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Coupling of Transverse and Longitudinal Collective Motions due to Closed Orbit Distortion and Dispersion in a Two-Particle Model

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Abstract

In order to explain the large discrepancy between the measured transverse coherent tune shifts and analytical ones in a short bunch in PETRA, the effects of the closed orbit distorsion y_{co} and the dispersion η on a beam instability is studied with a two-particle model. It follows the result which supports Kohaupt's previous results; they hardly contribute to real tune shift, while the momentum dependence of the wake force can make a beam unstable, with the growth rate which is proportional to the product of y_{co} and η.

1 INTRODUCTION

A transverse mode-coupling instability, which determines a limit of the stored current in recent large electron rings, was first observed in PETRA[1]. Since then, many studies have been made to understand the mechanism of this instability, by means of analytical methods[2,3,4] and computer simulations[4,5]. However, any attempt to quantitatively explain the experimental results in PETRA has lead to a large discrepancy between the measured threshold current and the analytical one: the former is $4 \sim 5$ times larger than the latter.

The conclusion was that a large part of the impedance was not taken into consideration, and an intensive search for "missing impedance" was made by Klatt, Kohaupt, and Weiland[6]. Although all possible beam components have been checked, a large discrepancy in the vertical coherent tune shifts was still left unexplained. Kohaupt[7] doubted the hypothesis: Is it not the impedance which is missing, but an effect which can cause the additional tune shifts, but was neglected in the former analytical consideration ? Some plausible candidates for such effects are the closed orbit distortion (C.O.D.) and the dispersion.

He investigated their contributions to a tune shift with a rigid bunch model[7], and the Vlasov equation[8], and then obtained no additional tune shifts, but damping or antidamping of the synchrotron and the betatron coherent oscillations. Whether they are damped or antidamped depends on the sign of the product of C.O.D. and the dispersion. In his model, only the coupling of the transverse and the longitudinal motions is considered. Namely, the interaction between particles in each phase space, which he calls the "transverse - transverse" mode-coupling, or the "longitudinal - longitudinal" mode-coupling, is neglected. Combination of those interactions may cause some new effects.

In order to get insight into the physical mechamism involved, the two-particle model[9,10] is quite a useful utility, which is simple, but includes all the essential effects. In this paper, we study a beam instability where the betatron and the synchrotron collective motions are coupled through C.O.D. and dispersion, with a two-particle model. We expect that an additional tune shifts will be produced by combination of these interactions. In Sec.2, we show the equation of motion for the two particles in both planes, and linearize them with some approximations. We then obtain the 8×8 eigenvalue matrix in Sec.3. The coherent tune shifts are calculated by solving the eigenvalues of this matrix. The formalism is applied to PETRA in Sec.4, where we find that no significant additional real tune shifts are produced by C.O.D. and dispersion. A beam can be stable or unstable, which depends on the sign of the product of C.O.D. and dispersion, as Kohaupt showed with his rigid-bunch model. We reconsider these stable or unstable solution in Sec.5, limiting the problem only in the longitudinal phase space. The paper is concluded in Sec.6.

2 EQUATION OF MOTION

The beam is represented as two macroparticles, each containing $N/2$ particles. The total transverse displacement y_i of particle i ($i = 1, 2$) from the beam axis consists of three terms: the closed orbit deviation y_{co}, the betatron oscillation $y_{i\beta}$ counted from y_{co}, and the dispersion term $\eta \cdot \delta_i$:

$$y_i = y_{co} + y_{i\beta} + \eta \cdot \delta_i, \tag{1}$$

where η is the dispersion, and δ_i is the momentum deviation of the particle i.

During time $0 < t < t_0$ when particle 1 proceeds particle 2 in the longitudinal phase space, the equation of motion for the transverse coordinates are written as

$$\frac{d^2}{dt^2} y_{1\beta} + \omega_\beta^2 y_{1\beta} = 0, \tag{2}$$

$$\frac{d^2}{dt^2} y_{2\beta} + \omega_\beta^2 y_{2\beta} = \frac{Ne^2c^2}{2E} W_\perp(\tau_1 - \tau_2)\,(y_{co} + y_{1\beta} + \delta_1\eta). \tag{3}$$

where $W_\perp(\tau_1 - \tau_2)$ is the wake potential as a function of the distance between two particles, $\tau_1 - \tau_2$, E is the beam energy, ω_β is the betatron angular frequency, e is the elementary charge, and c is the speed of light. For simplicity, the betatron function, the C.O.D., and the dispersion are assumed to be constant over the ring.

We define the longitudinal coordinate τ_i by the longitudinal position relative to the center of the bunch. The momentum deviation δ_i is given by the time derivative of τ_i:

$$\delta_i = -\frac{\dot{\tau}_i}{\alpha c}. \tag{4}$$

The equation of motion for the longitudinal coordinates are

$$\frac{d^2}{dt^2}\tau_1 + \omega_s^2 \tau_1 = \frac{Ne^2\alpha c^2}{2E} [\, W_\parallel(0) + W_\perp'(0)\,(y_{co} + y_{1\beta} + \delta_1\eta)^2\,], \tag{5}$$

$$\frac{d^2}{dt^2}\tau_2 + \omega_s^2\tau_2 = \frac{Ne^2\alpha c^2}{2E}[\,W_{\parallel}(0) + W_{\parallel}(\tau_1 - \tau_2) + W'_{\perp}(0)\,(y_{co} + y_{2\beta} + \delta_2\eta)^2$$
$$+W'_{\perp}(\tau_1 - \tau_2)(y_{co} + y_{1\beta} + \delta_1\eta)(y_{co} + y_{2\beta} + +\delta_2\eta)\,], \tag{6}$$

where $W_{\parallel}$ and $W'_{\perp}$ are the longitudinal wake potentials due to on-axis, and off-axis motion of particles, respectively, ω_s is the synchrotron angular frequency, and α is the momentum compaction factor. The terms $W_{\parallel}(0)$ and $W'_{\perp}(0)$ show the beam loading effects due to self-induced wake fields.

The longitudinal wake potential $W'_{\perp}(z)$ is related to the transverse one $W_{\perp}(z)$ by the relation (Panofsky-Wenzel theorem)[11]

$$W'_{\perp}(z) = \frac{d}{dz}W_{\perp}(z). \tag{7}$$

The transverse wake potential initially increases linearly with distance from zero, and then starts to decrease through the round top. For a short bunch, say $1 \sim 2$ cm long, the linear approximation of the transverse wake potential is a good approximation:

$$W_{\perp}(z) = W_{\perp}^0 \cdot z. \tag{8}$$

The corresponding longitudinal wake potential becomes a constant due to the relationship(7):

$$W'_{\perp}(z) = W_{\perp}^0. \tag{9}$$

For the same reason, the "pure" longitudinal wake potential can be approximated by a constant:

$$W_{\parallel}(z) = W_{\parallel}^0. \tag{10}$$

Inserting Eqs.(8)-(10) into eqs. of motion(2), (3), (5), and (6), we have

$$\frac{d^2}{dt^2}y_{1\beta} + \omega_\beta^2 y_{1\beta} = 0, \tag{11}$$

$$\frac{d^2}{dt^2}y_{2\beta} + \omega_\beta^2 y_{2\beta} = \frac{Ne^2c^2}{2E}W_{\perp}^0[\,(\tau_1 - \tau_2)y_{co} + (\tau_1 - \tau_2)(y_{1\beta} - \frac{\eta}{\alpha c}\dot{\tau}_1)\,], \tag{12}$$

$$\frac{d^2}{dt^2}\tau_1 + \omega_s^2\tau_1 = \frac{Ne^2\alpha c^2}{2E}\{\,\frac{1}{2}W_{\parallel}^0 + \frac{1}{2}W_{\perp}^0[\,y_{co}^2 + 2y_{co}(y_{1\beta} - \frac{\eta}{\alpha c}\dot{\tau}_1)$$
$$+(y_{1\beta} - \frac{\eta}{\alpha c}\dot{\tau}_1)^2\,]\}, \tag{13}$$

$$\frac{d^2}{dt^2}\tau_2 + \omega_s^2\tau_2 = \frac{Ne^2\alpha c^2}{2E}\{\,\frac{3}{2}W_{\parallel}^0 + \frac{1}{2}W_{\perp}^0[\,y_{co}^2 + 2y_{co}(y_{2\beta} - \frac{\eta}{\alpha c}\dot{\tau}_2) + (y_{2\beta} - \frac{\eta}{\alpha c}\dot{\tau}_2)^2\,]$$
$$-W_{\perp}^0 y_{co}^2 + W_{\perp}^0 y_{co}(y_{1\beta} + y_{2\beta} - \frac{\eta}{\alpha c}(\dot{\tau}_1 + \dot{\tau}_2))$$
$$-W_{\perp}^0\frac{\eta}{\alpha c}(y_{1\beta} + y_{2\beta})(\dot{\tau}_1 + \dot{\tau}_2)\,\}. \tag{14}$$

