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Abstract

In this paper we present a non-linear theory of coupled synchro-betatron oscillations which includes as special cases the well-known theory of non-linear betatron oscillations and the theory of satellite resonances. The method is based on a 6-dimensional canonical formalism in which, after introducing dispersion, all three unperturbed modes are described by Twiss parameters. These Twiss parameters have a more general form than those defined in the usual machine theory of Courant, Livingstone and Snyder. The resonances are then investigated using canonical perturbation theory in a way already familiar from its application to non-linear betatron motion. Selected applications are discussed.

Contents

1. Introduction

In the usual treatments of non-linear betatron motion in storage rings, energy variations due to synchrotron oscillations are ignored and the focussing properties of all magnets are regarded as static.

If the formulation is canonical, then in the uncoupled linearized theory, action-angle variables can be introduced and these are subsequently used to treat non-linear terms in the framework of canonical perturbation theory. This leads to the result that the stable region in the tune diagram for an ideal linear machine becomes criss-crossed by narrow strips where the full non-linear motion is unstable. These are the so-called non-linear resonances and they occur near[1,2,3,4,5,6]

$$m_x Q_x + m_z Q_z = n \ . \tag{1.1}$$

Q_x and Q_z are the horizontal and vertical tunes for the uncoupled linear machine and m and n are integers.

These resonance effects can also be seen in simulations with suitable tracking programs[7] and also, of course, while running machines. In reality, the betatron stop-bands (1.1) are accompanied by further stop-bands, the so-called satellite stop-bands defined by the relation

$$m_x Q_x + m_z Q_z + m_s Q_s = n \tag{1.2}$$

where Q_s is the synchrotron tune.[7]

These synchro-betatron resonance effects can be simulated by tracking programs in which energy variations due to synchrotron oscillations and their effect on the focussing properties of the magnets have been introduced.[8,9]

The mechanisms (energy dependence of the betatron frequency, dispersion in cavities, beam-beam interaction with a crossing angle) leading to excitation of synchro-betatron resonances have already been considered individually in references 10-15.

The aim of the present work is to develop the theory of non-linear synchro-betatron oscillations using methods analogous to those already used for non-linear betatron motion. Then the various effects can be handled in the same way as for the non-linear betatron case. This will be achieved using a dispersion formalism (Ref. 16) in which action-angle variables can be defined for describing linearized uncoupled synchrotron motion (with local cavities).

We are then in the position to carry over all the usual techniques of canonical perturbation theory to the general non-linear synchro-betatron problem. This treatment has the advantage that it is fully canonical, that it can be developed to any order in the perturbation expansion (i.e. to any order of satellite resonance) and that it treats all excitation mechanisms simultaniously. As a special case, it includes the known theory of non-linear betatron motion.

In this paper this programme will be illustrated by the investigation of satellite resonances. Numerical results for a HERA-optic will be presented.

2. Equations of motion

As the starting point for the study of non-linear synchro-betatron oscillations we write the (canonical) equations of motion in the form

$$\begin{cases} \frac{dx}{ds} = \frac{\partial H}{\partial p_x} \; ; & \frac{dp_x}{ds} = -\frac{\partial H}{\partial x} \; ; \\ \frac{dz}{ds} = \frac{\partial H}{\partial p_z} \; ; & \frac{dp_z}{ds} = -\frac{\partial H}{\partial z} \; ; \\ \frac{d\sigma}{ds} = \frac{\partial H}{\partial p_\sigma} \; ; & \frac{dp_\sigma}{ds} = -\frac{\partial H}{\partial \sigma} \; ; \end{cases} \tag{2.1}$$

$$\sigma \equiv s - c \cdot t(s)$$

where the Hamiltonian is (Ref. 8)

$$H = (1 + p_\sigma)\cdot\{1 - (1 + K_x\cdot x + K_z\cdot z)\cdot[1 - \frac{(p_x - \frac{e}{c}A_x)^2}{(1 + p_\sigma)^2} - \frac{(p_z - \frac{e}{c}A_z)^2}{(1 + p_\sigma)^2}]\}^{1/2}$$

$$- (1 + K_x\cdot x + K_z\cdot z) \cdot \frac{e}{E_0} A_s \; . \tag{2.2}$$

The quantity

$$\vec{A} = (A_x, A_z, A_s)$$

is the electromagnetic vector potential and the electric and magnetic fields are then given by [8]

$$\vec{\varepsilon} = \frac{\partial}{\partial \sigma} \vec{A} \quad ; \tag{2.3a}$$

$$\begin{cases} B_x = \frac{1}{h}\cdot\{\frac{\partial}{\partial z}(h \cdot A_s) - \frac{\partial}{\partial s} A_z\} \; ; \\ B_z = \frac{1}{h}\cdot\{\frac{\partial}{\partial s} A_x - \frac{\partial}{\partial x}(h \cdot A_s)\} \; ; \\ B_s = \frac{\partial}{\partial x} A_z - \frac{\partial}{\partial z} A_x \end{cases} \tag{2.3b}$$

where $\quad h = 1 + K_x\cdot x + K_z\cdot z \quad .$

In particular:

For a cavity:

$$\varepsilon(s,\sigma) = V(s) \cdot \sin[k \cdot \frac{2\pi}{L} \cdot \sigma + \varphi] \quad ; \tag{2.4a}$$

$$A_s = - \frac{L}{2\pi \cdot k} \cdot V(s) \cdot \cos[k \cdot \frac{2\pi}{L} \cdot \sigma + \varphi] \quad ; A_x = A_z = 0 \; ; \tag{2.4b}$$

For a quadrupole:

$$\begin{aligned} B_x &= z \cdot \left(\frac{\partial B_z}{\partial x}\right)_{x=z=o} \quad ; \\ B_z &= x \cdot \left(\frac{\partial B_z}{\partial x}\right)_{x=z=o} \quad ; \end{aligned} \tag{2.5a}$$

$$\frac{e}{E_o} A_s = \frac{1}{2} g_o \cdot (z^2 - x^2) \; ; \quad A_x = A_z = 0 \tag{2.5b}$$

$$\text{with} \quad g_o = \frac{e}{E_o} \cdot \left(\frac{\partial B_z}{\partial x}\right)_{x=z=o} \quad ; \tag{2.6}$$

For a dipole bending magnet:

$$(K_x, K_z) \neq (0, 0) \quad ; \quad K_x \cdot K_z = 0 \quad ; \tag{2.7}$$

$$\frac{e}{E_o} \cdot B_x = - K_z \quad ; \tag{2.8a}$$

$$\frac{e}{E_o} \cdot B_z = K_x \quad ; \tag{2.8b}$$

$$\frac{e}{E_o} A_s = - \frac{1}{2} \cdot (1 + K_x \cdot x + K_z \cdot z) \quad ;$$

For a sextupole:

$$\begin{cases} B_x = \left(\frac{\partial^2 B_z}{\partial x^2}\right)_{x=z=o} \cdot xz \quad ; \\ B_z = \left(\frac{\partial^2 B_z}{\partial x^2}\right)_{x=z=o} \cdot \frac{1}{2} (x^2 - z^2) \quad ; \end{cases} \tag{2.9a}$$

$$\frac{e}{E_o} A_s = - \lambda_o \cdot \frac{1}{6} (x^3 - 3xz^2) \tag{2.9b}$$

$$\text{with} \quad \lambda_o = \frac{e}{E_o} \cdot \left(\frac{\partial^2 B_z}{\partial x^2}\right)_{x=z=o} \quad . \tag{2.10}$$

Using (2.4b), (2.5b), (2.8b) and (2.9b) (we assume that the ring only contains cavities, quadrupoles, bending magnets and sextupoles and that there is no beam-beam interaction[11,13]), the Hamiltonian (2.2) becomes:

$$H = (1 + p_\sigma) \cdot \{1 - (1 + K_x \cdot x + K_z \cdot z) \cdot [1 - \frac{p_x^2 + p_z^2}{(1 + p_\sigma)^2}]^{1/2}\}$$

$$+ \frac{1}{2} (1 + K_x \cdot x + K_z \cdot z)^2 - \frac{1}{2} g_0 \cdot (z^2 - x^2) + \frac{\lambda_0}{6} (x^3 - 3xz^2)$$

$$+ \frac{L}{k \cdot 2\pi} \cdot \frac{eV(s)}{E_0} \cdot \{\cos[k \cdot \frac{2\pi}{L} \cdot \sigma + \varphi] - \cos\varphi + k \cdot \frac{2\pi}{L} \cdot \sigma \cdot \sin\varphi\}$$

$$- \sigma \cdot \frac{eV(s)}{E_0} \cdot \sin\varphi + \frac{L}{k \cdot 2\pi} \cdot \frac{eV(s)}{E_0} \cdot \cos\varphi \quad . \qquad (2.11)$$

This Hamiltonian can be expanded in a power series in the variables x, p_x, z, p_z, σ, p_σ. If the series is truncated at the third order we obtain:

$$H = \frac{1}{2} (p_x^2 + p_z^2) \cdot (1 - p_\sigma) - (K_x \cdot x + K_z \cdot z) \cdot p_\sigma +$$

$$+ \frac{1}{2} (K_x^2 + g_0) \cdot x^2 + \frac{1}{2} (K_z^2 - g_0) \cdot z^2 + \frac{\lambda_0}{6} (x^3 - 3xz^2)$$

$$+ \frac{eV(s)}{E_0} \cdot k \cdot \frac{2\pi}{L} \cdot \{ - \frac{1}{2} \cos\varphi \cdot \sigma^2 + \frac{1}{6} \sin\varphi \cdot k \cdot \frac{2\pi}{L} \cdot \sigma^3\}$$

$$- \sigma \cdot \frac{eV(s)}{E_0} \cdot \sin\varphi \quad . \qquad (2.12)$$

Constant terms in the Hamiltonian have no influence on the equations of motion and have been dropped.

This form for H is valid only for ultrarelativistic protons. Radiation effects are also neglected so that there is no energy uptake in the cavities and $\varphi = 0$.

For electrons, radiation effects must be taken into account. See appendix I.

In this discussion we will for brevity neglect vertical betatron oscillations.

Thus, finally the Hamiltonian in (2.12) becomes

$$H = \frac{1}{2} p_x^2 \cdot (1 - p_\sigma) - K_x \cdot x \cdot p_\sigma + \frac{1}{2} (K_x^2 + g_0) \cdot x^2 + \frac{\lambda_0}{6} \cdot x^3 -$$

$$- \frac{1}{2} \cdot \frac{eV(s)}{E_0} \cdot k \cdot \frac{2\pi}{L} \cdot \left\{ \cos\varphi \cdot \sigma^2 - \frac{1}{3} \sin\varphi \cdot k \cdot \frac{2\pi}{L} \cdot \sigma^3 \right\} \tag{2.13a}$$

and the canonical equations are

$$\begin{cases} \dfrac{dx}{ds} = \dfrac{\partial H}{\partial p_x} \; ; \quad \dfrac{dp_x}{ds} = - \dfrac{\partial H}{\partial x} \; ; \\[2ex] \dfrac{d\sigma}{ds} = \dfrac{\partial H}{\partial p_\sigma} \; ; \quad \dfrac{dp_\sigma}{ds} = - \dfrac{\partial H}{\partial \sigma} \; . \end{cases} \tag{2.13b}$$

3. Introduction of dispersion

In the equations of coupled synchro-betatron motion derived from equ. (2.13a,b) the coupling between the longitudinal and transverse motions is described in linear approximation by the term

$$- K_x \cdot x \cdot p_\sigma$$

in the Hamiltonian. This coupling therefore results from the curvature of the design orbit.

In order to rearrange (2.13) so that it is suitable for treatment by canonical perturbation theory it is useful to introduce the dispersion trajectories

$$\vec{D}^T = (D_x, D_x')$$

and instead of the variables x, p_x, σ, p_σ, to use new variables $\tilde{x}$, $\tilde{p}_x$, $\tilde{\sigma}$, $\tilde{p}_\sigma$, where $\tilde{x}$ and $\tilde{p}_x$ are defined by

$$\begin{cases} \tilde{x} = x - p_\sigma \cdot D_x \quad ; \\ \tilde{p}_x = p_x - p_\sigma \cdot D_x' \ . \end{cases} \tag{3.1}$$

This can be achieved by a canonical transformation

$$(x,\ p_x,\ \sigma,\ p_\sigma) \longrightarrow (\tilde{x},\ \tilde{p}_x,\ \tilde{\sigma},\ \tilde{p}_\sigma)$$

resulting from a generating function

$$F_2(x,\ \sigma,\ \tilde{p}_x,\ \tilde{p}_\sigma) = \tilde{p}_x \cdot (x - \tilde{p}_\sigma \cdot D_x) + \tilde{p}_\sigma \cdot D_x' \cdot x - \frac{1}{2} D_x \cdot D_x' \cdot \tilde{p}_\sigma^2 + \tilde{p}_\sigma \cdot \sigma \ . \tag{3.2}$$

The transformation formulae are then

$$\tilde{x} = \frac{\partial F_2}{\partial \tilde{p}_x} = x - \tilde{p}_\sigma \cdot D_x \quad ; \tag{3.3a}$$

$$p_x = \frac{\partial F_2}{\partial x} = \tilde{p}_x + \tilde{p}_\sigma \cdot D_x' \ ; \tag{3.3b}$$

$$\tilde{\sigma} = \frac{\partial F_2}{\partial \tilde{p}_\sigma} = - D_x \cdot \tilde{p}_x + D_x' \cdot \underbrace{(x - \tilde{p}_\sigma \cdot D_x)}_{\tilde{x}} + \sigma \quad ; \tag{3.3c}$$

$$p_\sigma = \frac{\partial F_2}{\partial \sigma} = \tilde{p}_\sigma \quad ; \tag{3.3d}$$

$$\tilde{H} = H + \frac{\partial F_2}{\partial s} \tag{3.4}$$

and one sees that (3.3a,b,d) indeed lead to equ. (3.1).

