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by

Y.H. Chin, K. Yokoya

Deutsches Elektronen-Synchrotron DESY, Hamburg

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Y.H. Chin and K. Yokoya*)

Deutsches Elektronen-Synchrotron DESY, Hamburg

*) On leave from National Laboratory for High Energy Physics KEK, Japan

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Y.H. Chin and K. Yokoya*

Deutsches Elektronen-Synchrotron DESY, Hamburg

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Abstract

A transverse multi-bunch instability in the presence of a split in the betatron tunes between individual bunches is studied. The formalism is summarized in a dispersion relation. It is found that the simple sinusoidal tune modulation by means of an RFQ produces no stable area in the stability diagram. Two cures are presented for expansion of the stable region. One of them, that using fractional filling of bunches, has an advantage that the width of stable band can be controlled by varying the number of missing bunches.

1 INTRODUCTION

In a multi-bunch ring, coherent oscillations of bunches can be coupled from bunch to bunch through structures which have impedances with long memory. A bunch leaves behind it a wake field in the structure, which perturbs the oscillation of the successive bunches. When the system makes a closed loop, all the bunches execute a coupled oscillation coherently with a certain phase difference between them.

The growth rate of the multi-bunch instability[1,2,3] is proportional to the number of bunches, i.e., the total bunch current. In a ring with few bunches, such as PETRA, a single bunch instability, which is attributed to a short-range wake field and depends only on the bunch's own current, is dominant and determines the intensity threshold[4]. In a ring filled with many bunches, such as DORIS and HERA, the multi-bunch instability gains in importance, and has an instability threshold lower than for the single bunch case[5].

One cure for the multi-bunch instability is to introduce a spread in the oscillation frequencies of individual bunches in order to destroy the coherence of the coupled oscillations. The spread can be provided, for example, by a modulation of the focussing force with an RF quadrupole (written as RFQ in what follows) in the transverse case[5], or with a subharmonic cavity in the longitudinal case[6]. In DORIS, an RFQ which produces a sinusoidal tune modulation with a maximum spread of .03 has been installed, and the threshold intensity has been increased by up to a factor of 6[5]. Good results are only obtained when the longitudinal

*On leave from National Laboratory for High Energy Physics KEK, Japan decoupler, which can shift the longitudinal position of each bunch, is also used. Damping is not observed when only the RFQ is switched on[7].

Many existing theories assume that bunches are equally spaced, equally populated, and have the same betatron oscillation frequency. If one naively extends these analyses to the case where M bunches have different oscillation frequencies, one has M simultaneous equations. Mode frequencies are obtained by solving the $M \times M$ eigenvalue matrix, given the impedance. However we can formulate the problem in a different way[8]: we rearrange the M simultaneous equations by expanding them with respect to the discrete frequencies in which the summation of the impedance is taken, with the result that the eigenvalue matrix is converted to a kind of dispersion relation. If an instability is attributed to the impedance in only some of the frequencies (say, L frequencies), e.g., the frequencies of parastic cavity modes with sharp resonance peaks, the size of the matrix can be reduced to $L \times L$. This formalism is much more useful if the number of bunches is large and the approximation of equal bunches does not hold. The purpose of this paper is to present this alternative formalism for the discussion of the stabilization of the transverse multi-bunch instability due to bunch-to-bunch betatron frequency spread. With some modifications the formalism is applicable to the longitudinal case. A bunch is assumed to oscillate rigidly, i.e., the particle distribution within a bunch does not change.

This paper is organized as follows. In Sec.2, we derive a dispersion relation from the equations of motion for rigid bunches. We then introduce some normalization in Sec.3, and as an example solve the dispersion relation for a two-bunch beam. In Sec.4, we show that under a certain condition, the summation in the dispersion relation can be replaced by an integral. We then present stability diagrams for some different ways of modulating the betatron frequencies, assuming that only one frequency of a single sharp resonator is responsible for the instability. From Sec.5 we focus on the study of the sinusoidal tune modulation using an RFQ. It is found that the stable region for a simple sinusoidal tune modulation is enclosed by the two coincident lines on the V - axis and thus has no area. This holds also for the case that two frequencies are involved. We investigate some plausible methods to expand the stable region in Sec.6. The paper is concluded in Sec.7.

2 DISPERSION RELATION

We consider M bunches which oscillate rigidly, each bunch being represented as a single macroparticle without internal structure. Let y_l be the transverse coordinate of the l -th bunch observed at a fixed location s. The l -th bunch arrives at this location at time t_l. The equation of motion for the l -th bunch is

$$\frac{d^2 y_l(s)}{ds^2} + k_l^2 y_l(s) = \frac{e^2}{m_0 \gamma c^2} \sum_{n=-\infty}^{\infty} \sum_{j=0}^{M-1} Q_j\, W(t_l - t_j)\, y_j(s - 2\pi R n), \tag{1}$$

where k_i and Q_i are the betatron wave number and the total charge of the i -th bunch, e is the elementary charge, m_0 is the rest mass of the particle, γ is the Lorentz factor, c is the speed of light and R is the average machine radius. The transverse wake function $W(t)$ is defined as required by causality such that $W(t)$ vanishes if $t < 0$.

If the bunches are equally spaced, the argument of the wake function is

$$t_l - t_j = T\,\frac{l-j}{M} + n\,T, \tag{2}$$

where $T(= \frac{2\pi}{\omega_0})$ is the revolution time. If their positions deviate from equally spaced positions by a fixed amount Δs_l ($l = 0 \sim M-1$), the argument becomes

$$t_l - t_j = T\,\frac{l-j}{M} + \frac{\Delta s_l - \Delta s_j}{c} + n\,T, \tag{3}$$

or, using the time deviation

$$\tau_l = \frac{\Delta s_l}{c}, \tag{4}$$

$$t_l - t_j = T\,\frac{l-j}{M} + \tau_l - \tau_j + n\,T. \tag{5}$$

We now change the independent variable from s to angular position θ defined by

$$\theta = \frac{s}{R}, \tag{6}$$

and Eq.(1) becomes

$$\frac{d^2 y_l}{d\theta^2} + \nu_l^2 y_l = \frac{e^2 R^2}{m_0 \gamma c^2} \sum_{n=-\infty}^{\infty} \sum_{j=0}^{M-1} Q_j\, W(T\frac{l-j}{M} + \tau_l - \tau_j + nT)\, y_l(\theta - 2\pi n), \tag{7}$$

where ν_l is the betatron tune of the l -th bunch:

$$\nu_l = \frac{k_l}{R}. \tag{8}$$

We solve Eq.(7) in the frequency domain and assume initially that all the bunches execute coherent coupled oscillations with tune ν:

$$y_l(\theta) = Y_l\, e^{-i\nu\theta}. \tag{9}$$

Note that Y_l gives the amplitude and the phase of the eigenmode concerned.

We introduce the impedance $Z(\omega)$ which is the Fourier transform of the wake function:

$$W(t) = \frac{i}{2\pi} \int_{-\infty}^{\infty} Z(\omega) e^{-i\omega t} d\omega. \tag{10}$$

Substituting Eqs.(9) and(10) into Eq.(7), we get

$$\begin{aligned}
(-\nu^2 + \nu_l^2)\, Y_l e^{-i\nu\theta} &= \frac{e^2 R^2}{m_0 \gamma c^2} \sum_{n=-\infty}^{\infty} \sum_{j=0}^{M-1} Q_j \frac{i}{2\pi} \int e^{-i\omega(T\frac{l-j}{M} + \tau_l - \tau_j + nT)} Z(\omega)\, Y_j e^{-i\nu(\theta - 2\pi n)} d\omega \\
&= \frac{e^2 R^2}{m_0 \gamma c^2} \sum_{j=0}^{M-1} Q_j \frac{i}{2\pi} \sum_{p=-\infty}^{\infty} \int \delta(\nu - \frac{\omega T}{2\pi} + p)\, Z(\omega) Y_j e^{-i\nu\theta - i\omega(T\frac{l-j}{M} + \tau_l - \tau_j)} d\omega \\
&= \frac{e^2 R^2}{m_0 \gamma c^2} \sum_{j=0}^{M-1} Q_j \frac{i}{T} \sum_{p} Z((p+\nu)\omega_0)\, Y_j e^{-i\nu\theta} e^{-2\pi i(p+\nu)(\frac{l-j}{M} + \frac{\tau_l - \tau_j}{T})},
\end{aligned} \tag{11}$$

where we have used the Poisson sum formula:

$$\sum_{k=-\infty}^{\infty} e^{-ikx} = \sum_{p=-\infty}^{\infty} \delta(\frac{x}{2\pi} - p). \tag{12}$$