The factor $1/2$ in front of $W_{\parallel}^0$ and $W_{\perp}^0$ in Eqs.(13) and (14) comes from the fundamental theorem of beam loading.

Our final goal is to derive an eigenvalue equation for coherent frequencies from Eqs.(11)-(14). For that purpose, we linearize Eqs.(11)-(14) by omitting all the nonlinear terms except for the second term in Eq.(12) which gives the usual transverse tune shift in the absence of C.O.D. and dispersion. In order to make this term linear with respect to $y_{1\beta} - \frac{\eta}{\alpha c}\dot{\tau}_1$, we replace the term $\tau_1 - \tau_2$ by the time average over the period $0 < t < t_0$:

$$(\tau_1 - \tau_2)(y_{1\beta} - \frac{\eta}{\alpha c}\dot{\tau}_1) \to \frac{1}{t_0}\int_0^{t_0}(\tau_1 - \tau_2)dt \cdot (y_{1\beta} - \frac{\eta}{\alpha c}\dot{\tau}_1). \tag{15}$$

This is not so unreasonable, because during the time $0 < t < t_0$, the quantity $\tau_1 - \tau_2$ does not change sign (particle 1 always proceeds particle 2). In addition, the formalism becomes equivalent to the usual two-particle model in the absence of C.O.D. and the dispersion[9,10], where the transverse wake potential is assumed to be constant. However, the relationship(7) between the transverse and the longitudinal wake potentials no longer holds exactly.

As usual, we eliminate the beam loading terms $W_{\parallel}^0/2$ and $W_{\perp}^0 y_{co}^2/2$, assuming that they are compensated by the manipulation of rf voltage in such a way that the resulting "renormalized" synchrotron frequency becomes ω_s. Added to that, we neglect the coordinate- independent longitudinal wake potential terms $W_{\parallel}^0$ and $W_{\perp}^0 y_{co}^2$ in Eq.(14). These terms are the source of the usual longitudinal tune shift which, however, we are not interested in in the present study.

The remaining terms in Eq.(13) and (14), i.e., $W_{\perp}^0 y_{co}(y_{i\beta} - \frac{\eta}{\alpha c}\dot{\tau}_i)$ $(i = 1, 2)$ should be treated with more attention, for their momentum dependence might change the stability of a beam. However, since we are looking for the additional real tune shift, we omit these terms from the analysis for the time being. We will come back to this problem later in Sec.4, and reformulate the longitudinal instability, including these terms, for a more precise discussion.

Now we have a new set of equations of motion, which we actually solve:

$$\frac{d^2}{dt^2}y_{1\beta} + \omega_\beta^2 y_{1\beta} = 0, \tag{16}$$

$$\frac{d^2}{dt^2}y_{2\beta} + \omega_\beta^2 y_{2\beta} = K y_{co}(\tau_1 - \tau_2) + K\bar{\Delta\tau}(y_{1\beta} - \frac{\eta}{\alpha c}\dot{\tau}_1), \tag{17}$$

$$\frac{d^2}{dt^2}\tau_1 + \omega_s^2 \tau_1 = 0, \tag{18}$$

$$\frac{d^2}{dt^2}\tau_2 + \omega_s^2 \tau_2 = K y_{co}\alpha(y_{1\beta} + y_{2\beta} - \frac{\eta}{\alpha c}(\dot{\tau}_1 - \dot{\tau}_2)). \tag{19}$$

where we use the abbreviations

$$K = \frac{Ne^2c^2}{2E}W_{\perp}^0, \tag{20}$$

and

$$\bar{\Delta\tau} = \frac{1}{t_0}\int_0^{t_0}(\tau_1 - \tau_2)dt. \tag{21}$$

3 EIGENVALUE EQUATION

We describe the solution for Eqs.(16)-(19) in terms of the phasors, instead of the coordinate and the momentum, which make the structure of the final eigenvalue matrix more regular. They are defined by

$$\hat{y}_{j\beta} = y_{j\beta} + \frac{i}{\omega_\beta}\frac{dy_{j\beta}}{dt}, \tag{22}$$

$$\tilde{y}_{j\beta}^{*} = y_{j\beta} - \frac{i}{\omega_\beta}\frac{dy_{j\beta}}{dt}, \tag{23}$$

$$\tilde{\tau}_j = \tau_j + \frac{i}{\omega_s}\frac{d\tau_j}{dt}, \tag{24}$$

$$\tilde{\tau}_j^{*} = \tau_j - \frac{i}{\omega_s}\frac{d\tau_j}{dt}. \qquad (j = 1, 2) \tag{25}$$

The solutions for $y_{1\beta}$ and τ_1 are simply free betatron and synchrotron oscillations:

$$\tilde{y}_{1\beta} = \tilde{y}_{1\beta}(0)\, e^{-i\omega_\beta t}, \tag{26}$$

$$\tilde{y}_{1\beta}^{*} = \tilde{y}_{1\beta}^{*}(0)\, e^{i\omega_\beta t}, \tag{27}$$

$$\tilde{\tau}_1 = \tilde{\tau}_1(0)\, e^{-i\omega_s t}, \tag{28}$$

$$\tilde{\tau}_1^{*} = \tilde{\tau}_1^{*}(0)\, e^{i\omega_s t}. \tag{29}$$

Equations(17) and (19) can be written in terms of phasors as follows:

$$\begin{aligned} \frac{\omega_\beta}{2i}(\dot{\tilde{y}}_{2\beta} - \dot{\tilde{y}}_{2\beta}^{*}) &+ \frac{\omega_\beta^2}{2}(\tilde{y}_{2\beta} + \tilde{y}_{2\beta}^{*}) + \frac{K y_{co}}{2}(\tilde{\tau}_2 + \tilde{\tau}_2^{*}) \\ &= \frac{K y_{co}}{2}(\tilde{\tau}_1 + \tilde{\tau}_1^{*}) + K\bar{\Delta\tau}\left[\,\frac{1}{2}(\tilde{y}_{1\beta} + \tilde{y}_{1\beta}^{*}) - \frac{\omega_s}{2i}\frac{\eta}{\alpha c}(\tilde{\tau}_1 - \tilde{\tau}_1^{*})\,\right], \end{aligned} \tag{30}$$

$$\begin{aligned} \frac{\omega_s}{2i}(\dot{\tilde{\tau}}_2 - \dot{\tilde{\tau}}_2^{*}) &+ \frac{\omega_s^2}{2}(\tilde{\tau}_2 + \tilde{\tau}_2^{*}) - \frac{K y_{co}\alpha}{2}(\tilde{y}_{2\beta} + \tilde{y}_{2\beta}^{*}) + \frac{K y_{co}\eta\omega_s}{2ic}(\tilde{\tau}_2 - \tilde{\tau}_2^{*}) \\ &= K y_{co}\alpha\left[\,\frac{1}{2}(\tilde{y}_{1\beta} + \tilde{y}_{1\beta}^{*}) - \frac{\omega_s}{2i}\frac{\eta}{\alpha c}(\tilde{\tau}_1 - \tilde{\tau}_1^{*})\,\right]. \end{aligned} \tag{31}$$

Besides, we have two supplementary equations which impose the relationship between the phasors:

$$\dot{\tilde{y}}_{2\beta} + \dot{\tilde{y}}_{2\beta}^{*} - \frac{\omega_\beta}{i}(\tilde{y}_{2\beta} - \tilde{y}_{2\beta}^{*}) = 0, \tag{32}$$

$$\dot{\tilde{\tau}}_2 + \dot{\tilde{\tau}}_2^{*} - \frac{\omega_s}{i}(\tilde{\tau}_2 - \tilde{\tau}_2^{*}) = 0. \tag{33}$$

Their matrix form is given in Appendix A for the convenience of the further explanation.