Taking account of the defining equations for the dispersion

$$D_x'' = -(K_x^2 + g_0)\cdot D_x + K_x \tag{3.5}$$

one obtains (equ. 3.4) the new Hamiltonian $\tilde{H}$ which we write in terms of three components:

$$\tilde{H} = H_{ox} + H_{o\sigma} + H_1 \quad ; \tag{3.6}$$

$$H_{ox} = \frac{1}{2}\tilde{p}_x^2 + \frac{1}{2}(K_x^2 + g_0)\cdot\tilde{x}^2 - $$
$$- \frac{1}{2}\,\frac{eV(s)}{E_0}\cdot k\cdot\frac{2\pi}{L}\cdot\cos\varphi\cdot[\tilde{p}_x\cdot D_x - \tilde{x}\cdot D_x']^2 \quad ; \tag{3.7a}$$

$$H_{o\sigma} = -\frac{1}{2}K_x\cdot D_x\cdot\tilde{p}_\sigma^2 - \frac{1}{2}\frac{eV(s)}{E_0}\cdot k\cdot\frac{2\pi}{L}\cdot\cos\varphi\cdot\tilde{\sigma}^2 \quad ; \tag{3.7b}$$

$$
\begin{aligned}
H_1 = \; & -\frac{eV(s)}{E_0}\cdot k\cdot\frac{2\pi}{L}\cdot\cos\varphi\cdot\tilde{\sigma}\cdot[\tilde{p}_x\cdot D_x - \tilde{x}\cdot D_x'] + && \text{(A)}\\
& + \frac{1}{6}\cdot\frac{eV(s)}{E_0}\cdot[k\cdot\frac{2\pi}{L}]^2\cdot\{\tilde{\sigma} + [\tilde{p}_x\cdot D_x - \tilde{x}\cdot D_x']\}^3\cdot\sin\varphi && \text{(B)}\\
& - \frac{1}{2}\tilde{p}_x^2\cdot\tilde{p}_\sigma - D_x'\cdot\tilde{p}_x\cdot\tilde{p}_\sigma^2 - \frac{1}{2}D_x'^2\cdot\tilde{p}_\sigma^3 + && \text{(C)}\\
& + \frac{\lambda_0}{6}\cdot[\tilde{x}^3 + 3D_x\cdot\tilde{x}^2\cdot\tilde{p}_\sigma + 3D_x^2\cdot\tilde{x}\cdot\tilde{p}_\sigma^2 + D_x^3\cdot\tilde{p}_\sigma^3] + \cdots . && \text{(D)}
\end{aligned}
\tag{3.8}
$$

The new canonical equations of motion are then

$$
\left\{
\begin{aligned}
\frac{d\tilde{x}}{ds} &= \frac{\partial\tilde{H}}{\partial\tilde{p}_x} \; ; \quad \frac{d\tilde{p}_x}{ds} = -\frac{\partial\tilde{H}}{\partial\tilde{x}} \; ;\\
\frac{d\tilde{\sigma}}{ds} &= \frac{\partial\tilde{H}}{\partial\tilde{p}_\sigma} \; ; \quad \frac{d\tilde{p}_\sigma}{ds} = -\frac{\partial\tilde{H}}{\partial\tilde{\sigma}} \; .
\end{aligned}
\right.
\tag{3.9}
$$

In the following, H_{ox} and $H_{o\sigma}$ will be used as the Hamiltonians for the unperturbed betatron and synchrotron motion resp. while H_1 will be regarded as a perturbation.

In linear approximation the coupling between synchrotron and betatron motion only takes place in the cavities (Ref. 8) through the term

$$\frac{eV(s)}{E_0} \cdot k \cdot \frac{2\pi}{L} \cdot \cos\varphi \cdot \tilde{\sigma}[\tilde{p}_x \cdot D_x - \tilde{x} \cdot D_x']$$

in H_1. For

$$V(s) \cdot D_x(s) = 0$$
$$V(s) \cdot D_x'(s) = 0 \qquad (3.10)$$

this term disappears (no dispersion in the cavities). In this case H_1 simplifies to:

$$H_1 = \frac{1}{6}\,\frac{eV(s)}{E_0} \cdot [k \cdot \frac{2\pi}{L}]^2 \cdot \sin\varphi \cdot \tilde{\sigma}^3$$
$$- \frac{1}{2}\,\tilde{p}_x^2 \cdot \tilde{p}_\sigma - D_x' \cdot \tilde{p}_x \cdot \tilde{p}_\sigma^2 - \frac{1}{2}\,D_x'^2 \cdot \tilde{p}_\sigma^3 +$$
$$+ \frac{\lambda_0}{6} \cdot [\tilde{x}^3 + 3D_x \cdot \tilde{x}^2 \cdot \tilde{p}_\sigma + 3D_x^2 \cdot \tilde{x} \cdot \tilde{p}_\sigma^2 + D_x^3 \cdot \tilde{p}_\sigma^2] + \dots \qquad (3.11)$$

4. Action-angle variables for the unperturbed oscillation modes

With equs. (3.6-8) we have a representation of the Hamiltonian which enables the use of canonical perturbation theory. The important point here is that the Hamiltonians H_{0x} and $H_{0\sigma}$ which describe the unperturbed motion can be written in the general form

$$H_0 = \frac{1}{2} F(s) \cdot p^2 + R(s) \cdot y \cdot p + \frac{1}{2} G(s) \cdot y^2 \qquad (4.1)$$

with

$$\begin{cases} F(s) = 1 - \dfrac{eV(s)}{E_0} \cdot k \cdot \dfrac{2\pi}{L} \cdot \cos\varphi \cdot D_x^2 \quad ; \\ R(s) = D_x \cdot D_x' \cdot \dfrac{eV(s)}{E_0} \cdot k \cdot \dfrac{2\pi}{L} \cdot \cos\varphi \quad ; \\ G(s) = (K_x^2 + g_0) - \dfrac{eV(s)}{E_0} \cdot k \cdot \dfrac{2\pi}{L} \cdot \cos\varphi \cdot D_x'^2 \end{cases} \qquad (4.2a)$$

for betatron motion

and

$$\begin{cases} F(s) = -K_x \cdot D_x \quad ; \\ R(s) = 0 \quad ; \\ G(s) = -\dfrac{eV(s)}{E_0} \cdot k \cdot \dfrac{2\pi}{L} \cdot \cos\varphi \end{cases} \qquad (4.2b)$$

for synchrotron motion.

With these forms (4.1) it is possible to define action-angle variables. This will be carried out in the following in several steps.

4.1 Twiss-Parameters

The canonical equations resulting from equ. (4.1) are

$$\frac{d}{ds} \begin{pmatrix} y \\ p \end{pmatrix} = \underline{A}(s) \cdot \begin{pmatrix} y \\ p \end{pmatrix} \qquad (4.3a)$$

with

$$\underline{A}(s) = \begin{pmatrix} R & F \\ -G & -R \end{pmatrix} \qquad (4.3b)$$

The solution to (4.3) can be written in the form

$$\begin{pmatrix} y(s) \\ p(s) \end{pmatrix} = \underline{M}(s,s_0) \begin{pmatrix} y(s_0) \\ p(s_0) \end{pmatrix} \tag{4.4}$$

where $\underline{M}(s,s_0)$ is a transfer matrix satisfying the equations

$$\frac{d}{ds}\underline{M}(s,s_0) = \underline{A}(s)\cdot\underline{M}(s,s_0) \tag{4.5a}$$

$$\underline{M}(s,s_0) = \underline{1} \ . \tag{4.5b}$$

It is well-known that the one turn matrix for pure betatron motion can be represented in terms of the Twiss-Parameters α, β, γ (Ref. 18). However, this representation may also be used to represent the more general case of transfer matrices (equ. (4.4)) resulting from the Hamiltonian (4.1). Once Twiss-Parameters have been established it is then straightforward to proceed to an action-angle variable representation. With this in mind we recall that the transfer matrix satisfies the symplecticity conditions

$$\underline{M}^T(s,s_0)\cdot\underline{S}\cdot\underline{M}(s,s_0) = \underline{S} \tag{4.6}$$

with

$$\underline{S} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} . \tag{4.7}$$

Denoting the left handside of (4.6) by $\underline{B}(s)$:

$$\underline{B}(s) = \underline{M}^T(s,s_0)\cdot\underline{S}\cdot\underline{M}(s,s_0) \quad ,$$

one can immediately prove the symplecticity conditions using the relations:

$$\underline{B}(s_0) = \underline{S} \quad \text{following equ. (4.5b)}$$

$$\frac{d}{ds}\underline{B}(s) = [\frac{d}{ds}\underline{M}(s,s_0)]^T\cdot\underline{S}\cdot\underline{M}(s,s_0) + \underline{M}^T(s,s_0)\cdot\underline{S}\cdot\frac{d}{ds}\underline{M}(s,s_0)$$

$$= \underline{M}^T(s,s_0)\cdot\underbrace{[A^T\cdot\underline{S} + \underline{S}\cdot\underline{A}]}_{=\,0 \text{ following equs. (4.3b) and (4.7)}}\cdot\underline{M}(s,s_0) = \underline{0} \ .$$

Since $\underline{M}^T \cdot \underline{S} \cdot \underline{M} = \underline{S} \cdot \det(\underline{M})$

equ. (4.6) is equivalent to the condition

$$\det(\underline{M}) = 1 \ . \tag{4.8}$$

On the basis of this relation which is also valid for the one turn matrix $\underline{M}(s+L,s)$, we may write the one turn matrix following Courant and Snyder (Ref. 16) in the form

$$\underline{M}(s+L,s) = \cos\mu \cdot \underline{1} + \sin\mu \cdot \underline{J}(s) \tag{4.9a}$$

with

$$\underline{J}(s) = \begin{pmatrix} \alpha(s) & \beta(s) \\ -\gamma(s) & -\alpha(s) \end{pmatrix} \ . \tag{4.9b}$$

Since $\det(\underline{M}) = 1$

we have

$$\gamma \cdot \beta - \alpha^2 = 1 \tag{4.9c}$$

where in addition we require

$$\beta \geq 0 \ . \tag{4.9d}$$

These equations (4.9) are the defining equations for the Twiss-Parameters α, β, γ for the general case. If the matrix elements are known:

$$\underline{M}(s+L,s) = \begin{pmatrix} M_{11}(s+L,s) & M_{12}(s+L,s) \\ M_{21}(s+L,s) & M_{22}(s+L,s) \end{pmatrix},$$

the quantities μ, α, β and γ are defined by the equations

$$\cos\mu = \frac{1}{2} \cdot \mathrm{Sp} \ \underline{M}(s+L,s) \tag{4.10a}$$

$$\sin\mu \begin{cases} > 0, & \text{if } M_{12} > 0 \ ; \\ < 0, & \text{if } M_{12} < 0 \ ; \end{cases} \quad \text{(since } \beta \geq 0\text{)} \tag{4.10b}$$

$$\alpha(s) = \frac{1}{2 \cdot \sin\mu} \cdot [M_{11}(s+L,s) - M_{22}(s+L,s)] \ ; \tag{4.10c}$$

$$\beta(s) = \frac{M_{12}(s+L,s)}{\sin\mu} \ ; \tag{4.10d}$$

$$\gamma(s) = -\frac{M_{21}(s+L,s)}{\sin\mu} \ . \tag{4.10e}$$

From the condition

$$\underline{M}(s+L,s) = \underline{M}(s+L,s_0+L) \cdot \underline{M}(s_0+L,s_0) \cdot \underline{M}(s_0,s)$$
$$= \underline{M}(s,s_0) \cdot \underline{M}(s_0+L,s_0) \cdot \underline{M}^{-1}(s,s_0) \qquad (4.11)$$

we also have

$$\cos\mu(s) = \frac{1}{2} \cdot \mathrm{Sp}\ \underline{M}(s+L,s) = \frac{1}{2} \cdot \mathrm{Sp}\ \underline{M}(s_0+L,s_0) = \cos\mu(s_0)$$
$$\text{i.e.} \quad \mu = \text{const.} \qquad (4.12)$$

and from (4.9a) and (4.12)

$$\underline{J}(s) = \underline{M}(s,s_0) \cdot J(s_0) \cdot \underline{M}^{-1}(s,s_0) \ . \qquad (4.13)$$

Using (4.13) we can now calculate $\alpha(s)$, $\beta(s)$, $\gamma(s)$ if the values $\alpha(s_0)$, $\beta(s_0)$, $\gamma(s_0)$ have been previously calculated (e.g. using (4.10)).

Furthermore, from (4.12)

$$\underline{J}(s_0+L) = \underline{M}(s_0+L,s_0) \cdot \underline{J}(s_0) \cdot \underline{M}^{-1}(s_0+L,s_0)$$
$$= [\underline{1} \cdot \cos\mu + \underline{J}(s_0) \cdot \sin\mu] \cdot \underline{J}(s_0) \cdot [\underline{1} \cdot \cos\mu + \underline{J}(s_0) \cdot \sin\mu]^{-1}$$
$$= \underline{J}(s_0) \cdot [\underline{1} \cdot \cos\mu + \underline{J}(s_0) \cdot \sin\mu] \cdot [\underline{1} \cdot \cos\mu + \underline{J}(s_0) \cdot \sin\mu]^{-1}$$
$$= \underline{J}(s_0) \ ,$$

so that

$$\begin{cases} \alpha(s_0+L) = \alpha(s_0) \ ; \\ \beta(s_0+L) = \beta(s_0) \ ; \\ \gamma(s_0+L) = \gamma(s_0) \ . \end{cases} \qquad (4.14)$$

Thus $\alpha(s)$, $\beta(s)$ and $\gamma(s)$ are periodic.

Finally, using (4.12), from which

$$\underline{J}'(s) = \begin{pmatrix} \alpha'(s) & \beta'(s) \\ -\gamma'(s) & -\alpha'(s) \end{pmatrix} \qquad (4.15)$$

we can obtain differential equations for α, β, γ which are needed for the construction of the action-angle variables:

$$\underline{J}'(s) = \lim_{\Delta s \to 0} \frac{1}{\Delta s} \cdot \{\underline{J}(s+\Delta s) - \underline{J}(s)\}$$

$$= \lim_{\Delta s \to 0} \frac{1}{\Delta s} \cdot \{\underline{M}(s+\Delta s,s) \cdot \underline{J}(s) \cdot \underline{M}^{-1}(s+\Delta s,s) - J(s)\}$$

(following equ. (4.12))

$$= \lim_{\Delta s \to 0} \frac{1}{\Delta s} \cdot \{[\underline{1} + \Delta s \cdot \underline{A}(s)] \cdot \underline{J}(s) \cdot [\underline{1} - \Delta s \cdot \underline{A}(s)] - \underline{J}(s)\}$$

(following equ. (4.3a) or (4.5))

$$= \underline{A}(s) \cdot \underline{J}(s) - \underline{J}(s) \cdot \underline{A}(s)$$

$$= \begin{pmatrix} -\gamma F + \beta G & 2(\beta R - \alpha F) \\ 2\cdot(-\alpha G + \gamma R) & -(-\gamma F + \beta G) \end{pmatrix} \tag{4.16}$$

(following equ. (4.3b) and (4.9b))

By comparing (4.15) and (4.16) one then finds that

$$\alpha'(s) = -\gamma F + \beta G \quad ; \tag{4.17a}$$

$$\beta'(s) = 2(\beta R - \alpha F) \quad ; \tag{4.17b}$$

$$\gamma'(s) = 2(\alpha G - \gamma R) \quad . \tag{4.17c}$$

Remark : In equ. (4.10b) it is assumed that the parameter μ is real. Only then can the motion remain stable since for k turns we can write

$$\underline{M}(s+L,s)^k = [\cos\mu \cdot \underline{1} + \sin 1 \cdot \underline{J}]^k$$

$$= \cos(k\mu) \cdot \underline{1} + \sin(k\mu) \cdot \underline{J} \tag{4.18}$$

where we have used the relation

$$\underline{J}^2(s) = -\underline{1} \quad .$$

For arbitrary k the matrix $\underline{M}^k$ remains finite only if μ is real[18].

In the following we assume that the stability conditions are satisfied. In addition, the quantities Q appearing in equ. (4.34), (4.24), (4.33) are also real.

4.2 Action-angle variables for the unperturbed Hamiltonian

Using the properties of the Twiss-parameters established in section 4.1 we are now in the position to construct the action-angle variables for the unperturbed Hamiltonian H_0.

This is achieved using the generating function

$$F_1(y,\Phi) = -\frac{y^2}{2\beta}\cdot[\mathrm{tg}(\Phi+\Phi_0)+\alpha] \tag{4.19}$$

to make the transformation $(y, p) \longrightarrow (\Phi, I)$.

