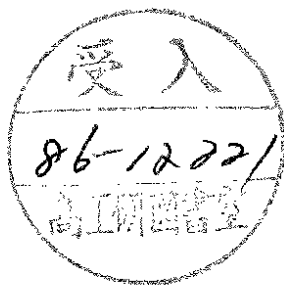


DESY 86-127
October 1986



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ACTIONS FROM THEIR ANALYTICITY PROPERTIES

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ISSN 0418-9833

NOTKESTRASSE 85 · 2 HAMBURG 52

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Restrictions on Effective Dielectric Lattice Gauge Theory Actions
 from their Analyticity Properties *

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Abstract: Dielectric lattice gauge theory actions $L(\vec{\Phi})$ depend on variables that are attached to the links $b = (x+e_\mu, x)$ of the lattice and take their values in the linear space \mathcal{L} which consists of real linear combinations of matrices in the gauge group G . L is determined by the fundamental gauge field theory through a block spin transformation. We argue that it will be an analytic function of $\vec{\Phi}$, at least in the small field region. It can therefore not depend on the dielectric factor $\sigma_\mu(x)$ and lattice gauge field $U(b) \in G$ in the polar decomposition $\vec{\Phi}(b) = U(b)\sigma_\mu(x)$ in an arbitrary way. For gauge group $SU(N)$ and low enough cutoff, L is believed to have a global minimum at $\vec{\Phi} = 0$. More generally, power series expansion in $\vec{\Phi}$ provides a convenient parametrization of L . The coefficients in this expansion are truncated expectation values in the non-critical auxiliary statistical mechanical system whose partition function is e^{-L} . This points the way to Monte Carlo calculation of L which avoids critical slowing down.

* Work supported by Deutsche Forschungsgemeinschaft.

1. Analyticity properties of effective actions

Effective actions L with lattice cutoff a play an important role in modern quantum field theory. Starting from a pure Yang Mills theory on the continuum, or on a lattice Λ' with arbitrarily small lattice spacing a' , an effective theory on a lattice Λ of lattice spacing a is determined through a block spin transformation. It will have an action $L(\vec{\Phi})$. To control the block spin transformation one may split it into a sequence of block spin transformations which increase the lattice spacing by small integer factors. But for us only the final result will be important which is obtained when a is not very much smaller than the typical length scale (correlation length or square root of string tension) of the theory.

We consider effective lattice actions $L(\vec{\Phi})$ for pure gauge theories which depend on field variables $\vec{\Phi}(b) \equiv \vec{\Phi}_\mu(x)$ that are attached to the links $b = (x+e_\mu, x)$ of the lattice Λ (*). In a dielectric effective theory, $\vec{\Phi}(b)$ does not take its values in the gauge group G , as in usual lattice gauge theory models, but in the linear space \mathcal{L} which consists of real linear combinations of matrices in the gauge group. If the gauge group is $G = SU(2)$ then $\vec{\Phi}(b)$ are real multiples of matrices in $SU(2)$, hence they are determined by a field with 4(real) components $\vec{\Phi}(b)$

$$\vec{\Phi}(b) = \vec{\varphi}^*(b) \cdot \vec{1} + i\vec{\varphi}(b) \cdot \vec{\sigma} \tag{1.1}$$

For gauge group $G = SU(N)$ with $N \geq 3$, $\vec{\Phi}(b)$ are arbitrary complex $N \times N$ matrices.

Such dielectric effective lattice gauge theories were introduced by the author in ref. /1/. (For earlier ideas see /2/.) They are obtained by a block spin transformation /3/ whose block spin $\vec{\Phi}(b)$ is a real linear combination of parallel transporters $U(\omega)$ in the original gauge theory

* e_μ is the vector of length a in μ -direction; $e_{-\mu} = -e_\mu$.

$$\Phi(b) = \sum_{\omega: x \rightarrow y} \rho(\omega) u(\omega) \quad (1.2)$$

x' and y' are the centers of the blocks which are endpoints of the link b on the block lattice Λ . The sum runs over paths ω from x' to y' . If ω consists of links b_1', \dots, b_n' on the original lattice Λ' and u is the original lattice gauge field then $u(\omega) = u(b_1') \dots u(b_n')$. A concrete example of a block spin definition of the form (1.2) is obtained from the $\sqrt{3}$ scheme of Cordery, Gupta and Novotny, see Appendix. It is convenient for numerical work.

