_1}{\nu_1} = \frac{d\Phi_2}{\nu_2} = \frac{d\Phi_3}{\nu_3}. \tag{5,18}$$

Owing to the periodicity , $\vec{n}$ can be expanded into a four-fold Fourier series

$$\vec{n}(I, \Phi, \theta) = \sum_{k,\mathbf{m}} \vec{n}_{k,\mathbf{m}}(I) \exp i(k\theta + \sum m_\lambda \Phi_\lambda). \tag{5,19}$$

If we trace the orbit of a particle, (5,17), then $\vec{n}$ becomes a fuction of $\theta$ only and its Fourier spectrum contains the frequencies $k + \sum m_\lambda \omega_\lambda$. A remarkable fact is that it does not contain the spin tune. This is one of the most important properties of $\vec{n}$.

The Fourier expansion not only helps us to get an understanding of $\vec{n}$ but it is also useful for applications. Since $\vec{w}$ in (5,10) is also periodic in $\Phi_\lambda$ and $\theta$, it can be expanded as

$$\vec{w}(I, \Phi, \theta) = \sum_{k,\mathbf{m}} \vec{w}_{k,\mathbf{m}}(I) \exp i(k\theta + \sum m_\lambda \Phi_\lambda). \tag{5,20}$$

Then the BMT equation (5,10) becomes a coupled linear equation

$$i(k + \sum_{\lambda=1}^{3} m_\lambda \nu_\lambda) \vec{n}_{k,\mathbf{m}}(I) = \sum_{k',\mathbf{m}'} \vec{w}_{k-k',\mathbf{m}-\mathbf{m}'}(I) \times \vec{n}_{k',\mathbf{m}'}(I). \tag{5,21}$$

We can obtain $\vec{n}$ by solving this linear equation. (Some caution is required, but it is not the aim of the present paper.) It has a unique solution unless the spin tune $\nu$ satisfies the resonant condition

$$\nu = k + \sum_{\lambda=1}^{3} m_\lambda \nu_\lambda . \tag{5,22}$$

Note that the spin resonance condition now has the integer term, which comes from the revolution frequency term in (4,14).

We have constructed an action-angle representation of the Hamiltonian with spin and elucidated the role of the vector $\vec{n}$ which is not familiar in accelerator physics. The description is rather abstract but it helps us, the author believes, to make further steps in the problem of radiative polarization.

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