Using the transformation relations

$$p = \frac{\partial F_1}{\partial y} = -\frac{y}{\beta}\cdot[\mathrm{tg}(\Phi+\Phi_0)+\alpha] \Longrightarrow \mathrm{tg}(\Phi+\Phi_0) = -\frac{\beta\cdot p}{y} - \alpha \; ; \tag{4.19a}$$

$$\begin{aligned} I = -\frac{\partial F_1}{\partial \Phi} &= \frac{y^2}{2\beta}\cdot\frac{1}{\cos^2(\Phi+\Phi_0)} \\ &= \frac{y^2}{2\beta}\cdot[1+\mathrm{tg}^2(\Phi+\Phi_0)] \\ &= \frac{1}{2\beta}\cdot[y^2+(\alpha\cdot y+\beta\cdot p)^2] \\ &= \frac{1}{2}\cdot[\gamma\cdot y^2+\beta\cdot p^2+2\alpha\cdot p\cdot y] \end{aligned} \tag{4.19b}$$

we obtain the representation

$$y(s) = \sqrt{2\beta(s)\cdot I}\cdot\cos[\Phi(s)+\Phi_0] \; ; \tag{4.20a}$$

$$p(s) = -\sqrt{\frac{2\cdot I}{\beta(s)}}\cdot\{\sin[\Phi(s)+\Phi_0]+\alpha(s)\cdot\cos[\Phi(s)+\Phi_0]\}. \tag{4.20b}$$

The new Hamiltonian $\overline{H}_0(\Phi, I, s)$ is

$$\overline{H}_0 = H_0 + \frac{\partial F_1}{\partial s} \quad .$$

Using (4.17) we obtain

$$\overline{H}_0 = \frac{F(s)}{\beta(s)}\cdot I \tag{4.21}$$

with the accompanying canonical equations

$$\frac{d}{ds}\Phi = \frac{\partial}{\partial I}\bar{H}_0 = \frac{F}{\beta} \Longrightarrow \Phi(s) = \Phi(s_0) + \int_{s_0}^{s} d\tilde{s}\,\frac{F(\tilde{s})}{\beta(\tilde{s})} \quad ; \qquad (4.22a)$$

$$\frac{d}{ds} I = -\frac{\partial}{\partial\Phi}\bar{H}_0 = 0 \Longrightarrow I = \text{const.} \qquad (4.22b)$$

Although the quantity I in (4.22) is a constant, $\Phi'(s)$ varies with s. This is in contrast to the usual case with action-angle variables, where $\Phi(s)$ would vary linearly with s. However, the usual action-angle variable representation can be immediately obtained by using one more canonical transformation

$$(\Phi, I) \longrightarrow (\Psi, \hat{I})$$

with the generating function

$$F_2(\Phi, \hat{I}, s) = \hat{I}\cdot\left[\frac{2\pi Q}{L}\cdot s - \int_0^s d\tilde{s}\cdot\frac{F(\tilde{s})}{\beta(\tilde{s})}\right] + \Phi\cdot\hat{I}\ , \qquad (4.23)$$

in which Q is given by

$$Q = \frac{1}{2\pi}\int_0^L d\tilde{s}\cdot\frac{F(\tilde{s})}{\beta(\tilde{s})} \quad . \qquad (4.24)$$

Then

$$\Psi = \frac{\partial F_2}{\partial\hat{I}} = \frac{2\pi Q}{L}\cdot s - \int_0^s d\tilde{s}\cdot\frac{F(\tilde{s})}{\beta(\tilde{s})} + \Phi \quad ; \qquad (4.25a)$$

$$I = \frac{\partial F_2}{\partial\Phi} = \hat{I} \quad ; \qquad (4.25b)$$

$$\bar{\bar{H}}_0 = \bar{H}_0 + \frac{\partial F_2}{\partial s} = \hat{I}\cdot\frac{2\pi Q}{L} \qquad (4.26)$$

and the variable Ψ has a linear dependence on s:

$$\frac{d\Psi}{ds} = \frac{\partial\bar{\bar{H}}_0}{\partial\hat{I}} = \frac{2\pi Q}{L} \Longrightarrow \Psi(s) = \Psi(s_0) + \frac{2\pi Q}{L}\cdot(s - s_0)\ . \qquad (4.27)$$

$\hat{I}$ is identical to I and is a constant of the motion.

Both Φ and Ψ can be used in canonical perturbation theory.
Our next aim is to use equs. (4.20), (4.22) and (4.26) to examine the particle motion. This will lead to an appreciation of the physical significance of I and Φ or of $\hat{I}$ and Ψ respectively.

4.3 Eigenvectors for the particle motion

In discussing particle motion we now note, that by using the representation (4.20) of the orbit vector

$$\vec{u}(s) = \begin{pmatrix} y(s) \\ p(s) \end{pmatrix}$$

for which I is constant and Φ_0 is a constant to be chosen, two linearly independent solutions to the equation of motion (4.3a) can be obtained. We could for example choose $\Phi_0 = 0$ and $\Phi_0 = -\frac{\pi}{2}$ (with $2I = \frac{1}{2}$):

$$\vec{u}_1(s) = \begin{pmatrix} \sqrt{\frac{\beta(s)}{2}} \cdot \cos\Phi(s) \\ -\frac{1}{\sqrt{2\beta(s)}} \cdot [\sin\Phi(s) + \alpha(s) \cdot \cos\Phi(s)] \end{pmatrix} \; ; \qquad (4.28a)$$

$$\vec{u}_2(s) = \begin{pmatrix} \sqrt{\frac{\beta(s)}{2}} \cdot \sin\Phi(s) \\ \frac{1}{\sqrt{2\beta(s)}} \cdot [\cos\Phi(s) - \alpha(s) \cdot \sin\Phi(s)] \end{pmatrix} \; . \qquad (4.28b)$$

The general solution is then a linear combination of $\vec{u}_1$ and $\vec{u}_2$ in the form

$$\vec{u} = c_1 \cdot \vec{u}_1 + c_2 \cdot \vec{u}_2 \; . \qquad (4.29)$$

Given these solution vectors it is possible to replace the quantity μ in (4.9a) by Φ. From the relation

$$(\vec{u}_1(s+L) \;\; \vec{u}_2(s+L)) = \underline{M}(s+L,s) \cdot (\vec{u}_1(s) \;\; \vec{u}_2(s))$$

we have

$$\underline{M}(s+L,s) = (\vec{u}_1(s+L) \;\; \vec{u}_2(s+L)) \cdot (\vec{u}_1(s) \;\; \vec{u}_2(s))^{-1} \qquad (4.30)$$

and by putting (4.28) into (4.30) and using the periodicity conditions (4.14) we get

$$\underline{M}(s+L,s) = \underline{1} \cdot \cos[\Phi(s+L) - \Phi(s)] + \underline{J}(s) \cdot \sin[\Phi(s+L) - \Phi(s)]. \qquad (4.31)$$

By comparing (4.31) and (4.9a) we find

$$\mu = \Phi(s+L) - \Phi(s) \; . \qquad (4.32)$$

Since from (4.22a) and (4.24)

$$\Phi(s+L) - \Phi(s) = \int_{s}^{s+L} d\tilde{s} \cdot \frac{F(\tilde{s})}{\beta(\tilde{s})}$$

$$= \int_{0}^{L} d\tilde{s} \cdot \frac{F(\tilde{s})}{\beta(\tilde{s})}$$

$$= 2\pi Q \qquad (4.33)$$

we can replace (4.32) by

$$\mu = 2\pi Q \quad . \qquad (4.34)$$

The quantity, Q, introduced in (4.24) can, according to (4.33), be interpreted as the number of betatron or synchrotron waves per revolution.

For further development it is useful to introduce in addition the solution vectors $\vec{\upsilon}_1$ and $\vec{\upsilon}_2$:

$$\vec{\upsilon}_I(s) = \vec{u}_1(s) - i \cdot \vec{u}_2(s) = \frac{1}{\sqrt{2\beta(s)}} \begin{pmatrix} \beta(s) \\ -[\alpha(s) + i] \end{pmatrix} \cdot e^{-i \cdot \psi(s)} \quad ; \qquad (4.35a)$$

$$\vec{\upsilon}_{-I}(s) = \vec{u}_1(s) + i \cdot \vec{u}_2(s) \equiv \vec{\upsilon}_I^{\,*}(s) \qquad (4.35b)$$

which are eigenvectors of the one turn matrix $\underline{M}(s+L,s)$:

$$\underline{M}(s+L,s)\, \vec{\upsilon}_I(s) = \lambda_I \cdot \vec{\upsilon}_I(s) \quad ; \qquad (4.36a)$$

$$\underline{M}(s+L,s)\, \vec{\upsilon}_{-I}(s) = \lambda_{-I} \cdot \vec{\upsilon}_{-I}(s) \qquad (4.36b)$$

with the eigenvalues

$$\lambda_I = e^{-i \cdot 2\pi Q} \qquad (4.37a)$$

$$\lambda_{-I} = e^{+i \cdot 2\pi Q} \qquad (4.37b)$$

and the normalization conditions

$$\vec{\upsilon}_I^{\,+}(s) \cdot \underline{S} \cdot \vec{\upsilon}_I(s) = i \quad ; \qquad (4.38a)$$

$$\vec{\upsilon}_{-I}^{\,+}(s) \cdot \underline{S} \cdot \vec{\upsilon}_{-I}(s) = -i \quad . \qquad (4.38b)$$

Equs. (4.36) and (4.38) can be verified by putting (4.35) into the left sides of (4.36) and (4.38) and using (4.31), (4.33) and (4.9c).

We also note that $\vec{\upsilon}_I$ and $\vec{\upsilon}_{-I}$ are orthogonal in the sense that

$$\vec{\upsilon}_I^{+}(s) \cdot \underline{S} \cdot \vec{\upsilon}_{-I}(s) = \vec{\upsilon}_{-I}^{+}(s) \cdot \underline{S} \cdot \vec{\upsilon}_I(s) = 0 \,. \qquad (4.39)$$

Finally, we point out that according to equ. (4.19b) the particles move in the (y,p) phase space on the ellipse:

$$\gamma \cdot y^2 + \beta \cdot p^2 + 2\alpha \cdot py = 2I \quad . \qquad (4.40)$$

Usually one writes the phase ellipse in the form

$$\gamma \cdot y^2 + \beta \cdot p^2 + 2\alpha \cdot py = \varepsilon \qquad (4.41)$$

where the area of the ellipse is

$$J = \pi \varepsilon \qquad (4.42)$$

and ε is called the emittance. Clearly we may put

$$\varepsilon = 2I \qquad (4.43)$$

and we then see that (4.22b) the area is a constant of the motion, independent of azimuth s. This is a manifestation of Liouville's Theorem.

More details on the properties of the ellipse and its representation in terms of conjugate trajectories can be found in ref. (17).

4.4 Special cases for unperturbed motion

To complete this chapter two special cases for the unperturbed Hamiltonian (4.1) will be investigated.

4.4.1 Betatron motion neglecting the influence of cavities

The general form for linear betatron motion is described by equs. (4.1) and (4.2a). If we neglect the effect of cavity fields on the transverse motion, H_0 becomes (with $y \equiv x$):

$$H_{0x} = \frac{1}{2} \cdot p_x^2 + \frac{1}{2} (K_x^2 + g_0) \cdot x^2 \quad . \tag{4.44}$$

This is even exact when there is no dispersion in the cavities i.e. if

$$V(s) \cdot D_x(s) = 0 \quad ;$$

$$V(s) \cdot D_x'(s) = 0 \quad . \tag{4.45}$$

Thus we have

$$F(s) = 1 \quad ; \tag{4.46a}$$

$$R(s) = 0 \quad ; \tag{4.46b}$$

$$G(s) = (K_x^2 + g_0) \quad , \tag{4.46c}$$

and according to (4.3a,b) the equations of motion become

$$x'(s) = p(s)$$

$$p_x'(s) = - G(s) \cdot x \tag{4.47a}$$

or equivalently

$$x''(s) = - G(s) \cdot x \tag{4.47b}$$

Note that in this case p(s) is just the derivative x'(s).

Furthermore, from (4.17), (4.22a) and (4.35) the differential equations for the Twiss parameters reduce to the well-known forms given by Courant and Snyder [18]:

$$\alpha'(s) = \beta \cdot G - \gamma \quad ; \tag{4.48a}$$

$$\beta'(s) = -2 \cdot \alpha(s) \quad ; \tag{4.48b}$$

$$\gamma'(s) = 2G \cdot \alpha(s) \quad ; \tag{4.48c}$$

$$\Phi'(s) = \frac{1}{\beta(s)} \tag{4.48d}$$

and from (4.20) and (4.43) the particle motion has the form:

$$x(s) = \sqrt{\varepsilon \cdot \beta_x(s)} \cdot \cos[\Phi(s) + \Phi_0] \quad ; \tag{4.49a}$$

$$p_x(s) \equiv x'(s) = -\sqrt{\frac{\varepsilon}{\beta(s)}} \cdot \{\sin[\Phi(s) + \Phi_0] + \alpha(s) \cdot \cos[\Phi(s) + \Phi_0]\} \tag{4.49b}$$

where $\beta_x(s)$ and $\Phi_x(s)$ are the usual amplitude and phase functions.

4.4.2 Synchrotron motion using the "oscillator model"

According to (4.1) and (4.2b) the unperturbed Hamiltonian $H_{0\sigma}$ for synchrotron motion is

$$H_{0\sigma} = -\frac{1}{2} K_x \cdot D_x \cdot p_\sigma^2 - \frac{1}{2} \frac{eV(s)}{E_0} \cdot k \cdot \frac{2\pi}{L} \cdot \cos\varphi \cdot \sigma^2 \quad . \tag{4.50}$$

If the cavities are treated as pointlike at positions s_μ, we put

$$V(s) = \hat{V} \cdot \sum_\mu \delta(s - s_\mu) \quad . \tag{4.51}$$

We now replace the coefficients of p_σ^2 and σ^2 in (4.50) by their averages over one revolution and make the replacements:

$$K_x \cdot D_x \longrightarrow \hat{\varkappa} = \frac{1}{L} \int_0^L ds \cdot K_x(s) \cdot D_x(s) \tag{4.52}$$

(momentum compaction factor) ;

$$\frac{eV(s)}{E_0} \cdot k \cdot \frac{2\pi}{L} \cdot \cos\varphi \longrightarrow \frac{1}{L} \int_0^L ds \cdot \frac{eV(s)}{E_0} \cdot k \cdot \frac{2\pi}{L} \cdot \cos\varphi$$

$$= \frac{1}{L} \cdot k \cdot \frac{2\pi}{L} \cdot \frac{\cos\varphi}{\sin\varphi} \cdot \int_0^L ds \cdot \frac{eV(s)}{E_0} \cdot \sin\varphi$$

$$= \frac{\Omega^2}{\hat{\varkappa}} \quad . \tag{4.53}$$

(As in Appendix I, for electrons we would have:

$$\Omega^2 = \frac{\hat{\varkappa}}{L} \cdot k \cdot \frac{2\pi}{L} \cdot \operatorname{ctg}\varphi \cdot \frac{U_0}{E_0} \quad ; \tag{4.53a}$$

$$U_0 = \int_0^L ds \cdot eV(s) \cdot \sin\varphi \tag{4.53b}$$

= average energy gained by a particle in one turn

$$= E_0 \cdot \int_0^L ds \cdot C_1 \cdot K^2(s)$$

= average energy radiated by a particle in one turn.)