Given the action $L_f(u)$ of the original theory on lattice Λ' , an effective action $L(\Phi)$ can be defined by

$$e^{-L(\Phi)} = \text{const.} \int_{b \in \Lambda} \delta_x(\Phi(b) - \bar{u}(b)) e^{-L_f(u)} \mathcal{D}u \quad (1.3)$$

The constant is so chosen that $\min L(\Phi) = 0$. $\bar{u}(b)$ is defined by the right hand side of eq. (1.2). δ_x can be a δ -function which constrains $\bar{u}(b)$ to equal $\Phi(b)$, but it is advantageous to use a normalized Gaussian instead $/4/$

$$\delta_x(\Phi(b) - \bar{u}(b)) = (2\pi\kappa)^{-m} \exp\left[-\frac{x}{2} \text{tr} \left[(\Phi(b) - \bar{u}(b)) (\Phi(-b) - \bar{u}(-b)) \right] \right] \quad (1.4)$$

$-b = (x, x+e_\mu)$ is link b with reversed direction. Definition (1.3) of the effective action can be rewritten as

$$e^{-L(\Phi)} = \int e^{-L_{\text{eff}}(\Phi, u)} \mathcal{D}u$$

$$L_{\text{eff}}(\Phi, u) = L_f(u) + \frac{x}{2} \text{tr} \left\{ (\Phi(b) - \bar{u}(b)) (\Phi(-b) - \bar{u}(-b)) \right\} + \text{const}$$

Physical block spins will be such that

$$\Phi(-b) = \Phi(b)^* \quad (1.5)$$

but for the purpose of discussing analyticity properties we may consider $L(\Phi)$ as defined by eq. (1.3) for fields which do not satisfy this generalized reality condition.

The integrand in eq. (1.3) depends on Φ only through the Gaussian factors (1.4). These factors are entire analytic functions of the complex matrices $\Phi(b)$. Reasonable actions $L_f(u)$ are bounded below, and u -integration runs over a compact space. Therefore also $e^{-L(\Phi)}$ will be an entire analytic function of the complex matrices $\Phi(b)$.

This implies analyticity properties of $L(\Phi)$. $L(\Phi)$ need not be an entire function of Φ because $e^{-L(\Phi)}$ may have zeroes for fields Φ which do not obey the reality property (1.5). However, we can expect that $L(\Phi)$ will be holomorphic in Φ in a complex neighborhood of the small field region. The small field region consists of fields Φ that satisfy the reality condition (1.5) and for which the action $L(\Phi)$ is not very large locally. These are the most important field configurations. A more detailed discussion of this point will be given in section 4.

For simple gauge groups G , such as $G = \text{SU}(N)$, it is expected that the effective action $L(\Phi)$ will have its absolute nondegenerate minimum at $\Phi = 0$ when the cutoff a^{-1} is low enough. Confinement is supposed to follow from this. [It was proven that it does for a class of models with local action L that satisfy Osterwalder Schrader positivity, see ref. /1/.] $\Phi = 0$ will certainly be in the small field region when it is the global minimum, but also for appreciably larger cutoff where this situation is not quite reached yet. Therefore it will follow from the analyticity properties that $L(\Phi)$ will admit a convergent power series expansion in Φ around $\Phi = 0$, at least when the cutoff a^{-1} is low enough. Such power series expansions will be studied in the next section. Power series expansions around different configurations Φ in the small field region are also useful.

The radius of convergence of the power series expansion in Φ is determined by singularities for complex Φ . We cannot prove it, but we can hope that it will converge throughout the small field region when the constant κ in the definition (1.3), (1.4) of the block spin transformation is suitably chosen. That power series expansions (in the fields) for effective action around suitable points of the small field region should converge throughout the small field region is supported by results of rigorous renormalization group studies for non-gauge theories /5/, and for gauge theories but with block spin in the group /4/. There is a chance that this property remains true for $\kappa = \infty$, i.e. if $\delta_{\mathbf{x}}$ is a true δ -function.

The fields $\Phi(b)$ admit a generalized polar decomposition /1/

$$\Phi(b) = U(b)\sigma_{\mu}(x) \quad \text{for} \quad b = (x + \epsilon_{\mu}, x)$$

where $U(b) \in G$ is an ordinary lattice gauge field, and

$$\sigma_{\mu}(x) \geq 0, \quad \text{real, if } G = \text{SU}(2)$$

$$\sigma_{\mu}(x) = e^{iA_{\mu}(x)}, \quad (\text{positive hermitean } N \times N \text{ matrix})$$

with real $A_{\mu}(x)$, for $G = \text{SU}(N)$, $N \geq 3$.

$$\text{The matrix } \epsilon(x) = (\epsilon_{ij}(x))_{i,j=1,2,3},$$

$$\epsilon_{ij}(x) = \sigma_i(x)\sigma_j(x)^* \delta_{ij}$$

has the physical meaning of dielectric constant, or rather field. We shall call $\sigma_{\mu}(x)$ the dielectric factor. Although it has one index, it is not a vector field because it transforms differently under lattice rotations (see section 5).

The analyticity properties of the effective action L imply that it is not an arbitrary function of the lattice gauge field $U(b)$ and dielectric factor $\sigma_{\mu}(x)$ separately, but is a sum of terms which are proportional to integer powers of their product $\Phi(b)$. For instance, in a $\text{SU}(2)$ theory,

$$\sigma_{\mu}(x) = \left[\frac{1}{2} \text{tr} \Phi(b)\Phi(-b) \right]^{1/2}$$

is not analytic in Φ at $\Phi = 0$, nor is $U(b) = \Phi(b)\sigma_{\mu}(x)^{-1}$.