Equations(11) are M simultaneous equations for the amplitude Y_l. If we introduce a function defined by

$$F_p = \sum_j Q_j Y_j e^{2\pi i(p+\nu)(\frac{j}{M}+\frac{\tau_j}{T})}, \tag{13}$$

we can rearrange Eqs.(11) into an infinite set of equations for F_p. By dividing both sides of Eq.(11) by $(-\nu^2+\nu_l^2)$, multiplying by $Q_l e^{2\pi i(q+\nu)(\frac{l}{M}+\frac{\tau_l}{T})}$, and taking the summation over l, we obtain

$$F_q = \frac{e^2R^2}{m_0\gamma c^2}\frac{i}{T}\sum_{p=-\infty}^{\infty}\sum_{l=0}^{M-1}\frac{Q_l}{\nu_l^2-\nu^2}Z((p+\nu)\omega_0)F_p e^{2\pi i(q-p)(\frac{l}{M}+\frac{\tau_l}{T})}. \tag{14}$$

Given the impedance, the eigenvalue ν is obtained by solving the equation

$$\det[\,\mathbf{I}-\mathbf{M}\,]=0, \tag{15}$$

where $\mathbf{I}$ is a unit matrix, and $\mathbf{M}$ is a matrix with elements

$$M_{qp} = \frac{e^2R^2 i}{m_0\gamma c^2 T}Z((p+\nu)\omega_0)\sum_{l=0}^{M-1}\frac{Q_l}{\nu_l^2-\nu^2}e^{2\pi i(q-p)(\frac{l}{M}+\frac{\tau_l}{T})}. \tag{16}$$

Equation(15) is in fact a dispersion relation.

This formalism has a great advantage if the impedance is due to several sharp resonators and the frequencies $(p+\nu)\omega_0$ coincide with some of those peaks at L p's. Then the size of matrix is reduced from $M\times M$ to $L\times L$. For instance, in the case of DORIS, only two transverse deflecting cavity modes are significant[9], while the ring is filled with 480 bunches[5]. Thus we gain a reduction by a factor of 240 in the size of the matrix.

3 NORMALIZATION

The simplest case is that the frequency $(p+\nu)\omega_0$ coincides with only one sharp resonator peak. Equation(15) then becomes

$$1 = \frac{e^2R^2 i}{m_0\gamma c^2 T}Z((p+\nu)\omega_0)\sum_{l=0}^{M-1}\frac{Q_l}{\nu_l^2-\nu^2}. \tag{17}$$

We now introduce some normalizations and rewrite Eq.(17) in a more convenient form. For the moment, we assume that bunches are equally spaced, and equally populated:

$$\tau_l = 0, \qquad Q_l = Q. \tag{18}$$

Without losing generality, we assume that the tunes ν_l are distributed around the unperturbed tune ν_0:

$$\nu_l = \nu_0 + \frac{\Delta\nu}{2}x_l, \tag{19}$$

where x_l is a function determined by the modulation method and which varies between -1 and 1. We call $\Delta\nu$ the full spread in the betatron tunes between bunches. Each term in the summation can be decomposed into two terms:

$$\frac{1}{\nu_l^2-\nu^2} = \frac{1}{2\nu}\left[\frac{1}{\nu_l-\nu}-\frac{1}{\nu_l+\nu}\right]. \tag{20}$$

We are looking for the solution ν close to ν_0. We therefore neglect the second term in Eq.(20) and make an approximation

$$\frac{1}{\nu_l^2-\nu^2} = \frac{1}{2\nu_0}\cdot\frac{1}{\nu_l-\nu}. \tag{21}$$

Putting everything together, the dispersion relation can be written in the normalized form

$$1 = -(\,U+iV\,)\cdot\frac{\Delta\nu}{M}\sum_{l=0}^{M-1}\frac{1}{\nu_l-\nu}, \tag{22}$$

where we have defined U and V by

$$U+iV = -i\frac{e^2R^2Q}{m_0\gamma c^2 T}Z((p+\nu)\omega_0)\frac{M}{2\nu_0\Delta\nu}. \tag{23}$$

If U and V are known and if the small ν dependence of the impedance can be neglected, M solutions are obtained for ν by solving a polynomial equation of order M given by Eq.(22).

Equation(22) is readily solved for a beam with two bunches. Let us consider what happens if we separate two tunes gradually. In the first-order approximation, the solution to Eq.(22) is given by

$$\delta\nu = \frac{\Delta\nu}{4\nu_0}(U+iV)\left(1+\sqrt{1+\frac{4\nu_0^2}{(U+iV)^2}}\right), \tag{24}$$

where

$$\nu = \nu_0 + \delta\nu, \tag{25}$$

and

$$\nu_0 = \frac{\nu_1+\nu_2}{2} \tag{26}$$

is the midpoint between two tunes, ν_1 and ν_2. Note that $\Delta\nu = \nu_2-\nu_1$ is a measure of the separation of the two tunes. The quantity $\frac{\Delta\nu}{4\nu_0}(U+iV)$ does not depend on $\Delta\nu$ since $\Delta\nu$ appears also in the denominator of the definition of U and V and is therefore cancelled. See Eq.(23). In order to avoid confusion, we define new normalized coefficients

$$U' + iV' = \frac{\Delta\nu}{4\nu_0}(\,U+iV\,) \tag{27}$$

which are explicitly independent of $\Delta\nu$. For simplicity, consider the special case in which the impedance is purely resistive, i.e., $U=0$. Then Eq.(24) becomes

$$\delta\nu = iV'\left(1+\sqrt{1-\frac{\Delta\nu^2}{4V'^2}}\right). \tag{28}$$

The real and the imaginary parts of the complex tune shift $\delta\nu$ are represented as a function of $\Delta\nu$ in Fig. 1 by the solid and the broken lines, respectively. The Fourier spectrum of a beam oscillation is sketched in Fig. 2. The incoherent tunes of two bunches are represented by line spectra. When two tunes are degenerate, the spectrum of coherent oscillation coincides with the two line spectra, and its width is $2V'$. As the two tunes move away from each other, the spectrum narrows, remaining centered on ν_0. So the growth rate can be reduced by separating the tunes. From the moment that the separation of the two tunes is equal to the initial width of the spectrum for $\Delta\nu=0$, the decrease of the width stops, and the spectrum is split into two spectra. If they are far enough apart so that there is no interference, we have two spectra, each being centered on the tunes ν_1 and ν_2, and their width is V', i.e., half of the initial width. The physical picture for the last case is that of two bunches oscillating independently of each other. Thus, the problem is transformed into one of two single-bunch instabilities. The growth rate, which is now determined by the single bunch current, cannot be made lower than V' by decoupling tunes no matter how large the tune spread is.

4 STABILITY DIAGRAM

The summation appearing in Eq.(22)

$$S(\nu) = \sum_{l=0}^{M-1} \frac{1}{\nu_l - \nu} \tag{29}$$

has the singularities that result when the denominator of each term vanishes, as is illustrated in Fig. 3. Obviously if the eigenfrequency, ν, is further from the real axis than the distance between neighbouring singular points, the summation can be replaced by an integral to a good approximation:

$$\begin{aligned} S(\nu) &\longrightarrow \frac{M}{\Delta\nu} \int_{\nu_0-\frac{\Delta\nu}{2}}^{\nu_0+\frac{\Delta\nu}{2}} \frac{f(\nu_l)}{\nu_l - \nu} d\nu_l \\ &= \frac{M}{\Delta\nu} \int_{-1}^{1} \frac{f(x)}{x - x_1}\, dx. \end{aligned} \tag{30}$$

Here $f(x)$ is the normalized distribution function of tune so that $\int_{-1}^{1} f(x)dx = 1$, and

$$x_1 = \frac{\nu - \nu_0}{\Delta\nu/2}. \tag{31}$$

The dispersion relation becomes

$$1 = -(U + iV) \int_{-1}^{1} \frac{f(x)}{x - x_1}\, dx, \tag{32}$$

which is much easier to handle analytically. Roughly speaking, the distance between neighbouring singular points is $\Delta\nu/M$, so that the condition for the replacement of the summation by the integral is written as

$$\Im\nu \gtrsim \frac{\Delta\nu}{M}. \tag{33}$$

(The symbols $\Re$ and $\Im$ stand for the real and the imaginary parts of the following argument.)