Then the Hamiltonian in (4.50) reduces to:

$$H_{o\sigma} \longrightarrow \overline{H}_{o\sigma} = -\frac{1}{2}\hat{\varkappa} \cdot p_\sigma^2 - \frac{1}{2}\frac{\Omega^2}{\hat{\varkappa}} \cdot \sigma^2 \tag{4.54}$$

so that in (4.1)

$$\begin{cases} F_\sigma(s) = -\hat{\varkappa} \quad ; & (4.55a) \\ R_\sigma(s) = 0 \quad ; & (4.55b) \\ G_\sigma(s) = -\dfrac{\Omega^2}{\hat{\varkappa}} & (4.55c) \end{cases}$$

and from (4.17) and (4.22a) we have

$$\alpha'_\sigma(s) = \hat{\varkappa} \cdot \gamma_\sigma(s) - \frac{\Omega^2}{\hat{\varkappa}} \cdot \beta_\sigma(s) \quad ; \tag{4.56a}$$

$$\beta'_\sigma(s) = 2\hat{\varkappa} \cdot \alpha_\sigma(s) \quad ; \tag{4.56b}$$

$$\gamma'(s) = -2 \cdot \frac{\Omega^2}{\hat{\varkappa}} \cdot \alpha_\sigma(s) \quad ; \tag{4.56c}$$

$$\Phi'(s) = -\frac{\hat{\varkappa}}{\beta(s)} \quad . \tag{4.56d}$$

The equations of motion are now

$$\begin{cases} \dfrac{d}{ds}\sigma = -\hat{\varkappa} \cdot p_\sigma \quad ; & (4.57a) \\ \dfrac{d}{ds} p_\sigma = \dfrac{\Omega^2}{\hat{\varkappa}} \cdot \sigma & (4.57b) \end{cases}$$

or

$$\begin{cases} \dfrac{d^2\sigma}{ds^2} = -\Omega^2 \cdot \sigma \quad ; & (4.58a) \\ \dfrac{d^2 p_\sigma}{ds^2} = -\Omega^2 \cdot p_\sigma \quad . & (4.58b) \end{cases}$$

These have the form of a simple oscillator equation - hence the term "oscillator model" for the approximation (4.54). The solution to (4.57) has the form:

$$\begin{pmatrix} \sigma(s) \\ p_\sigma(s) \end{pmatrix} = \underline{M}\,(s,s_0) \begin{pmatrix} \sigma(s_0) \\ p_\sigma(s_0) \end{pmatrix} \quad ; \tag{4.59a}$$

$$\underline{M}_\sigma(s,s_0) = \begin{pmatrix} \cos\Omega(s-s_0) & -\dfrac{\hat{\varkappa}}{\Omega} \cdot \sin\Omega(s-s_0) \\ \dfrac{\Omega}{\hat{\varkappa}} \cdot \sin\Omega(s-s_0) & \cos\Omega(s-s_0) \end{pmatrix} \tag{4.59b}$$

where $\underline{M}_\sigma(s,s_0)$ is the transfer matrix for the equations (4.57).
In particular for the one turn matrix **one** has

$$\underline{M}\,(s+L,s) = \begin{pmatrix} \cos\Omega L & -\dfrac{\hat{\varkappa}}{\Omega} \cdot \sin\Omega L \\ \dfrac{\Omega}{\hat{\varkappa}} \cdot \sin\Omega L & \cos\Omega L \end{pmatrix} \quad . \tag{4.60}$$

By comparing with (4.9a,b), the forms for α, β and γ can be read off directly:

$$\beta_\sigma = \frac{\hat{\varkappa}}{\Omega} \quad ; \qquad (4.61a)$$

$$\alpha_\sigma = 0 \quad ; \qquad (4.61b)$$

$$\gamma_\sigma = \frac{\Omega}{\hat{\varkappa}} \quad . \qquad (4.61c)$$

These also satisfy the differential equations (4.56a,b,c).

The phase function $\Phi_\sigma(s)$ can be evaluated using (4.56a) and (4.61) as:

$$\Phi_\sigma(s) = \Phi_\sigma(s_0) - \Omega \cdot (s - s_0) \quad , \qquad (4.62)$$

where

$$Q_\sigma = -\frac{\Omega \cdot L}{2\pi} \quad . \qquad (4.63)$$

This relation can also be obtained from (4.24) using (4.55a) and (4.61a).

As an example, for a typical HERA electron optic the oscillator model gives

$$\alpha_\sigma = 0 \quad ;$$

$$\beta_\sigma = 12.505 \text{ m}$$

$$Q_\sigma = -0.0537 \quad .$$

An exact calculation of these quantities using localized cavities (SLICK[24]) dispersion version) gives

$$\text{Max } |\alpha_\sigma| = 0.03 \quad ;$$

$$\text{Max } \beta_\sigma = 12.480 \text{ m} \quad ; \quad \text{Min } \beta_\sigma = 12.476 \text{ m} \quad ;$$

$$Q_\sigma = -0.0537 \quad .$$

Thus the oscillator model is a very good approximation in this case.

5. Canonical perturbation theory

5.1 The perturbed Hamiltonian in terms of the action-angle variables of the unperturbed problem

Now that the equations for unperturbed betatron and synchrotron motion have been written in the above canonical forms we may use familiar techniques for investigating the nonlinear coupled synchro-betatron motion described by the full Hamiltonian $\tilde{H}$ (3.6).

We first of all write equ. (3.6) in the form

$$\begin{aligned}\tilde{H} = \; & \frac{1}{2} F_x(s) \cdot \tilde{p}_x^2 + R_x(s) \cdot \tilde{x} \cdot \tilde{p}_x + \frac{1}{2} G_x(s) \cdot \tilde{x}^2 + \\ & + \frac{1}{2} F_\sigma(s) \cdot \tilde{p}_\sigma^2 + R_\sigma(s) \cdot \tilde{\sigma} \cdot \tilde{p}_\sigma + \frac{1}{2} G_\sigma(s) \cdot \tilde{\sigma}^2 + \\ & + \sum_{\mu_1,\mu_2,\mu_3,\mu_4} A_{\mu_1\mu_2\mu_3\mu_4}(s) \cdot \tilde{x}^{\mu_1} \, \tilde{p}_x^{\mu_2} \tilde{\sigma}^{\mu_3} \, \tilde{p}_\sigma^{\mu_4} \; , \end{aligned} \qquad (5.1)$$

whereby the quantities F_y, R_y and G_y ($y \equiv x, \sigma$) are to be obtained from equs. (4.2a) and (4.2b) and the coefficients $A_{\mu_1\mu_2\mu_3\mu_4}$ from equ. (3.8). Then we introduce into (5.1) the action-angle variables of the unperturbed motion.

In analogy to section (4.2) we achieve this via two canonical transformations. In the first transformation

$$(\tilde{x}, \tilde{p}_x, \tilde{\sigma}, \tilde{p} \,) \longrightarrow (\Phi_x, I_x, \Phi_\sigma, I_\sigma) \qquad (5.2)$$

we choose as generating function corresponding to (4.19) (where for simplicity Φ_0 can be set to zero) the function

$$F_1(\tilde{x}, \tilde{\sigma}, \Phi_x, \Phi_\sigma) = - \frac{\tilde{x}^2}{2\beta_x} \cdot [\mathrm{tg}\Phi_x + \alpha_x] - \frac{\tilde{\sigma}^2}{2\beta_\sigma} \cdot [\mathrm{tg}\Phi_\sigma + \alpha_\sigma] \quad . \qquad (5.3)$$

Following equ. (4.20) this results in the representation

$$\begin{cases} \tilde{x}(s) = \sqrt{2\beta_x(s)\cdot I_x}\cdot\cos\Phi_x(s) \quad ; \\ \tilde{p}_x(s) = -\sqrt{\dfrac{2I_x}{\beta_x(s)}}\cdot[\sin\Phi_x(s) + \alpha_x(s)\cdot\cos\Phi_x(s)] \quad ; \end{cases} \tag{5.4a}$$

$$\begin{cases} \tilde{\sigma}(s) = \sqrt{2\beta_\sigma(s)\cdot I_\sigma}\cdot\cos\Phi_\sigma(s)] \quad ; \\ \tilde{p}_\sigma(s) = -\sqrt{\dfrac{2I_\sigma}{\beta_\sigma(s)}}\cdot[\sin\Phi_\sigma(s) + \alpha_\sigma(s)\cdot\cos\Phi_\sigma(s)] \ . \end{cases} \tag{5.4b}$$

The new Hamiltonian is

$$\bar{H} = \tilde{H} + \frac{\partial F_1}{\partial s}$$

which becomes

$$\bar{H} = \frac{F_x(s)}{\beta_x(s)}\cdot I_x + \frac{F_\sigma(s)}{\beta_\sigma(s)}\cdot I_\sigma + W(\Phi_x, \Phi_\sigma, I_x, I_\sigma, s) \tag{5.5}$$

where

$$\begin{aligned} W = \sum_{\mu_1,\mu_2,\mu_3,\mu_4} A_{\mu_1\mu_2\mu_3\mu_4}(s)\cdot\{\sqrt{2\beta_x(s)\cdot I_x}\cdot\cos\Phi_x\}^{\mu_1} \\ \times\ \{-\sqrt{\frac{2I_x}{\beta_x(s)}}\cdot[\sin\Phi_x + \alpha_x\cdot\cos\Phi_x]\}^{\mu_2} \\ \times\ \{\sqrt{2\beta_\sigma(s)\cdot I_\sigma}\cdot\cos\Phi_\sigma\}^{\mu_3} \\ \times\ \{-\sqrt{\frac{2I_\sigma}{\beta_\sigma(s)}}\cdot[\sin\Phi_\sigma + \alpha_\sigma\cdot\cos\Phi_\sigma]\}^{\mu_4} \ . \end{aligned} \tag{5.6}$$

The canonical equations are

$$\frac{d\Phi_x}{ds} = \frac{\partial\bar{H}}{\partial I_x} \quad ; \quad \frac{dI_x}{ds} = -\frac{\partial\bar{H}}{\partial\Phi_x} \quad ;$$

$$\frac{d\Phi_\sigma}{ds} = \frac{\partial\bar{H}}{\partial I_\sigma} \quad ; \quad \frac{dI_\sigma}{ds} = -\frac{\partial\bar{H}}{\partial\Phi_\sigma} \ .$$

The final form for the Hamiltonian is obtained from a further transformation on $\bar{H}$

$$(\Phi_x, I_x, \Phi_\sigma, I_\sigma) \longrightarrow (\Psi_x, \hat{I}_x, \Psi_\sigma, \hat{I}_\sigma)$$

using the generating function

$$F_2(\Phi_x, \Phi_\sigma, \hat{I}_x, \hat{I}_\sigma, s) = \hat{I}_x \cdot \left[\frac{2\pi Q_x}{L} \cdot s - \int_0^s d\tilde{s} \cdot \frac{F_x(\tilde{s})}{\beta_x(\tilde{s})}\right] + \Phi_x \cdot \hat{I}_x + \hat{I}_\sigma \cdot \left[\frac{2\pi Q_x}{L} \cdot s - \int_0^s d\tilde{s} \cdot \frac{F_\sigma(\tilde{s})}{\beta_\sigma(\tilde{s})}\right] + \Phi_\sigma \cdot \hat{I}_\sigma$$

in analogy to equ. (4.23).

Q_x and Q_σ are defined as in (4.24) by

$$Q_x = \frac{1}{2\pi} \cdot \int_0^L d\tilde{s} \cdot \frac{F_x(\tilde{s})}{\beta_x(\tilde{s})} \quad ; \tag{5.7a}$$

$$Q_\sigma = \frac{1}{2\pi} \cdot \int_0^L d\tilde{s} \cdot \frac{F_\sigma(\tilde{s})}{\beta_\sigma(\tilde{s})} \quad . \tag{5.7b}$$

In correspondence with equ. (4.25) one finds

$$\Psi_x = \frac{\partial F_2}{\partial \hat{I}_x} = \Phi_x - X_x(s) \quad ; \quad I_x = \frac{\partial F_2}{\partial \Phi_x} = \hat{I}_x \quad ; \tag{5.8a}$$

$$\Psi_\sigma = \frac{\partial F_2}{\partial \hat{I}_\sigma} = \Phi_\sigma - X_\sigma(s) \quad ; \quad I_\sigma = \frac{\partial F_2}{\partial \Phi_\sigma} = \hat{I}_\sigma \tag{5.8b}$$

with

$$X_x(s) = \int_0^s d\tilde{s} \cdot \frac{F_x(\tilde{s})}{\beta_x(\tilde{s})} - \frac{2\pi Q_x}{L} \cdot s \quad ; \tag{5.9a}$$

$$X_\sigma(s) = \int_0^s d\tilde{s} \cdot \frac{F_\sigma(\tilde{s})}{\beta_\sigma(\tilde{s})} - \frac{2\pi Q_\sigma}{L} \cdot s \tag{5.9b}$$

For the corresponding Hamiltonian

$$\bar{\bar{H}} = \bar{H} + \frac{\partial F_2}{\partial s}$$

one obtains (putting $\hat{I} = I$)

$$\bar{\bar{H}} = I_x \frac{2\pi Q_x}{L} + I_\sigma \cdot \frac{2\pi Q_\sigma}{L} + \underbrace{V(\Psi_x, \Psi_\sigma, I_x, I_\sigma, s)}_{= W(\Psi_x + X_x, \Psi_\sigma + X_\sigma, I_x, I_\sigma, s)} \tag{5.10}$$

The canonical equations are

$$\left\{\begin{array}{llll} \dfrac{d\psi_x}{ds} = \dfrac{\partial\bar{\bar{H}}}{\partial I_x} & ; & \dfrac{dI_x}{ds} = -\dfrac{\partial\bar{\bar{H}}}{\partial\psi_x} & ; \qquad (5.10a) \\ \dfrac{d\psi_\sigma}{ds} = \dfrac{\partial\bar{\bar{H}}}{\partial I_\sigma} & ; & \dfrac{dI_\sigma}{ds} = -\dfrac{\partial\bar{\bar{H}}}{\partial\psi_\sigma} & . \qquad (5.10b) \end{array}\right.$$

The form (5.10) can now be used as the starting point for the application of canonical perturbation theory.

Before applying it to this particular case we will however, as a preparation, describe the general technique of canonical perturbation theory.

5.2 Perturbation theory

5.2.1 Far from any resonance

The version of perturbation theory presented here is based on that given by Courant, Ruth and Weng (CRW) (Ref. 5). The starting point is the Hamiltonian

$$\tilde{H} = \tilde{H}_0(I_1, I_2) + \tilde{V}(\psi_1, \psi_2, I_1, I_2, s) \qquad (5.11)$$

where the unperturbed part, $\tilde{H}_0$, depends only on I_1 and I_2 as for example in (5.10) which becomes

$$\tilde{H}_0 = \frac{2\pi Q_1}{L} \cdot I_1 + \frac{2\pi Q_2}{L} \cdot I_2 \quad . \qquad (5.12)$$

Here we identify index 1 with x and index 2 with σ. The Term $V(\psi_1, \psi_2, I_1, I_2, s)$, which is generally non-linear, describes the perturbation and is periodic in s and in ψ_1, ψ_2:

(5.13a)

$$\tilde{V}(\psi_1, \psi_2, I_1, I_2, s+L) = \tilde{V}(\psi_1, \psi_2, I_1, I_2, s) \qquad (5.13a)$$

$$\tilde{V}(\psi_1 + 2\pi, \psi_2, I_1, I_2, s) = \tilde{V}(\psi_1, \psi_2, I_1, I_2, s) \quad ;$$

$$\tilde{V}(\psi_1, \psi_2 + 2\pi, I_1, I_2, s) = \tilde{V}(\psi_1, \psi_2, I_1, I_2, s) \qquad (5.13b)$$

(see equ. (5.6) and (5.10)).