2. Power series expansion in Φ .

Given that $L(\Phi)$ is analytic at $\Phi = 0$, we can write down a power series expansion

$$L(\Phi) = \sum_n \sum_{\mu_1, \dots, \mu_n} \frac{1}{n!} r_{\mu_1, \dots, \mu_n}(x_1, \dots, x_n) \Phi_{\mu_1}(x_1) \dots \Phi_{\mu_n}(x_n) \quad (2.1)$$

with complex coefficients r . Colour indices are suppressed. Since $\Phi(b) \equiv \Phi_{\mu}(x)$ may enter together with $\Phi(-b)$, summation over μ_1 must be extended through $\pm 1, \pm 2, \pm 3, \pm 4$. For physical fields which satisfy the reality constraints (1.5), viz.

$$\Phi_{-\mu}(x + \epsilon_{\mu}) = \Phi_{\mu}(x)^* \quad (2.2)$$

we may use these to reexpress $L(\Phi)$ so that the μ -summations run over $1, \dots, 4$,

but Φ and Φ^* appear.

Further constraints follow from gauge invariance of L . $\Phi(b)$ transforms under gauge transformations V just like an ordinary lattice gauge field

$$\Phi_{\mu}(x) \rightarrow V(x + \epsilon_{\mu})^{-1} \Phi_{\mu}(x) V(x) \quad (2.3)$$

Let us determine, as an example, the most general form of $L(\Phi)$ to 4-th order in Φ . The elementary invariants of at most 4-th order in Φ are

$$\frac{1}{N} \text{tr} \Phi(b)\Phi(-b) \equiv \|\Phi_{\mu}(x)\|^2 \quad (2.4a)$$

$$\text{tr} \Phi(b_4)\Phi(b_3)\Phi(b_2)\Phi(b_1) \equiv \text{tr} \Phi(\dot{p}) \quad (2.4b)$$

where links $b_1 \dots b_4$ form the boundary of a plaquette p , and possibly

$$\det \Phi(b). \quad (2.4c)$$

The last expression is invariant under $G = \text{SU}(N)$ and is of N -th order in Φ . When

$N = 2$ it is not independent. When $N \geq 3$ it is the only one among these invariants which is not invariant under the larger gauge group $U(N)$.

The plaquette term can be rewritten in another way using the generalized field strengths introduced in ref. 1.

Given Φ , one introduces a kind of covariant derivative \mathcal{D}_μ which acts on

$$\xi \text{-valued functions on links according to}$$

$$\mathcal{D}_\mu \psi_\nu(x) = \psi_\nu(x + e_\mu) \Phi_\mu(x) - \Phi_\mu(x + e_\nu) \psi_\nu(x) \quad (2.5)$$

One defines

$$\mathcal{F}_{\mu\nu}(x) = \mathcal{D}_\mu \Phi_\nu(x) - \mathcal{D}_\nu \Phi_\mu(x). \quad (2.6)$$

One checks that

$$\text{tr } \Phi(\dot{p}) = \frac{1}{2} \text{tr } \mathcal{F}_{\mu\nu}(x) \mathcal{F}_{\mu\nu}(x)^* + \text{products of invariants (2.4a)} \quad (2.7)$$

for physical Φ satisfying (1.15). To h -th order in Φ , the most general expression for $L(\Phi)$ in a $SU(N)$ theory is therefore

$$L(\Phi) = \sum_{\substack{x_1, \mu, \nu \\ \mu \neq \pm \nu}} \frac{1}{16f} \text{tr } \mathcal{F}_{\mu\nu}(x) \mathcal{F}_{\mu\nu}(x)^* + \sum_{x_1, \mu_1, x_2, \mu_2} \|\Phi_{\mu_1}(x_1)\|^2 k_{\mu_1 \mu_2}(x_1, x_2) \|\Phi_{\mu_2}(x_2)\|^2 - \sum_{x, \mu} \gamma_\mu \text{cled } \Phi_\mu(x) \quad (2.8)$$

with real coefficients $k_{\mu_1 \mu_2}(x_1, x_2)$ and complex $\gamma_\mu = \bar{\gamma}_{-\mu}$. Summation over μ, ν is extended over values $\pm 1, \pm 2, \pm 3, \pm 4$ in 4 dimensions.

