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Convergent Weak Coupling Expansions for Lattice Field Theories that look like Perturbation Series*

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Abstract:

We propose a new form of convergent weak coupling expansion for lattice field theory. It has the advantage that it is very similar to standard (Feynman) perturbation theory. Convergence is proven for sufficiently weak local coupling, i.e. when the theory is close to a free field theory. In the proof, use of analyticity in field variables, as pioneered by Kupiainen and Gawedzki, is supplemented with techniques for handling derivations with respect to free propagator.

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1. Introduction

High temperature expansions are popular, and low temperature expansions are standard for theories whose field variables assume a discrete set of values [1]. But convergent weak coupling expansions for nearly free lattice field theories appear not to be so widely known although related cluster expansions have been extensively used by constructive field theorists [2-4]. Here we propose a new form of such an expansion. It has the advantage that it leads to an expansion for Greens functions which is very similar to standard (Feynman) perturbation theory [5].

Feynman perturbation theory produces a series representation for Greens functions. For instance, the full (disconnected) 2-point function

$$G(y_1,y_2) = \sum_{n\geq 1} \int_{x_1}\cdots\int_{x_n} \Big\{ \mathcal{F}(x_1\dots x_n; y_1,y_2) + \sum_{m=1}^{n-1} \binom{n}{m} \mathcal{F}(x_1\dots x_m; y_1)\,\mathcal{F}(x_{m+1}\dots x_n; y_2) \Big\} . \tag{1.1}$$

On a ν-dimensional hypercubic lattice Λ of lattice spacing a one reads

$$\int_x (\cdots) = a^\nu \sum_{x\in\Lambda} (\cdots) .$$

$\mathcal{F}(x_1\dots x_n; y_1\dots y_k)$ are given as sums of connected Feynman diagrams on vertices $x_1,\dots,x_n$. For standard polynomial field theories like $\lambda\varphi^4$-theory they are constructed from free propagators $v(x,y)$ and "polynomial" vertices (Polynomial means that the number of lines emerging from a vertex is bounded). Some of the vertices x_j are identified as external through δ-functions $\delta(y_i - x_j)$.

Our expansion for the same Greens functions will read

$$G(y_1,y_2) = \sum_{n\geq 1} \underset{\text{distinct}}{\int_{x_1}\cdots\int_{x_n}} \Big\{ \mathcal{M}(x_1\dots x_n | y_1,y_2) + \sum_{m=1}^{n-1} \binom{n}{m} \mathcal{M}(x_1\dots x_m | y_1)\,\mathcal{M}(x_{m+1}\dots x_n | y_2) \Big\} \tag{1.2}$$

and similarly for higher Greens functions. It will be convenient to introduce generating functions so that

$$\mathcal{M}(x_1\dots x_n | y_1\dots y_k) = \frac{\delta}{\delta\psi(y_1)}\cdots\frac{\delta}{\delta\psi(y_k)}\, \mathcal{M}(x_1\dots x_n|\psi)\Big|_{\psi=0} . \tag{1.3}$$

They will be called Mayer amplitudes. On a finite lattice Λ, the sums in eq. (1.2) are finite sums because there exists only a finite number of distinct points, and validity of the expansion follows then straight from the elementary definition of Mayer amplitudes, see section 2. Our main results will be embodied in two theorems, stated in section 2.

Theorem 1 gives a representation of Mayer amplitudes $\mathcal{M}(x_1 \dots x_n | \psi)$ as a sum of contributions from tree graphs on vertices $1, \dots, n$. Each contribution is determined by $(2n-1)$ dimensional integrals. The need for the vacuum energy counter terms $\delta e(X|v)$ that appear in this representation will be explained below. Derivatives of the Gaussian integrals with respect to the free propagator v can be evaluated with the help of the change of covariance lemma, eq. (4.18) of section 4. [For low order calculations, the more elementary formulae of Appendix B are also available.]

Theorem 2 gives bounds on sums of Mayer amplitudes, for background field ψ in a complex neighborhood of zero. These bounds hold for a general class of local interactions, and free propagators $v(x,y)$ that are massive or at least absolutely integrable

$$\sup_y \int_x |v(x,y)| \equiv m^{-2} < \infty . \tag{1.4}$$

It follows that expansions (1.2) and their generalization for (free-propagator-amputated) Greens functions converge on the infinitely extended lattice, for sufficiently weak coupling, i.e. when the theory is close to a free field theory.

We will refer to the literature for the proof of some combinatorial identities, but otherwise the paper will be self contained. For additional reading, D. Brydges' Les Houches lectures '84 are recommended [6].

One may also write down expansions for connected Greens functions G_c. In their Feynman perturbation expansion, the second term $\mathcal{F}\mathcal{F}$ in (1.1) is absent. Because of the restriction to distinct points $x_1, \dots, x_n$, the second term $\mathcal{M}\mathcal{M}$ in the Mayer expansion (1.2) will leave a remnant, involving products of two amplitudes $\mathcal{M}$ with some shared arguments. One can define augmented Mayer amplitudes $\tilde{\mathcal{M}}(x_0, \dots, x_n | \psi)$ such that e.g.

$$G_c(y_1, y_2) = \sum_{n \geq 1} \int_{x_1} \cdots \int_{x_n} \frac{\delta}{\delta\psi(y_1)\delta\psi(y_2)} \tilde{\mathcal{M}}(x_1 \dots x_n | \psi) \Big|_{\psi=0} . \tag{1.5}$$

There is no restriction to distinct arguments $x_1 \dots x_n$ here. For distinct arguments $x_1 \dots x_n$, $\tilde{\mathcal{M}}(x_1 \dots x_n | \psi) = \mathcal{M}(x_1 \dots x_n | \psi)$, while in general they are finite sums of

products of Mayer amplitudes $\mathcal{M}$ with n arguments altogether, multiplied with combinatorial coefficients - cp. Appendix D.

Mayer amplitudes are very similar to Feynman amplitudes (with superpropagators [8]) for nonpolynomial field theories, except for the need for vacuum energy counter terms. It is convenient to consider also "unnormalized" amplitudes $\mathcal{M}_u(x_1 \dots x_n | \psi)$ with zero counter terms. Let us consider a lattice field theory of one scalar field φ with Euclidean action

$$S(\varphi) = S_{free}(\varphi) + \lambda \int_x V(\varphi(x)) . \tag{1.6}$$

S_{free} determines the free propagator $v(x,y)$. Both the Feynman amplitudes and the unnormalized Mayer amplitudes $\mathcal{M}_u$ can be represented as truncated Gaussian (= free field) expectation values

$$\mathcal{F}(x_1 \dots x_n) = \frac{(-\lambda)^n}{n!} \langle V(\varphi(x_1)); \dots ; V(\varphi(x_n)) \rangle_v \tag{1.7}$$

while

$$\mathcal{M}_u(x_1 \dots x_n) = \frac{1}{n!} a^{-\nu n} \langle e^{-a^\nu \lambda V(\varphi(x_1))}; \dots ; e^{-a^\nu \lambda V(\varphi(x_n))} \rangle_v . \tag{1.8}$$

We use the costumary semicolons to indicate that the expectation value is a truncated one. One may imagine that V depends parametrically on an external source or background field ψ, so that eq. (1.7) yields a generating function for Feynman amplitudes, and eq. (1.8) yields $\mathcal{M}_u(x_1 \dots x_n | \psi)$.

Now we can give an idea why the Mayer expansions (1.2) converge for weak coupling, while standard Feynman perturbation theory does not converge, for interactions $V(\varphi(x))$ like $\varphi(x)^4$ which grow (faster than $\varphi(x)^2$) for large φ. The product of $V(\varphi)$'s in (1.7) is badly behaved (growing fast) for large field φ. This produces combinatorial factors which make the series (1.1) divergent. For instance, on a lattice of a single point, integrals like

$$\int_{\mathbb{R}} d\varphi \, e^{-\frac{1}{2}v^{-1}\varphi^2} \varphi^{4n} \propto (2n)!$$

appear. In contrast, the product of factors $e^{-a^\nu \lambda V(\varphi(x))} - 1$ is bounded for large fields when $\mathrm{Re}\,\lambda \geq 0$. Our Mayer amplitudes $\mathcal{M}$ admit a representation as a generalized or "renormalized" truncated expectation value, see Appendix A. It will be shown in this paper that they admit essentially the same bounds as $\mathcal{M}_u$, at least for weak coupling.

If V is a polynomial, then the (Euclidean) Wick's theorem expresses the truncated expectation value $\mathcal{F}$, eq. (1.7), as a sum of connected Feynman graphs. Mayer graphs are like Feynman graphs except that there can be no multiple lines between the same pair of vertices, and no self lines (= lines that connect a vertex to itself). The nonpolynomial expressions (1.8) for $\mathcal{M}_u$ admit a representation as sums over Mayer graphs G. With each line (x,y) of G a superpropagator

$$e^{-\frac{1}{2}a^\nu q(x)v(x,y)q(y)} - 1 \tag{1.9}$$

is associated, and with each vertex x a vertex function

$$\tilde{\mathcal{F}}_x(q) = \frac{1}{2\pi}\int_{\mathbb{R}} d\xi \, \exp\{-a^\nu[\lambda V(\xi) + iq\xi]\} . \tag{1.10}$$

There is one variable $q_i = q(x_i)$ for each vertex x_i $(i = 1,\dots,n)$. These variables are integrated over with weight $dq_i \exp[-\frac{1}{2}a^\nu v(x_i,x_i)q_i^2]$. The corresponding expressions for the Mayer amplitudes $\mathcal{M}$ involve also vacuum energy counter terms and are described in Appendix B. The tree formula of theorem 1 may be thought of as obtained by resummation of all graphs which contain a given tree, with correction factors included to correct for multiple counting.