The first step is to solve the homogeneous equations of Eqs.(30)-(33). The four eigenfrequencies ω_j $(j = 1, 4)$ for the vector $\vec{x} = (\tilde{y}_{2\beta}, \tilde{y}_{2\beta}^{*}, \tilde{\tau}_2, \tilde{\tau}_2^{*})^T$ are determined by the condition that the determinant of the matrix $\mathbf{D}$ is zero:

$$\det\mathbf{D}(\omega_j) = -\omega_\beta\omega_s\{\,(\omega_\beta^2 - \omega_j^2)[\,\omega_s^2 - \omega_j^2 - i\omega_j K y_{co}\frac{\eta}{c}\,] + K^2 y_{co}^2\alpha\,\} = 0. \tag{34}$$

From the structure of this equation, we can say that there are only two independent solutions, which we call ω_1 and ω_2. The other solutions are given by the negative of the complex conjugate of these two solutions: so the four solutions are actually written as

$$\omega_1, \quad -\omega_1^{*}, \quad \omega_2, \quad -\omega_2^{*}.$$

We use this property implicitly in the formulation.

Then a solution of the homogeneous equations can be written as

$$\vec{x} = \sum_{j=1}^{4} \vec{a}_j \, e^{-i\omega_j t}. \tag{35}$$

where the vector $\vec{a}_j$ has the form

$$\vec{a}_j = a_j \times \left(\omega_\beta + \omega_j, \omega_\beta - \omega_j, \frac{\omega_\beta(\omega_j^2 - \omega_\beta^2)(\omega_s + \omega_j)}{K y_{co}^2 \omega_s}, \frac{\omega_\beta(\omega_j^2 - \omega_\beta^2)(\omega_s - \omega_j)}{K y_{co}^2 \omega_s}\right)^T. \tag{36}$$

The general solution to the complete differential eqs. is obtained by summing the contribution from the inhomogeneous parts:

$$\begin{aligned} \vec{x} &= \sum_j \vec{a}_j \, e^{-i\omega_j t} + \mathbf{D}^{-1}(-i\omega_\beta)\, \vec{b}\, \tilde{y}_{1\beta} + \mathbf{D}^{-1}(i\omega_\beta)\, \vec{b}\, \tilde{y}_{1\beta}^* \\ & \quad + \mathbf{D}^{-1}(-i\omega_s)\, \vec{c}\, \tilde{\tau}_1 + \mathbf{D}^{-1}(i\omega_s)\, \vec{d}\, \tilde{\tau}_1^*. \end{aligned} \tag{37}$$

The definitions of the vectors $\vec{b}$, $\vec{c}$, and $\vec{d}$ are given in Appendix A. The explicit expressions for the inhomogeneous parts are given in Appendix B.

The coefficients a_j in Eq.(37) are determined by the initial conditions at $t = 0$: $\tilde{y}_{2\beta\,t=0} = \tilde{y}_{2\beta}(0)$, etc. The result is summarized in Appendix C. Substituting those coefficients into Eq.(37), we finally obtain the complete solution to Eqs.(16)-(19) in matrix form:

$$\begin{pmatrix} \tilde{y}_{1\beta} \\ \tilde{y}_{1\beta}^* \\ \tilde{y}_{2\beta} \\ \tilde{y}_{2\beta}^* \\ \tilde{\tau}_1 \\ \tilde{\tau}_1^* \\ \tilde{\tau}_2 \\ \tilde{\tau}_2^* \end{pmatrix} = \underbrace{\begin{pmatrix} e^{-i\omega_\beta t} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{i\omega_\beta t} & 0 & 0 & 0 & 0 & 0 & 0 \\ M_{31} & M_{41}^* & M_{33} & M_{43}^* & M_{35} & M_{45}^* & M_{37} & M_{47}^* \\ M_{41} & M_{31}^* & M_{43} & M_{33}^* & M_{45} & M_{35}^* & M_{47} & M_{37}^* \\ 0 & 0 & 0 & 0 & e^{-i\omega_s t} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{i\omega_s t} & 0 & 0 \\ M_{71} & M_{81}^* & M_{73} & M_{83}^* & M_{75} & M_{85}^* & M_{77} & M_{87}^* \\ M_{81} & M_{71}^* & M_{83} & M_{73}^* & M_{85} & M_{75}^* & M_{87} & M_{77}^* \end{pmatrix}}_{\mathbf{M}_1} \begin{pmatrix} \tilde{y}_{1\beta}(0) \\ \tilde{y}_{1\beta}^*(0) \\ \tilde{y}_{2\beta}(0) \\ \tilde{y}_{2\beta}^*(0) \\ \tilde{\tau}_1(0) \\ \tilde{\tau}_1^*(0) \\ \tilde{\tau}_2(0) \\ \tilde{\tau}_2^*(0) \end{pmatrix}. \tag{38}$$

All the elements are summarized in Appendix D.

During the rest of the synchrotron oscillation period, $t_0 < t < 2t_0$, in which paricle 2 leads particle 1, the transformation matrix is given by

$$\begin{pmatrix} \tilde{y}_{1\beta} \\ \tilde{y}_{1\beta}^* \\ \tilde{y}_{2\beta} \\ \tilde{y}_{2\beta}^* \\ \tilde{\tau}_1 \\ \tilde{\tau}_1^* \\ \tilde{\tau}_2 \\ \tilde{\tau}_2^* \end{pmatrix} = \underbrace{\begin{pmatrix} M_{33} & M_{43}^* & M_{31} & M_{41}^* & M_{37} & M_{47}^* & M_{35} & M_{45}^* \\ M_{43} & M_{33}^* & M_{41} & M_{31}^* & M_{47} & M_{37}^* & M_{45} & M_{35}^* \\ 0 & 0 & e^{-i\omega_\beta(t-t_0)} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{i\omega_\beta(t-t_0)} & 0 & 0 & 0 & 0 \\ M_{73} & M_{83}^* & M_{71} & M_{81}^* & M_{77} & M_{87}^* & M_{75} & M_{85}^* \\ M_{83} & M_{73}^* & M_{81} & M_{71}^* & M_{87} & M_{77}^* & M_{85} & M_{75}^* \\ 0 & 0 & 0 & 0 & 0 & 0 & e^{-i\omega_s(t-t_0)} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{i\omega_s(t-t_0)} \end{pmatrix}}_{\mathbf{M}_2} \begin{pmatrix} \tilde{y}_{1\beta}(0) \\ \tilde{y}_{1\beta}^*(0) \\ \tilde{y}_{2\beta}(0) \\ \tilde{y}_{2\beta}^*(0) \\ \tilde{\tau}_1(0) \\ \tilde{\tau}_1^*(0) \\ \tilde{\tau}_2(0) \\ \tilde{\tau}_2^*(0) \end{pmatrix}, \tag{39}$$

where $M_{ij} = M_{ij}(t - t_0)$.