Assuming that our hopes concerning convergence throughout the small field region are justified, the power series representation (2.1) provides a convenient parametrization of $L(\Phi)$. It is in particular also very suitable for numerical calculation of $L(\Phi)$, starting from a fundamental action L_f . The coefficients r in power series expansion (2.1) are derivatives of $L(\Phi)$ at $\Phi = 0$,

$$r_{\mu_1 \dots \mu_n}(x_1 \dots x_n) = \frac{\partial}{\partial \Phi_{\mu_1}(x_1)} \dots \frac{\partial}{\partial \Phi_{\mu_n}(x_n)} L(\Phi) \Big|_{\Phi=0} \quad (2.9)$$

For each Φ , $-L(\Phi)$ is logarithm of the partition function $Z(\Phi) = e^{-L(\Phi)}$ of an auxiliary statistical system with integration variables u and probability distribution

$$Z(\Phi)^{-1} e^{-L(\Phi)} \mathcal{Q}u \quad (2.10)$$

Corresponding expectation values will be denoted by $\langle \rangle_\Phi$. Thus the coefficients (2.9) are truncated expectation values of this auxiliary statistical mechanical system (2.10) with $\Phi = 0$. Explicit formulae can be obtained by differentiating the Gaussian (1.4). For instance, if all the directed links $(x_i + e_{\mu_i}, x_i)$ are distinct then

$$r_{\mu_1 \dots \mu_n}(x_1 \dots x_n) = -\left(\frac{x}{2}\right)^n \langle \bar{u}(x_1, x_1 + e_{\mu_1}); \dots; \bar{u}(x_n, x_n + e_{\mu_n}) \rangle_\Phi = 0$$

The semicolons indicate truncated expectation values

$$\langle A; B \rangle = \langle AB \rangle - \langle A \rangle \langle B \rangle,$$

etc. $\bar{u}(y, x)$ is again defined to equal the right hand side of eq. (1.2).

3. On Monte Carlo Calculations of L

It is a central point in Wilsons renormalization group philosophy that the auxiliary statistical mechanical system (2.10) is noncritical and has correlations of order one block lattice spacing a only, assuming that the block spin definition is well chosen. Rigorous studies of weakly coupled models support this picture, at least for Φ in the small field region /4, 5/.

This offers the possibility of computing effective actions by Monte Carlo simulation of the noncritical auxiliary system. Because the system is noncritical, there will be no critical slowing down (so long as the block size a/a' is not extremely large). Such a direct computation was first suggested by S.K. Ma (for ferromagnets) /6/ but his procedure was declared impractical in Wilson's 1979 Cargèse lectures /7/, and very nearly all Monte Carlo renormalization group calculations so far simulated the full theory, which is nearly critical, as opposed to the auxiliary system, which is not*. The reason is that one would otherwise have needed simulations for at least as many block spin configurations Φ as there are parameters in the effective action. There could be "thousands or even millions of them ... /7/".

* A recent exception is Petronzio and Callaway /13/. Our method has been applied to the 2-dimensional Heisenberg ferromagnet /14/. Simulations with $\kappa = \infty$ and with $\kappa < \infty$ were done. In the first case, pairs of spins in the same block were updated, with one partner chosen at random. In this way extremely fast convergence was obtained up to the largest feasible block sizes a/a' . If $\kappa < \infty$ (and not too large) updating of single spins suffices. The fast convergence can be understood in an elementary way by noting that the term $\|\phi \cdot \bar{u}\|^2$ in L_{tot} is nonlocal in u and aids the diffusion of perturbations of the system created by updating u at single sites (links).

Our method avoids this problem. It requires simulation for a single block spin configuration Φ only, and proceeds by computation of the truncated expectation values r of the auxiliary system which figure as expansion coefficients in the action. In practise one may want to do calculations for some more configurations Φ . This would permit to calculate coefficients of Taylor expansions of $L(\Phi)$ around some other points Φ , again through computation of truncated expectation values. In this way, the accuracy of the approximation (truncated Taylor expansion) could be tested.

A great advantage of the method is that it offers a ready possibility to check whether the effective action has good locality properties. This will be the case if the truncated expectation values that one computes decay over a distance of order one block lattice spacing.

So far we discussed the effective action $L(\Phi)$ in the small field region, i.e. for Φ such that $L(\Phi)$ is not very large (locally). Other configurations Φ need a separate discussion - see the next section. This is known as the large field problem. At the present state of the art in numerical renormalization group work, the large field problem is more or less ignored. One determines an approximate action that fits well in the small field region, and uses it for all fields Φ . One elementary precaution must certainly be taken, though. One must make sure that the approximate action is not small or negative in the large field region, i.e. where the true $L(\Phi)$ is large. Otherwise, simulations using the approximate $L(\Phi)$ would not suppress contributions from all unlikely Φ 's. Because of this it is important to exhibit positivity properties of $L(\Phi)$. This is why we insisted on writing the first term in the fourth order expression (2.8) in manifestly positive form. This term contains the 1-plaquette contribution.