Let us discuss the meaning of an instability when $\Im\nu$ is smaller than $\Delta\nu/M$. Usually, in order to damp an instability with growth rate $\Im\nu_i$, one tries a spread, $\Delta\nu$, of the same order as $\Im\nu_i$, i.e., $\Delta\nu \sim \Im\nu_i$. Then one can predict the growth rate $\Im\nu_f$ in the presence of the spread from the integral form(32) unless

$$\Im\nu_f \lesssim \frac{\Delta\nu}{M} \sim \frac{\Im\nu_i}{M}. \tag{34}$$

Indeed, the growth rate $\Im\nu_i/M$ is the minimum which can be damped by giving a bunch-to-bunch tune spread, since it is the contribution of the single bunch current. (Remember the

argument in Sec.3 for a two-bunch beam). It is desirable that some other damping systems such as an incoherent tune spread within a bunch are used in addition so as to stabilize the beam after the growth rate has been reduced to the level at which the eigen properties of individual bunch determine the instability. The fact is, however, this is often not the case in real machines. If the growth rate $\Im\nu_i/M$ is still not acceptable from the point of view of operation of machine, we reluctantly have to give up our goal, namely, to have a stable beam. In this case, the main concern is shifted to the problem of how much reduction of the growth rate is obtainable. Its optimum value is, as is clear from the discussions above, approximately equal to the total number of bunches.

The condition(33) might be easier to understand if one imagines how the frequency spectrum of a coherent beam oscillation looks in relation to those of individual bunches. Figure 4 shows those spectra schematically. The condition(33) says that if the spectrum of a beam oscillation bridges more than two individual spectrum lines, the summation can be replaced by an integral, just as if all the frequency region is continuously filled with line spectra. If the spectrum of a beam oscillation overlaps only one or no spectrum line, then since the contribution of that spectrum line is dominant in the summation of tune in Eq.(29), the integral is too poor an approximation to be employed.

The dispersion relation either in the summation form or in the integral form can be interpreted as a mapping from the complex ν (or x_1) plane to the $U - V$ plane, or the other way around. Figure 5 shows an example of mapping for the rectangular distribution $f(x) = 1/2$ by means of the dispersion relation in the integral form(32), whose explicit form after the integration is carried out is

$$1 = -(U + iV) \cdot \frac{1}{2} \log \frac{1 - x_1}{-1 - x_1}. \tag{35}$$

The curve $\Im x_1 = 0$ is called the stability limit inside of which a beam is stable.

Figure 6 shows the mapping for the same distribution with use of the summation form. The number of bunches M is chosen to be 480. As anticipated, the mapping curves of Fig. 5 agree quite well with those of Fig. 6 for $\Im x_1$ larger than $2/M$. The mapping curves for smaller $\Im x_1$ than $2/M$, which are not drawn in the figure, are not confined to the band area surrounded by the $\Im x_1 = 0$ and $\Im x_1 = 2/M$ curves in Fig. 5, but fill up the whole of the $U - V$ plane. At each U, V point, there are M curves crossing.

We should mention the relationship between the M solutions for ν. We consider the case where the solution to Eq.(32) is determined uniquely for a given U, V pair. As is clear from the argument made so far, only one solution has an imaginary part larger than $\Delta\nu/M$, and the $M - 1$ remaining solutions are scattered in the narrow band enclosed by the real axis and the $\Im\nu = \Delta\nu/M$ line, which is schematically illustrated in Fig. 7. The outstanding solution corresponds to that which is obtained from the integral form. The stability of the system is determined by the growth rate of this solution.

Figure 8 shows the inverse mapping from the $U - V$ plane to the complex ν plane (actually the x_1 plane in the figure) for the $M = 5$ case. Varying U with V = constant, we obtain one curve somewhere in the upper region of the regular mapping and $M - 1$ curves in the region occupied by the short lines between the singular points.

We conclude the present section by briefly mentioning how to calculate the reduction of the growth rate from the figure. Suppose that the working point lies on the point w marked by the closed circle in Fig.6. The value of V at the point gives the growth rate in the absence of the spread: .73 in the example. The reduction of the growth rate is given by

$$r = \frac{V}{\Im x_1} = \frac{.73}{.2} = 3.7 \quad (36)$$

5 SINUSOIDAL MODULATION

The most frequently used tune modulation pattern is a sinusoidal one generated by means of an RFQ. In this and following sections, we concentrate on the sinusoidal modulation, and consider in detail the damping of multi-bunch instabilities due to this method.

Firstly, we again take up the case where the frequency $(p + \nu)\omega_0$ coincides with only one sharp resonator peak as in the last section. The distribution function for the sinusoidal modulation with harmonic 1 (modulo M) is

$$f(x) = \frac{2}{\pi} \frac{1}{\sqrt{1 - x^2}}. \quad (37)$$

As the modulation pattern becomes complicated, it becomes difficult to express the distribution function in terms of an analytical function. In our case, since the modulation pattern is given artificially, the form for the dispersion integral in which the modulation pattern is specified directly is simpler to describe and often makes the integration easier. Such a form is given by

$$1 = -(U + iV)\frac{1}{\pi}\int_{-\pi}^{\pi} \frac{d\theta}{g(\theta) - x_1}, \quad (38)$$

where $g(\theta)$ is the modulation pattern; e.g.,

$$g(\theta) = \sin\theta \quad (39)$$

for the sinusoidal modulation with harmonic 1.

A few comments should be made on the integral(38). The original summation over bunch tunes with harmonic h is given by

$$S = \sum_{k=0}^{M-1} \frac{1}{e^{2\pi i \frac{hk}{M}} - x_1}. \quad (40)$$

Since the denominator may oscillate rapidly, the summation may not be replaced directly by the integral over k. If h and M have no common divisor, we can find the number n which makes the modulus of hn with respect to M equal to unity:

$$1 = mod(hn, M). \quad (41)$$

Together with n we define j which satisfies $k = nj$ in order to reorder the summation as

$$S = \sum_{j=0}^{M-1} \frac{1}{e^{2\pi i \frac{j}{M}} - x_1}. \quad (42)$$

This is equivalent to the sinusoidal modulation with harmonic 1. If h and M have a common divisor, degeneracy of the index j occurs, and the summation is written as

$$S = d\sum_{j=0}^{M-1} \frac{1}{e^{2\pi i \frac{j}{M}} - x_1}, \quad (43)$$

where d is the greatest common devisor, and the summation is taken over multiples of d.

The most effective distribution of bunch tunes is such that the degeneracy of tunes becomes minimum. That is, the modulation period is chosen to be the revolution period. This is satisfied if h and M have no common divisor.

The integration is readily carried out (see Appendix A), with the result,

$$1 = \pm(U + iV)\frac{2}{\sqrt{x_1^2 - 1}}, \quad \Re x_1 \gtrless \pm 1 \quad (44)$$

$$= -(U + iV)\frac{2i}{\sqrt{1 - x_1^2}}, \quad |\Re x_1| < 1 \quad (45)$$

Equation(45) states that the stability limit curve $\Im x_1 = 0$ moves on the V - axis back and forth once as x_1 varies in the range ± 1. As a result, the stability region is enclosed by the two coincident lines and has no area. This is seen more clearly from the stability diagram shown in Fig. 9. We can draw the interesting conclusion from this result that a beam cannot in principle be stabilized by means of a simple sinusoidal tune modulation, no matter how large the tune spread is, unless the impedance is purely resistive, which is rare. Of course, the growth rate can be reduced to some extent by the sinusoidal tune modulation. Thus, in the presence of other damping mechanisms such as the radiation damping in an electron ring, it may be possible to reduce the growth rate of the instability to a point where the damping due to these mechanisms gives a stable beam.