As a first step we separate off the average of $\tilde{V}$

$$<\tilde{V}(I_1, I_2)> = \frac{1}{(2\pi)^2 \cdot L} \cdot \int_0^L ds \cdot \int_0^{2\pi} d\psi_1 \cdot \int_0^{2\pi} d\psi_2 \cdot \tilde{V}(\psi_1, \psi_2, I_1, I_2, s) \quad (5.14)$$

and add it to $\tilde{H}_0$ so that

$$H = H_0(I_1, I_2) + V(\psi_1, \psi_2, I_1, I_2, s) \quad (5.15)$$

with

$$H_0(I_1, I_2) = \tilde{H}_0(I_1, I_2) + <\tilde{V}(I_1, I_2)> ; \quad (5.15a)$$

$$V(\psi_1, \psi_2, I_1, I_2, s) = \tilde{V}(\psi_1, \psi_2, I_1, I_2, s) - <\tilde{V}(I_1, I_2)> . \quad (5.15b)$$

(We comment further on this separation at the end of this section.)

As is clear from equs. (5.12) and (5.15a) the term $<\tilde{V}(I_1, I_2)>$ results in a tune shift of the form

$$\delta Q_1^{(1)} = \frac{L}{2\pi} \cdot \frac{\partial}{\partial I_1} <\tilde{V}(I_1, I_2)> ; \quad (5.16a)$$

$$\delta Q_2^{(1)} = \frac{L}{2\pi} \cdot \frac{\partial}{\partial I_2} <\tilde{V}(I_1, I_2)> . \quad (5.16b)$$

If $<\tilde{V}>$ depends nonlinearly on I_1, I_2, the tune shift is amplitude dependent.

In a second step we make a further canonical transformation

$$(\psi_1, \psi_2, I_1, I_2) \longrightarrow (\hat{\psi}_1, \hat{\psi}_2, \hat{I}_1, \hat{I}_2)$$

using the generating function

$$F_2(\psi_1, \psi_2, \hat{I}_1, \hat{I}_2, s) = \psi_1 \cdot \hat{I}_1 + \psi_2 \cdot \hat{I}_2 + G(\psi_1, \psi_2, \hat{I}_1, \hat{I}_2, s)$$

so as to introduce new variables $\hat{\psi}_1, \hat{\psi}_2, \hat{I}_1, \hat{I}_2$:

$$\begin{aligned} \hat{\psi}_1 &= \frac{\partial F_2}{\partial \hat{I}_1} = \psi_1 + G_{\hat{I}_1} ; \\ \hat{\psi}_2 &= \frac{\partial F_2}{\partial \hat{I}_2} = \psi_2 + G_{\hat{I}_2} ; \\ I_1 &= \frac{\partial F_2}{\partial \psi_1} = \hat{I}_1 + G_{\psi_1} ; \\ I_2 &= \frac{\partial F_2}{\partial \psi_2} = \hat{I}_2 + G_{\psi_2} \end{aligned} \quad (5.17)$$

for which the corresponding Hamiltonian

$$\hat{H} = H + \frac{\partial F_2}{\partial s}$$

$$= H_0(\hat{I}_1 + G_{\psi_1}\ , \hat{I}_2 + G_{\psi_2}\ ) + V(\psi_1,\ \psi_2, \hat{I}_1 + G_{\psi_1}\ , \hat{I}_2 + G_{\psi_2}\ , s) + G_s \qquad (5.18a)$$

is in first order only dependent on $\hat{I}_1$ and $\hat{I}_2$. In equ. (5.17) and below we use the notation $G_\psi = \frac{\partial G}{\partial \psi}$ etc.

For this purpose, following CRW, we rewrite (5.18a) as

$$\begin{aligned}
\hat{H} = & H_0(\hat{I}_1,\ \hat{I}_2) + \\
& + \{H_0(\hat{I}_1 + G_{\psi_1}\ ,\ \hat{I}_2 + G_{\psi_2}\ ) - H_0(\hat{I}_1,\ \hat{I}_2) - \\
& - \frac{2\pi}{L} \cdot Q_1(\hat{I}_1,\ \hat{I}_2) \cdot G_{\psi_1} - \frac{2\pi}{L} \cdot Q_2(\hat{I}_1,\ \hat{I}_2) \cdot G_{\psi_2} \} + \\
& + \{V(\ \psi_1,\ \psi_2,\ \hat{I}_1 + G_{\psi_1}\ ,\ \hat{I}_2 + G_{\psi_2}\ ,\ s) - V(\ \psi_1,\ \psi_2,\ \hat{I}_1,\ \hat{I}_2,\ s)\} + \\
& + \{\frac{2\pi}{L} \cdot Q_1(\hat{I}_1,\ \hat{I}_2) \cdot G_{\psi_1} + \frac{2\pi}{L} \cdot Q_2(\hat{I}_1,\ \hat{I}_2) \cdot G_{\psi_2} + G_s + \\
& + V(\ \psi_1,\ \psi_2,\ \hat{I}_1,\ \hat{I}_2,\ s)\} \qquad (5.18b)
\end{aligned}$$

where for brevity we have written (see (5.12), (5.15a), (5.16))

$$\frac{\partial H_0(I_1\ ,\ I_2)}{\partial I_1} = \frac{2\pi}{L} \cdot Q_1(I_1,\ I_2) \quad ; \qquad (5.19a)$$

$$\frac{\partial H_0(I_1\ ,\ I_2)}{\partial I_2} = \frac{2\pi}{L} \cdot Q_2(I_1,\ I_2) \quad . \qquad (5.19b)$$

We now require that the generating function G satisfies the partial differential equation

$$\frac{2\pi}{L} \cdot Q_1(\hat{I}_1,\ \hat{I}_2) \cdot G_{\psi_1} + \frac{2\pi}{L} \cdot Q_2(\hat{I}_1,\ \hat{I}_2) \cdot G_{\psi_2} + G_s + \\ + V(\ \psi_1,\ \psi_2,\ \hat{I}_1,\ \hat{I}_2,\ s) = 0 \qquad (5.20)$$

so that

$$\hat{H} = H_0(\hat{I}_1,\ \hat{I}_2) + \tilde{V}' \qquad (5.21)$$

where

$$\tilde{V}' = \frac{1}{2}\,\frac{\partial^2 H_0(\hat{I}_1,\ \hat{I}_2)}{\partial \hat{I}_1^2}\cdot G_{\psi_1}^2 + \frac{1}{2}\,\frac{\partial^2 H_0(\hat{I}_1,\ \hat{I}_2)}{\partial \hat{I}_2^2}\cdot G_{\psi_2}^2 +$$

$$+ \frac{\partial^2 H_0(\hat{I}_1,\ \hat{I}_2)}{\partial \hat{I}_1\ \partial \hat{I}_2}\cdot G_{\psi_1}\cdot G_{\psi_2} +$$

$$+ G_{\psi_1}\cdot \frac{\partial}{\partial \hat{I}_1}\ V(\psi_1,\ \psi_2,\ \hat{I}_1,\ \hat{I}_2,\ s) +$$

$$+ G_{\psi_2}\cdot \frac{\partial}{\partial \hat{I}_2}\ V(\psi_1,\ \psi_2,\ \hat{I}_1,\ \hat{I}_2,\ s) + \cdots \quad . \tag{5.22}$$

If the perturbation, V, in (5.15) is small compared to H_0, then we expect according to (5.20) that G is small. It is then clear from (5.22) that $\tilde{V}'$ in (5.21) is only a second order correction compared to H_0.

For convenience we also require that the solution of (5.20) is periodic in s

$$G(\psi_1,\ \psi_2,\ \hat{I}_1,\ \hat{I}_2,\ s+L) = G(\psi_1,\ \psi_2,\ \hat{I}_1,\ \hat{I}_2,\ s) \tag{5.23}$$

so that $\tilde{V}'$ is also periodic with the same period L. In this case, the calculation embodied in equs. (5.14-5.20)(and in the Fourier expansion below) can be repeated in a second iteration step, in which $\tilde{V}'$ replaces V in equ. (5.14).

In particular we can use the average of $\tilde{V}'$

$$<\tilde{V}'(\hat{I}_1,\ \hat{I}_2)> = \frac{1}{L\cdot(2\pi)^2}\int_0^L ds \int_0^{2\pi} d\psi_1 \int_0^{2\pi} d\psi_2 \cdot \tilde{V}'(\psi_1,\ \psi_2,\ \hat{I}_1,\ \hat{I}_2,\ s)$$

to calculate the contribution to the Q-shift in the next order

$$\begin{aligned} \delta Q_1^{(2)} &= \frac{L}{2\pi}\cdot\frac{\partial}{\partial \hat{I}_1} <\tilde{V}'(\hat{I}_1,\ \hat{I}_2)> \quad ; \\ \delta Q_2^{(2)} &= \frac{L}{2\pi}\cdot\frac{\partial}{\partial \hat{I}_2} <\tilde{V}'(\hat{I}_1,\ \hat{I}_2)> \quad . \end{aligned} \tag{5.24}$$

A periodic solution to equ. (5.20) can be obtained by writing V and G as

$$V(\psi_1,\ \psi_2,\hat{I}_1,\hat{I}_2,s) = \sum_{m_1,m_2} \upsilon_{m_1 m_2}(\hat{I}_1,\hat{I}_2,s)\cdot e^{i[m_1\ \psi_1+m_2\ \psi_2]} \ ; \tag{5.25}$$

$$G(\psi_1,\ \psi_2,\ \hat{I}_1,\hat{I}_2,s) = \sum_{m_1,m_2} g_{m_1 m_2}(\hat{I}_1,\hat{I}_2,s)\cdot e^{i[m_1\ \psi_1+m_2\ \psi_2]} \tag{5.26a}$$

where $\upsilon_{m_1m_2}$ is periodic in s and where according to (5.23) we require

$$g_{m_1m_2}(\hat{I}_1, \hat{I}_2, s+L) = g_{m_1m_2}(\hat{I}_1, \hat{I}_2, s). \tag{5.26b}$$

On substituting (5.25) and (5.26a) in (5.20) we get the differential equation connecting the coefficients g and υ:

$$\{i\cdot[\frac{2\pi}{L} Q_1(\hat{I}_1, \hat{I}_2)\cdot m_1 + \frac{2\pi}{L} Q_2(\hat{I}_1, \hat{I}_2)\cdot m_2] + \frac{\partial}{\partial s}\} g_{m_1m_2}(\hat{I}_1, \hat{I}_2, s) =$$

$$= - \upsilon_{m_1m_2}(\hat{I}_1, \hat{I}_2, s) . \tag{5.27}$$

This may also be rewritten as

$$\frac{\partial}{\partial s} \{e^{i\cdot\frac{2\pi}{L}[m_1Q_1+m_2Q_2]\cdot s} \cdot g_{m_1m_2}(\hat{I}_1, \hat{I}_2, s)\} =$$

$$= - e^{i\cdot\frac{2\pi}{L}[m_1Q_1+m_2Q_2]\cdot s} \cdot \upsilon_{m_1m_2}(\hat{I}_1, \hat{I}_2, s) . \tag{5.28}$$

By integrating (5.28) from s to (s+L) and using (5.26b) we then obtain

$$g_{m_1m_2}(\hat{I}_1, \hat{I}_2, s)\cdot\{e^{i\cdot\frac{2\pi}{L}[m_1Q_1+m_2Q_2]\cdot(s+L)} - e^{i\cdot\frac{2\pi}{L}[m_1Q_1+m_2Q_2]\cdot s}\}$$

$$= - \int_s^{s+L} d\tilde{s}\cdot e^{i\cdot\frac{2\pi}{L}[m_1Q_1+m_2Q_2]\cdot\tilde{s}} \cdot \upsilon_{m_1m_2}(\hat{I}_1, \hat{I}_2, \tilde{s}) \tag{5.29}$$

from which

$$g_{m_1m_2}(\hat{I}_1, \hat{I}_2, s) = \frac{i}{2\cdot\sin\pi[m_1Q_1 + m_2Q_2]}$$

$$\times \int_s^{s+L} d\tilde{s}\cdot \upsilon_{m_1m_2}(\hat{I}_1, \hat{I}_2, \tilde{s})$$

$$\times\ e^{i\cdot\frac{2\pi}{L}[m_1Q_1+m_2Q_2]\cdot(\tilde{s}-s-\frac{L}{2})} \tag{5.30}$$

so that (5.26a) finally:

$$G(\psi_1, \psi_2, \hat{I}_1, \hat{I}_2, s) = \sum_{m_1,m_2} \frac{i}{2\cdot\sin\pi[m_1Q_1 + m_2Q_2]}$$

$$\times \int_s^{s+L} d\tilde{s}\cdot \upsilon_{m_1m_2}(\hat{I}_1, \hat{I}_2, \tilde{s})$$

$$\times\ e^{i\cdot\{[m_1\psi_1+m_2\psi_2] + \frac{2\pi}{L}[m_1Q_1+m_2Q_2](\tilde{s}-s-\frac{L}{2})} . \tag{5.31}$$

If the function $\upsilon_{m_1 m_2}$ in (5.31) is furthermore expanded as a Fourier series in s:

$$\upsilon_{m_1 m_2}(\hat{I}_1, \hat{I}_2, s) = \sum_n \upsilon_{m_1 m_2 n}(\hat{I}_1, \hat{I}_2) \cdot e^{-i \cdot n \cdot \frac{2\pi}{L} \cdot s} \qquad (5.32)$$

then G takes the form

$$G(\psi_1, \psi_2, \hat{I}_1, \hat{I}_2, s) = i \cdot \frac{L}{2\pi} \cdot \sum_{m_1 m_2 n} \frac{\upsilon_{m_1 m_2 n}(\hat{I}_1, \hat{I}_2) \cdot e^{i \cdot [m_1 \psi_1 + m_2 \psi_2 - n \frac{2\pi}{L} \cdot s]}}{[m_1 Q_1 + m_2 Q_2 - n]} \qquad (5.33)$$

Since $\hat{H}$ is in first order independent of $\hat{\psi}_1$ and $\hat{\psi}_2$, the canonical equations

$$\frac{d\hat{I}_1}{ds} = - \frac{\partial \hat{H}}{\partial \hat{\psi}_1} \quad ; \quad \frac{d\hat{I}_2}{ds} = - \frac{\partial \hat{H}}{\partial \hat{\psi}_2}$$

predict that $\hat{I}_1$, $\hat{I}_2$ are constants of motion which together with equ. (5.17):

$$\begin{cases} I_1 = \hat{I}_1 + \dfrac{\partial}{\partial \psi_1} G(\psi_1, \psi_2, \hat{I}_1, \hat{I}_2, s) \ ; \\ I_2 = \hat{I}_2 + \dfrac{\partial}{\partial \psi_2} G(\psi_1, \psi_2, \hat{I}_1, \hat{I}_2, s) \end{cases} \qquad (5.34)$$

define an invariant surface.

<u>Remark</u>: In separating off the average of $\tilde{V}$

$$< \tilde{V}(\hat{I}_1, \hat{I}_2) > \ \equiv \ \upsilon_{000}$$

in equ. (5.15b) we have ensured that the term (5.33) for which m_1, m_2 and n in the denominator, $m_1 Q_1 + m_2 Q_2 - n$, are all zero, does not appear.