4. Small field region

Consider pairs (ϕ, u) , where u is a lattice gauge field on the original lattice Λ' , and ϕ is a dielectric lattice gauge field on the block lattice which obeys the physical reality condition (1.5). We may define a probability distribution for such pairs

$$\begin{aligned} dP(\phi, u) &= e^{-L_{\text{tot}}(\phi, u)} \mathcal{D}_u \mathcal{D}\phi \\ &= Z(\phi)^{-1} e^{-L_f(u)} \prod_{b \in \Lambda'} d u(b) \prod_{b \in \Lambda} [d\phi(b) \delta_{\mathbf{x}}(\phi(b) - \bar{u}(b))] \end{aligned} \quad (4.1)$$

$d\phi(b)$ is the ordinary (Euclidean) measure on the real-linear space \mathbb{R}^m . Because the Gaussian factor integrates to unity,

$$\int d\phi(b) \delta_{\mathbf{x}}(\phi(b) - \bar{u}(b)) = 1, \quad (4.2)$$

the expectation value of observables in the old theory can be rewritten as

$$\langle O \rangle = \int_{\phi, u} dP(\phi, u) O(u) \quad (4.3)$$

Integration is over variables ϕ and u here. The effective action was defined as

$$e^{-L(\phi)} \mathcal{D}\phi \propto \int_u dP(\phi, u) \quad [\mathcal{D}\phi = \prod_{b \in \Lambda} d\phi(b)] \quad (4.4)$$

where integration is only over u .

Suppose we define effective actions $L_{\Lambda}(\phi)$ for block lattices Λ of arbitrary shape and size in this way. Then we may consider the part of the action $\mathcal{Z}(X|\phi)$ that is "spread throughout the subset X of Λ ". It is uniquely defined by validity of the following formula for arbitrary Λ

$$L_{\Lambda}(\phi) = \sum_{X: X \subseteq \Lambda} \mathcal{Z}(X|\phi) \quad (4.5)$$

$\mathcal{Z}(X|\phi)$ will depend on $\phi(b)$ only if $b \in X$ [We regard Λ' and X as sets of links.] We assume that additive constants are so chosen that $\min_{\phi} L_{\Lambda}(\phi) = 0$.

Consider a large, or infinitely extended lattice Λ again. Define a Lagrangean (density)

$$\bar{\mathcal{Z}}(b|\phi) = \sum_{X: b \in X} |\Lambda|^{-1} \mathcal{Z}(X|\phi) \quad (|\Lambda| = \text{no. Links in } X)$$

so that $L(\phi) = \sum_b \bar{\mathcal{Z}}(b|\phi)$. Given a configuration ϕ on Λ , we may divide Λ into small field region and large field region as follows. A link b will be in the small field region if $\bar{\mathcal{Z}}(b|\phi) < \gamma$

where γ is a suitably chosen number appreciably bigger than 1, so that $e^{-\gamma}$ is small. The large field region is the complement of the small field region. The name "small field region" is merely historical. It depends on L whether $\phi(b)$ is actually small in the small field region. In our context, we expect that L will have a minimum at $\bar{\phi} = 0$ when the cutoff a^{-1} is low enough, and fields in the small field region are then really small. But for very large cutoff a^{-1} , L will have a sharp minimum for fields ϕ with values $\phi(b)$ in the group G , or nearby. In the small field region, $\phi(b)$ will then be close to the subset G of \mathcal{F} .

On the basis of rigorous studies of weakly coupled models /4, 5/, we expect that the following picture emerges for a generic configuration ϕ (figure 1).

Nearly all of Λ will be small field region. The complement decomposes into connected components, which we call large field islands, or islands for short. Typically these large field islands will be small (one or few lattice spacings across), while large islands will be very rare, i.e. dilute.

Instead of referring to a generic configuration ϕ , we may consider the probability distribution $e^{-L(\phi)} \mathcal{D}\phi$ for ϕ . In this language, the probability of finding ϕ such that there is a large island containing a given link is expected to be very small.

Given ϕ , let us look at the auxiliary statistical mechanical system whose variables are u , and whose partition function is $e^{-L(\phi)}$. We expect that the

correlations of the auxiliary system will be short range, of order one block lattice spacing a , in the small field region, and $Z(X|\Phi)$ will become very small when X contains many points or points that are far apart*. In this sense the effective action will be nearly local in the small field region.

In the large field region, the situation could possibly be different. There could sometimes be correlations throughout individual islands; in large islands correlations could be over large distances. But large islands will be very rare.

The following discussion will be based on the hypothesis that this picture is correct. In principle this could be checked by suitable Monte Carlo simulations (and ought to be checked ...).

Let us now turn to the discussion of analyticity properties in the small field region. The essential point will be that $Z(X|\Phi)$ will be very small unless X contains few points close together. As a result, the sum in expression (4.5) for the effective action can be truncated, and only a finite number m of sets X need be considered, modulo translations. Consider the partition functions $Z_\Lambda(\Phi) = e^{-L_\Lambda(\Phi)}$ for small lattices Λ . They are entire functions of variables $\Phi(b)$, $b \in \Lambda$.