Finally we will comment on the need for (finite) vacuum energy counter terms. The unnormalized Mayer amplitudes $\mathcal{M}_u$ make their appearance in standard cluster expansions [6]. These are not of the form (1.2) but involve ratios of partition functions as an extra factor (see eq. (2.14')). To aid our understanding of this, let us look at the relation with Feynman perturbation theory. For unbounded V, the amplitudes $\mathcal{M}_u(x_1 \dots x_n | y_1 \dots y_k)$ are not analytic at $\lambda = 0$ (they are ill defined for $\mathrm{Re}\,\lambda < 0$) but they admit an expansion in a asymptotic (Borel summable) power series in λ. It is readily obtained from (1.8) and is given by a sum over all "point connected" Feynman diagrams whose vertices occupy the n distinct sites $x_1 \dots x_n$ [8]. A Feynman diagram is called point connected if it is either connected or becomes upon identification of vertices that occupy the same sites . Feynman diagrams of arbitrarily high order occur because every one of the distinct sites x_i may be occupied by arbitrarily many vertices. Among the point connected Feynman diagrams will be some composed of vacuum

diagrams and diagrams with external legs, with vertices that occupy some shared sites. These contributions will have to be cancelled if formula (1.2) is to be valid. We show how to do this in a nonperturbative way by introducing suitable vacuum energy counter terms in the action.

Convergence of the expansions discussed here depends in an essential way on the lattice, and the range of permissible values of the coupling constant shrinks to 0 in the continuum limit $a \to 0$. However, similar expansions can be written down for continuum theories, upon reformulating them as living on a "staggered lattice" (= multigrid) [9]. Computable expansions whose n-th order term is given by O(n)-dimensional integrals have been studied by Battle and Federbush [10] and by the authors [9, 11]. Their convergence was shown for some superrenormalizable φ^4-models [12].

Other "intrinsic" cluster expansions for lattice theories were proposed by Battle [13]. In contrast with our's, they do not fit into a polymer frame work, and they do not terminate on a finite lattice.

2. Definition of Mayer amplitudes, and their properties

It is appropriate to begin with a discussion of the general

Question: What do expansion methods of (classical) statistical mechanics really do?

Answer: They express observable quantities (e.g. free energies as functions of external fields or sources, or correlation functions) of an infinite system in terms of properties of small subsystems:

Expansion = sequence of approximate answers that are determined by partition functions of ("small") subsystems of increasing size. Different expansions can be obtained by different choice of subsystems.

A systematic procedure is provided by the theory of polymer systems [14, 1]: The expansions involve activities of a polymer system. They are determined by corresponding partition functions through a process of truncation. Our Mayer amplitudes will be defined as activities of this kind.

The polymers considered here are a mathematical abstraction. To specify a polymer system one must, first of all. specify a <u>set Λ of sites</u> which may be occupied by the polymers. In our application $\Lambda \subseteq (a\mathbb{Z})^\nu$ is a ν-dimensional hypercubic lattice of lattice spacing a. We admit arbitrary finite nonempty subsets P of Λ as <u>polymers</u>. Polymers occupying a single site are called monomers. Two polymers P_1, P_2 are said to be <u>compatible</u>, if they don't intersect. We write $P_1 \sim P_2$ in this case

$$P_1 \sim P_2 \quad \text{if} \quad P_1 \cap P_2 = \phi .$$

A (real) activity A(P) will be assigned to every polymer P.

Once a polymer system is specified in this way, partition functions of finite subsets X of Λ are defined. Let $\Pi(X)$ be the set of all partitions of X into nonempty subsets (polymers) P. Then one sets

$$Z(X) = \sum_{\mathcal{P} \in \Pi(X)} \prod_{P \in \mathcal{P}} A(P) . \tag{2.1a}$$

It is required that $Z(X) > 0$ for all finite X. If follows from the definition (2.1a) that

$$Z(\emptyset) = 1 . \tag{2.1b}$$

Conversely, a collection of partition functions Z(X) satisfying eq. (2.1b) will specify a unique set of activities A(P) such that eq. (2.1a) holds. This simple fact will be of central importance. Let us therefore recall its proof. Suppose the assertion is true for (no. of sites in P) $\leq$ n. Consider a polymer X with n+1 sites. Then $A(X) = Z(X) - \sum \prod$(activities of polymers with $\leq$ n sites) is determined by partition functions Z(Y) with $Y \subsetneq X$ q.e.d. A well known formula says that

$$A(P) = \sum_{n \geq 1} (-1)^{n-1}(n-1)! \sum_{\mathcal{P} \in \Pi_n(P)} \prod_{X \in \mathcal{P}} Z(X) . \tag{2.1c}$$

$\Pi_n(P)$ is the set of partitions of P into exactly n nonempty subsets.

Let us now consider a lattice field theory with Euclidean action

$$S(\varphi) = \tfrac{1}{2}(\varphi, v^{-1}\varphi) + \int_x V(\varphi(x)) . \tag{2.2}$$

We will find it convenient to consider families of interactions V that are parametrized by a dimensionless coupling constant g

$$a^\nu V(\varphi(x)) = W(g\varphi_x) . \tag{2.3}$$

We use lattice notation and dimensionless fields φ_x etc. as explained in the footnote *. In a $\lambda\varphi^4$-theory one would set $g = (\lambda a^{4-\nu})^{1/4}$

The free propagator v has to be positive semidefinite (as an operator) and it will be assumed that it is absolutely integrable

$$\sup_y \int_{x \in (a\mathbb{Z})^\nu} |v(x,y)| \equiv m^{-2} < \infty . \tag{2.4}$$

For instance, v may be a Yukawa potential with mass m

$$(-\Delta_x + m^2)\, v(x,y) = \delta(x-y) . \tag{2.5}$$

In this case

$$(\varphi, v^{-1}\varphi) = \int_x \varphi(x)[-\Delta + m^2]\varphi(x) . \tag{2.6}$$

Expectation values in the free field theory with propagator v are computed with the help of the normalized Gaussian measure

$$d\mu_v(\varphi) = (\det 2\pi v)^{-\frac{1}{2}} e^{-\frac{1}{2}(\varphi, v^{-1}\varphi)} \prod_{x \in \Lambda} d\varphi(x) . \tag{2.7}$$

Gaussian measures remain meaningful when the covariance operator v has a zero eigenspace. The simplest example is the Dirac δ-measure $d\mu_0(\xi) = \delta(\xi)d\xi$ on $\mathbb{R}$.

We define partition functions for finite subsets X of Λ by

$$Z(X|\psi) = \int d\mu_v(\varphi) \exp\left[\delta e(X|v) - \int_{x \in X} V(\varphi(x) + \psi(x))\right] . \tag{2.8}$$

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* <u>Lattice notations</u>: $\int_{x \in \Lambda} (\cdots) = a^\nu \sum_{x \in \Lambda} (\cdots)$; $\nabla_\mu f(x) = a^{-1}\{f(x+e_\mu) - f(x)\}$

$e_\mu = -e_{-\mu}$ = lattice vector in μ-direction; $-\Delta = \sum_{\mu=1}^{\nu} \nabla_{-\mu}\nabla_\mu$ in ν dimensions.

<u>Dimensionless quantities</u>: $\varphi_x = a^d \varphi(x)$, $v_{xy} = a^{2d} v(x,y)$, $a^\nu \delta(x-y) = \delta_{xy}$.

$a^{\nu-d} \frac{\delta}{\delta\psi(x)} = \frac{\partial}{\partial\psi_x}$ with $d = \frac{\nu-2}{2}$. The Fourier conjugate variable $q_x = a^{\nu-d} q(x)$.

They depend on a background field Ψ. The vacuum energy counter term $\delta e(X|v)$ is determined by the requirement that

$$Z(X|0) = 1 \qquad \text{for all } X. \tag{2.9}$$

Since obviously $Z(\phi|\Psi) = 1$, $Z(X|\Psi)$ may be regarded as partition functions of a polymer system. This defines activities $A(P|\Psi)$ as we have seen. Our Mayer amplitudes $\mathcal{M}$ are symmetric in their arguments $x_1, \ldots, x_n$, and are defined by

$$\begin{aligned} A(P|\Psi) &= \delta_{1,n} + M(P|\Psi) \\ M(P|\Psi) &= n!\, a^{\nu n} \mathcal{M}(x_1 \ldots x_n|\Psi) \qquad \text{if} \quad P = (x_1 \ldots x_n)\,. \end{aligned} \tag{2.10}$$

If follows from the normalization condition (2.9) that

$$A(P|0) = \delta_{1,n}\,. \tag{2.11a}$$

That is,

$$\mathcal{M}(x_1 \ldots x_n|0) = 0\,. \tag{2.11b}$$

Since activities are uniquely determined by partition functions, validity of eq. (2.11a) is established by verifying that it reproduces normalization condition (2.9) when inserted in polymer representation (2.1a).