We would like to make some remarks here, relating to the original summation form. In the above discussion, M is assumed to be a fairly large number so that the curve $\Im\nu = \Delta\nu/M$ defining the limit of validity of the integral form only deviates very slightly from the $\Im\nu = 0$ curve, and a beam can be regarded to be stable enough in the inside. We use the word "stable" in this extended sense. In practice, that a beam is within the stable region means no more than that the growth rate is smaller than $\Delta\nu/M$. If this condition is not satisfied, the crux of problem turns out to be that the region where the large reduction of the growth rate is obtained is strongly limited to the narrow area near the "stable line". We want a larger volume for the stable region in order to secure the large area of growth rate reduction.

Let us see if the above conclusion still holds for the case where the frequency $(p + \nu)\omega_0$ coincides with two resonator peaks. Again we have the problem of reordering the summation to obtain the integral form. Occasionally, the off-diagonal terms in the matrix have a phase factor in the numerator:

$$S = \sum_{k=0}^{M-1} \frac{e^{2\pi i \frac{(q-p)k}{M}}}{e^{2\pi i \frac{hk}{M}} - x_1}. \quad (46)$$

Using n defined by Eq.(41), we reorder the summation in such a manner that the denominator varies slowly, with the result that the oscillation of the numerator is also modified:

$$S = \sum_{j=0}^{M-1} \frac{e^{2\pi i \frac{rj}{M}}}{e^{2\pi i \frac{j}{M}} - x_1}, \quad (47)$$

where

$$r = (q - p)n. \quad (48)$$

If the modulus of r with respect to M, $mod(r, M)$, is a small number, the numerator and the denominator oscillate in phase, yielding a large value for the summation. Otherwise, the summation nearly vanishes since the individual terms cancel.

The dispersion relation is written in the matrix form:

$$\left| \begin{array}{cc} 1+(U_p+iV_p)\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{d\theta}{g(\theta)-x_1}, & (U_q+iV_q)\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{e^{ir\theta}d\theta}{g(\theta)-x_1} \\ (U_p+iV_p)\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{e^{-ir\theta}d\theta}{g(\theta)-x_1}, & 1+(U_q+iV_q)\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{d\theta}{g(\theta)-x_1} \end{array} \right| = 0, \qquad (49)$$

where

$$U_s + iV_s = -i\frac{e^2R^2Q}{m_0\gamma c^2 T}Z((s+\nu)\omega_0)\frac{M}{2\nu_0\Delta\nu}, \qquad (50)$$

and

$$r = \begin{cases} mod((q-p)n, M) & \text{if } mod((q-p)n, M) \le M/2 \\ mod((q-p)n, M) - M. & \text{otherwise} \end{cases} \qquad (51)$$

The integrals in the off-diagonal terms are evaluated in the Appendix A. The result is

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{e^{ir\theta}d\theta}{\sin\theta - x_1} = 2i\frac{e^{ir\arcsin x_1}}{\sqrt{1-x_1^2}}, \qquad r \ge 0 \qquad (52)$$

$$= 2i(-1)^{|r|}\frac{e^{i|r|\arcsin x_1}}{\sqrt{1-x_1^2}}, \quad r<0 \qquad (53)$$

where we note that

$$|e^{i\arcsin x_1}| \le 1 \quad \text{if } \Im x_1 \ge 0 \qquad (54)$$

and

$$e^{i\arcsin x_1} \to 0 \quad \text{as } |x_1| \to \infty. \qquad (55)$$

From Eqs.(52) and (53), it can be seen that if

$$\Im x_1 > \frac{1}{|r|}, \qquad (56)$$

the off-diagonal terms can be neglected. The diagonal terms give two independent dispersion relations for each impedance pair (U_p,V_p) and (U_q,V_q) so that we have two stability diagrams as in Fig. 9. Instabilities due to two impedances can be treated separately. In this case, which is the case that occurs most frequently, the stable region is still the straight line.

If the condition(56) is not satisfied, the effect of the two impedances couples. The simple mapping from the complex x_1 plane to the 4 dimensional (U_p, V_p, U_q, V_q) space is then useless. However, for some cases, we can still plot meaningful stability diagrams. Let us look at it for the case where the off-diagonal terms are least negligible; the $r = 0$, and $r = 1$ cases.

If $r = 0$, the dispersion relation is readily evaluated, and is given by

$$1 = -(U_p + U_q + i(V_p + V_q))\frac{2i}{\sqrt{1-x_1^2}} \qquad (57)$$

which has the same form as Eq.(45). If we regard Eq.(57) as the mapping from the complex x_1 plane to the $(U_p + U_q)$-$(V_p + V_q)$ plane, we can make a stability diagram whose stable region is enclosed by two coincident lines as we have seen. Since it is the summation of the impedance which gives the coherent tune shift in the absence of the spread, this stability

diagram makes practical sense. This result is also good for the case that $\Im x_1$ is so small that the phase factor $e^{ir\arcsin x_1}$ can be approximated by unity.

The next case is that of $r = 1$. When $U_p, U_q, V_p, V_q \ll 1$, the dispersion relation(49) leads to

$$x_1 = \pm\{1 + 2[\,U_p + U_q + i(V_p + V_q)\,]^2\}, \qquad (58)$$

$$(\textit{the sign in front is } \pm \textit{ for } U_p + U_q \gtrless 0)$$

which is equivalent to Eq.(57) when solved for x_1 neglecting terms higher than the second-order terms in the impedance. Therefore, the stability diagram has the same shape near the origin: the stable region is confined by the two coincident lines and has no area.

We have seen that the stable region has zero area in some simple cases. If many high-Q resonator are responsible for the instability, many complicated cases might be imagined in which various effects are entangled, with the result that the non-zero area is produced in the stable region. Thus, we cannot generalize the above statement. However it can be said that the sinusoidal tune modulation alone which creates no stable region even in simple cases is not favoarable from the point of view of removing an instabilty.

6 EXPANSION OF THE STABLE REGION

In this section we consider various ways to obtain a stable region with non-zero area. One idea is to add higher harmonic RFQ's to the main RFQ in such a way that the modulation pattern becomes similar to a sawtooth wave form which gives the rectangular tune distribution and the stability diagram shown in Fig. 5. Another idea is to make a gap in the distribution of bunches around a ring in such a way that the rapid rise of the distribution function(37) near $x_1 = \pm 1$, which is the cause of the coincidence of the two stability curves on the V - axis, is removed. We consider these two ideas.

6.1 Higher harmonic RFQ

The sawtooth wave form

$$f(\theta) = \begin{cases} \frac{2}{\pi}(-\pi-\theta), & -3\pi/2 \le \theta \le -\pi/2 \\ \frac{2}{\pi}\theta, & -\pi/2 \le \theta \le \pi/2 \\ \frac{2}{\pi}(\pi-\theta), & \pi/2 \le \theta \le 3\pi/2 \end{cases} \qquad (59)$$

can be Fourier decomposed as follows:

$$f(\theta) = \frac{8}{\pi^2}\sum_{n=0}^{\infty}(-1)^n\frac{\sin(2n+1)\theta}{(2n+1)^2}. \qquad (60)$$

Figure 10 shows the stability diagram where a third-harmonic RFQ is superimposed on top of the main RFQ. The area between the two coincident lines on the V - axis is expanded and is split into three sections; main area and two loops with non-zero area. When the fifth-harmonic RFQ is further included, the number of loops becomes three, while the inner regions lose their area. If we include more and more higher order harmonics, the main area becomes disk shaped, and the loops get smaller. The convergence to the circle is, however, very slow. The number of loops is equal to that of the harmonic included.

From the practical point of view, the double RFQ system (3rd and main harmonic RFQ) looks feasible. However, the peculiar triangular stable region does not seem to help very much in stabilizing a beam.