The total transformation matrix for one full synchrotron period is the product of two matrixes:

$$\begin{pmatrix} \tilde{y}_{1\beta} \\ \tilde{y}_{1\beta}' \\ \tilde{y}_{2\beta} \\ \tilde{y}_{2\beta}' \\ \tilde{\tau}_1 \\ \tilde{\tau}_1' \\ \tilde{\tau}_2 \\ \tilde{\tau}_2' \end{pmatrix} = \mathbf{M}_2(t_0) \times \mathbf{M}_1(t_0) \times \begin{pmatrix} \tilde{y}_{1\beta}(0) \\ \tilde{y}_{1\beta}'(0) \\ \tilde{y}_{2\beta}(0) \\ \tilde{y}_{2\beta}'(0) \\ \tilde{\tau}_1(0) \\ \tilde{\tau}_1'(0) \\ \tilde{\tau}_2(0) \\ \tilde{\tau}_2'(0) \end{pmatrix}. \tag{40}$$

The stability of the system is determined by the eigenvalue of the matrix $\mathbf{M} = \mathbf{M}_2 \times \mathbf{M}_1$.

We do not know yet how long one full synchrotron period $2t_0$ lasts, because this is one of the eigenvalues of the matrix which we are solving. Note that the period is a coherent synchrotron oscillation period. not an incoherent one. However, since a small coherent tune shift is expected in our case. we simply take the incoherent synchrotron period T_s $(= \frac{2\pi}{\omega_s})$ as $2t_0$. This approximation would be justified if the resulting eigenvalues for the coherent synchrotron frequency are not far from the incoherent ones.

4 APPLICATION TO PETRA

We have applied the formalism to PETRA. and have calculated the vertical coherent frequency shifts as a function of C.O.D. and dispersion. The vertical C.O.D. y_{co}, and the vertical dispersion η in PETRA are correlated with each other by the relation[12]

$$\sqrt{\langle\eta^2\rangle} = 140 \cdot \sqrt{\langle y_{co}^2\rangle}. \tag{41}$$

The results of the calculations are summarized in Table 1. The parameters used are listed in Table 2. The wake potential $W_\perp^0$ is determined in such a way that the vertical frequency shift is equal to 250 Hz at $y_{co} = \eta = 0$. The bunch length σ_τ is the amplitude of particles in the longitudinal phase space. Looking at the eigenvector for each eigenvalue, we find that the first 4 eigenmodes include many more transverse components than the later ones, therefore they should be observed experimentally as transverse oscillations. and as longitudinal oscillations for the last 4 eigenmodes.

From Table 2. we can see no significant change in the real tune shift. while some eigenmodes start to have non- negligible damping rates when the C.O.D. and the dispersion are considerably large. To be sure. this damping was anticipated from Kohaupt's analysis, although its rate is half the value obtaind according his formula[7]. The discrepancy probably comes from the fact that we omitted the self-induced terms $W_\perp^0 y_{co}(y_{i\beta} - \frac{\eta}{\alpha c}\dot{\tau}_i)$ out of the eqations of motion. After all, it is numerically found that the real part of the transverse coherent frequency is hardly influenced by the coupling with the longitudinal motion. We thus turn our attention to the detailed examination of the damping solutions.

5 RECONSIDERATION OF INSTABILITY

The source of damping is the momentum dependent terms on the right hand side of Eq.(19), namely, the terms which give the momentum dependence to the wake field kick via the dispersion. In Sec.2, we left the self-induced terms $W'_{-} y_{co}(y_{i\beta} - \frac{\eta}{\alpha c}\dot{\tau}_i)$ out of the equations of motion. which obviously contribute to the stability of the oscillations. Here we restore these terms. and solve the equations of motion again. However. since little effect is expected from the coupling to the transverse motion. we limit the motion only to the longitudinal one.

During the time $0 < t < T_s/2$ when particle 1 proceeds particle 2, the equations of motion are

$$
\begin{aligned}
\frac{d^2}{dt^2}\tau_1 + \omega_s^2\tau_1 &= -K y_{co}\frac{\eta}{c}\dot{\tau}_1, &(42)\\
\frac{d^2}{dt^2}\tau_2 + \omega_s^2\tau_2 &= -K y_{co}\frac{\eta}{c}\dot{\tau}_2 - K y_{co}\frac{\eta}{c}(\dot{\tau}_1 + \dot{\tau}_2). &(43)
\end{aligned}
$$

As in Sec. 2, we write down Eqs.(42) and (43) in terms of the phasors:

$$
\begin{aligned}
\frac{\omega_s}{2i}(\dot{\tilde{\tau}}_1 - \dot{\tilde{\tau}}_1^*) &+ \frac{\omega_s^2}{2}(\tilde{\tau}_1 + \tilde{\tau}_1^*) + \frac{K y_{co}}{2}\frac{\eta}{c}\frac{\omega_s}{i}(\tilde{\tau}_1 - \tilde{\tau}_1^*) = 0, &(44)\\
(\dot{\tilde{\tau}}_1 + \dot{\tilde{\tau}}_1^*) &- \frac{\omega_s}{i}(\tilde{\tau}_1 - \tilde{\tau}_1^*) = 0. &(45)\\
\frac{\omega_s}{2i}(\dot{\tilde{\tau}}_2 - \dot{\tilde{\tau}}_2^*) &+ \frac{\omega_s^2}{2}(\tilde{\tau}_2 + \tilde{\tau}_2^*) + K y_{co}\frac{\eta}{c}\frac{\omega_s}{i}(\tilde{\tau}_2 - \tilde{\tau}_2^*) = -\frac{K y_{co}}{2}\frac{\eta}{c}\frac{\omega_s}{i}(\tilde{\tau}_1 - \tilde{\tau}_1^*), &(46)\\
(\dot{\tilde{\tau}}_2 + \dot{\tilde{\tau}}_2^*) &- \frac{\omega_s}{i}(\tilde{\tau}_2 - \tilde{\tau}_2^*) = 0. &(47)
\end{aligned}
$$

Assuming the exponential time dependence $\tilde{\tau}_1 = \tilde{\tau}_1(0)\, e^{\lambda t}$, the solution $\lambda_\pm$ for the differential equations (44) and (45) are

$$
\lambda_\pm = -\frac{K y_{co}\eta}{2c} \pm \sqrt{(\frac{K y_{co}\eta}{2c})^2 - \omega_s^2}. \tag{48}
$$

The solutions $\omega_\pm$ to the inhomogeneous equations (46) and (47) are

$$
\omega_\pm = -\frac{K y_{co}\eta}{c} \pm \sqrt{(\frac{K y_{co}\eta}{c})^2 - \omega_s^2}. \tag{49}
$$

Assuming that y_{co} and η are small, we retain only the first-order terms with respect to $y_{co}\,\eta$ in $\lambda_\pm$ and $\omega_\pm$:

$$
\begin{aligned}
\lambda_\pm &= -\delta \pm i\omega_s, &(50)\\
\omega_\pm &= -2\delta \pm i\omega_s, &(51)
\end{aligned}
$$

where

$$
\delta = \frac{K y_{co}\eta}{2c}. \tag{52}
$$

In the further procedure, we retain only the first-order terms of δ.