The full (disconnected) free-propagator-amputated [15] Greens functions are given by

$$G(y_1 \ldots y_n) = Z(\Lambda|0)^{-1} \frac{\delta}{\delta\Psi(y_1)} \cdots \frac{\delta}{\delta\Psi(y_n)} Z(\Lambda|\Psi)\Big|_{\Psi=0}\,. \tag{2.12}$$

We consider a finite lattice Λ to begin with, and take the ∞ -volume limit $\Lambda \to (a\mathbb{Z})^\nu$ in the end.

Let us insert the polymer representation (2.1a) for $Z(\Lambda|\Psi)$, and carry out the differentations using the product rules. The nondifferentiated activities are all zero at $\Psi = 0$ by eq. (2.11a), except for monomer activities which are 1. Since $Z(\Lambda|0)^{-1} = 1$ by eq. (2.9) it follows that

$$\begin{aligned} G(y_1, y_2) = {} & \sum_P \frac{\delta^2}{\delta\Psi(y_1)\delta\Psi(y_2)} A(P|\Psi)\Big|_{\Psi=0} \\ & + \sum_{P_1, P_2 :\, P_1 \cap P_2 = \phi} \frac{\delta A(P_1|\Psi)}{\delta\Psi(y_1)} \frac{\delta A(P_2|\Psi)}{\delta\Psi(y_2)}\Big|_{\Psi=0}\,. \end{aligned} \tag{2.13}$$

Summation over polymers P with n sites is equivalent to summing over sequences $x_1, \ldots, x_n$ of n distinct sites, and dividing by n! to correct for multiple counting. Therefore, expansion (1.2) of the two-point function in terms of Mayer amplitudes is obtained when the derivatives of activities in eq. (2.13) are expressed in terms of Mayer amplitudes (2.10), (1.3).

Let us note that $Z(X|\Psi)$, and therefore also $A(X|\Psi)$, depend on $\Psi(y)$ only if $y \in X$. Therefore the sum over P in eq. (2.13) extends only over polymers with $y_1 \in P$ and $y_2 \in P$, and similarly for P_1, P_2.

The generalization of eq. (2.13) to higher Greens functions is obvious. The m-point function is expressed as sum of products of up to m Mayer amplitudes.

The expansions for the full disconnected Greens functions will determine expansions for the connected Greens functions G_c in sums of products of Mayer amplitudes.

Let us for comparison also mention the standard cluster expansion formula for the (full, disconnected) Greens functions which involves the unnormalized activities. One defines unnormalized partition functions Z_u by the same formula (2.8) but with $\delta e(X|v) \equiv 0$. Let $A_u(P|\Psi) = \delta_{1,n} + M_u(P|\Psi)$ be the corresponding activities. One substitutes Z_u for Z in the formula (2.12) for the Greens function and inserts the polymer representation for Z_u. After differentation a partial resummation can be carried out with the result*

* Here and later in this paper, the symbols Σ, + are used for union of disjoint sets.

$$G(x_1,x_2) = \sum_X \rho_\Lambda(X) \Big\{ \frac{\delta^2}{\delta\psi(x_1)\delta\psi(x_2)} A_u(X|\psi)\Big|_{\psi=0}$$

$$+ \sum_{P_1,P_2 : X = P_1+P_2} \frac{\delta A_u(P_1|\psi)}{\delta\psi(x_1)} \frac{\delta A_u(P_2|\psi)}{\delta\psi(x_2)}\Big|_{\psi=0} \Big\} \tag{2.14'}$$

with

$$\rho_\Lambda(X) = \frac{Z_u(\Lambda - X|0)}{Z_u(\Lambda|0)} .$$

To use this formula for computations one must insert an expansion like (5.7b) for ρ_Λ in powers of M_u. This will produce an expansion unlike ours which does not terminate on a finite lattice Λ. It is known that it will converge for sufficiently weak coupling, both for finite and infinite volume [6]. This follows e.g. from our bounds (4.17) on A_u and proposition 8 for ρ_Λ (with $T = \emptyset$).

Let us now return to our Mayer amplitudes $\mathcal{M}(x_1 \ldots x_n|\psi)$. They are proportional to the activities $A(X|\psi)$ by eq. (2.10).

The Gaussian integral in formula (2.8) for $Z(X|\psi)$ is equal to an n-dimensional integral if $X = \{x_1, \ldots, x_n\}$ because of the following property of Gaussian measures.

Suppose that $f(\varphi)$ depends on φ only through $\varphi_{x_i} \equiv \varphi_i$ ($i = 1 \ldots n$). Consider the $n \times n$ matrix $v = (v_{ij})$ with entries

$$v_{ij} = v_{x_i x_j} . \tag{2.14a}$$

Then

$$\int d\mu_v(\varphi) f(\varphi) = (\det 2\pi v)^{-\frac{1}{2}} \int \prod_{i=1}^n d\varphi_i \; f(\varphi_1, \ldots, \varphi_n) \cdot$$

$$\cdot \exp\Big(-\tfrac{1}{2} \sum_{i,j} \varphi_i (v^{-1})_{ij} \varphi_j\Big) \tag{2.14b}$$

see Appendix A.

It follows that the activity $A(X|\psi)$ is expressed via eq. (2.1c) as sum of products of altogether n-dimensional integrals. There can be large cancellations in this sum, however, and the fact that $A(X|\psi)$ tends to be small when the points of X are far apart is not apparent. Therefore this defining formula for activities A is not suitable for the purpose of estimation, and alternative formulas for the same quantity need to be derived.

This completes the presentation of the definition of Mayer amplitudes, and the derivation of expansions for Greens functions on a finite lattice. Next we state our main results, theorems 1 and 2.

Theorem 1 gives an alternative formula for the Mayer amplitudes. To state it, we need to introduce trees and interpolating propagators in the standard way [18].

An n-tree η is an integer valued function, with $\eta(l)$ defined for $l = 2, \ldots, n$ and such that

$$1 \le \eta(l) < l . \tag{2.15}$$

This determines a tree graph on vertices $1, \ldots, n$ with links $(i, \eta(i))$, see figure 1 (p.23) One introduces n-1 auxiliary integration variables $(s_1, \ldots, s_n) = s$ in the real interval $0 \ldots 1$, and defines

$$f(\eta|s) = \prod_{i=2}^n [s_{i-2}\, s_{i-3} \cdots s_{\eta(i)}] . \tag{2.16}$$

Empty products are to be read as 1.

Given $x_1, \ldots, x_n$, one defines an interpolating propagator $v[s]$ with kernel $v(s|x,y)$ by

$$v(s|x_i,x_j) = v(s|x_j,x_i) = s_i\, s_{i+1} \cdots s_{j-1}\, v(x_i,x_j) \tag{2.17}$$

and $v(s|x,y) = 0$ unless $x = x_i$, $y = x_j$ for some $i, j = 1, \ldots, n$. Eq. (2.17) specializes for diagonal elements to $v(s|x_i,x_i) = v(x_i,x_i)$. It is a well known fact that $v[s]$ is positive semidefinite when v is . When the points $x_1 \ldots x_n \in X$ are identified by a labelling $\underline{x}$ of X we will write $v[\underline{x},s]$ in place of $v[s]$.

<u>Theorem 1</u>: We use the notations (2.15) - (2.17), setting $\lambda = 1$ in the action (1.6)

$$a^{-\nu}\delta_{1,n} + \mathcal{M}(x_1 \ldots x_n) = \frac{1}{n} a^{\nu(n-2)} \mathcal{S}\Big[\sum_\eta \int ds\, f(\eta|s) \int d\mu_{v[s]}(\varphi) \cdot$$

$$\Big\{\prod_{i=2}^n \Big[v(x_i, x_{\eta(i)}) \Big(\frac{\delta}{\delta\varphi(x_i)} \frac{\delta}{\delta\varphi(x_{\eta(i)})} + \frac{\delta}{\delta w(x_i, x_{\eta(i)})}\Big)\Big]\Big\}$$

$$\exp\Big(\delta e(X|w) - \int_{x \in X} V(\varphi(x) + \psi(x))\Big)\Big]_{w = v[s]} \tag{2.18}$$

with $X = \{x_1 \ldots x_n\}$.

Sum over η is over all n-trees, and $\mathcal{S}$ stands for symmetrization in the arguments $x_2, \dots, x_n$. The vacuum energy counter term is determined by the unnormalized partition function Z_u,

$$e^{\delta e(X|v)} = Z_u(X|0)^{-1} \tag{2.19a}$$

$$Z_u(X|\psi) = \int d\mu_v(\varphi) \exp\left(-\int_{x\in X} V(\varphi(x)+\psi(x))\right) . \tag{2.19b}$$

The Gaussian integrals in eqs. (2.18), (2.19) become n-dimensional integrals by using formula (2.14) for Gaussian measures.