6.2 Fractionally filled beam

Our formalism can be readily applied to a fractionally filled beam, by pretending that a gap is still filled with bunches which consist of, say, only one particle. The number of bunches M should be counted including these pseudo bunches in the gap. As mentioned in the first paragraph of this section, the fact that the distribution function(37) diverges at $\theta = \pm\frac{\pi}{2}$ causes the curve of the stability limit to be folded onto the V - axis. We therefore make gap, Δ, around $\theta = \pm\frac{\pi}{2}$; between $-\frac{\pi}{2} - \frac{\Delta}{2}$ and $-\frac{\pi}{2} + \frac{\Delta}{2}$, and between $\frac{\pi}{2} - \frac{\Delta}{2}$ and $\frac{\pi}{2} + \frac{\Delta}{2}$.

The dispersion integral for $x_1 = x_1 + 0_+$ is now

$$I = \frac{2}{\pi}\int_{-\frac{\pi}{2}+\frac{\Delta}{2}}^{\frac{\pi}{2}-\frac{\Delta}{2}} \frac{d\theta}{\sin\theta - (x_1 + 0_+)}, \tag{61}$$

which on evaluation gives

$$\begin{aligned} I &= \frac{2}{\pi\sqrt{1-x_1^2}}\left[\log\left|\frac{\sqrt{(1-x_1)(2-\delta)} - \sqrt{(1+x_1)\delta}}{\sqrt{(1-x_1)(2-\delta)} + \sqrt{(1+x_1)\delta}} \cdot \frac{\sqrt{(1+x_1)(2-\delta)} + \sqrt{(1-x_1)\delta}}{\sqrt{(1+x_1)(2-\delta)} - \sqrt{(1-x_1)\delta}}\right|\right] \\ &\quad + \frac{2i}{\sqrt{1-x_1^2}}\Theta(1-\delta-|x_1|), \text{ for } |x_1| < 1 &(62) \\ &= \mp\frac{2}{\pi}\left[\sqrt{\frac{2-\delta}{\delta}} - \sqrt{\frac{\delta}{2-\delta}}\right], \text{ for } x_1 = \pm 1 &(63) \\ &= \frac{4}{\pi\sqrt{x_1^2-1}}\left[\arctan\sqrt{\frac{(x_1-1)(2-\delta)}{(x_1+1)\delta}} - \arctan\sqrt{\frac{(x_1-1)\delta}{(x_1+1)(2-\delta)}}\right], \text{ for } |x_1| > 1 &(64) \end{aligned}$$

where

$$\delta = 1 - \cos(\frac{\Delta}{2}). \tag{65}$$

The stability diagram for $\Delta = \pi/6$ is plotted in Fig. 11. The stable region forms a band whose full width is approximately (see Appendix B)

$$\Delta U \cong \frac{\Delta}{\pi}. \tag{66}$$

To the extent that gaps are allowed, this cure is a rather promising method for enabling the sinusoidal tune modulation to work effectively.

7 CONCLUSIONS

We have studied the "Landau damping"[1] of the multi-bunch instability due to the betatron tune spread of individual bunches. The formalism is summarized in the dispersion relation Eq.(15). We have found that the summation can be replaced by an integral unless the imaginary part of the complex tune is smaller than the spread divided by the total number of bunches. This is a great help in handling the problem analytically. Decoupling of tunes cannot reduce the growth rate below that given by the single bunch instability. From a different point of view, this single bunch growth rate gives the limit to the possible reduction of the growth rate: the optimum rate is approximately equal to the total number of bunches.

The result of the analysis for the sinusoidal tune modulation by means of an RFQ shows that for some basic cases, the stability limit curve is folded onto the V - axis (see Fig. 9), which yields a stable area of zero. Thus, a beam cannot be stabilized by means of a sinusoidal tune modulation alone. This conclusion holds if a beam has fairly many bunches. Even if this is not the case, that the region where the growth rate is strongly reduced is extremely narrow near the V - axis is a fatal problem. A positive cure must be found in order to expand the stable region.

Two cures are presented. For the former one, that using higher harmonic RFQ's, it turns out that it is necessary to include RFQ's with rather high order harmonics in order to get a regular stable region. The latter one, that using fractional filling of bunches, seems to be more promising, and the width of the stable "band" can be controlled by varying the number of missing bunches.

Instead of adding higher harmonic RFQ's, we also suggest giving the voltage of the main RFQ a sawtooth shape by using a feed-back loop. If a gap is undesirable for reasons not connected with these problems, this method is an alternative solution.

The main role of the longitudinal decoupler in the experiment at DORIS using an RFQ seems to be to break the degeneracy of the bunch tunes (harmonic number of RFQ is 16, while the ring is filled with 480 bunches). The expansion of the stable region resulting from the asymmetry of filling is too small to be considered as the reason of the increase in the reduction of the growth rate, since the bunch position cannot be shifted by more than one bucket.

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APPENDIX A

Dispersion integral for the sinusoidal tune modulation

[1] We use this term in an extended sense to describe a system with finite number of oscillators.

The integral to be evaluated is

$$I = \int_{-\pi}^{\pi} \frac{e^{ir\theta}}{\sin\theta - x_1}. \tag{67}$$

Let us first consider the case $r \geq 0$. We introduce the complex variable z as

$$z = e^{i\theta} \tag{68}$$

to get

$$\begin{aligned} I &= -i \oint \frac{z^{r-1}}{\frac{1}{2i}(z - \frac{1}{z}) - x_1} \\ &= 2 \oint \frac{z^r dz}{(z - (ix_1 + \sqrt{1-x_1^2}))(z - (ix_1 - \sqrt{1-x_1^2}))}, \end{aligned} \tag{69}$$

where the integration is carried out anticlockwise along the unit circle centered on the origin. If $\Im x_1 > 0$, the poles $ix_1 + \sqrt{1-x_1^2}$ and $ix_1 - \sqrt{1-x_1^2}$ are outside and inside of the circle, respectively. From the residue theorem,

$$I = 2\pi i \, \frac{(ix_1 + \sqrt{1-x_1^2})^r}{\sqrt{1-x_1^2}}. \tag{70}$$

If $r < 0$, we define z as

$$z = e^{-i\theta}, \tag{71}$$

and the integration results in

$$I = 2\pi i \, (-1)^{|r|} \frac{(ix_1 + \sqrt{1-x_1^2})^{|r|}}{\sqrt{1-x_1^2}}. \tag{72}$$

Equations(70) and(72) can be expressed differently using trigonometric functions as

$$I = 2\pi i \, \frac{e^{ir \arcsin x_1}}{\sqrt{1-x_1^2}}, \tag{73}$$

and

$$I = 2\pi i \, (-1)^{|r|} \frac{e^{i|r| \arcsin x_1}}{\sqrt{1-x_1^2}}, \tag{74}$$

respectively.

APPENDIX B

Width of the stable band

If we retain only the first order terms in δ and $\delta_1 \equiv 1 - x_1$ in Eq.(62), we obtain

$$I = \frac{2}{\pi} \cdot \frac{\log\left|\frac{\sqrt{\delta_1}-\sqrt{\delta}}{\sqrt{\delta_1}+\sqrt{\delta}}\right| + \pi i \Theta(\delta_1 - \delta)}{\sqrt{2\delta_1}}. \tag{75}$$

If we take δ_1 such that $\frac{\delta_1}{\delta} = a \gg 1$ but $\delta_1 \ll 1$, then

$$\begin{aligned} \frac{1}{I} &= \frac{\pi}{2} \cdot \frac{\sqrt{2a\delta}}{\log\frac{\sqrt{a}-1}{\sqrt{a}+1} + \pi i} \\ &\approx \frac{\pi}{2} \cdot \frac{\sqrt{2a\delta}}{-\frac{2}{\sqrt{a}} + \pi i} \\ &\approx \frac{2}{\pi^2}(-\sqrt{2\delta} + \pi i \sqrt{2a\delta}). \end{aligned} \tag{76}$$

Therefore,

$$\begin{aligned} \Delta U &= 2|\Re\frac{1}{I}| \\ &\approx \frac{4}{\pi}\sqrt{\frac{\delta}{2}} \\ &\approx \frac{\Delta}{\pi}. \end{aligned} \tag{77}$$