The general solution for $\tilde{\tau}_2$ and $\tilde{\tau}_2^*$ are given by

$$
\begin{aligned}
\tilde{\tau}_2 &= a e^{\omega_- t} + b e^{\omega_+} - (1 + \frac{i\delta}{2\omega_s})\tilde{\tau}_1(0)e^{\lambda_- t} + \frac{i\delta}{2\omega_s}\tilde{\tau}_1^*(0)e^{\lambda_+ t}, &(53)\\
\tilde{\tau}_2^* &= \frac{i\delta}{\omega_s}a e^{\omega_- t} - (\frac{\omega_s}{i\delta} + 1)b e^{\omega_+ t} - \frac{i\delta}{2\omega_s}\tilde{\tau}_1(0)e^{\lambda_- t} - (1 - \frac{i\delta}{2\omega_s})\tilde{\tau}_1^*(0)e^{\lambda_+ t}. &(54)
\end{aligned}
$$

From the initial conditions $\tilde{\tau}_2 = \tilde{\tau}_2(0)$ and $\tilde{\tau}_2^* = \tilde{\tau}_2^*(0)$ at $t = 0$, the coefficients a and b are determined as

$$a = (1 + \frac{i\delta}{2\omega_s})\tilde{\tau}_1(0) - \frac{i\delta}{2\omega_s}\tilde{\tau}_1^*(0) + \tilde{\tau}_2(0) + \frac{i\delta}{\omega_s}\tilde{\tau}_2^*(0), \quad (55)$$

$$b = \frac{1}{-(\frac{\omega_s}{i\delta}+1)}[-\frac{i\delta}{2\omega_s}\tilde{\tau}_1(0) + (1 - \frac{i\delta}{2\omega_s})\tilde{\tau}_1^*(0) - \frac{i\delta}{\omega_s}\tilde{\tau}_2(0) + \tilde{\tau}_2^*(0)]. \quad (56)$$

Substituting these coefficients into Eqs.(53) and (54), and expanding the exponential terms by polynomials, and taking only the first-order terms:

$$e^{\lambda\pm t} = e^{\pm i\omega_s t}(1 - \delta t), \quad (57)$$

$$e^{\omega\pm t} = e^{\pm i\omega_s t}(1 - 2\delta t), \quad (58)$$

we finally get the solution for $\tilde{\tau}_2$ and $\tilde{\tau}_2^*$

$$\begin{aligned}\tilde{\tau}_2 &= -\delta t e^{-i\omega_s t}\tilde{\tau}_1(0) + \frac{\delta}{\omega_s}\sin\omega_s t\tilde{\tau}_1^*(0)\\ &+ (1 - 2\delta t)e^{-i\omega_s t}\tilde{\tau}_2(0) + \frac{2\delta}{\omega_s}\sin\omega_s t\tilde{\tau}_2^*(0).\end{aligned} \quad (59)$$

$$\begin{aligned}\tilde{\tau}_2^* &= -\delta t e^{i\omega_s t}\tilde{\tau}_1^*(0) + \frac{\delta}{\omega_s}\sin\omega_s t\tilde{\tau}_1(0)\\ &+ (1 - 2\delta t)e^{i\omega_s t}\tilde{\tau}_2^*(0) + \frac{2\delta}{\omega_s}\sin\omega_s t\tilde{\tau}_2(0).\end{aligned} \quad (60)$$

Written in the matix form together with the solutions for $\tilde{\tau}_1$ and $\tilde{\tau}_1^*$, we have

$$\begin{pmatrix}\tilde{\tau}_1\\ \tilde{\tau}_1^*\\ \tilde{\tau}_2\\ \tilde{\tau}_2^*\end{pmatrix} = \begin{pmatrix}(1-\delta t)e^{-i\omega_s t} & 0 & 0 & 0\\ 0 & (1-\delta t)e^{i\omega_s t} & 0 & 0\\ -\delta t e^{-\omega_s t} & \frac{\delta}{\omega_s}\sin\omega_s t & (1-2\delta t)e^{-i\omega_s t} & \frac{2\delta}{\omega_s}\sin\omega_s t\\ \frac{\delta}{\omega_s}\sin\omega_s t & -\delta t e^{i\omega_s t} & \frac{2\delta}{\omega_s}\sin\omega_s t & (1-2\delta t)e^{i\omega_s t}\end{pmatrix}\begin{pmatrix}\tilde{\tau}_1(0)\\ \tilde{\tau}_1^*(0)\\ \tilde{\tau}_2(0)\\ \tilde{\tau}_2^*(0)\end{pmatrix}. \quad (61)$$

At the time $t = T_s/2$, the matrix becomes drastically simple as

$$\begin{pmatrix}\tilde{\tau}_1\\ \tilde{\tau}_1^*\\ \tilde{\tau}_2\\ \tilde{\tau}_2^*\end{pmatrix} = \underbrace{\begin{pmatrix}-(1-\frac{\delta T_s}{2}) & 0 & 0 & 0\\ 0 & -(1-\frac{\delta T_s}{2}) & 0 & 0\\ \frac{\delta T_s}{2} & 0 & -(1-\delta T_s) & 0\\ 0 & \frac{\delta T_s}{2} & 0 & -(1-\delta T_s)\end{pmatrix}}_{\mathbf{L}_1}\begin{pmatrix}\tilde{\tau}_1(0)\\ \tilde{\tau}_1^*(0)\\ \tilde{\tau}_2(0)\\ \tilde{\tau}_2^*(0)\end{pmatrix}. \quad (62)$$

The transfer matrix for another half synchrotron period, in which particle 2 proceeds particle 1. is obtained by interchanging the index 1 and 2. At $t = T_s$, we have

$$\begin{pmatrix} \tilde{r}_1 \\ \tilde{r}_1' \\ \tilde{r}_2 \\ \tilde{r}_2' \end{pmatrix} = \underbrace{\begin{pmatrix} -(1-\delta T_s) & 0 & \frac{\delta T_s}{2} & 0 \\ 0 & -(1-\delta T_s) & 0 & \frac{\delta T_s}{2} \\ 0 & 0 & -(1-\frac{\delta T_s}{2}) & 0 \\ 0 & 0 & 0 & -(1-\frac{\delta T_s}{2}) \end{pmatrix}}_{\mathbf{L}_2} \begin{pmatrix} \tilde{r}_1(0) \\ \tilde{r}_1'(0) \\ \tilde{r}_2(0) \\ \tilde{r}_2'(0) \end{pmatrix}. \tag{63}$$

The total transfer matrix for one full synchrotron period is then given by the product of $\mathbf{L}_2 \times \mathbf{L}_1 = \mathbf{L}$

$$\mathbf{L} = \begin{pmatrix} 1-\frac{3}{2}\delta T_s & 0 & -\frac{\delta T_s}{2} & 0 \\ 0 & 1-\frac{3}{2}\delta T_s & 0 & 0 \\ -\frac{\delta T_s}{2} & 0 & 1-\frac{3}{2}\delta T_s & 0 \\ 0 & -\frac{\delta T_s}{2} & 0 & 1-\frac{3}{2}\delta T_s \end{pmatrix}. \tag{64}$$

The eigenvalues of this matrix are

$$\lambda = 1 - 2\delta T_s, \tag{65}$$

$$1 - \delta T_s, \tag{66}$$

which correspond to the following four mode frequencies:

$$\omega_1 = \omega_s - i\frac{K y_{co}\eta}{c}, \tag{67}$$

$$\omega_2 = -\omega_s - i\frac{K y_{co}\eta}{c}, \tag{68}$$

$$\omega_3 = \omega_s - i\frac{K y_{co}\eta}{2c}, \tag{69}$$

$$\omega_4 = -\omega_s - i\frac{K y_{co}\eta}{2c}. \tag{70}$$

The eigenvector for each eigenvalue (67)-(70) is, respectively,

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \mathbf{v}_4 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}. \tag{71}$$

In the first two modes, the two particles move together in the same phase, and the last two modes show the out-of-phase motion. The damping rate of the in-phase motion, which is only motion allowed in the rigid bunch model, now agrees with that of Kohaupt's formula.

If we again neglect the self-induced terms, and repeat the algebra, mode frequencies are changed to the following:

$$\omega_1 = \omega_s - i\frac{K y_{co}\eta}{2c}, \tag{72}$$

$$
\begin{aligned}
\omega_2 &= -\omega_s - i\frac{K y_{co}\eta}{2c}, &(73)\\
\omega_3 &= \omega_s, &(74)\\
\omega_4 &= -\omega_s. &(75)
\end{aligned}
$$

The first two modes correspond to the in-phase motion. The reason why the out-of-phase motions are stationary is obvious on inspection of Eq.(43). If $\dot{\tau}_1 = -\dot{\tau}_2$, the right hand side of Eq.(43) vanishes, so that there is no wake field effect left. Besides, it becomes more clear that the self-induced terms attribute to half of the damping rate in the two-particle model.