5.2.2 Resonances

The above treatment of perturbation theory relies on the assumption that the perturbation G in (5.17) is small. From (5.33) it is clear that this condition is not valid if

$$m_1 Q_1 + m_2 Q_2$$

is close to an integer.

At these resonances the above treatment cannot be used.

In order to obtain results at isolated resonances

$$\mu_1 Q_1 + \mu_2 Q_2 \approx n_0 \tag{5.35}$$

we use equs. (5.15), (5.25) and (5.32) to write the Hamiltonian in the form

$$H = H_0(I_1, I_2) + \sum_{m_1 m_2 n} \upsilon_{m_1 m_2 n}(I_1, I_2) \; e^{i[m_1 \psi_1 + m_2 \psi_2 - n \cdot \frac{2\pi}{L} \cdot s]} \tag{5.36}$$

where (by inserting (5.25) and (5.32))

$$\upsilon_{m_1 m_2 n} = \frac{1}{(2\pi)^2 \cdot L} \cdot \int_0^{2\pi} d\psi_1 \cdot \int_0^{2\pi} d\psi_2 \cdot \int_0^L ds \cdot V(\psi_1, \psi_2, I_1, I_2, s) \times$$

$$\times \; e^{-i[m_1 \psi_1 + m_2 \psi_2 - n \cdot \frac{2\pi}{L} \cdot s]} \quad . \tag{5.37}$$

For the exponent in (5.36) we find

$$\frac{d}{ds}[m_1 \psi_1 + m_2 \psi_2 - n \cdot \frac{2\pi}{L} \cdot s] = m_1 \cdot \frac{d\psi_1}{ds} + m_2 \cdot \frac{d\psi_2}{ds} - n \cdot \frac{2\pi}{L}$$

$$\approx \quad m_1 \cdot \frac{dH_0(I_1, I_2)}{dI_1} + m_2 \cdot \frac{dH_0(I_1, I_2)}{dI_2} - n \cdot \frac{2\pi}{L} \quad \text{according to (5.15)}$$

$$= \quad \frac{2\pi}{L} \cdot [m_1 Q_1 + m_2 Q_2 - n] \qquad \text{according to (5.19)} . \tag{5.38}$$

The terms in (5.38) for which $m_1 Q_1 + m_2 Q_2 - n \neq 0$ lead to rapid oscillations whose influence averages to zero (Bogoliubov's averaging technique).

It is then only necessary to take into account the slowly varying terms

$$(m_1, m_2, n) = (\mu_1, \mu_2, n_0) \quad ;$$
$$(m_1, m_2, n) = (-\mu_1, -\mu_2, -n_0)$$

in (5.36). Thus, the Hamiltonian H simplifies to the form

$$H(\psi_1, \psi_2, I_1, I_2, s) = H_0(I_1, I_2) + \{\upsilon_{\mu_1\mu_2 n_0} \cdot e^{i[\mu_1 \psi_1 + \mu_2 \psi_2 - n_0 \cdot \frac{2\pi}{L} \cdot s]}$$
$$+ \upsilon^*_{\mu_1\mu_2 n_0} \cdot e^{-i[\mu_1 \psi_1 + \mu_2 \psi_2 - n_0 \cdot \frac{2\pi}{L} \cdot s]}\} \; . \tag{5.39}$$

Because the perturbation V is of the form

$$V = \sum_{n_1 n_2 n_3 n_4} C_{n_1 n_2 n_3 n_4}(s) \cdot \tilde{x}_1^{n_1} \tilde{p}_{x_1}^{n_2} \tilde{x}_2^{n_3} \tilde{p}_{x_2}^{n_4} \tag{5.40}$$

(see equs. (5.6) and (5.10)) the quantity $\upsilon_{\mu_1\mu_2 n_0}$ can now be represented as

$$\upsilon_{\mu_1\mu_2 n_0} = \frac{1}{2} \varkappa \cdot f(I_1, I_2) \cdot e^{i\alpha} \tag{5.41}$$

with

$$f(I_1, I_2) = I_1^{\frac{\mu_1}{2}} \quad I_2^{\frac{\mu_2}{2}} \; . \tag{5.42}$$

The factor $\varkappa$ is called the "driving term" of the resonance.

Then we have (5.15a):

$$H(\psi_1, \psi_2, I_1, I_2, s) = \tilde{H}_0(I_1, I_2) + \langle \tilde{V}(I_1, I_2) \rangle +$$
$$+ \varkappa \cdot \sqrt{I_1 \cdot I_2} \cdot \cos[\mu_1 \psi_1 + \mu_2 \psi_2 - n_0 \cdot \frac{2\pi}{L} \cdot s + \alpha] \; . \tag{5.43}$$

Two canonical transformations will now be applied to this Hamiltonian.

In the first

$$(\psi_1, \psi_2, I_1, I_2) \longrightarrow (\varphi_1, \varphi_2, I_1, I_2)$$

with the generating function

$$F_2(\psi_1, \psi_2, I_1, I_2, s) = \psi_1 \cdot I_1 + \psi_2 \cdot I_2 - \tilde{H}_0(I_1, I_2) \cdot s + \frac{\alpha}{2} \cdot \left(\frac{I_1}{\mu_1} + \frac{I_2}{\mu_2}\right) \tag{5.44}$$

one has

$$\varphi_1 = \frac{\partial F_2}{\partial I_1} = \psi_1 - \frac{\partial \tilde{H}_0}{\partial I_1} \cdot s + \frac{\alpha}{2} \cdot \frac{1}{\mu_1} \quad ;$$

$$\varphi_2 = \frac{\partial F_2}{\partial I_2} = \psi_2 - \frac{\partial \tilde{H}_0}{\partial I_2} \cdot s + \frac{\alpha}{2} \cdot \frac{1}{\mu_2} \quad ;$$

$$H \longrightarrow \bar{H} = H + \frac{\partial F_2}{\partial s}$$

$$= \varkappa f(I_1, I_2) \cdot [\cos \mu_1 \varphi_1 + \mu_2 \varphi_2 + \frac{2\pi}{L} \cdot \Delta \cdot s] + \langle \tilde{V} \rangle \tag{5.45}$$

where we have used the notation

$$\Delta = \mu_1 \cdot \frac{L}{2\pi} \frac{\partial \tilde{H}_0}{\partial I_1} + \mu_2 \cdot \frac{L}{2\pi} \frac{\partial \tilde{H}_0}{\partial I_2} - n_0 \tag{5.46}$$

to represent the distance from the resonance.

A further transformation

$$(\varphi_1, \varphi_2, I_1, I_2) \longrightarrow (\tilde{\varphi}_1, \tilde{\varphi}_2, I_1, I_2)$$

using the generating function

$$F_2(\varphi_1, \varphi_2, I_1, I_2, s) = \varphi_1 \cdot I_1 + \varphi_2 \cdot I_2 + \frac{1}{2} \Delta \cdot \left(\frac{I_1}{\mu_1} + \frac{I_2}{\mu_2}\right) \cdot \frac{2\pi}{L} \cdot s \tag{5.47}$$

then gives

$$\tilde{\varphi}_1 = \frac{\partial F_2}{\partial I_1} = \varphi_1 + \frac{1}{2} \Delta \cdot s \cdot \frac{2\pi}{L} \cdot \frac{1}{\mu_1} \quad ;$$

$$\tilde{\varphi}_2 = \frac{\partial F_2}{\partial I_2} = \varphi_2 + \frac{1}{2} \Delta \cdot s \cdot \frac{2\pi}{L} \cdot \frac{1}{\mu_2} \quad ;$$

$$\bar{H} \longrightarrow \bar{\bar{H}}(\tilde{\varphi}_1, \tilde{\varphi}_2, I_1, I_2) = \bar{H} + \frac{\partial F_2}{\partial s}$$

$$= \frac{1}{2} \Delta \cdot \left(\frac{I_1}{\mu_1} + \frac{I_2}{\mu_2}\right) \cdot \frac{2\pi}{L} + \varkappa f(I_1, I_2) \cdot \cos[\mu_1 \tilde{\varphi}_1 + \mu_2 \tilde{\varphi}_2] + \langle \tilde{V} \rangle \tag{5.48}$$

whereby the canonical equations for $\bar{\bar{H}}$ are

$$\frac{d\tilde{\varphi}_1}{ds} = \frac{\partial \bar{\bar{H}}}{\partial I_1} = \frac{1}{2}\{\Delta \cdot \frac{2\pi}{L} \cdot \frac{1}{\mu_1} + \varkappa f_{I_1} \cdot \cos[\mu_1 \cdot \tilde{\varphi}_1 + \mu_2 \cdot \tilde{\varphi}_2]\} + \frac{\partial < \tilde{V} >}{\partial I_1} ; \qquad (5.49a)$$

$$\frac{d\tilde{\varphi}_2}{ds} = \frac{\partial \bar{\bar{H}}}{\partial I_2} = \frac{1}{2}\{\Delta \cdot \frac{2\pi}{L} \cdot \frac{1}{\mu_2} + \varkappa f_{I_2} \cdot \cos[\mu_1 \cdot \tilde{\varphi}_1 + \mu_2 \cdot \tilde{\varphi}_2]\} + \frac{\partial < \tilde{V} >}{\partial I_2} ; \qquad (5.49b)$$

$$\frac{dI_1}{ds} = -\frac{\partial \bar{\bar{H}}}{\partial \tilde{\varphi}_1} = \mu_1 \cdot \varkappa f(I_1, I_2) \cdot \sin(\mu_1 \cdot \tilde{\varphi}_1 + \mu_2 \cdot \tilde{\varphi}_2) \quad ; \qquad (5.49c)$$

$$\frac{dI_2}{ds} = -\frac{\partial \bar{\bar{H}}}{\partial \tilde{\varphi}_2} = \mu_2 \cdot \varkappa f(I_1, I_2) \cdot \sin(\mu_1 \cdot \tilde{\varphi}_1 + \mu_2 \cdot \tilde{\varphi}_2) \quad . \qquad (5.49d)$$

Since $\bar{\bar{H}}$ is not explicitly s dependent:

$$\bar{\bar{H}}(\tilde{\varphi}_1, \tilde{\varphi}_2, I_1, I_2) = \text{const.} \qquad (5.50)$$

From equs. (5.49c) and (5.49d) it follows also that

$$\frac{I_1}{\mu_1} - \frac{I_2}{\mu_2} = \text{const.} \qquad (5.51)$$

From the last equation we see that for a "difference resonance" for which

$$\text{sgn}(\mu_1) = -\,\text{sgn}(\mu_2)$$

the motion remains stable whereas for the "sum resonance" for which

$$\text{sgn}(\mu_1) = \text{sgn}(\mu_2)$$

the variables I_1 and I_2 may in principle become arbitrarily large so that the stability of the particle motion is no longer guaranteed.

We note here that other treatments (e.g. ref. 10) find that for ultrarelativistic particles, it is the difference resonances which turn out to be unstable. However, there is in fact no inconsistency with our treatment: the frequency Q_S used here is negative (see equ. (4.63)). We recall also that below transition the stability conditions become interchanged[12].

Equations (5.48), (5.49) and (5.51) contain all the information necessary for a detailed investigation of the synchro-betatron phase plane. For example, for the unstable resonances, the stability limits (i.e. the resonance widths) for the amplitudes I_1, I_2 can be investigated by applying the fixed point conditions:

$$\frac{\partial \bar{\bar{H}}}{\partial I_{1,2}} = \frac{\partial \bar{\bar{H}}}{\partial \tilde{\varphi}_{1,2}} = 0 \ . \tag{5.52}$$

This correspond to setting the right hand side of equs. (5.49) to zero and solving.

From the resonance width one easily derives a growth rate on the stability limit.

Furthermore, we can easily take into account the effect of the amplitude dependent tunes (detuning) which are produced by the term $< \tilde{V} >$ in the Hamiltonian.

The presence of weak detuning leads to an asymmetric resonance width. For stronger detuning the resonance is stabilized and if the detuning becomes very strong the stabilized resonant trajectories begin to overlap. This in general leads to chaotic behaviour[19].

<u>Remark</u>:

We have used Bogoliubov's averaging technique to cancel the high frequency terms in equ. (5.36)[19]. Another way to eliminate these terms is to modify the perturbation technique described in chapter 5.2.1[1]. In this case we write the perturbation term V in equ. (5.15) in the form

$$V = V_0 + \upsilon_{\mu_1\mu_2 n_0} \cdot e^{i[\mu_1 \psi_1 + \mu_2 \psi_2 - n_0 \cdot \frac{2\pi}{L} \cdot s]}$$
$$+ \ \upsilon_{-\mu_1,-\mu_2,-n_0} \cdot e^{-i[\mu_1 \psi_1 + \mu_2 \psi_2 - n_0 \cdot \frac{2\pi}{L} \cdot s]}$$

where we have excluded from V the dangerous terms $\upsilon_{\mu_1\mu_2 n_0}$ and $\upsilon_{-\mu_1,-\mu_2,-n_0}$. Then, using the function V_0 instead of V in equ. (5.20) the generator G in (5.33) remains small (even in the neighbourhood of this resonance:

$\mu_1 Q_1 + \mu_2 Q_2 = n_0$) and for the Hamiltonian $\hat{H}$ in (5.18b) we get in first order

$$\hat{H} = H_0(\hat{I}_1, \hat{I}_2) + \upsilon_{\mu_1\mu_2 n_0}(\hat{I}_1, \hat{I}_2) \cdot e^{-i[\mu_1 \Psi_1 + \mu_2 \Psi_2 - n_0 \cdot \frac{2\pi}{L} \cdot s]}$$
$$+ \upsilon_{-\mu_1,-\mu_2,-n_0}(\hat{I}_1, \hat{I}_2) \cdot e^{-i[\mu_1 \Psi_1 + \mu_2 \Psi_2 - n_0 \cdot \frac{2\pi}{L} \cdot s]} .$$

We then proceed in the same way as in equ. (5.43), but writing $\hat{\Psi}_1$, $\hat{\Psi}_2$, $\hat{I}_1$, $\hat{I}_2$ instead of Ψ_1, Ψ_2, I_1, I_2. In this way we arrive again at equ. (5.49) and the subsequent development of this equation proceeds as before.

5.2.3 A special case: linear sum and difference resonances

The equations (5.49) can be easily solved for a linear resonance ($|\mu_1| = |\mu_2| = 1$)[20]. To achieve this we note that:

$$f(I_1, I_2) = \sqrt{I_1 \cdot I_2} \quad , \quad <\widetilde{V}> = 0$$

and we introduce the intermediate quantities

$$u_1 = \sqrt{I_1} \cdot \cos\widetilde{\varphi}_1 \quad ; \quad u_2 = \sqrt{I_2} \cdot \cos\widetilde{\varphi}_2 \quad ;$$
$$\upsilon_1 = \sqrt{I_1} \cdot \sin\widetilde{\varphi}_1 \quad ; \quad \upsilon_2 = \sqrt{I_2} \cdot \sin\widetilde{\varphi}_2 \quad . \tag{5.53}$$

We treat the sum and difference cases separately.