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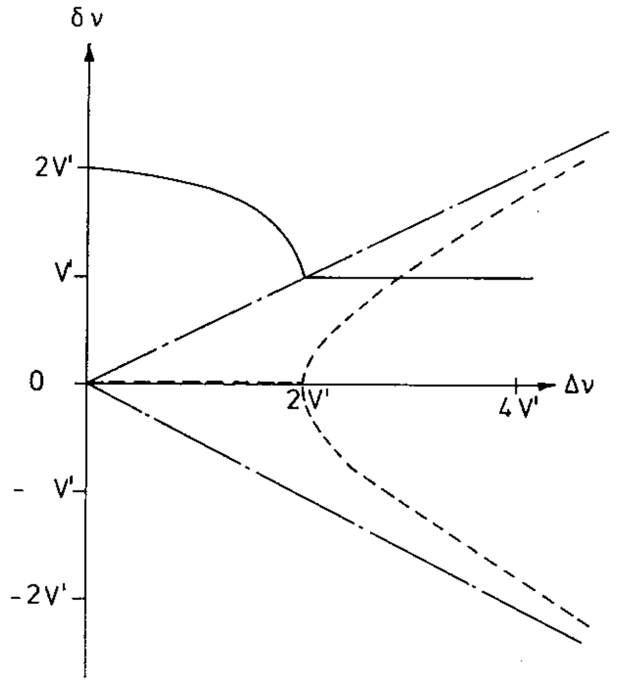

Figure 1: The complex tune shift $\delta\nu$ versus the spread $\Delta\nu$ for a two-bunch beam.

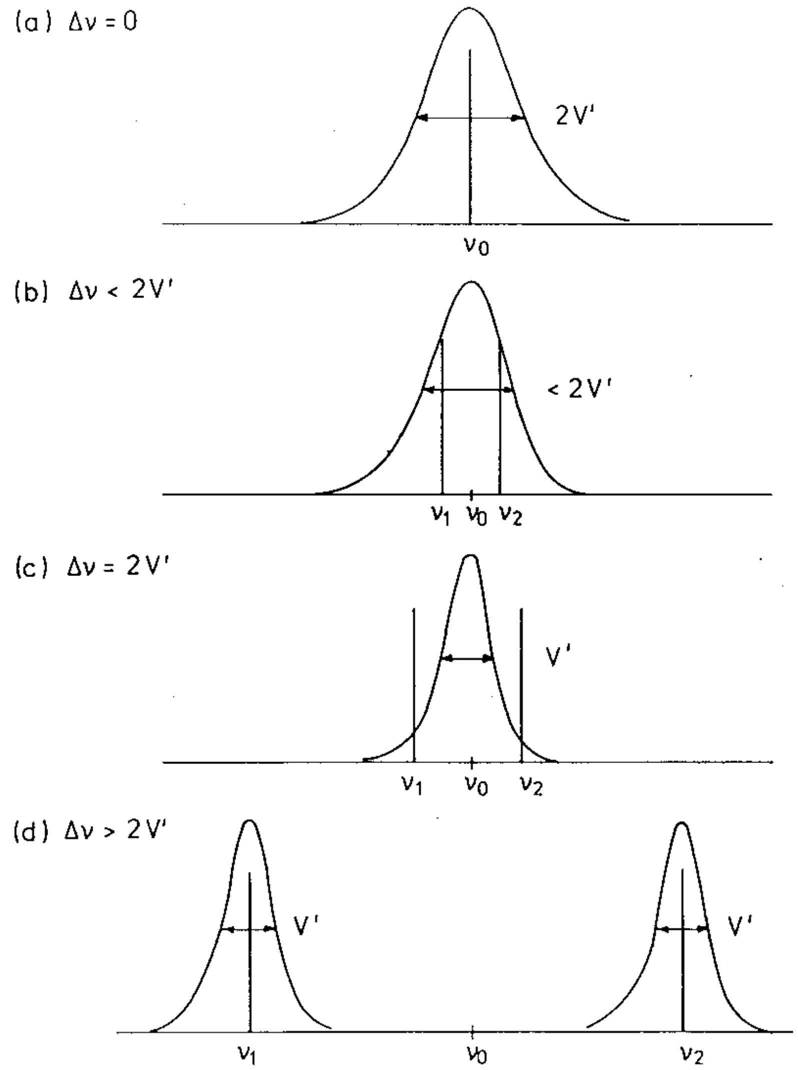

Figure 2: Fourier spectra of a beam oscillation for various spreads $\Delta\nu$.

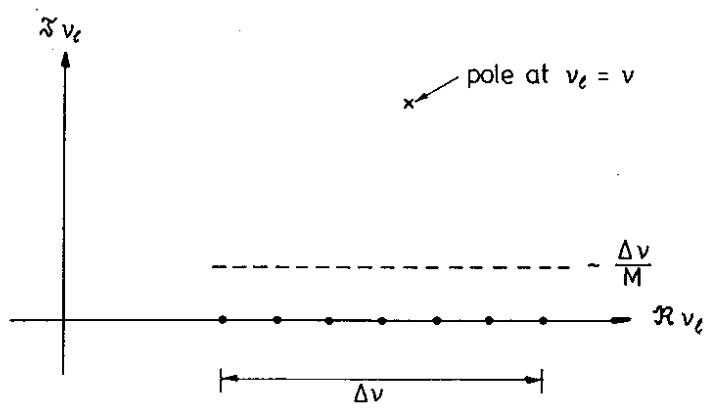

Figure 3: Location of the pole of $S(\nu)$ in the ν_l plane.

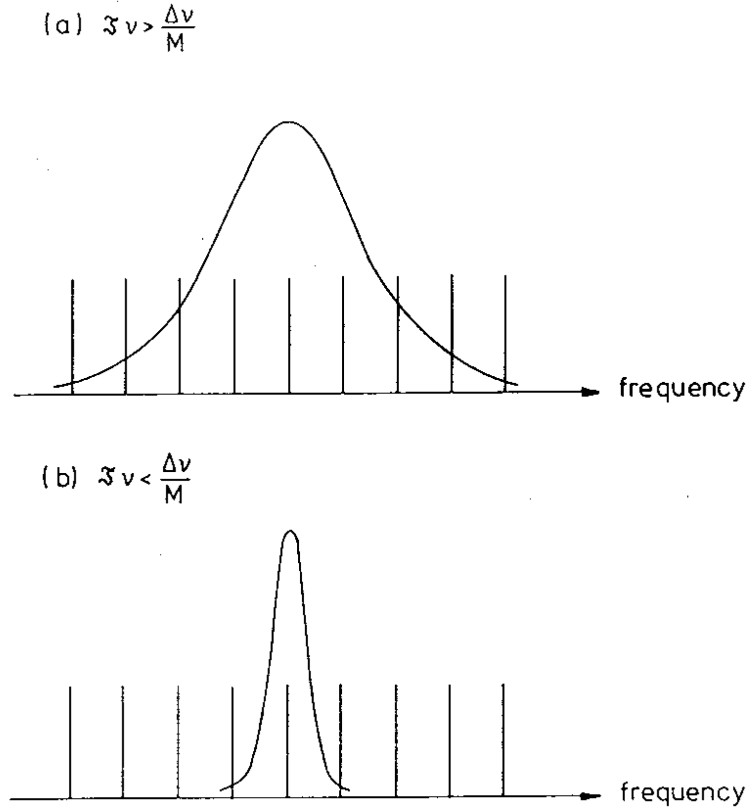

Figure 4: Sketches of frequency spectra of a coherent beam oscillation for various $\Im\nu$.