6 CONCLUSIONS

In order to explain the large discrepancy between the measured coherent tune shifts and analytical ones in a short bunch in PETRA, we have studied the effects of the closed orbit distorsion y_{co} and the dispersion η on a beam instability, with a two-particle model. We obtained the result which supports Kohaupt's previous results: they hardly contribute to real tune shift, while the momentum dependence of the wake force can make a beam unstable, with the growth rate which is proportional to the product of y_{co} and η. The numerical examples for constant y_{co} and η over the PETRA ring shows that the growth rate can be large enough to be observed. However in the real machine, since y_{co} and η change sign frequently in the ring, the effect might be very weak.

Kohaupt's model[7] says that the damping (antidamping) of transverse oscillation should happen in conjunction with the longitudinal instability (stability), with a damping rate of the same strength, but opposite sign. In the present model, we do not have such a counter effect in the transverse oscillation. The reason is that the Panofsky-Wenzel relation(7) between the transverse and the longitudinal wake potentials was no longer fulfilled after linearizing the nonlinear term $(\tau_1 - \tau_2)(y_{1\beta} - \frac{\eta}{\alpha c}\dot{\tau}_1)$ in Eq.(12).

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APPENDIX A

Matrix form of equations of motion

$$
\begin{pmatrix}
D - \frac{\omega_\beta}{i} & D + \frac{\omega_\beta}{i} & 0 & 0 \\
\frac{\omega_\beta}{2i}D + \frac{\omega_\beta^2}{2} & -\frac{\omega_\beta}{2i}D + \frac{\omega_\beta^2}{2} & \frac{K y_{co}}{2} & \frac{K y_{co}}{2} \\
0 & 0 & D - \frac{\omega_s}{i} & D + \frac{\omega_s}{i} \\
-\frac{K y_{co}\alpha}{2} & -\frac{K y_{co}\alpha}{2} & \frac{\omega_s}{2i}D + \frac{\omega_s^2}{2} + K y_{co}\frac{\eta\omega_s}{c2i} & -\frac{\omega_s}{2i}D + \frac{\omega_s^2}{2} - K y_{co}\frac{\eta\omega_s}{c2i}
\end{pmatrix}
\begin{pmatrix}
\tilde{y}_{2\beta} \\ \dot{\tilde{y}}_{2\beta} \\ \tilde{\tau}_2 \\ \dot{\tilde{\tau}}_2
\end{pmatrix}
$$

$$= \underbrace{\begin{pmatrix} 0 \\ \frac{K\bar{\Delta\tau}}{2} \\ 0 \\ \frac{K y_{co}\alpha}{2} \end{pmatrix}}_{\bar{b}} \tilde{y}_{1\beta} + \underbrace{\begin{pmatrix} 0 \\ \frac{K\bar{\Delta\tau}}{2} \\ 0 \\ \frac{K y_{co}\alpha}{2} \end{pmatrix}}_{\bar{b}} \tilde{y}^*_{1\beta} - \underbrace{\begin{pmatrix} 0 \\ \frac{K y_{co}}{2} - \frac{K\bar{\Delta\tau}\eta}{\alpha c}\frac{\omega_s}{2i} \\ 0 \\ -\frac{K y_{co}\eta\omega_s}{2ic} \end{pmatrix}}_{\bar{c}} \tilde{\tau}_1 + \underbrace{\begin{pmatrix} 0 \\ \frac{K y_{co}}{2} + \frac{K\bar{\Delta\tau}\eta}{\alpha c}\frac{\omega_s}{2i} \\ 0 \\ \frac{K y_{co}\eta\omega_s}{2ic} \end{pmatrix}}_{\bar{d}} \tilde{\tau}^*_1, \tag{76}$$

where

$$D = \frac{d}{dt}. \tag{77}$$

APPENDIX B

The contributions from inhomogeneous parts

$$\tilde{y}_{2\beta}\ \text{inhomo} = T_1(\omega_\beta,\omega_s)\tilde{y}_{1\beta} + L_1(\omega_\beta,\omega_s)\tilde{\tau}_1 + L_1(\omega_\beta,-\omega_s)\tilde{\tau}^*_1, \tag{78}$$

$$\tilde{y}^*_{2\beta}\ \text{inhomo} = T^*_1(\omega_\beta,\omega_s)\tilde{y}^*_{1\beta} + L^*_1(\omega_\beta,-\omega_s)\tilde{\tau}_1 + L^*_1(\omega_\beta,\omega_s)\tilde{\tau}^*_1, \tag{79}$$

$$\tilde{\tau}_2\ \text{inhomo} = T_2(\omega_\beta,\omega_s)\tilde{y}_{1\beta} + T_2(-\omega_\beta,\omega_s)\tilde{y}^*_{1\beta} - L_2(\omega_\beta,\omega_s)\tilde{\tau}_1, \tag{80}$$

$$\tilde{\tau}^*_2\ \text{inhomo} = T^*_2(-\omega_\beta,\omega_s)\tilde{y}_{1\beta} + T^*_2(\omega_\beta,\omega_s)\tilde{y}^*_{1\beta} + L^*_2(\omega_\beta,\omega_s)\tilde{\tau}^*_1, \tag{81}$$

where

$$T_1(\omega_\beta,\omega_s) = \frac{\bar{\Delta\tau}}{K y_{co}^2\alpha}\left(\omega_s^2 - \omega_\beta^2 - i\omega_\beta K y_{co}\frac{\eta}{c}\right) - 1, \tag{82}$$

$$L_1(\omega_\beta,\omega_s) = -\frac{iK\omega_s(\omega_\beta+\omega_s)\eta\left(y_{co} - \frac{\bar{\Delta\tau}\eta\omega_s}{2\alpha ci}\right)}{\omega_\beta\left(K y_{co} - i(\omega_\beta^2-\omega_s^2)\omega_s\frac{\eta}{\alpha c}\right)\alpha c}, \tag{83}$$

$$T_2(\omega_\beta,\omega_s) = \frac{(\omega_\beta+\omega_s)\bar{\Delta\tau}}{2\omega_s y_{co}}, \tag{84}$$

$$L_2(\omega_\beta,\omega_s) = -1 + 2K\frac{\left(y_{co} - \frac{\bar{\Delta\tau}\eta\omega_s}{2\alpha ci}\right)}{K y_{co} - i(\omega_\beta^2-\omega_s^2)\omega_s\frac{\eta}{\alpha c}}. \tag{85}$$

APPENDIX C

Coefficients a_j

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} -H_T(\omega_\beta,\omega_s,\omega_1) & -H^*_T(\omega_\beta,\omega_s,-\omega^*_1) & F_s(\omega_\beta,\omega_s,\omega_1,\omega_2) & F^*_s(\omega_\beta,\omega_s,-\omega^*_1,\omega_2) \\ -H_T(\omega_\beta,\omega_s,-\omega^*_1) & -H^*_T(\omega_\beta,\omega_s,\omega_1) & F_s(\omega_\beta,\omega_s,-\omega^*_1,\omega_2) & F^*_s(\omega_\beta,\omega_s,\omega_1,\omega_2) \\ -H_T(\omega_\beta,\omega_s,\omega_2) & -H^*_T(\omega_\beta,\omega_s,-\omega^*_2) & F_s(\omega_\beta,\omega_s,\omega_2,\omega_1) & F^*_s(\omega_\beta,\omega_s,-\omega^*_2,\omega_1) \\ -H_T(\omega_\beta,\omega_s,-\omega^*_2) & -H^*_T(\omega_\beta,\omega_s,\omega_2) & F_s(\omega_\beta,\omega_s,-\omega^*_2,\omega_1) & F^*_s(\omega_\beta,\omega_s,\omega_2,\omega_1) \end{pmatrix}$$