Derivatives of unnormalized partition functions Z_u with respect to the free propagator can be evaluated with the help of the "change of covariance lemma" (eq. (4.18) in section 3) for Gaussian integrals like (2.19b) [2]. Amplitudes with few arguments $x_1 \dots x_n$ can also be calculated by more elementary means, see Appendix B.

Next we state our bounds for Mayer amplitudes.

Theorem 2: Consider local interactions of the form $V(\varphi(x)) = a^{-\nu} W(g\varphi_x)$ with $V(o) =$ and suppose that bounds of the following form hold for fields in a complex strip

$$|e^{-W(\zeta)}| \leq b < \infty \qquad \text{for } \zeta \in \mathbb{C} \ , \ |\Im m \zeta| \leq 1 \ , \tag{2.20}$$

and that the free propagator is absolutely integrable

$$\sup_x \int_y |v(x,y)| \equiv m^{-2} < \infty \ . \tag{2.21}$$

If $g^2(m^2a^2)^{-1} < (384e^2b^2)^{-1}$ and $0 \leq \varepsilon < 1$ then the Mayer amplitudes are holomorphic in ψ in a complex strip,

$$|\Im m \psi_x| \leq g^{-1}\varepsilon \qquad \text{for all } x, \tag{2.22}$$

and obey the following bound for all such complex ψ, and $n \geq 1$.

$$\left| a^{-\nu}\delta_{1,n} + \int_{x_2} \dots \int_{x_n} \mathcal{M}(x_1 \dots x_n|\psi) \right| \leq a^{-\nu}(2b)^n \frac{1}{n} \left(\frac{cg}{ma(1-\varepsilon)}\right)^{2(n-1)}$$

with $c = \left(\frac{1840}{6}\right)^{\frac{1}{2}} b$. a is the lattice spacing.

The hypotheses of the theorem are fulfilled for a weakly coupled massive $\lambda\varphi^4$-theory because

$$|e^{-\lambda\zeta^4}| \leq e^{8\lambda\varkappa^4} \qquad \text{for } |\Im m \zeta| \leq \varkappa \ .$$

The use of analyticity in field space was pioneered by Kupiainen and Gawedzki [19]. Bounds on derivatives of $\mathcal{M}$ with respect to fields ψ follow from theorem 2 by use of the Cauchy formula. Since $\mathcal{M}(x_1 \dots x_n|\psi)$ depends on $\psi(y)$ only if $y = x_i$ for some i, it follows that the expansions for Greens functions such as (1.2) will converge if

$$\sum_{n\geq 1} \int_{x_2} \dots \int_{x_n} |\mathcal{M}(x_1 \dots x_n|\psi)| < \infty$$

for arbitrary x_1 and in a complex neighborhood of 0. Therefore, theorem 2 implies

Corally 3. The expansions for Greens functions converge if $\frac{g^2}{m^2a^2} b^3$ is sufficiently small.

In applications one normalizes g such that b = O(1).

3. Derivation of the tree formula, theorem 1

Let us consider the unnormalized partition functions Z_u which determine the vacuum energy counter terms (2.19a). We introduce the vertex function

$$2\pi \tilde{F}_x(k) = \int_{\mathbb{R}} d\xi \ \exp\left[-W(g\xi) - ik\xi\right] \ . \tag{3.1}$$

This differs from expression (1.10) with $\lambda = 1$ by a substitution $q = a^{d-\nu}k$. Using formula (A.4) for the Fourier transform of a Gaussian measure we obtain Z_u in the form of partition function for an "unbounded spin system" [20].

$$Z_u(X|\psi) = \int \prod_{x\in X} \left[dq_x \tilde{F}_x(q_x)\right] \exp\left(-\tfrac{1}{2}\sum_{x,y} q_x v_{xy} q_y + i\sum_x q_x \psi_x\right) . \tag{3.2}$$

Therefore in particular

$$e^{-\delta e(X|v)} = \int \prod_{x\in X} \left[dq_x \tilde{F}_x(q_x)\right] \exp\left(-\tfrac{1}{2}\sum_{x,y} q_x v_{xy} q_y\right) . \tag{3.3}$$

We deduce properties of the vacuum energy counter terms. Define

$$v_X(x,y) = \begin{cases} v(x,y) & \text{if } x,y \in X \\ 0 & \text{otherwise} \end{cases} .$$

Lemma 4. (i) $\delta e(X|v) = \delta e(X|v_X) = \delta e(\Lambda|v_X)$ when $X \subset \Lambda$

(ii) Suppose that $X_1 \cap X_2 = \emptyset$ and $v_{xy} = 0$ whenever $x \in X_1, y \notin X_2$. Then

$$\delta e(X_1 + X_2|v) = \delta e(X_1|v) + \delta e(X_2|v) .$$

Both assertions are immediate from eq. (3.3).

Next we state a general identity relating partition functions and activities of a polymer system. Write X^* for the set of pairs (x,y) of distinct points in X.

Proposition 5: Consider partition functions $Z(X)$ of a polymer system on Λ of the form

$$\mathcal{Z}(X) = \exp \mathcal{E}(X) \tag{3.4}$$

with $\mathcal{E}(\emptyset) = 0$. Consider interpolating interactions $\mathcal{E}(X|t)$ which depend on positive real variables $t = (t_{xy})_{(x,y)\in\Lambda^*}$ such that

$$\begin{aligned} &\text{i) } \mathcal{E}(X|t) = \mathcal{E}(X) \quad \text{when } t_{xy} = 1 \\ &\text{ii) } \mathcal{E}(X|t) \text{ depends only on } t_{xy} \text{ with } (x,y)\in X^* \\ &\text{iii) } \mathcal{E}(X_1+X_2|t) = \mathcal{E}(X_1|t)+\mathcal{E}(X_2|t) \text{ if } X_1\cap X_2=\emptyset \text{ and } t_{xy}=0 \text{ whenever } x\in X_1, y\notin X_1. \end{aligned} \tag{3.5}$$

Retain the notations (2.15), (2.16) for trees. Given $x_1, \ldots, x_n$ define

$$t(s)_{x_i x_j} = s_i s_{i+1} \cdots s_{j-1} \quad \text{if } i<j . \tag{3.6}$$

The activities $\mathcal{A}(P)$ associated with partition functions $\mathcal{Z}(X)$ equal

$$\mathcal{A}(P) = (n-1)!\,\mathcal{S}\left\{\sum_\eta \int ds\, f(\eta|s)\Big[\prod_{i=2}^n \frac{\partial}{\partial t_{x_i x_{\eta(i)}}}\Big] e^{\mathcal{E}(P|t)}\Big|_{t=t(s)}\right\} \tag{3.7}$$

if $P = (x_1, \ldots, x_n)$. $\mathcal{S}$ stands for symmetrization with respect to $x_2, \ldots, x_n$; any element of P can be chosen as x_1. The η-summation runs over all n-trees, and $s_1 \ldots s_{n-1}$ are integrated from 0 to 1.

The proof of proposition 5 will be postponed to the end of this section. We will now apply lemma 4 and proposition 5 to prove our theorem 1. A variety of other formulas for the Mayer amplitudes can be deduced from proposition 5 by chosing different interpolations $\mathcal{E}(X|t)$. For instance,

$$\mathcal{E}(X|t) = \Big[\prod_{(x,y)\in X^*} t_{xy}\Big]\mathcal{E}(X) \tag{3.8}$$

satisfies the hypotheses.

According to eq. (3.2), $Z(X|\psi) = e^{\delta e(X|v)} Z_u(X|\psi)$ equals

$$Z(X|\psi) = \int \prod_{x\in X} \mathcal{D}q_x\, \mathcal{Z}(X|q) \tag{3.9}$$

with

$$\mathcal{D}q_x = dq_x\, \tilde{F}_x(q_x)\, e^{iq_x\psi_x} \tag{3.10a}$$

and

$$\mathcal{Z}(X|q) = \exp\Big[\delta e(X|v) - \frac{1}{2}\sum_{x,y\in X} q_x v_{xy} q_y\Big] \equiv \exp \mathcal{E}(X) . \tag{3.10b}$$

Define the interpolating interaction $\mathcal{E}(X|t)$ by the substitution

$$v \to tv = (t_{xy} v_{xy}) \quad , \quad t_{xx} \equiv 1 . \tag{3.11}$$

We pass over some mathematical subtleties * associated with the fact that tv might not be positive semidefinite in general - the resulting formula for $\mathcal{A}(P)$ will only involve positive semidefinite tv so that $\delta e(X|tv)$ is uniquely defined by normalization conditions. Lemma 4 assures that the interpolating interaction $\mathcal{E}(X|t)$ will satisfy the hypotheses of proposition 5. Therefore $\mathcal{Z}(X|q)$ admits the polymer representation

$$\mathcal{Z}(X|q) = \sum_{\mathcal{P}\in\Pi(X)} \prod_{P\in\mathcal{P}} \mathcal{A}(P|q) \tag{3.12a}$$

with

$$\begin{aligned} \mathcal{A}(P|q) &= (n-1)!\,\mathcal{S}\Big\{\sum_\eta \int ds\, f(\eta|s)\Big[\prod_{i=2}^n \frac{\partial}{\partial t_{x_i x_{\eta(i)}}}\Big] \cdot \exp\Big[\delta e(P|tv) - \frac{1}{2}\sum_{x,y\in P} q_x t_{xy} v_{xy} q_y\Big]\Big|_{t=t(s)}\Big\} \\ &= (n-1)!\,\mathcal{S}\Big\{\sum_\eta \int ds\, f(\eta|s) \exp\Big[-\frac{1}{2}\sum_{i,j} q_{x_i} v[s]_{x_i x_j} q_{x_j}\Big] \cdot \prod_{i=2}^n \Big[v_{x_i x_{\eta(i)}}\Big(-q_{x_i} q_{x_{\eta(i)}} + \frac{\partial}{\partial w_{x_i x_{\eta(i)}}}\Big)\Big] e^{\delta e(P|w)}\Big|_{w=v(s)}\Big\} \end{aligned} \tag{3.12b}$$

for $P = (x_1, \ldots, x_n)$.