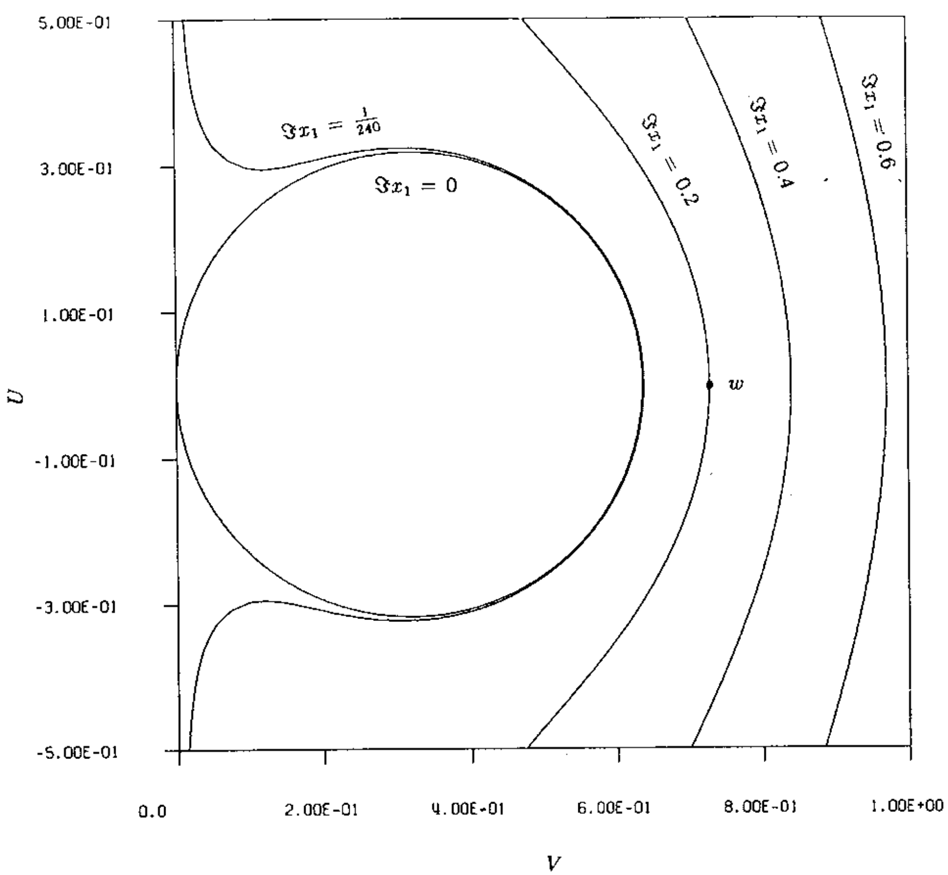

Figure 5: The stability diagram for the rectangular distribution in terms of the integral form of the dispersion relation.

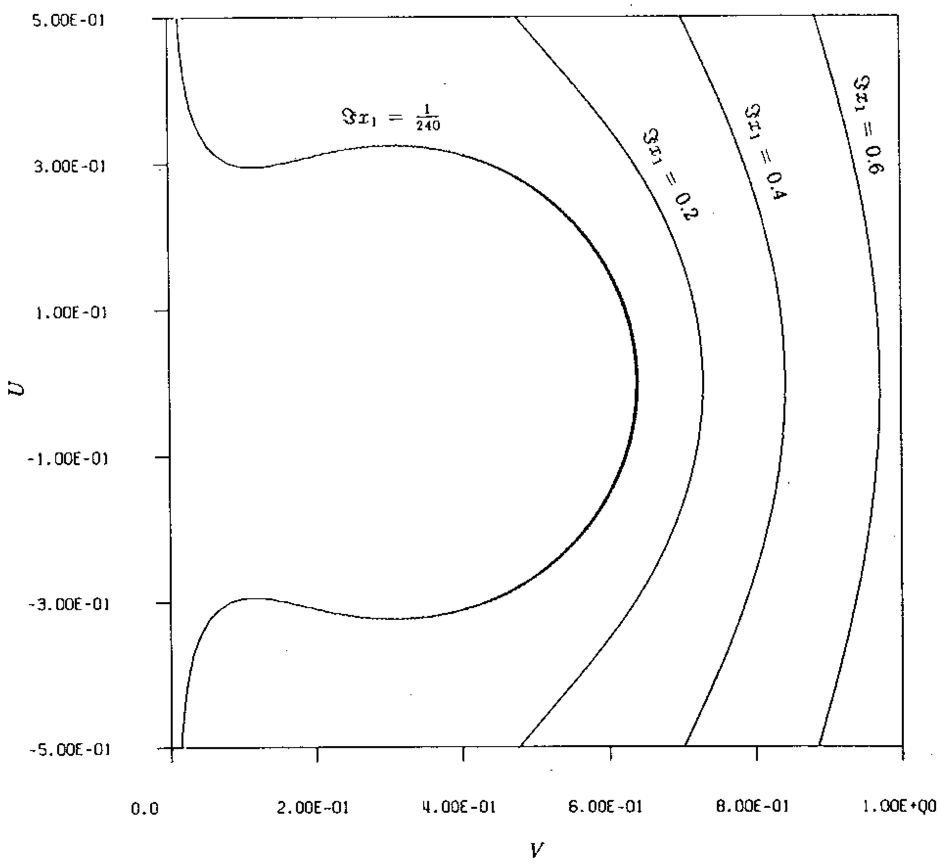

Figure 6: The stability diagram for the rectangular distribution in terms of the summation form for the dispersion relation.

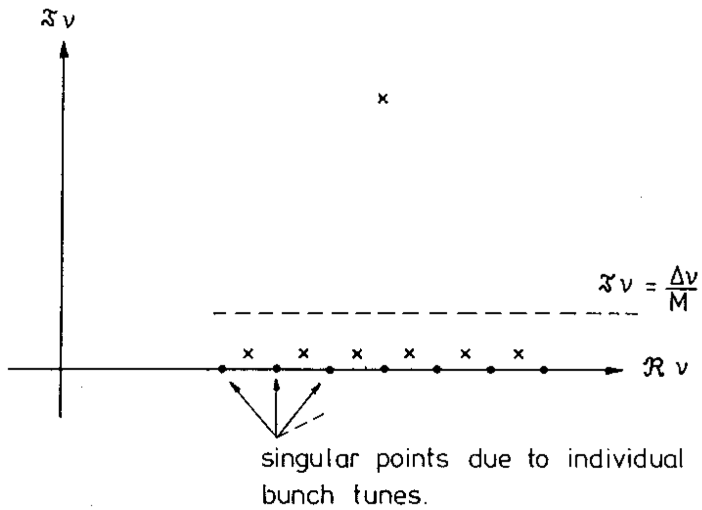

Figure 7: Sketches depicting typical locations of M solutions for ν in the complex plane.

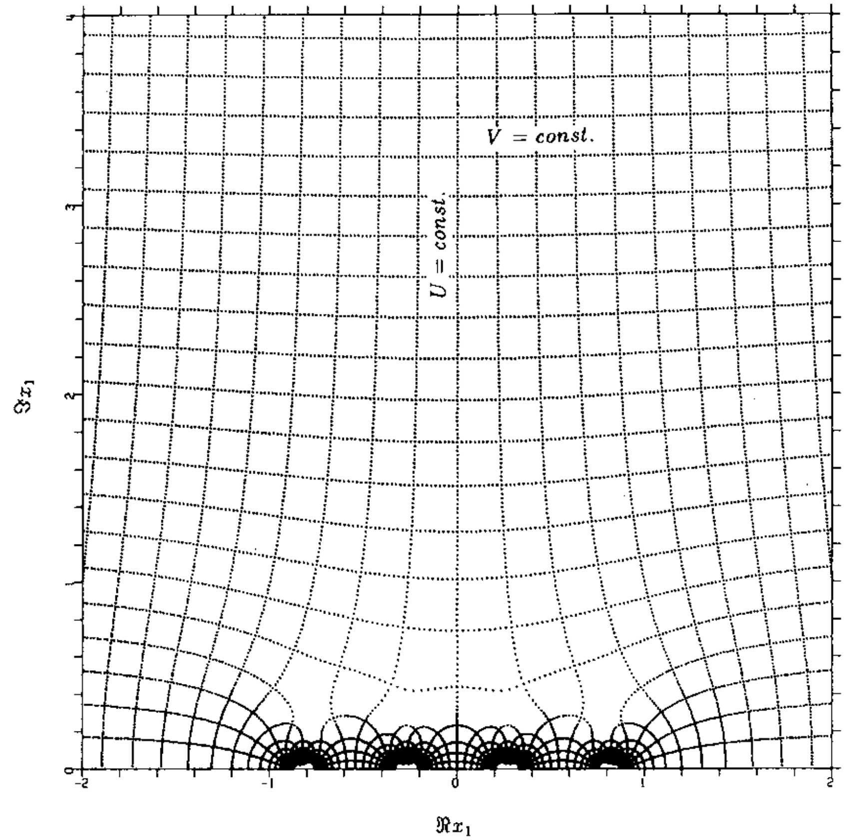

Figure 8: The mapping from the $U - V$ plane to the complex x_1 plane for $M = 5$.