They are strictly positive for fields that obey the physical reality condition. The number of variables and the zeroes of $Z_\Lambda(\Phi)$ for complex fields will depend on Λ . But there exists only a finite number of subsets Λ of the m sets X which need to be considered. Therefore, there will be a common complex neighborhood \mathcal{O} of $\Phi(b) = 0$ such that for all such Λ , $Z_\Lambda(\Phi)$ is free from zeroes when all $\Phi(b) \in \mathcal{O}$. As a result $L_\Lambda(\Phi) = -\ln Z_\Lambda(\Phi)$ will be holomorphic. Inverting** eq. (4.5) we see

* The relevant quantity which governs the decay is the length of the shortest tree that connects all points /16, 4, 5/.

** This is known as a Moebius inversion /9/. The inversion formula is proven by noting that a set with n elements has $\binom{n}{k}$ subsets with k elements, and

$$\sum_{k=0}^n \binom{n}{k} (-1)^k = (1-1)^n = \delta_{n,0}$$

that the same will be true for

$$-Z(X|\Phi) = \sum_{\Lambda: \Lambda \subseteq X} (-1)^{|\Lambda|} \ln Z_\Lambda(\Phi) \quad (|\Lambda| = \text{no. of points in } \Lambda)$$

This implies analyticity of the truncated sum of contributions $Z(X|\Phi)$ which defines the (approximate) effective action, for $\Phi(b) \in \mathcal{O}$ and arbitrarily large lattice. This completes the argument for analyticity of $L(\Phi)$ in the small field region.

To satisfy the reader who is curious, we will explain below how the large field region is treated in rigorous analytical work. But before that we should like to add a trivial but important remark. In the large field region the action is large (locally). Since e^{-1000} is the same as e^{-100} for all practical purposes, what one really needs there is not an accurate representation of the action itself, but of the Boltzmann factor. Thus, if one determines parameters in a parametrized Ansatz for an approximate action, one should be careful not to enforce a good fit to the action in the large field region at the cost of a good fit in the small field region. For this reason, the "Improved Monte Carlo Renormalization Group" of Gupta and Cordery /8/ would have to be used with great caution for systems with continuous spins.

Let us finally discuss representations of the Boltzmann factor $Z(\Phi)$ for the whole lattice, including the large field region. One uses series representations like (4.5) for the action in the small field region $\mathcal{Y}(\Phi)$ and a polymer representation for the Boltzmann factor in the large field region. The result is something like

$$Z(\Phi) = \left[\exp - \sum_{X: X \subseteq \mathcal{Y}(\Phi)} Z(X|\Phi) \right] \prod_{P \in \mathcal{J}(\Phi)} A(P|\Phi)$$

$\mathcal{J}(\Phi)$ is the set of large field islands. This formula is still somewhat oversimplified. In particular, the presence of a large field island disturbs the system in

some neighborhood. Therefore P are not actually taken as the islands themselves, but as some neighborhood of these, and there is a sum over such neighborhoods.

The technique of splitting large and small field region was first introduced by Gallavotti and coworkers /10/, it was further developed by Kupiainen and Gawędzki /5/. Similar techniques are used in the theory of disordered systems, see e.g. /11/. The use of analyticity in fields was pioneered by Kupiainen and Gawędzki /5/.

5. Breaking of U(N) gauge symmetry to SU(N).
Gauge invariant fields in the effective action.

Suppose that the fundamental theory is an SU(N) gauge theory with $N \geq 3$. Then the block spins $\Phi(b)$ can be arbitrary complex $N \times N$ matrices. Therefore it is possible to define an action of U(N) gauge transformations on these fields

$$\Phi(b) \rightarrow V(x_2) \Phi(b) V(x_1)^{-1} \quad \text{for } b = (x_2, x_1), \quad V(x_i) \in U(N) \quad (5.1)$$

The effective action $I(\Phi)$ will be SU(N) invariant, but it need not be U(N) invariant. It could happen, however, that it is either approximately invariant, or even exactly in the limit of low cutoff. Such a phenomenon is called spontaneous symmetry generation. It means that the long distance behavior of the theory shows a larger symmetry than its action. An example was found by Fröhlich and Spencer /12/. In expression (2.8) for the effective action to h-th order in Φ , only the determinant term breaks the gauge symmetry from U(N) to SU(N). [This term is actually of higher order than h-th for $N > 4$. But because it is the lowest order term that breaks a larger gauge symmetry, its presence ought not to be ignored anyway.]

Let us now discuss the consequences of the extra U(1) symmetry and of its breaking.

Let us start from the generalized polar decomposition of $\Phi(b)$ into a lattice gauge field $U(b) \in SU(N)$ and a dielectric factor $\sigma_\mu(x)$ [$b = (x + \epsilon_\mu, x)$]

$$\Phi(b) = U(b) \sigma_\mu(x)$$

For a SU(N)-gauge theory with $N \geq 3$,

$$\sigma_\mu(x) = \rho_\mu(x) e^{i A_\mu(x)}, \quad \rho_\mu(x) = \rho_\mu(x)^* \geq 0$$

A_μ is a real field, while $\rho_\mu(x)$ is a positive hermitean 3×3 matrix. According to the discussion in ref. /1/,

$$\xi_{ij}(x) = \sigma_i(x) \sigma_j(x)^* \delta_{ij} = \rho_i(x) \rho_j(x)^* \delta_{ij}$$

plays the physical role of dielectric constant, or rather field. It does not depend on A_μ . A_μ is the vector potential for the (broken) extra $U(1)$ gauge symmetry.