$$
\begin{pmatrix}
-H_L(\omega_\beta,\omega_s,\omega_1) & -H_L^*(\omega_\beta,\omega_s,-\omega_1^*) & G_s(\omega_\beta,\omega_s,\omega_1,\omega_2) & G_s^*(\omega_\beta,\omega_s,-\omega_1^*,\omega_2) \\
-H_L(\omega_\beta,\omega_s,-\omega_1^*) & -H_L^*(\omega_\beta,\omega_s,\omega_1) & G_s(\omega_\beta,\omega_s,-\omega_1^*,\omega_2) & G_s^*(\omega_\beta,\omega_s,\omega_1,\omega_2) \\
-H_L(\omega_\beta,\omega_s,\omega_2) & -H_L^*(\omega_\beta,\omega_s,-\omega_2^*) & G_s(\omega_\beta,\omega_s,\omega_2,\omega_1) & G_s^*(\omega_\beta,\omega_s,-\omega_2^*,\omega_1) \\
-H_L(\omega_\beta,\omega_s,-\omega_2^*) & -H_L^*(\omega_\beta,\omega_s,\omega_2) & G_s(\omega_\beta,\omega_s,-\omega_2^*,\omega_1) & G_s^*(\omega_\beta,\omega_s,\omega_2,\omega_1)
\end{pmatrix}
\begin{pmatrix}
\tilde{y}_{1\beta}(0) \\ \tilde{y}_{1\beta}^*(0) \\ \tilde{y}_{2\beta}(0) \\ \tilde{y}_{2\beta}^*(0) \\ \tilde{\tau}_1(0) \\ \tilde{\tau}_1^*(0) \\ \tilde{\tau}_2(0) \\ \tilde{\tau}_2^*(0)
\end{pmatrix}
\tag{86}
$$

APPENDIX D

Elements of matrix $\mathbf{M}_1$

$$
\begin{aligned}
M_{31}(t) &= -J_{T1}^R(\omega_\beta,\omega_s,t) - J_{T1}^I(\omega_\beta,\omega_s,t) + T_1(\omega_\beta,\omega_s)e^{-i\omega_\beta t}, & (87)\\
M_{41}(t) &= -J_{T1}^R(\omega_\beta,\omega_s,t) + J_{T1}^I(\omega_\beta,\omega_s,t), & (88)\\
M_{33}(t) &= f_1^R(\omega_\beta,\omega_s,t) + f_1^I(\omega_\beta,\omega_s,t), & (89)\\
M_{43}(t) &= f_1^R(\omega_\beta,\omega_s,t) - f_1^I(\omega_\beta,\omega_s,t), & (90)\\
M_{35}(t) &= -J_{L1}^R(\omega_\beta,\omega_s,t) - J_{L1}^I(\omega_\beta,\omega_s,t) + L_1(\omega_\beta,\omega_s)e^{-i\omega_s t}, & (91)\\
M_{45}(t) &= -J_{L1}^R(\omega_\beta,\omega_s,t) + J_{L1}^I(\omega_\beta,\omega_s,t) + L_1(-\omega_\beta,\omega_s)e^{-i\omega_s t}, & (92)\\
M_{37}(t) &= g_1^R(\omega_\beta,\omega_s,t) + g_1^I(\omega_\beta,\omega_s,t), & (93)\\
M_{47}(t) &= g_1^R(\omega_\beta,\omega_s,t) - g_1^I(\omega_\beta,\omega_s,t), & (94)
\end{aligned}
$$

where

$$
\begin{aligned}
J_{T1}^R(\omega_\beta,\omega_s,t) &= \sum_{j=1}^{4}\omega_\beta H_T(\omega_\beta,\omega_s,\omega_j)e^{-i\omega_j t}, & (95)\\
J_{T1}^I(\omega_\beta,\omega_s,t) &= \sum_{j=1}^{4}\omega_j H_T(\omega_\beta,\omega_s,\omega_j)e^{-i\omega_j t}, & (96)\\
J_{L1}^R(\omega_\beta,\omega_s,t) &= \sum_{j=1}^{4}\omega_\beta H_L(\omega_\beta,\omega_s,\omega_j)e^{-i\omega_j t}, & (97)\\
J_{L1}^I(\omega_\beta,\omega_s,t) &= \sum_{j=1}^{4}\omega_j H_L(\omega_\beta,\omega_s,\omega_j)e^{-i\omega_j t}, & (98)\\
f_1^R(\omega_\beta,\omega_s,t) &= \sum_{j=1}^{4}\omega_\beta F(\omega_\beta,\omega_s,\omega_j)e^{-i\omega_j t}, & (99)\\
f_1^I(\omega_\beta,\omega_s,t) &= \sum_{j=1}^{4}\omega_j F(\omega_\beta,\omega_s,\omega_j)e^{-i\omega_j t}, & (100)\\
g_1^R(\omega_\beta,\omega_s,t) &= \sum_{j=1}^{4}\omega_\beta G(\omega_\beta,\omega_s,\omega_j)e^{-i\omega_j t}, & (101)\\
g_1^I(\omega_\beta,\omega_s,t) &= \sum_{j=1}^{4}\omega_j G(\omega_\beta,\omega_s,\omega_j)e^{-i\omega_j t}, & (102)
\end{aligned}
$$

and

$$M_{71}(t) = -J_{T2}^R(\omega_\beta,\omega_s,t) - J_{T2}^I(\omega_\beta,\omega_s,t) - T_2(\omega_\beta,\omega_s)e^{-i\omega_\beta t}, \quad (103)$$
$$M_{81}(t) = -J_{T2}^R(\omega_\beta,\omega_s,t) + J_{T2}^I(\omega_\beta,\omega_s,t) - T_2(\omega_\beta,-\omega_s)e^{-i\omega_\beta t}, \quad (104)$$
$$M_{73}(t) = f_2^R(\omega_\beta,\omega_s,t) + f_2^I(\omega_\beta,\omega_s,t), \quad (105)$$
$$M_{83}(t) = f_2^R(\omega_\beta,\omega_s,t) - f_2^I(\omega_\beta,\omega_s,t), \quad (106)$$
$$M_{75}(t) = -J_{L2}^R(\omega_\beta,\omega_s,t) - J_{L2}^I(\omega_\beta,\omega_s,t) + L_2(\omega_\beta,\omega_s)e^{-i\omega_s t}, \quad (107)$$
$$M_{85}(t) = -J_{L2}^R(\omega_\beta,\omega_s,t) + J_{L2}^I(\omega_\beta,\omega_s,t), \quad (108)$$
$$M_{77}(t) = g_2^R(\omega_\beta,\omega_s,t) + g_2^I(\omega_\beta,\omega_s,t), \quad (109)$$
$$M_{87}(t) = g_2^R(\omega_\beta,\omega_s,t) - g_2^I(\omega_\beta,\omega_s,t), \quad (110)$$