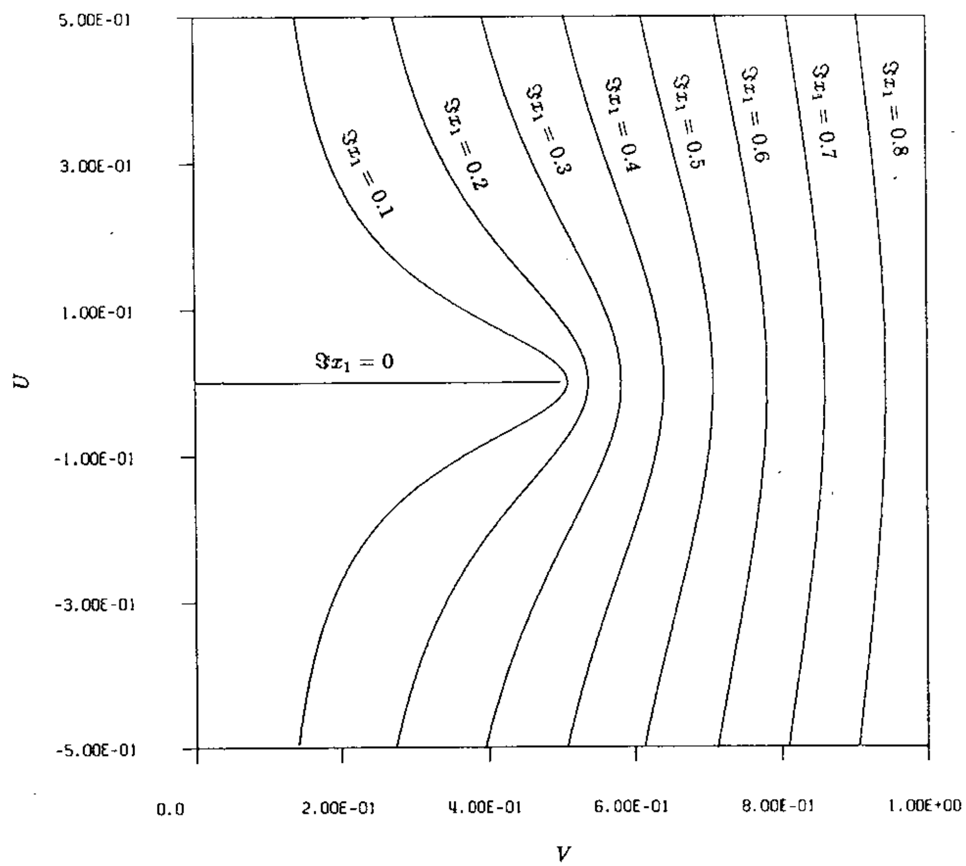

Figure 9: The stability diagram for the sinusoidal tune modulation.

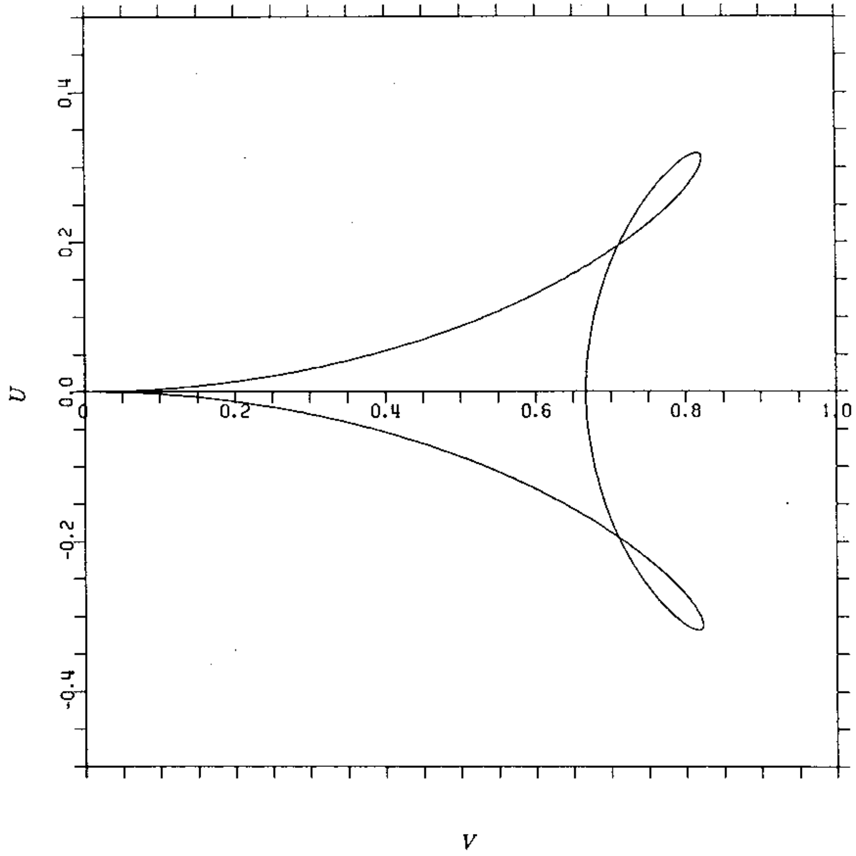

Figure 10: The stability diagram where the 3rd harmonic RFQ is included.

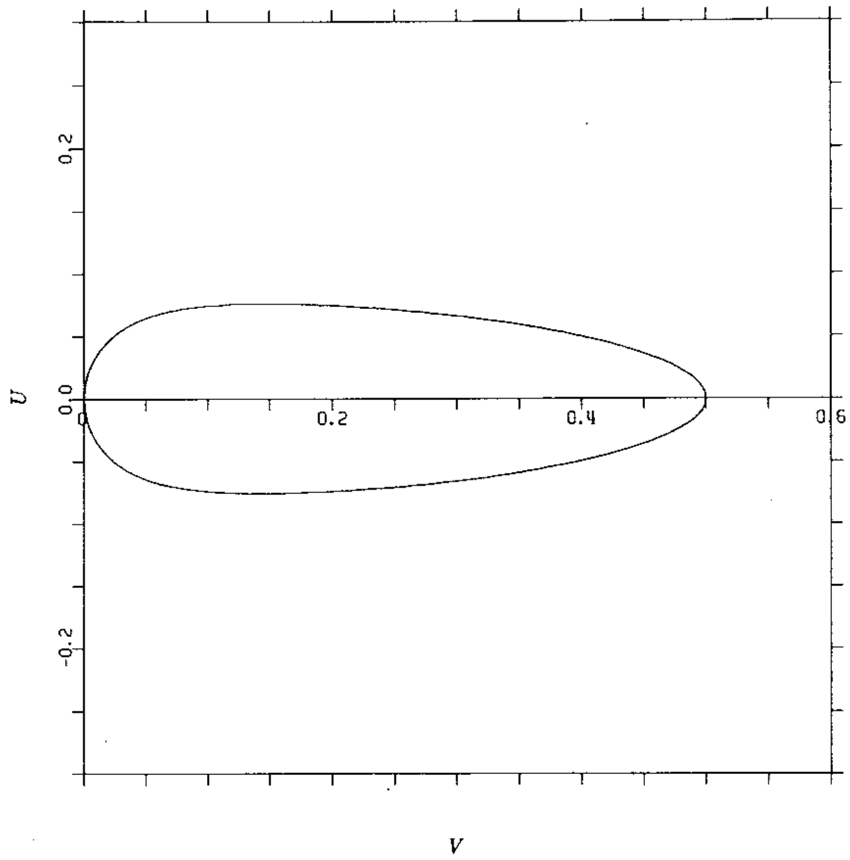

Figure 11: The stability diagram for the fractionally filled beam. The gap parameter is $\Delta = \pi/6$.