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* One sets $e^{\delta e(P|v)} = 0$ when v is not positive semidefinite.

Evidently, $\mathcal{A}(P|q)$ depends on q_x only when $x \in P$. Therefore, inserting eqs. (3.12) into formula (3.9) for Z one obtains

$$Z(X|\psi) = \sum_{P \in \Pi(X)} \prod_{P \in P} A(P|\psi)$$

with

$$A(P|\psi) = \int \Big[\prod_{x \in P} Dq_x\Big] \mathcal{A}(P|q) \qquad \text{for } P = (x_1, \dots, x_n) \ .$$

According to eq. (3.10a), Dq_x depends parametrically on ψ_x. The formula (A.4) for the Fourier transform of a Gaussian measure yields

$$\exp\Big[-\tfrac{1}{2}\sum_{i,j} q_{x_i} v[s]_{x_i x_{\eta(i)}} q_{x_j}\Big] = \int d\mu_{v[s]}(\varphi) \exp i \sum_j q_{x_j}\varphi_{x_j} \ .$$

Upon inserting this, the q-integrations can be performed by inserting eq. (3.1). The result is theorem 1, expressed in terms of dimensionless quantitites. Activities $A(P|\psi)$ and Mayer amplitudes are related by eq. (2.10).

It remains to establish proposition 5. It will be deduced from another general combinatorial identity [20].

A Mayer graph G on vertex set X is a collection of pairs (x,y) of distinct elements of X such that the graph whose vertices are $x \in X$ and whose links are $(x,y) \in G$ is connected. If X contains only one point, then there is only one Mayer graph G. It contains zero links.

Lemma 6 (Abstract tree formula) Let $\mathcal{G}_n$ be the set of all Mayer graphs on vertex set $\{1, \dots, n\}$, and $w = (w_{ij})$ an arbitrary symmetric nxn matrix. Retain the notations (2.15) ... (2.16) for trees.

i) $$\sum_{G \in \mathcal{G}_n} \prod_{(ij) \in G} \big[e^{w_{ij}} - 1\big] = \sum_{\pi} \sum_{\eta} \int ds\, f(\eta|s) \Big[\prod_{k=2}^{n} w_{\pi(k)\,\eta(\pi(k))}\Big] \cdot \exp \sum_{1 \le i < j \le n} w_{\pi(i)\pi(j)}[s]$$

with

$$w_{ij}[s] = w_{ji}[s] = \begin{cases} s_i s_{i+1} \cdots s_{j-1} w_{ij} & \text{if } i<j \\ w_{ii} & \text{if } i=j \ . \end{cases}$$

The sum over π runs over all (n-1)! permutations of 1,...,n with $\pi(1) = 1$, and the sum over η runs over all n-trees η .

ii) If $-w \geq 0$ as a matrix then also the matrix $-w[s] \geq 0$.

A proof of formula (i) is found in Appendix B of ref. 20. (The possibility of fixing $\pi(1) = 1$ is verified by inspecting the proof.) The derivation of a related formula is also sketched in [4] and in Glimm and Jaffe's book [18], and the positivity property is also proven there.

Let us now begin with the derivation of proposition 5. Set

$$\mathcal{Z}(X|t) = \exp \mathcal{E}(X|t) \ . \tag{3.13}$$

Consider partitions of X into disjoint subsets Y_i such that $t_{xy} = 0$ whenever $x \in Y_i$, $y \in Y_j$ $(i \neq j)$. Then

$$\mathcal{Z}(X|t) = \prod_i \mathcal{Z}(Y_i|t) \tag{3.14}$$

by hypothesis iii), and $Z(Y_i|t)$ depends only on t_{xy} for $(x,y) \in Y_i^*$ by hypothesis ii). Introduce translation operators T(t) which act on functions F of t by

$$(T(t)F)(t') = F(t+t') \ . \tag{3.15}$$

For transparency we write

$$T(t) = \prod_{(x,y) \in \Lambda^*} \exp\Big(t_{xy}\frac{\partial}{\partial t_{xy}}\Big)$$

but analyticity or differentiability of F in t need not be assumed. Since $\mathcal{Z}(X) = \mathcal{Z}(X|1)$ we have

$$\mathcal{Z}(X) = \Big[\prod_{(x,y) \in X^*} e^{\partial/\partial t_{xy}}\Big] \mathcal{Z}(X|t)\Big|_{t=0} \ .$$

Write

$$e^{\partial/\partial t_{xy}} = 1 + f_{xy}$$

and expand in powers of f. The terms in the resulting sum will be labelled by graphs which are specified by sets of links (x,y). The connected parts of these graphs are Mayer graphs G on vertex sets $Y \subset X$. Therefore the sum over graphs can be rewritten in the form

$$\mathcal{Z}(X)=\sum_{\mathcal{P}\in\Pi(X)}\Big\{\prod_{Y\in\mathcal{P}}\Big(\sum_{G\in\mathcal{G}_Y}\prod_{(x,y)\in G}\big[e^{\partial/\partial t_{xy}}-1\big]\Big)\Big\}\mathcal{Z}(X|t)\Big|_{t=0}$$

Empty products are read as 1. Now we use the factorization property (3.14) to conclude that the expression $\{\}\mathcal{Z}$ factorizes

$$\mathcal{Z}(X)=\sum_{\mathcal{P}\in\Pi(X)}\prod_{Y\in\mathcal{P}}\mathcal{A}(Y)$$

with

$$\mathcal{A}(Y)=\sum_{G\in\mathcal{G}_Y}\Big(\prod_{(x,y)\in G}\big[e^{\partial/\partial t_{xy}}-1\big]\Big)\mathcal{Z}(Y|t)\Big|_{t=0} .$$

We may imagine that a Fourier transformation in t_{xy} is performed in an interval containing $[0, 1]$ and read $\partial/\partial t_{xy}$ as a Fourier conjugate variable iw_{xy}. The abstract tree formula, lemma 6 may now be applied with the result

$$\mathcal{A}(Y)=(n-1)!\,\mathcal{S}\Big\{\sum_{\eta}\int ds\,f(\eta|s)\Big[\prod_{i=2}^{n}\frac{\partial}{\partial t_{x_i x_{\eta(i)}}}\Big]$$
$$\cdot\exp\Big(\sum_{1\le i<j\le n}s_i s_{i+1}\cdots s_{j-1}\frac{\partial}{\partial t_{x_i x_{\eta(i)}}}\Big)\Big\}\mathcal{Z}(Y|t)\Big|_{t=0} .$$

The translation operator exp( ) translates t into t + t(s), and $\mathcal{Z}(Y|t)=\exp\mathcal{E}(Y|t)$ by definition. Therefore the assertion of proposition 5 obtains.

4. Bounds on Mayer amplitudes

Bounds on Mayer amplitudes and convergence of expansions for Greens functions can be established in different ways. We choose to start from the tree formula, theorem 1. This formlua involves $e^{\delta e(X|v)} = Z_u(X|0)^{-1}$ and its derivatives with respect to the free propagator v . Techniques for handling such derivatives with respect to propagators are of independent interest.

The unnormalized partition functions Z_u are defined by eq. (2.19b), i.e.

$$Z_u(X|\psi)=\int d\mu_v(\varphi)\prod_{x\in X}e^{-W(g[\varphi_x+\psi_x])} \qquad (4.1)$$

We regard them as partition functions of a polymer system and denote the corresponding activities by $A_u(P|\psi)$. Thus

$$Z_u(X|\psi)=\sum_{\mathcal{P}\in\Pi(X)}\prod_{P\in\mathcal{P}}A_u(P|\psi) . \qquad (4.2)$$

The partition functions, and therefore also the activities A_u, depend on the free propagator v and on the coupling constant g.

Our first step will be to establish bounds on unnormalized activities $A_u(P|0)$ and their derivatives with respect to v . From these, bounds on $Z_u(X|0)^{-1}$ and its derivatives are deduced, for sufficiently weak coupling. Details of this second step are in section 5. Given these bounds on derivatives of $e^{\delta e(X|v)}$, bounds on the Mayer amplitudes are obtained by the same method as used in step 1 for A_u.