Now we come back to the theme of this paper. A_μ will enter the effective action only through ϕ , therefore in the combination $\rho_\mu e^{iA_\mu}$. This implies that ξ_{ij} acts as a dielectric field not only for the $SU(N)$ gauge fields, but also for the $U(1)$ gauge field A_μ . Thus, the abelian electric induction \mathcal{E} can be nonzero only where $\xi \neq 0$. If $L(\phi)$ has its unique minimum at $\phi = 0$, then $\xi = 0$ is the classical ground state and $\xi \neq 0$ only in the "bag".

Now suppose that the $U(N)$ symmetry is exact. Then Gauss' law is valid for the $U(1)$ subgroup also, so that abelian field lines can only terminate in sources. As a result, any source that transforms nontrivially under the $U(1)$ gauge group will be confined through a flux tube that carries the abelian electric induction. There will be no massless vector particle, in spite of the extra $U(1)$ symmetry, again because the abelian electric induction can exist only in the bag.

Let us now turn to the effect of symmetry breaking. We remind the reader that the groups $U(N)$ and $U(1) \times SU(N)$ are not the same, although they have the same Lie algebra. The $U(1)$ subgroup of $U(N)$ intersects the $SU(N)$ subgroup in a subgroup Z_N which is the center of $SU(N)$. Therefore the breaking of $U(1)$ will be down to Z_N . In the presence of symmetry breaking, Gauss' law for the $U(1)$ gauge field will not remain valid. Therefore abelian strings can break. As a result, only sources that transform nontrivially under Z_N - i.e. fractionally charged sources - will remain confined. The others get screened - see the discussion and Fig. 1 at the beginning of ref. 1.

The determinant term in eq. (2.8) breaks the $U(1)$ gauge symmetry down to Z_N .

Assuming γ_μ is positive real, it reads

$$\gamma_\mu [i \xi \rho_\mu(x)] \left(1 - \frac{N}{2} A_\mu(x)^2 + \dots \right)$$

for small A_μ , and resembles therefore a mass term for the A_μ field. The expectation value of $\gamma_\mu M \det \rho_\mu(x)$ inside the bag will determine the range of the force mediated by the abelian gauge field inside a bag. Such a force might conceivably be important in nuclear matter. It is therefore important to determine the symmetry breaking term in the effective action by Monte Carlo simulations as described before.

Because the abelian electric induction can only exist in a bag, there is no reason why a glueball with the quantum numbers $J^{PC} = 1^{--}$ of A_μ should be particularly light. Nevertheless it deserves attention, being described by one of the $SU(N)$ -gauge invariant fields (A_μ) on which the effective action depends.

There is another $SU(N)$ -gauge invariant field B_μ^0 in the action. Write

$$\rho_\mu = \exp \left[B_\mu^0(x) + \sum_{a=1}^{N-1} \lambda^a B_\mu^a \right]$$

with real B : Under $SU(N)$ -gauge transformations, ρ_μ transforms covariantly as

$$\rho_\mu(x) \rightarrow V(x) \rho_\mu(x) V(x)^{-1}$$

Therefore B_μ^0 is invariant. It is not a vector field because it transforms differently under lattice rotations by π . Consider the rotation which takes the link

$$b = (x + e_\mu, x) \text{ into } (x, x + e_\mu). \text{ Then}$$

$$\sigma_\mu(x) \rightarrow U(b) \sigma_\mu(x)^* U(b)^{-1}$$

Therefore

$$B_\mu^0(x) \rightarrow + B_\mu^0(x)$$

In contrast, a vector field transforms as

$$A_\mu(x) \rightarrow -A_\mu(x)$$

As a result, B_μ^0 will be interpolating fields for particles with even spin J (if any).

Because $f_\mu \gtrsim 0$ there must be a $J = 0$ component present.

If it is possible to lower the UV cutoff a^{-1} sufficiently before losing good locality properties of the effective action $L(\phi)$, one may hope to end up with an action in which only terms that depend on the $SU(N)$ gauge invariant fields A_μ, B_μ^0 alone are nonnegligible. One may call this a phenomenological effective action. In the 4-th order expression (2.8), only the $\mathcal{F}_{\mu\nu}^2$ term is not a function of gauge invariant fields A_μ, B_μ^0 alone.