where

$$J_{T2}^R(\omega_\beta,\omega_s,t) = \sum_{j=1}^{4}\frac{\omega_\beta(\omega_j^2-\omega_\beta^2)}{Ky_{co}\omega_s}\omega_s H_T(\omega_\beta,\omega_s,\omega_j)e^{-i\omega_j t}, \quad (111)$$
$$J_{T2}^I(\omega_\beta,\omega_s,t) = \sum_{j=1}^{4}\frac{\omega_\beta(\omega_j^2-\omega_\beta^2)}{Ky_{co}\omega_s}\omega_j H_T(\omega_\beta,\omega_s,\omega_j)e^{-i\omega_j t}, \quad (112)$$
$$J_{L2}^R(\omega_\beta,\omega_s,t) = \sum_{j=1}^{4}\frac{\omega_\beta(\omega_j^2-\omega_\beta^2)}{Ky_{co}\omega_s}\omega_s H_L(\omega_\beta,\omega_s,\omega_j)e^{-i\omega_j t}, \quad (113)$$
$$J_{L2}^I(\omega_\beta,\omega_s,t) = \sum_{j=1}^{4}\frac{\omega_\beta(\omega_j^2-\omega_\beta^2)}{Ky_{co}\omega_s}\omega_j H_L(\omega_\beta,\omega_s,\omega_j)e^{-i\omega_j t}, \quad (114)$$
$$f_2^R(\omega_\beta,\omega_s,t) = \sum_{j=1}^{4}\frac{\omega_\beta(\omega_j^2-\omega_\beta^2)}{Ky_{co}\omega_s}\omega_s F(\omega_\beta,\omega_s,\omega_j)e^{-i\omega_j t}, \quad (115)$$
$$f_2^I(\omega_\beta,\omega_s,t) = \sum_{j=1}^{4}\frac{\omega_\beta(\omega_j^2-\omega_\beta^2)}{Ky_{co}\omega_s}\omega_j F(\omega_\beta,\omega_s,\omega_j)e^{-i\omega_j t}, \quad (116)$$
$$g_2^R(\omega_\beta,\omega_s,t) = \sum_{j=1}^{4}\frac{\omega_\beta(\omega_j^2-\omega_\beta^2)}{Ky_{co}\omega_s}\omega_s G(\omega_\beta,\omega_s,\omega_j)e^{-i\omega_j t}, \quad (117)$$
$$g_2^I(\omega_\beta,\omega_s,t) = \sum_{j=1}^{4}\frac{\omega_\beta(\omega_j^2-\omega_\beta^2)}{Ky_{co}\omega_s}\omega_j G(\omega_\beta,\omega_s,\omega_j)e^{-i\omega_j t}, \quad (118)$$

where

$$\begin{aligned} H_T(\omega_\beta,\omega_s,\omega_j) &= F(\omega_\beta,\omega_s,\omega_j)T_1(\omega_\beta,\omega_s) \\ &+ G(\omega_\beta,\omega_s,\omega_j)T_2(\omega_\beta,\omega_s) \\ &+ G(\omega_\beta,-\omega_s,\omega_j)T_2(\omega_\beta,-\omega_s). \quad (119) \\ H_L(\omega_\beta,\omega_s,\omega_j) &= F(\omega_\beta,\omega_s,\omega_j)L_1(\omega_\beta,\omega_s) \\ &- F(-\omega_\beta,\omega_s,\omega_j)L_1(-\omega_\beta,\omega_s) \\ &+ G(\omega_\beta,\omega_s,\omega_j)L_2(\omega_\beta,\omega_s), \quad (120) \end{aligned}$$

where

$$\begin{aligned} F(\omega_\beta,\omega_s,\omega_j) &= F_s(\omega_\beta,\omega_s,\omega_j,\omega_2) \quad \text{for } \omega_j = \omega_1 \text{ or } -\omega_1^* \\ &= F_s(\omega_\beta,\omega_s,\omega_j,\omega_1) \quad \text{for } \omega_j = \omega_2 \text{ or } -\omega_2^*, \end{aligned} \quad (121)$$

$$
\begin{aligned}
G(\omega_\beta, \omega_s, \omega_j) &= G_s(\omega_\beta, \omega_s, \omega_j, \omega_2) \quad \text{for } \omega_j = \omega_1 \text{ or } -\omega_1^* \\
&= G_s(\omega_\beta, \omega_s, \omega_j, \omega_1) \quad \text{for } \omega_j = \omega_2 \text{ or } -\omega_2^*,
\end{aligned} \tag{122}
$$

where

$$
F_s(\omega_\beta, \omega_s, \omega_1, \omega_2) = \frac{1}{2\omega_\beta} \frac{(\omega_\beta + \omega_1^*)(\omega_2 - \omega_\beta)(\omega_\beta + \omega_2^*)}{(\omega_2^* + \omega_1)(\omega_2 - \omega_1)(\omega_1^* + \omega_1)}, \tag{123}
$$

$$
G_s(\omega_\beta, \omega_s, \omega_1, \omega_2) = \frac{K y_{co}}{2\omega_\beta} \frac{\omega_2 - \omega_2^* - \omega_s - \omega_1^*}{(\omega_2^* + \omega_1)(\omega_2 - \omega_1)(\omega_1^* + \omega_1)}. \tag{124}
$$

References

[1] R.D. Kohaupt. *Proc. 11th Int. Conf. on High Energy Accelerators*, CERN (Birkhäuser Verlag, Basel, 1980), p.562.

[2] R.D. Kohaupt, DESY Report 80-22, (1980).

[3] K. Satoh, and Y.H. Chin, Nucl. Instr. Methods **207**, 309(1983).

[4] G. Besnier, D. Brandt, and B. Zotter, CERN LEP-TH/84-11, 1984.

[5] R.H. Siemann, Nucl. Instr. Methods, **221**, 293(1984).

[6] R. Klatt, R.D. Kohaupt, and T.Weiland, Trans. IEEE **NS-32** (1985) p.2356.

[7] R.D. Kohaupt, DESY Report 85-59 (1985).

[8] R.D. Kohaupt, DESY Report 85-140 (1985).

[9] R.D. Kohaupt, DESY Report M-80/19 (1980).

[10] D. Brandt, and B. Zotter, CERN-LEP-TH/84-2 (1984).

[11] W.K.H. Panofsky, and W.A. Wenzel, Rev. Sci. Instrum. **27**, 967(1956).

[12] J. Rossbach, private communications.

Table 1

The vertical tune shift in PETRA for various y_{co} and η.

$y_{co} = 0.05mm, \eta = 0.007m$

mode	$\frac{\Delta\omega}{2\pi}$ (Hz)/mA Real	Imaginary
1	−250.	-0.33×10^{-10}
2	250.	-0.33×10^{-10}
3	−250.	-0.358×10^{-9}
4	250.	-0.358×10^{-9}
5	-0.459×10^{-9}	-0.138×10^{-2}
6	0.459×10^{-9}	-0.138×10^{-2}
7	-0.962×10^{-5}	0.288×10^{-6}
8	0.962×10^{-5}	0.288×10^{-6}

$y_{co} = 0.5mm, \eta = 0.07m$

mode	$\frac{\Delta\omega}{2\pi}$ (Hz)/mA Real	Imaginary
1	−250.	-0.111×10^{-3}
2	250.	-0.111×10^{-3}
3	−250.	-0.979×10^{-4}
4	250.	-0.979×10^{-4}
5	-0.726×10^{-20}	−0.138
6	-0.78×10^{-20}	−0.138
7	-0.289×10^{-2}	0.145×10^{-3}
8	0.289×10^{-2}	0.145×10^{-3}

$y_{co} = 5.mm, \eta = 0.7m$

mode	$\frac{\Delta\omega}{2\pi}$ (Hz)/mA Real	Imaginary
1	−250.	-0.13×10^{-5}
2	250.	-0.13×10^{-5}
3	−250.	-0.443×10^{-5}
4	250.	-0.443×10^{-5}
5	-0.11×10^{-1}	−13.8
6	0.11×10^{-1}	−13.8
7	-0.963×10^{-1}	0.28×10^{-2}
8	0.963×10^{-1}	0.28×10^{-2}

$y_{co} = 20.mm, \eta = 2.8m$

mode	$\frac{\Delta\omega}{2\pi}$ (Hz)/mA Real	Imaginary
1	−250.	-0.536×10^{-5}
2	250.	-0.536×10^{-5}
3	−250.	-0.564×10^{-4}
4	250.	-0.564×10^{-4}
5	−2.8	−220.
6	2.8	−220.
7	−1.54	0.265×10^{-1}
8	1.54	0.265×10^{-1}

Table 2

Parameters for PETRA.

E	:	beam enrgy (GeV)	7
ω_β/ω_0	:	betatron tune	18.4
ω_s/ω_0	:	synchrotron tune	0.0665
$\omega_0/2\pi$	:	revolution frequency (kHz)	130.
α	:	momentum compaction factor	0.0027
$W_\perp^0$	:	transverse wake potential ($\Omega \cdot m^{-2} \cdot \mathrm{Hz}^2$)	$3. \times 10^{14}$
σ_τ	:	bunch length (cm)	1