To state the auxiliarly bounds, we introduce some terminology. A <u>tree</u> T_1 on a vertex set $X \subset \Lambda$ with n elements is a set of n-1 pairs $\ell = (x,y)$ of distinct elements of X such that the graph with vertices $x\in X$ and links $\ell\in T_1$ is connected (hence a tree graph). If n = 1 there is only one tree T_1, it contains zero links ℓ. A <u>possibly disconnected tree</u> T (on X) is a possibly empty subset $T \subseteq T_1$ of some tree T_1 (on X). We use the notation

$$\partial_T A_u(P)=\Big(\prod_{(x,y)\in T}\frac{\partial}{\partial v_{xy}}\Big)A_u(P|0) \qquad (4.3)$$

and

$$d_T(x) = \text{no. of pairs } \ell\in T \text{ with } x\in\ell . \qquad (4.4)$$

Graphically, $d_T(x)$ is the number of links incident at vertex x; it may be zero.

Our first step will establish

<u>Proposition 7</u>. Suppose that the interaction V and the free propagator v satisfy the hypotheses of theorem 2. Let T be a possibly disconnected tree. Then for any real coupling constant g, the derivatives (4.3) of unnormalized activities obey the bounds

$$\sum_{\substack{P \\ |P|=n,\, x_1\in P}} |\partial_T A_u(P)| \leq (2g_1)^{2t}\left[\prod_{x\in P} d_T(x)!\right] b^n \left(\frac{4eg}{ma}\right)^{2(n-1-t)} \tag{4.5}$$

with $g_1 = g\left(1+\frac{8e^2g^2}{m^2a^2}\right)^{1/2}$ and e = exp(1); t = no. of links in T. If $T \neq 0$ then the restriction $x_1 \in P$ in the sum is redundant, and $\partial_T A_u(P) = 0$ unless $\ell \in T$ consist of points $x, y \in P$ only (hence $t \leq n-1$).

These bounds continue to be valid in the presence of a background field Ψ in the complex strip $|\mathrm{Im}\,\Psi_x| < g^{-1}\varepsilon$, if $g/(1-\varepsilon)$ is substituted for g. For $T = \emptyset$ the slightly better bound (4.17) is known to hold [6, 22].

We will also obtain a better bound for the monomer activity (n=1) if V(o) = 0.

$$|A_u(\{x\}|0) - 1| \leq v_{xx}\, g^2 b \leq \frac{g^2}{m^2a^2}\, b \quad . \tag{4.6}$$

This bound does not generalize to arbitrarily large real background field Ψ, however.

The second step will establish

Proposition 8. Suppose the hypotheses of proposition 7 are fulfilled and $g^2(m^2a^2)^{-1} < (384e^2b^2)^{-1}$. If X contains n points and T is a possibly disconnected tree on X with t links, then the reduced correlation functions $\rho_\Lambda(X) = \frac{Z_u(\Lambda\setminus X|0)}{Z_u(\Lambda|0)}$ obey the bounds

$$\left|\left(\prod_{(x,y)\in T} \frac{\partial}{\partial v_{xy}}\right)\rho_\Lambda(X)\right| \leq (32b^2g_1^2)^t \left[\prod_{x\in X} d_T(x)!\right] 2^n \tag{4.7}$$

for all Λ, with g_1 as in proposition 7. In particular this bound holds for $\rho_X(X) = Z_u(X|0)^{-1}$.

The third step will establish theorem 2.

The tree formula, theorem 1, is valid for arbitrary counter terms $\delta e(X|v)$ in the definition of $Z(X|\Psi)$, so long as they obey the decoupling relation ii) of lemma 4. Therefore it holds in particular for $\delta e \equiv 0$, i.e. for the unnormalized activities. In this special case it reduces to the known formula [3, 6]

$$A_u(P|\Psi) = (n-1)!\, S \sum_\eta \int ds\, f(\eta|s) \int d\mu_{v[s]}(\varphi)\, \cdot \left[\prod_{i=2}^n \left(\frac{\delta}{\delta\varphi_{x_i}} v_{x_i x_{\eta(i)}} \frac{\delta}{\delta\varphi_{x_{\eta(i)}}}\right)\right] \prod_{x\in P} e^{-W(g[\varphi_x+\Psi_x])} \tag{4.8}$$

for $P = (x_1, \ldots, x_n)$. Summation over η is again over all n-trees.

A chief ingredient in the proof of bounds like (4.5) is a bound on sums over n-trees η that was discovered by Battle and Federbush [23]. Let $\tilde{d}_\eta(i)$ be the number of integers $j \in [2 \ldots n]$ with $\eta(j) = i$. η determines a tree graph on vertices 1 ... n, the number of links in this tree graph that are incident at i is

$$d_\eta(i) = \begin{cases} \tilde{d}_\eta(i) + 1 & \text{if } 2 \leq i \leq n \\ \tilde{d}_\eta(i) & \text{if } i = 1 \end{cases} \tag{4.9}$$

Their sum is twice the number of links,

$$\Sigma_{i=1}^n\, d_\eta(i) = 2(n-1)$$

Let $f(\eta|s)$ be defined as in eq. (2.16). A variant of the Battle Federbush bound for the sum over all n-trees η reads

$$\sum_\eta \int ds\, f(\eta|s) \prod_{i=1}^n [d_\eta(i)!] \leq 8^{n-1} \tag{4.10a}$$

In fact this bound had been used implicity in the earlier work of Göpfert and Mack [21, 24]. It played a crucial role in their proof that static quarks in 3-dimensional U(1) lattice gauge theory are confined for all values of the coupling constant [24]. They used the estimate

$$\sum_\eta \int ds\, f(\eta|s) \prod_{i=2}^n [\mu(\eta(i))] \leq \exp \sum_{i=1}^{n-1} \mu(i) \tag{4.10b}$$

for arbitrary $\mu(i) \geq 0$. Inequality (4.10a) is obtained as a corollary from inequality (4.10b): Multiply both sides with $e^{-2\mu(1)} \prod_{i=2}^n [\mu(i) e^{-2\mu(i)}]$ and integrate over $\mu(i)$ from $0 \ldots \infty$, for $i = 1, \ldots, n$. The proof of inequality (4.10b) is a simple extension of the proof of the "old" tree estimate of Glimm and Jaffe [18] in which factors μ or $d_\eta(i)!$ are absent - see Appendix of [21]. Meanwhile, equalities to replace the inequalities (4.10) are also known [24, 25]. (They follow from identity (4.12) and Cayley's formula.)

Battle and Federbush derived their bound in another way which yielded a useful combinatorial identity. Let $X \subset \Lambda$ be a subset of the lattice Λ with n points, among them an arbitrarily selected first element x_1. A labelling $\underset{\sim}{x}$ of the pointed set X is a bijective map of the set $\{1, \ldots, n\}$ into X such that $\underset{\sim}{x}(1) = x_1$

$$i \to \underset{\sim}{x}(i) \equiv x_i \in X \quad . \tag{4.11}$$

We will usually write x_i in place of $\underline{x}(i)$. A labelling of X together with a n-tree η determines a tree $T_1 = T(\underline{x},\eta)$ on vertex set X, with links $(\underline{x}(i),\underline{x}(\eta(i)))_{i=2\ldots n}$. Let T_1 be an arbitrary tree on vertex set X. The Battle Federbush identity says that

$$\sum_{\substack{\underline{x} \\ T(\underline{x},\eta)=T_1}} \sum_{\eta} \int ds\, f(\eta|s) = 1 \ . \tag{4.12}$$

The outer sum runs over all labellings of X with $\underline{x}(1) = x_1$. Inequalities like (4.10a) can be obtained from this identity by using Cayley's formula for the number of tree graphs T_1 with given incidence numbers [1]. We will see an illustration below.

The usefulness of the improved form (4.10) of the Glimm Jaffe tree estimate comes from the fact that the extra factorials $d_\eta(i)!$ can be used to absorb factors k! in the Cauchy formula for the k-th derivative of a complex analytic function

$$\frac{d^k}{dz^k} f(z) = \frac{k!}{2\pi i} \oint \frac{f(w)\,dw}{(w-z)^{k+1}} \ . \tag{4.13}$$

Such derivatives with resepct to field variables appear in the tree formula (4.8) and theorem 1. In our derivation of theorem 2, it will be equally important that the factors $\prod_x d_T(x)!$ coming from the bound of proposition 8 for v-derivatives of $e^{\delta e(X|v)} = {}^{x}Z_U(X|0)^{-1}$ can be absorbed in the same way. In the application to the U(1)-confinement problem, the factors $\mu(\cdot)$ in inequality (4.10b) were used to control the sums over the arbitrarily large charges of particles that occur in the Coulomb gas representation of the U(1) lattice gauge theory model [24].