1) Sum resonance

For the linear sum resonance with

$$\mu_1 = \mu_2 = 1$$

we have ((5.52))

$$\begin{aligned} u_1' &= \frac{1}{2} \cdot [\varkappa \cdot \upsilon_2 - \Delta \cdot \frac{2\pi}{L} \cdot \upsilon_1]; \\ u_2' &= \frac{1}{2} \cdot [\varkappa \cdot \upsilon_1 - \Delta \cdot \frac{2\pi}{L} \cdot \upsilon_2]; \\ \upsilon_1' &= \frac{1}{2} \cdot [\varkappa \cdot u_2 + \Delta \cdot \frac{2\pi}{L} \cdot u_1]; \\ \upsilon_2' &= \frac{1}{2} \cdot [\varkappa \cdot u_1 + \Delta \cdot \frac{2\pi}{L} \cdot u_2] \end{aligned} \tag{5.54}$$

so that for the quantity

$$\omega = u_1 + i \cdot u_2 \tag{5.55}$$

we get

$$\omega' = u_1' + i \cdot u_2'$$

$$= \frac{1}{2}\,[i \cdot \varkappa \cdot \bar{\omega}^* - \Delta \cdot \frac{2\pi}{L} \cdot \bar{\omega}]$$

$$\omega'' = \frac{1}{2}\,[i \cdot \varkappa \cdot (\bar{\omega}')^* - \Delta \cdot \frac{2\pi}{L} \cdot \bar{\omega}'] \quad . \tag{5.56}$$

with

$$\bar{\omega} = \upsilon_1 + i \cdot \upsilon_2 \quad ;$$

$$\bar{\omega}' = \upsilon_1' + i \cdot \upsilon_2'$$

$$= \frac{1}{2}\,[i \cdot \varkappa \cdot \omega^* + \Delta \cdot \frac{2\pi}{L} \cdot \omega] \, . \tag{5.57}$$

By substituting (5.57) in (5.56) this results in

$$\omega'' = -\,\Omega^2 \cdot \omega \tag{5.58}$$

with

$$\Omega^2 = \frac{1}{4}\,[(\frac{2\pi}{L} \cdot \Delta)^2 - \varkappa^2] \quad . \tag{5.59}$$

The solution for equ. (5.58) is

$$\omega = A \cdot e^{i\Omega \cdot s} + B \cdot e^{-i\Omega \cdot s} \quad . \tag{5.60}$$

However, from equ. (5.59) and (5.60) we see that if

$$|\Delta| > \frac{L}{2\pi} \cdot \varkappa$$

the motion is stable and if

$$|\Delta| < \frac{L}{2\pi} \cdot \varkappa$$

the motion is unstable.

The quantity

$$\Delta_o^{(+)} = \frac{L}{2\pi} \cdot \varkappa \equiv \frac{L}{\pi} \; \frac{|\upsilon_{11\,n_o}|}{\sqrt{I_1 \cdot I_2}} \tag{5.61}$$

thus provides a measure of the stopband width of the linear sum resonance.

2) Difference-resonance

For the linear difference resonance with

$$\mu_1 = 1 \quad , \quad \mu_2 = -1$$

equ. (5.52) gives:

$$
\begin{aligned}
u_1' &= -\frac{1}{2}\cdot[\varkappa\cdot\upsilon_2 + \Delta\cdot\upsilon_1] \; ; \\
u_2' &= -\frac{1}{2}\cdot[\varkappa\cdot\upsilon_1 - \Delta\cdot\upsilon_2] \; ; \\
\upsilon_1' &= \frac{1}{2}\cdot[\varkappa\cdot u_2 - \Delta\cdot u_1] \; ; \\
\upsilon_2 &= \frac{1}{2}\cdot[\varkappa\cdot u_1 - \Delta\cdot u_2] \; .
\end{aligned}
\qquad (5.62)
$$

In this case we obtain for the quantities

$$
\begin{aligned}
\omega_1 &= u_1 + i\cdot\upsilon_1 \equiv \sqrt{I_1}\cdot e^{i\tilde{\varphi}_1} \; ; \\
\omega_2 &= u_2 + i\cdot\upsilon_2 \equiv \sqrt{I_2}\cdot e^{i\tilde{\varphi}_2}
\end{aligned}
\qquad (5.63)
$$

the differential equation

$$\frac{d}{ds}\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix} = \frac{i}{2}\cdot\begin{pmatrix}\frac{2\pi}{L}\cdot\Delta & \varkappa \\ \varkappa & -\frac{2\pi}{L}\cdot\Delta\end{pmatrix}\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix}$$

with the solution

$$\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix} = A\cdot\begin{pmatrix}\varkappa \\ 2\cdot\Omega - \frac{2\pi}{L}\cdot\Delta\end{pmatrix} e^{i\Omega\cdot s} + B\cdot\begin{pmatrix}-\varkappa \\ 2\cdot\Omega + \frac{2\pi}{L}\cdot\Delta\end{pmatrix} e^{-i\Omega\cdot s} \qquad (5.64)$$

where Ω is given by

$$\Omega^2 = \frac{1}{4}\,[(\frac{2\pi}{L}\cdot\Delta)^2 + \varkappa^2] \quad . \qquad (5.65)$$

To extract information on the resonance behaviour we consider the case

$$\omega_1(0) = 0 \quad ; \qquad \omega_2(0) \neq 0 \tag{5.66}$$

so that

$$A = B = \frac{\omega_2(0)}{4\Omega} \quad ,$$

and we obtain

$$\text{Max}\,|\omega_1(s)| \equiv \text{Max}\sqrt{I_1} = \frac{\varkappa}{2\Omega}\cdot|\omega_2(0)| \quad .$$

$$\equiv \frac{\varkappa}{\sqrt{(\frac{2\pi}{L}\cdot\Delta)^2 + \varkappa^2}}\cdot\sqrt{I_2(0)} \quad . \tag{5.67}$$

From (5.67) it is however clear that the quantity Max $\sqrt{I_1}$ which is equal to $\sqrt{I_2(0)}$ at $\Delta = 0$, approaches zero monotonically as the distance from the resonance, $|\Delta|$, becomes large and has the value $\frac{1}{2}\sqrt{I_2(0)}$ at $|\Delta| = \frac{L}{2\pi}\sqrt{3}\cdot\varkappa$. The oscillations for the linear difference resonance thus remain stable in agreement with equ. (5.43) and the quantity

$$\Delta_0^{(-)} = \frac{L}{2\pi}\cdot\varkappa \equiv \frac{L}{\pi}\;\frac{|\upsilon_{1,-1,n_0}|}{\sqrt{I_1\cdot I_2}} \tag{5.68}$$

can be interpreted as the width of the resonance.

The starting condition

$$\omega_1(0) \neq 0 \quad , \quad \omega_2(0) = 0 \quad ,$$

of course leads to the same result.

This is an example of a coupling resonance (at $\Delta = 0$, Max $\sqrt{I_1(s)}$ = Max $\sqrt{I_2(s)}$; see equ. (5.64)) in which there is a large exchange of energy between both degrees of freedom but in which no instability appears.

5.3 Consequences for Synchro-betatron oscillations

We now apply these results of canonical perturbation theory to the special case of synchro-betatron motion. For this purpose, we consider that the function V of equ. (5.11) is given as in (5.6) and (5.10) by

$$\tilde{V}(\psi_1, \psi_2, I_1, I_2, s) = W(\psi_1 + \chi_1, \psi_2 + \chi_2, I_1, I_2, s)$$

$$= \sum_{\nu_1,\nu_2,\nu_3,\nu_4} A_{\nu_1 \nu_2 \nu_3 \nu_4}(s) \quad \times$$

$$\times \quad \{\sqrt{2\beta_x(s) \cdot I_1} \cdot \cos(\psi_1 + \chi_1)\}^{\nu_1} \quad \times$$

$$\times \quad \{-\sqrt{\frac{2 I_1}{\beta_x(s)}} \cdot [\sin(\psi_1 + \chi_1) + \alpha_x(s) \cdot \cos(\psi_1 + \chi_1)\}^{\nu_2} \quad \times$$

$$\times \quad \{\sqrt{2\beta_\sigma(s) \cdot I_2} \cdot \cos(\psi_2 + \chi_2)\}^{\nu_3} \quad \times$$

$$\times \quad \{\sqrt{\frac{2 I_2}{\beta_\sigma(s)}} \cdot [\sin(\psi_2 + \chi_2) + \alpha_\sigma(s) \cdot \cos(\psi_2 + \chi_2)\}^{\nu_4}$$

$$\equiv H_1(\tilde{x}, \tilde{p}_x, \tilde{\sigma}, \tilde{p}_\sigma, s) \qquad (5.69)$$

with the coefficients $A_{\nu_1 \nu_2 \nu_3 \nu_4}(s)$ defined by equ. (3.8).

Using equ. (3.8) it can be shown that the average, $<\tilde{V}(I_1, I_2)>$, is zero

$$<\tilde{V}(I_1, I_2)> = 0 \quad , \qquad (5.70)$$

so that the function V (5.15b) is identical with $\tilde{V}$ and the Q-shifts $\delta Q_1^{(1)}$ and $\delta Q_2^{(1)}$ (5.10) disappear:

$$\delta Q_x^{(1)} = 0 \quad ; \qquad \delta Q_\sigma^{(1)} = 0 \quad . \qquad (5.71)$$

Thus the Q-values remain as defined in equ. (5.7):

$$Q_v = \frac{1}{2\pi} \int_0^L d\tilde{s} \cdot \frac{F_v(\tilde{s})}{\beta_v(\tilde{s})} \quad ; \qquad (5.72)$$

$$(v = x, \sigma).$$

For synchro-betatron resonances the integer coefficients μ_1 and μ_2 in the relation (5.35)

$$\mu_1 \cdot Q_x + \mu_2 \cdot Q_\sigma = n_0$$

are confined by the condition

$$|\mu_1| + |\mu_2| \leq 3$$

since the Hamiltonian (2.11) was developed only up to third order (see equ. (3.8)) in this treatment.

For the Fourier coefficients υ_{11n_0} which determine (5.61) the stopband width $\Delta_0^{(+)}$ of the linear sum resonance we obtain ((5.37), (5.69), (3.8)) after carrying out the integrals in ψ_1, ψ_2:

$$\begin{aligned} \upsilon_{11n_0} = & \frac{1}{L} \int_0^L ds \cdot e^{i n_0 \cdot \frac{2\pi}{L} \cdot s} \cdot \frac{1}{2} \sqrt{I_1 \cdot I_2} \quad \times \\ & \times \; [\cos X_1(s) + i \cdot \sin X_1(s)] \cdot [\cos X_2(s) + i \cdot \sin X_2(s)] \times \\ & \times \; \{A_{1010}(s) \cdot \sqrt{\beta_x(s) \cdot \beta_\sigma(s)} \; - \\ & - A_{0110}(s) \cdot \sqrt{\frac{\beta_0(s)}{\beta_x(s)}} \cdot [-i + \alpha_x(s)]\} \end{aligned} \tag{5.73}$$

with

$$\begin{aligned} A_{1010}(s) &= \frac{eV(s)}{E_0} \cdot k \cdot \frac{2\pi}{L} \cdot \cos\varphi \cdot D_x'(s) \quad , \\ A_{0110}(s) &= - \frac{eV(s)}{E_0} \cdot k \cdot \frac{2\pi}{L} \cdot \cos\varphi \cdot D_x(s) \quad . \end{aligned} \tag{5.74}$$

Note that only the cavity term (3.8A) contributes here.

In a similar way, for the Fourier coefficients $\upsilon_{1,-1,n_0}$ of the linear difference resonance with resonance width $\Delta_0^{(-)}$:

$$\begin{aligned} \upsilon_{1,-1,n_0} = & \frac{1}{L} \int_0^L ds \cdot e^{i n_0 \cdot \frac{2\pi}{L} \cdot s} \cdot \frac{1}{2} \sqrt{I_1 \cdot I_2} \quad \times \\ & \times \; [\cos X_1(s) + i \cdot \sin X_1(s)] \cdot [\cos X_2(s) - i \cdot \sin X_2(s)] \quad \times \\ & \times \; \{A_{1010}(s) \cdot \sqrt{\beta_x(s) \cdot \beta_\sigma(s)} \; - \\ & - A_{0110}(s) \cdot \sqrt{\frac{\beta_0(s)}{\beta_x(s)}} \cdot [-i + \alpha_x(s)]\} \quad . \end{aligned} \tag{5.75}$$

As an example, for a typical HERA electron optic:

$$\Delta_0^{(+)} \approx 6 \cdot 10^{-3} \quad ;$$

$$\Delta_0^{(-)} \approx 6 \cdot 10^{-3} \quad .$$

We have chosen to illustrate the perturbation theory in some detail only for the case $|\mu_1| = |\mu_2| = 1$. We should point out, however, that for this case, which has to do with the term (3.8A), the equations of motion are linear; hence the use of the term. Thus, for this example the problem can also be described non-perturbatively and hence more precisely by using the eigenvalues of transport matrices as was originally done in Ref. 10.

As has been pointed out previously[10,12)], the Fourier coefficients $\upsilon_{\mu_1\mu_2 n_0}$ and the accompanying resonances can in principle be made to vanish by suitable choice of phases between the cavities and of dispersion values in the cavities. See for example equs. (5.37), (5.73), (5.75).

Higher order synchrotron sidebands can receive two kinds of contribution:

a) those originating in the sinusoidal (non-linear) form for the cavity voltage which is expressed by the series expansion in equ. (3.8) of the cosine term in equ. (2.4b), and

b) those caused by chromatic effects as for example in terms (3.8C) and (3.8D).

Remark:

Particle motion may be described ((3.6), (3.7)) using the variables $\tilde{x}$, $\tilde{p}_x$, $\tilde{\sigma}$, $\tilde{p}_\sigma$. Since the variables $\tilde{\sigma}$ and $\tilde{p}_\sigma$ vary slowly in comparison to $\tilde{x}$ and $\tilde{p}_x$, we can make the approximation that $\tilde{p}_\sigma$ is constant so that the perturbation terms

$$-\frac{1}{2}\tilde{p}_x^2 \cdot \tilde{p}_\sigma \quad \text{and} \quad \frac{\lambda_0}{2} D_x \cdot \tilde{x}^2 \, \tilde{p}_\sigma$$

in H_1 (3.8) lead to a Q-shift for the x-coordinate, ΔQ_x, proportional to $\tilde{p}_\sigma$. From equs. (5.16), (5.14), (5.4a) and (5.8a) we obtain

$$\begin{aligned}\Delta Q_x &= \tilde{p}_\sigma \cdot \frac{L}{2\pi} \cdot \frac{\partial}{\partial I_x}\left\{\frac{1}{2\pi \cdot L}\int_0^L ds \int_0^{2\pi} d\psi_x \cdot \left[-\frac{1}{2}\tilde{p}_x^2 + \frac{1}{2}\lambda_0 \cdot D_x \cdot \tilde{x}^2\right]\right\} \\ &= \tilde{p}_\sigma \cdot \frac{1}{2\pi} \cdot \frac{\partial}{\partial I_x}\left\{\int_0^L ds \cdot \left[-\frac{1}{2} \cdot \frac{2I_x}{\beta_x}\left(\frac{1}{2} + \frac{1}{2}\alpha_x^2\right) + \frac{1}{2}\lambda_0 \cdot D_x \cdot \beta_x \cdot I_x\right]\right\} \\ &= \tilde{p}_\sigma \cdot \frac{1}{4\pi}\int_0^L ds \cdot [-\gamma_x + \lambda_0 \cdot D_x \cdot \beta_x] \quad . \end{aligned} \tag{5.76}$$

Furthermore, from equ. (4.17a), if we put F = 1, so that the influence of cavity fields on the betatron motion is neglected (equ. (4.2a)), then:

$$\begin{aligned}\int_0^L ds \cdot [-\gamma_x + \beta_x \cdot G] &= \int_0^L ds \cdot \alpha_x'(s) \\ &= \alpha_x(L) - \alpha_x(0) = 0\end{aligned}$$

where

$$G = K_x^2 + g_0$$

Then instead of (5.76) we can also put

$$\frac{\Delta Q_x}{\tilde{p}_\sigma} = \frac{1}{4\pi}\int_0^L ds \cdot [-(K_x^2 + g_0) + \lambda_0 \cdot D_x] \cdot \beta_x \quad . \tag{5.77}$$

This is the usual expression for the linear chromaticity[21,22].