Appendix: The $\sqrt{3}$ scheme of Cordery, Gupta and Novotny /15/

One begins with a map \mathcal{K} of the original lattice Λ' of lattice spacing a into a sublattice $\mathcal{K}\Lambda'$ of Λ' with lattice spacing $\sqrt{3}a$. The lattice point $x' = 0$ is mapped into itself, and the lattice vectors, e_1', \dots, e_4' of length a in Λ' are mapped into

$$\begin{aligned} e_1' &= e_2' + e_3' + e_4' \\ e_2' &= e_1' + e_3' - e_4' \\ e_3' &= e_1' - e_2' + e_4' \\ e_4' &= e_1' + e_2' - e_3' \end{aligned} \tag{A.1}$$

respectively. The lattice $\mathcal{K}\Lambda'$ is oblique relative to Λ' . A block lattice of lattice spacing $a = 3^{n/2} a'$ is obtained by applying the map \mathcal{K} n times

$$\Lambda = \mathcal{K}^n \Lambda'$$

Next we define a block spin of the form (1.2). Given a path C on a lattice Σ which consists of links b_1, \dots, b_n , and a gauge field ϕ on Σ , dielectric or otherwise, one defines the (generalized) parallel transporter

$$\Phi(C) = \phi(b_n) \dots \phi(b_1) \in \mathcal{G} \tag{A.2}$$

We consider formal sums $\sum_{C_i} p_i$ of finite numbers of paths C_i with real weights p_i . We extend the definition (A.2) of parallel transporters to arguments which are such formal sums by setting

$$\Phi(\sum p_i C_i) = \sum p_i \Phi(C_i) \tag{A.3}$$

We will now introduce a map \mathcal{A} of formal sums of paths on $\mathcal{K}\Sigma$ into formal sums of paths on Σ . Let $b = (y, x)$ be a link in $\mathcal{K}\Sigma$. Then x and y are at the same time points of Σ , and there exist six 3-link-paths C_i on Σ which connect x to y . We set

Acknowledgement: The authors are very grateful to K. Gawędzki for emphasizing the advantages of block spin transformations with Gaussian factors, and also to H. Pirner for stimulating discussions.

$$\mathcal{A}b = \frac{1}{6} \sum_{i=1}^6 C_i$$

The map \mathcal{A} extends to paths made of several links in an obvious way, and to formal sums of paths by linearity, as follows. If C_i^1 are paths from x to y , and C_j^2 are paths from y to z we write $C_j^2 \circ C_i^1$ for the path obtained by juxtaposition of C_i^1 and C_j^2 , and define

$$\left(\sum_i \rho_i C_i^2 \right) \circ \left(\sum_j \sigma_j C_j^1 \right) = \sum_{i,j} \rho_i \sigma_j C_i^2 \circ C_j^1$$

If $C = b_n \circ \dots \circ b_1$ is a path made up from links b_1, \dots, b_n we define

$$\mathcal{A}C = \mathcal{A}b_n \circ \dots \circ \mathcal{A}b_1$$

We define

$$\mathcal{A} \left(\sum_i \rho_i C_i \right) = \sum_i \rho_i \mathcal{A}(C_i)$$

For a block lattice $\Lambda = \mathcal{A}^n \Lambda'$, a block spin $\bar{\Phi}$ on Λ can be defined in terms of a lattice gauge field u on Λ' by

$$\bar{\Phi}(b) = u(\mathcal{A}^n b)$$

This is an example of a block spin of the form (1.2).

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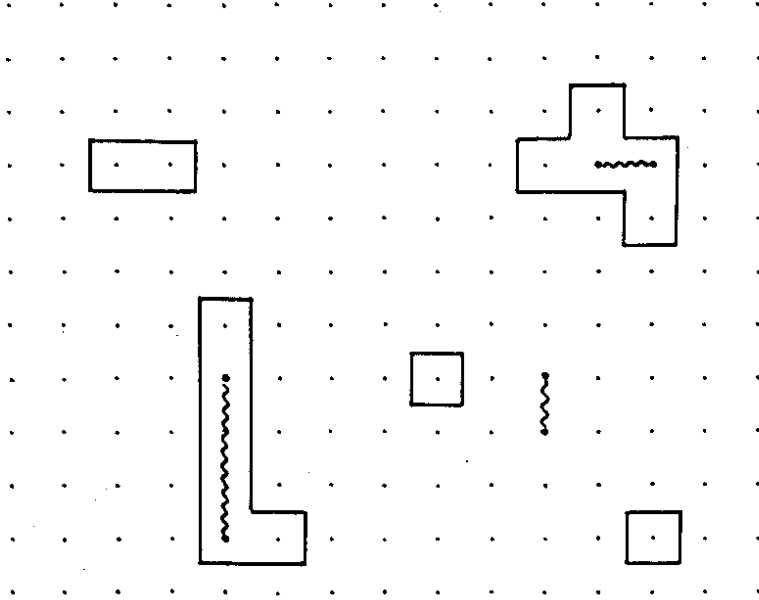


Fig. 1 Large field islands. Inside the islands there could in exceptional cases be correlations ~~~ between distant points. But large islands are very rare (dilute).