As a warming up exercise we will first derive the bounds (4.5) of proposition 7 for unnormalized activities $A_u(P|\Psi)$ in the special case without v-derivatives ($T = \emptyset$). Let $x_1 \in P$ be chosen arbitrarily among the n points in P. We exhibit the symmetrization $\mathcal{S}$ in the tree formula (4.8) as $1/(n-1)!$ times a sum over labellings $\underline{x}$ of P with $\underline{x}(1) = x_1$.

$$A_u(P|\Psi) = \sum_{\underline{x}} \sum_{\eta} \int ds\, f(\eta|s) \Big[\prod_{i=2}^{n} v_{x_i x_{\eta(i)}}\Big] \Big\langle \prod_{j=1}^{n} \Big(\frac{\partial}{\partial\varphi_{x_j}}\Big)^{d_\eta(j)} \prod_{k=1}^{n} e^{-W(g[\varphi_{x_i}+\psi_{x_i}])} \Big\rangle_{v[\underline{x},s]} \ . \tag{4.14}$$

The interpolating propagator is given by eq. (2.17). We write $v[\underline{x},s]$ in place of $v[s]$ to indicate its dependence on the labelling $\underline{x}$ which is present because now

$x_i \equiv \underline{x}(i)$. The Cauchy formula (4.13) and the bounds (2.20) on e^{-W} in the complex strip yield $(\varphi_i \equiv \varphi_{x_i}\ ,\ d_i \equiv d_\eta(i))$

$$\Big(\frac{\partial}{\partial\varphi_i}\Big)^{d_i} e^{-W(g[\varphi_i+\psi_i])} \le b\Big(\frac{g}{1-\varepsilon}\Big)^{d_i} d_i! \quad \text{for } |\Im m\,\psi_i| \le g^{-1}\varepsilon \ . \tag{4.15}$$

The basic property of an expectation value , $|\langle F\rangle| \le \sup|F|$, can now be used to obtain

$$|A_u(P|\Psi)| \le \sum_{\underline{x}} \sum_{\eta} \int ds\, f(\eta|s) \Big(\prod_{i=2}^{n} |v_{x_i x_{\eta(i)}}|\Big) b^n \Big(\frac{g}{1-\varepsilon}\Big)^{2(n-1)} .$$

We wish to estimate the sum over polymers P with n elements containing x_1. When $x_2 \ldots x_n$ are summed over, the sum over permutations $\underline{x}$ becomes redundant (it gives a factor $(n-1)!$)

$$\sum_{\substack{P \\ x_1\in P,\, |P|=n}} |A_u(P|\Psi)| = \frac{1}{(n-1)!} \sum_{\substack{x_2 \ldots x_n \\ \text{distinct}}} |A_u(P|\Psi)|$$

$$\le b^n \Big(\frac{g}{1-\varepsilon}\Big)^{2(n-1)} \sum_{x_2} \ldots \sum_{x_n} \sum_{\eta} \int ds\, f(\eta|s) \Big(\prod_{i=1}^{n} d_\eta(i)!\Big) \prod_{j=2}^{n} |v_{x_j x_{\eta(j)}}| \ .$$

Now the sums over $x_n \ldots x_2$ are done in the indicated order - x_n first, then x_{n-1} etc. Since $\eta(i) < i$, each of these sums can be done with the help of inequality (2.21). One may picture the process of sucessive integrations which removes factors $v_{x_i x_{\eta(i)}}$ by "trimming the tree" as in figure 1.

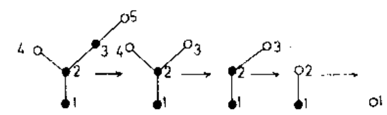

figure 1 How to trim a tree

As a result

$$\sum_{\substack{P \\ |P|=n,\, x_1\in P}} |A_u(P|\Psi)| \le b^n \Big(\frac{g}{ma(1-\varepsilon)}\Big)^{2(n-1)} \sum_{\eta} \int ds\, f(\eta|s) \prod_{i=1}^{n} d_\eta(i)! \ . \tag{4.16}$$

The Battle Federbush inequality (4.10a) leads to the final bound

$$\sum_{\substack{P \\ |P|=n,\, x_1\in P}} |A_u(P|\Psi)| \le b^n \Big(\frac{8g^2}{m^2a^2(1-\varepsilon)^2}\Big)^{n-1} \tag{4.17}$$

for $|\Im m\,\psi_x| \le g^{-1}\varepsilon$.

This also completes the proof of proposition 7 for the special case $T = \emptyset$. This special bound was known before [6], we reviewed its derivation to prepare the reader for the following generalizations which involve added combinatorial complexity.

We will now consider derivatives with respect to propagators v. Starting point is the following well known "change of covariance" lemma for Gaussian measures [26]. Suppose the covariance (propagator) C of a Gaussian measure $d\mu_{C(t)}$ depends on a real parameter t. Then

$$\frac{d}{dt}\int d\mu_{C(t)}(\varphi)f(\varphi) = \frac{1}{2}\int d\mu_{C(t)}(\varphi)\left(\frac{\delta}{\delta\varphi}, \dot{C}(t)\frac{\delta}{\delta\varphi}\right)f(\varphi)$$
$$= \frac{1}{2}\int d\mu_{C(t)}(\varphi)\sum_{x,y}\left[\frac{\partial}{\partial\varphi_x}\dot{C}(t)_{xy}\frac{\partial}{\partial\varphi_y}\right]f(\varphi) \qquad (4.18)$$

with $\dot{C}(t) = dC(t)/dt$. It suffices to verify this for $f(\varphi) = e^{i(q,\varphi)}$. In this special case validity of the formula follows straight from eq. (A.4) for the Fourier transform of a Gaussian measure.

Let T be a possibly disconnected tree graph and consider derivatives $\partial_T A_u(P)$ of the activities $A_u(P|0)$ with respect to propagators as defined in eq. (4.3). Because of the property (A.2) of Gaussian measures, $Z_u(X|\Psi)$, and therefore also $A_u(X|\Psi)$ depend on the propagator v only through its restriction v_X to X. Therefore $\partial_T A_u(P) = 0$ unless all links in T are pairs of points in P. Let us therefore assume that this is so.

Given a labelling $\underline{x}$ of the vertices in P, set

$$v_{ij} = v_{x_i x_j} \quad , \quad \varphi_i = \varphi_{x_i} \qquad (x_i \equiv \underline{x}(i))$$
$$t_T = \underline{x}^{-1}(T) \quad , \quad t_\eta = \{(i, \eta(i))_{i=2\ldots n}\}$$

$\underline{x}^{-1}(T)$ is a subset of a tree graph on vertices $1,\ldots,n$. In this notation tree formula (4.8) gives

$$\partial_T A_u(P) = \sum_{\underline{x}}\sum_\eta \int ds\, f(\eta|s)\left(\prod_{(ij)\in t_T}\frac{\partial}{\partial v_{ij}}\right)$$
$$\left\langle \prod_{(k\ell)\in t_\eta}\left(\frac{\partial}{\partial\varphi_k}v_{k\ell}\frac{\partial}{\partial\varphi_\ell}\right)\prod_{i=1}^n F_i(\varphi)\right\rangle_{v[\underline{x},s]}$$

with

$$F_i(\varphi) = e^{-W(g\varphi_i)} .$$

We have

$$v[\underline{x},s]_{ij} = s[ij]\,v_{ij}$$
$$s[ij] = s[ji] \equiv s_i s_{i+1}\cdots s_{j-1} \quad \text{if} \quad i<j .$$

Therefore

$$\frac{\partial}{\partial v_{ij}} v[\underline{x},s]_{k\ell} = \delta_{ij,k\ell}\, s[ij] . \qquad (4.19)$$

In the expression $\langle\,,\rangle$ in the above formulas for $\partial_T A_u$ there appears v-dependence in two places: the explicit propagators $v_{k\ell}$, and the propagator used to compute the expectation value. Let $t \subset t_T \cap t_\eta$ contain the pairs (ij) such that $\partial/\partial v_{k\ell}$ acts on $v_{k\ell}$ factors. We use the change of covariance lemma, eq. (4.18) for the other derivatives to obtain

$$\partial_T A_u(P) = \sum_{\underline{x}}\sum_\eta \int ds\, f(\eta|s)\sum_{t\subset t_T\cap t_\eta}\prod_{(ij)\in t_T\setminus t} s[ij]$$
$$\cdot\left\langle \prod_{(k\ell)\in t_\eta\setminus t}\left(\frac{\partial}{\partial\varphi_k}v_{k\ell}\frac{\partial}{\partial\varphi_\ell}\right)\prod_{(mp)\in t_T}\left(\frac{\partial}{\partial\varphi_m}\frac{\partial}{\partial\varphi_p}\right)\prod_i F_i(\varphi)\right\rangle_{v[\underline{x},s]} .$$

Summation over t is over sets of pairs (ij) that are both in t_T and in t_η . (t is a possibly disconnected tree on $1,\ldots,n$.) Set

$$n_t(i) = \text{no. of pairs in t which contain } i$$

Bounding derivatives of $F_i(\varphi) = \exp[-W(g\varphi_i)]$ with the help of the Cauchy formula as before (with $\varepsilon = 0$ since $\Psi = 0$), and bounding $s[ij]$ by 1 we obtain

$$|\partial_T A_u(P)| \le \sum_{\underline{x}}\sum_\eta \int ds\, f(\eta|s)\sum_{t\subset t_T\cap t_\eta} b^n g^{2(n-1-|T|-|t|)}$$
$$\prod_{(k\ell)\in t_\eta\setminus t}|v_{k\ell}|\prod_{i=1}^n\left[d_\eta(i) + n_{t_T\setminus t}(i)\right]! .$$

$|t|$ is the number of pairs in t.