6. Influence of cavity fields on betatron motion

In separating the Hamiltonian $\tilde{H}$ into unperturbed parts H_{ox} and $H_{o\sigma}$ and the perturbation H_1 (equ. (3.6)-(3.8)) we have included the term

$$U(s) = -\frac{1}{2}\frac{eV(s)}{E_o}\cdot k\cdot\frac{2\pi}{L}\cdot[\tilde{p}_x\cdot D_x - \tilde{x}\cdot D_x']^2 \qquad (6.1)$$

in H_{ox}. The equation of motion for unperturbed betatron motion is then

$$\frac{d}{ds}\begin{pmatrix}\tilde{x}\\ \tilde{p}_x\end{pmatrix} = [\underline{A}(s) + \delta\underline{A}(s)]\cdot\begin{pmatrix}\tilde{x}\\ \tilde{p}_x\end{pmatrix} \qquad (6.2)$$

with

$$\underline{A}(s) = \begin{pmatrix}0 & 1\\ -(K_x^2+g_o) & 0\end{pmatrix} \quad ; \qquad (6.2a)$$

$$\delta\underline{A}(s) = \frac{eV(s)}{E_o}\cdot\frac{2\pi}{L}\cdot k\cdot\cos\varphi\cdot\begin{pmatrix}D_x(s)\cdot D_x'(s) & -D_x(s)^2\\ D_x'(s)^2 & -D_x(s)\cdot D_x'(s)\end{pmatrix}. \qquad (6.2b)$$

The usual treatments of uncoupled betatron motion are based on the equation of motion

$$\frac{d}{ds}\begin{pmatrix}\tilde{x}\\ \tilde{p}_x\end{pmatrix} = \underline{A}(s)\cdot\begin{pmatrix}\tilde{x}\\ \tilde{p}_x\end{pmatrix} \quad . \qquad (6.3)$$

The term $\delta\underline{A}(s)$ in (6.2) would then be treated as a linear perturbaton.

We demonstrate this here for the calculation of the Q-shift. The first step is to write the one turn transfer matrix as

$$\underline{M}(s_o+L,\ s_o) + \delta\underline{M}(s_o+L,\ s_o)$$

where $\underline{M}(s_o+L,\ s_o)$ refers to the unperturbed motion (equ. (6.3)).

This is achieved by considering the perturbation $\delta\underline{A}(s)$ in the intervals

$$s_\mu \leqq s \leqq s_\mu + \Delta s_\mu \qquad (\mu = 1,2,\ldots,p)$$

so that we can write

$$\underline{M}(s_0+L,\ s_0) + \underline{M}(s_0+L,\ s_0) =$$

$$= \underline{M}(s_0+L,\ s_p)\cdot \underline{M}^{-1}(s_\mu+\Delta s_p,\ s_p)\cdot [\underline{M}(s_p+\Delta s_p,\ s_p) + \delta\underline{M}(s_p+\Delta s_p,\ s_p)] \times$$

$$\times\ \underline{M}(s_p,\ s_{(p-1)}\cdot \underline{M}^{-1}(s_{(p-1)}+\Delta s_{(p-1)},\ s_{(p-1)})\cdot[\underline{M}(s_{(p-1)}+\Delta s_{(p-1)},\ s_{(p-1)}) +$$

$$+\ \delta\underline{M}(s_{(p-1)}+\Delta s_{(p-1)},\ s_{(p-1)})]$$

*
*
*

$$\times\ \underline{M}(s_{(\mu+1)},\ s_\mu)\cdot \underline{M}^{-1}(s_\mu+\Delta s_\mu,\ s_\mu)\cdot[\underline{M}(s_\mu+\Delta s_\mu,\ s_\mu) + \delta\underline{M}(s_\mu+\Delta s_\mu,\ s_\mu)]$$

*
*
*

$$\times\ \underline{M}(s_2,\ s_1)\cdot \underline{M}^{-1}(s_1+\Delta s_1,\ s_1)\cdot[\underline{M}(s_1+\Delta s_1,\ s_1) + \delta\underline{M}(s_1+\Delta s_1,\ s_1)]$$

$$\times\ \underline{M}(s_1,\ s_0). \tag{6.4}$$

According to equs. (6.2) and (6.3) we have

$$\underline{M}(s+\Delta s,\ s) = \underline{1} + \Delta s\cdot\underline{A}(s) \quad ;$$

$$\underline{M}(s+\Delta s,\ s) + \delta\underline{M}(s+\Delta s,\ s) = \underline{1} + \Delta s\cdot[\underline{A}(s) + \delta\underline{A}(s)] \ ,$$

so that the factor

$$\underline{M}^{-1}(s_\mu+\Delta s_\mu,\ s_\mu)\cdot[\underline{M}(s_\mu+\Delta s_\mu,\ s_\mu) + \delta\underline{M}(s_\mu+\Delta s_\mu,\ s_\mu)]$$

can be written as

$$\underline{M}^{-1}(s+\Delta s,\ s)\cdot[\underline{M}(s+\Delta s,\ s) + \delta M(s+\Delta s,\ s)]$$

$$= [\underline{1} - \Delta s\cdot\underline{A}(s)\cdot[\underline{1} + \Delta s\cdot\underline{A}(s) + \Delta s\cdot\delta\underline{A}(s)]$$

$$= \underline{1} + \Delta s\cdot\delta\underline{A}(s) \quad .$$

Then equ. (6.4) becomes

$$\underline{M}(s_0+L,\ s_0) + \delta\underline{M}(s_0+L,\ s_0)$$

$$= \underline{M}(s_0+L.\ s_p)\cdot\{\underline{1} + \Delta s_p\cdot\delta\underline{A}(s_p)\}$$

$$\times\ \underline{M}(s_p,\ s_{(p-1)})\cdot\{\underline{1} + \Delta s_{(p-1)}\cdot\delta\underline{A}(s_{(p-1)})\}$$

*
*
*

$$\times\ \underline{M}(s_{(\mu+1)},\ s_\mu)\cdot\{\underline{1} + \Delta s_\mu\cdot\delta\underline{A}(s_\mu)\}$$

*
*
*

$$\times\ \underline{M}(s_2,\ s_1)\cdot\{\underline{1} + \Delta s_1\cdot\delta\underline{A}(s_1)\}$$

$$\times\ \underline{M}(s_1,\ s_0)$$

and the expression for $\delta\underline{M}$ is in first approximation:

$$\delta\underline{M}(s_0+L,\ s_0) = \int_{s_0}^{s_0+L} d\tilde{s}\cdot\underline{M}(s_0+L,\ \tilde{s})\cdot\delta\underline{A}(\tilde{s})\cdot\underline{M}(\tilde{s},\ s_0)$$

$$= \underline{M}(s_0+L,\ s_0)\cdot\int_{s_0}^{s_0+L} d\tilde{s}\cdot\underline{M}^{-1}(\tilde{s},\ s_0)\cdot\delta\underline{A}(\tilde{s})\cdot\underline{M}(\tilde{s},\ s_0)\ . \qquad (6.5)$$

The Q-shift is then calculated by considering the eigenvalues of the perturbed one turn matrix $\underline{M} + \delta\underline{M}$ (see equ. (4.36))[23]:

$$(\underline{M} + \delta\underline{M})(\vec{v}_I + \delta\vec{v}_I) = (\lambda_I + \delta\lambda_I)(\vec{v}_I + \delta\vec{v}_I)\quad .$$

Using the fact that $\underline{M}\vec{v}_I = \lambda_I\vec{v}_I$ we obtain in first order

$$\underline{M}\cdot\delta\vec{v}_I + \delta\underline{M}\cdot\vec{v}_I = \lambda_I\cdot\delta\vec{v}_I + \delta\lambda_I\cdot\vec{v}_I\quad . \qquad (6.6)$$

The tune shift is defined by (equ. (4.37)):

$$\Delta Q_x = \frac{i}{2\pi\cdot\lambda_I}\cdot\delta\lambda_I\quad . \qquad (6.7)$$

$\delta\vec{v}_I$ can be written in terms of the eigenvectors $\vec{v}_I$ and $\vec{v}_{-I}$:

$$\delta\vec{v}_I = a_I \cdot \vec{v}_I + a_{-I} \cdot \vec{v}_{-I}$$

and by multiplying equ. (6.6) from the left by $\vec{v}_I^+ \underline{S}$ and by using equs. (4.38a), (4.40) and (4.36) we obtain

$$\vec{v}_I^+ \underline{S} \cdot \delta\underline{M} \cdot \vec{v}_I = \delta\lambda_I \cdot i \quad .$$

Recalling

$$\underline{M}^T(s_1, s_2) \cdot \underline{S} \cdot \underline{M}(s_1, s_2) = \underline{S}$$

$$\Longrightarrow \quad \vec{v}_I^+(s_0) \cdot \underline{S} \cdot \underline{M}(s_0 + L, s_0) = \vec{v}_I^+(s_0) \cdot [\underline{M}^{-1}(s_0 + L, s_0)]^T \cdot \underline{S}$$

$$= [\underline{M}^{-1}(s_0 + L, s_0)\vec{v}_I(s_0)]^+ \cdot \underline{S}$$

$$= [\lambda_I^{-1} \cdot \vec{v}_I(s_0)]^+ \cdot \underline{S}$$

$$= \lambda_I \cdot \vec{v}_I^+(s_0) \cdot \underline{S}$$

we find that equs. (6.5) and (6.7) give

$$\Delta Q_x = \frac{1}{2\pi \cdot \lambda_I} \cdot \vec{v}_I^+(s_0) \cdot \underline{S} \cdot \underline{M}(s_0 + L, s_0) \cdot \int_{s_0}^{s_0+L} d\tilde{s} \cdot \underline{M}^{-1}(\tilde{s}, s_0) \cdot \delta\underline{A}(\tilde{s}) \cdot \underline{M}(\tilde{s}, s_0) \cdot \vec{v}_I(s_0)$$

$$= \frac{1}{2\pi} \cdot \vec{v}_I^+(s_0) \int_{s_0}^{s_0+L} d\tilde{s} \cdot \underline{M}^{-1}(\tilde{s}, s_0) \cdot \underline{S} \cdot \delta\underline{A}(\tilde{s}) \cdot \vec{v}_I(\tilde{s})$$

$$= \frac{1}{2\pi} \cdot \int_{s_0}^{s_0+L} d\tilde{s} \cdot \vec{v}_I^+(\tilde{s}) \cdot \underline{S} \cdot \delta\underline{A}(\tilde{s}) \cdot \vec{v}_I(\tilde{s}) \qquad (6.8)$$

ΔQ_x is real:

$$\Delta Q_x = \Delta Q_x^*$$

since (6.2b)

$$\underline{S} \cdot \delta\underline{A} = - \delta\underline{A}^T \cdot \underline{S} \quad .$$

By substituting these expressions in equs. (4.35a) and (6.2b) the final expression for the Q-shift generated by the cavities is:

$$\Delta Q_x = -\frac{1}{2}\frac{k}{L} \cdot \cos\varphi \int_{s_0}^{s_0+L} ds \cdot \frac{eV(s)}{E_0} \cdot [\beta_x \cdot D_x'^2 + \gamma_x \cdot D_x^2 + 2\alpha_x \cdot D_x D_x'] \quad (6.8)$$

The same result is obtained when the prescription in equ. (5.16)

$$\Delta Q_x = \frac{L}{2\pi} \frac{\partial}{\partial I_x} < U(I_x) >$$

is used in conjunction with the average of the perturbation in equ. (6.1)

$$< U(I_x) > = \frac{1}{2\pi \cdot L} \int_0^L ds \int_0^{2\pi} d\psi_x \left(-\frac{1}{2}\frac{eV(s)}{E_0} \cdot k \cdot \frac{2\pi}{L}\right) \times$$

$$\times \left\{ -\sqrt{\frac{2I_x}{\beta_x}} \cdot [\sin(\psi_x + \chi_x) + \alpha_x \cdot \cos(\psi_x + \chi_x)] \cdot D_x \right.$$

$$\left. - \sqrt{2\beta_x \cdot I_x} \cdot \cos(\psi_x + \chi_x) \cdot D_x' \right\}^2 \quad .$$

As is already clear from equ. (6.1) the Q_x-shift vanishes if the dispersion vector (D_x, D_x') in the cavities is zero.

Even for the case when $(D_x, D_x') \neq (0,0)$ in the cavities, ΔQ_x is normally so small that it can be neglected.

7. Summary

We have demonstrated how the well-known techniques for applying canonical perturbation theory to non-linear betatron oscillations can be generalized to the case of coupled synchro-betatron motion.

For this purpose we have utilized a dispersion formalism for the unperturbed oscillation modes. This enables us to introduce action-angle variables which can be represented in terms of appropriate Twiss parameters.

The Twiss parameters themselves have a more general form than those used in usual machine theory (Ref. 16). For numerical calculations, however, it is usually sufficient to use Courant-Livingstone-Snyder forms for the transverse motion and to use an oscillator model (equ. (4.70)) to describe the longitudinal motion.

In order to simplify the presentation we have suppressed the z-motion and only considered transverse motion in the x-direction. Inclusion of motion in the z-direction presents no difficulty and proceeds just as for the x-direction.[25]

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Appendix I

The Hamiltonian for Electrons in the Presence of Radiation Losses

For electrons one needs the extra-term in the Hamiltonian

$$\sigma = C_1 \cdot (K_x^2 + K_z^2)$$

$$\left(\text{where } C_1 = \frac{2}{3} e^2 \frac{\gamma_0^4}{E_0}\right)$$

to describe the energy loss by radiation in the bending magnets. In this case, the cavity phase in (2.11) is determined by the need to replace the energy radiated in the bending magnets. Thus

$$\underbrace{\int_{s_0}^{s_0+L} ds \cdot eV(s) \cdot \sin\varphi}_{\substack{\text{average energy uptake}\\ \text{in the cavities ;}}} = \underbrace{\int_{s_0}^{s_0+L} ds \cdot E_0 \cdot C_1 \cdot [K_x^2(s) + K_z^2(s)]}_{\substack{\text{average energy loss due to radiation}\\ \text{in the bending magnets.}}} \quad \text{(I.1)}$$

In addition to the quadratic and cubic terms in x, p_x, z, p_z, σ, p_σ in H, a linear term

$$- \sigma \cdot c_0(s)$$

appears with

$$c_0(s) = \frac{1}{E_0} \cdot \{eV(s) \cdot \sin\varphi - E_0 \cdot C_1 \cdot [K_x^2 + K_z^2]\} \quad \text{(I.2)}$$

The additional linear term only produces a closed orbit shift. The average of the coefficient $c_0(s)$ disappears according to equ. (I.1):

$$\int_{s_0}^{s_0+L} ds \cdot c_0(s) = 0 \quad \text{(I.3)}$$

The $\sigma \cdot c_0(s)$ term only accounts for the average energy loss. Deviations from this average due to stochastic radiation effects and damping introduce non-symplectic terms into the equation of motion and cannot be handled within a canonical framework.

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