This time we will choose to use the Battle Federbush identity (4.12), together with Cayley's formula in place of inequality (4.10a). Let $T(\underline{x},\eta)$ be the tree on vertex set P with links $(\underset{\sim}{x}(i), \underset{\sim}{x}(\eta(i)))$ and let $T_1 = \underset{\sim}{x}(t)$. We may rewrite the above inequality as

$$|\partial_T A_u(P)| \le \sum_{T_1 \subseteq T} \sum_{T_2 \supseteq T_1} \left\{ \sum_{\underset{\sim}{x}, \eta \,:\, T(\underset{\sim}{x}, \eta) = T_2} \int ds\, f(\eta|s) \right\}$$
$$b^n g^{2(n-1+|T|-|T_1|)} \prod_{(yz) \in T_2 \setminus T_1} |v_{yz}| \prod_{x \in P} \left[d_{T_2+T-T_1}(x)! \right]$$

where $d_{T'}(z)$ = no. of pairs in T' containing z. Sum over T_2 is over trees whose vertex set is all of P. Therefore $|T_2| = n-1$. The sum over T_1 runs over subsets of T. According to the Battle Federbush identity (4.12) the sum in $\{\}$ is 1. Moreover

$$d_{T_2+T-T_1}(x)! \le 2^{|T_2|+|T|-|T_1|}\, d_{T_2-T_1}(x)!\, d_T(x)! \,.$$

Thus

$$|\partial_T A(P)| \le \alpha_n(T) \sum_{T_1 \subseteq T} \sum_{T_2 \supseteq T_1} (2g^2)^{n-1-|T_1|} \Big/ \Big(\prod_x d_{T_2-T_1}(x)! \Big) \prod_{(yz) \in T_2 - T_1} |v_{yz}|$$

with

$$\alpha_n(T) = b^n (2g^2)^{|T|} \prod_x d_T(x)! \,.$$

Let us temporarily (for the rest of this section) reserve the name "vertex of T" to points that occur in at least one pair in T. We wish to obtain estimates on $\sum_{P:|P|=n} |\partial_T A_u(P)|$ when T is nonempty. Sets $P = (x_1, \ldots, x_n)$ will contribute 0 unless all the vertices of T are among $x_1, \ldots, x_n$. Select as x_1 a point which is vertex of T, and also vertex of T_1 if T_1 is nonempty. Designate x_1 to be the root of the tree T_2. (Remember that T_2 is a tree while T_1, T are possibly disconnected trees.) This induces a direction in links $\ell \in T_2$ - if one climbs down the tree starting from the initial point of ℓ one passes the final point of ℓ.

Among the vertices of T are x_1 and all vertices of T_1, therefore all points that are not initial points of links in T_2-T_1. Let these be the points $x_1, \ldots, x_{n-p}$ of P. We sum over the remaining points $x_{n-p+1} \cdots x_n \in P$ which are initial points of links in T_2-T_1. We must divide by p! to correct for multiple counting of sets P.

Let us do the summations over the points $x_{n-p+1} \cdots x_n$ in an order compatible with the partial order determined by the tree T_2 with root x_1. Then these summmations can be performed with the help of inequality (2.21) similarly as before, producing a factor $(ma)^{-2}$ for each sum. Thus

$$\sum_{P:|P|=n} |\partial_T A_u(P)| \le \alpha_n(T) \sum_{T_1 \subseteq T} \sum_{T_2 \supseteq T_1} \frac{1}{p!} \left(\frac{2g^2}{m^2 a^2}\right)^{n-1-|T_1|} \prod_{i=1}^{n} d_{T_2-T_1}(i)! \,.$$

In this formula, T_2 should now be regarded as a tree on vertex set $\{1, \ldots, n\}$ with root 1, and similarly for the possibly disconnected trees T. What was before x_i is now i. Given T_1 we decompose it into trees T_{1a}. Let P_a be the vertex set of T_{1a}. We add trees T_{1b} with no link and exactly one vertex if necessary so that the P_a exhaust the vertex set $P \equiv [1, \ldots, n]$ of T_2.

$$P = \sum P_a \quad , \quad T_1 = \sum T_{1a} \qquad \text{(as set of pairs)} \,.$$

Let us now consider the tree T_2/T_1 whose vertices are the sets P_a and whose links are pairs of sets (P_a, P_b) that are connected by the tree T_2. T_2/T_1 can be thought of as being obtained by shrinking the tree graphs T_{1a} to points. Now we will sum over all trees $T_2 \supseteq T_1$ with prescribed incidence numbers $d_{T_2-T_1}(i) = m(i)$ and prescribed $S = T_2/T_1$. The number of such trees is

$$\prod_a d_S(P_a)! \Big/ \prod_i d_{T_2-T_1}(i)! \,.$$

The incidence numbers must satisfy

$$\sum_{i \in P_a} m(i) = d_S(P_a)$$

$d_S(P_a)$ is the number of links in S incident at P_a. Thus

$$\sum_{P:|P|=n} |\partial_T A_u(P)| \le \alpha_n(P) \sum_{T_1 \subseteq T} \sum_{S} \sum_{\substack{\{m(i)\} \\ \sum_{i \in P_a} m(i) = d_S(P_a)}} \frac{1}{p!} \left(\frac{2g^2}{m^2 a^2}\right)^{n-1-|T_1|} \prod_a d_S(a)! \,.$$

Tree S has $p = n-1-|T_1|$ links. According to Cayleys formula, the number of trees on p+1 vertices a = 1,...,p+1 with prescribed incidence numbers d_a is

$$\frac{p!}{\prod (d_a - 1)!} \le e^{\sum d_a} \frac{p!}{\prod d_a!} \,. \tag{4.20}$$

We have

$$\sum d_S(P_a) = 2(n-1-|T_1|) \,.$$

Summation over incidence numbers $d_a = d_S(P_a)$ removes the constraints on $m(j)$'s, except for

$$\sum_{j=1}^{n} m(j) = 2(n-1-|T_1|) .$$

Thus the summation over S produces

$$\sum_{P:|P|=n} |\partial_T A_U(P)| \leq \alpha_n(T) \sum_{T_1 \subseteq T} \sum_{\substack{\{m(i)\} \\ \Sigma m(i) = 2(n-1-|T_1|)}} \left(\frac{2e^2 g^2}{m^2 a^2}\right)^{n-1-|T_1|}$$

The number of sequences of n nonnegative integers m(i) with prescribed sum $\Sigma m(i) = k$ is

$$\binom{n+k-1}{n-1} \leq 2^{n+k-1} ,$$

The number of subsets T_1 of T with r elements is $\binom{|T|}{r}$. Thus

$$\sum_{P:|P|=n} |\partial_T A_U(P)| \leq \alpha_n(T)\, 2^{n-1} \left(\frac{8e^2 g^2}{m^2 a^2}\right)^{n-1-|T|} \sum_{r=1}^{|T|} \binom{|T|}{r} \left(\frac{8e^2 g^2}{m^2 a^2}\right)^{|T|-r}$$

$$= \left(2g^2\left[1 + \frac{8e^2 g^2}{m^2 a^2}\right]\right)^{|T|} \left[\prod_x d_T(x)!\right] 2^{n-1} b^n \left(\frac{8e^2 g^2}{m^2 a^2}\right)^{n-1-|T|}$$

with e = exp(1).

This completes the proof of proposition 7.

Next we establish the bound (4.6) on the monomeractivity. We have

$$A_U(\{x\}|0) - 1 = \int d\mu_v(\varphi)\left[e^{-W(g\varphi_x)} - 1\right] .$$

Since $W(0) = 0$ by assumption this can be written as

$$\int d\mu_v(\varphi) \int_0^1 ds \frac{d}{ds} e^{-W(gs\varphi_x)} = \int d\mu_v(\varphi) \int ds\, \varphi_x \frac{\partial}{\partial \psi_x} e^{-W(g\psi_x)} \Big|_{\psi = s\varphi} .$$

Next one uses the integration by parts formula for Gaussian measures [27] which follows from eq. (A.4)

$$\int d\mu_v(\varphi)\, \varphi_x f(\varphi) = \sum_y \int d\mu_v(\varphi)\, v_{xy} \frac{\partial}{\partial \varphi_y} f(\varphi) . \qquad (4.21)$$

This gives

$$A_U(\{x\}|0) - 1 = \int ds\, s \left\langle \frac{\delta^2}{\delta \psi_x^2} e^{-W(g\psi_x)} \right\rangle_v v_{xx} .$$

Bounding the derivatives with the help of Cauchy formula as before one obtains the desired bound $|A_U(\{x\}|0) - 1| \leq v_{xx} g^2 b$. By inequality (2.21), $v_{xx} \leq (ma)^{-2}$. This proves inequality (4.6).

Proposition 8 will be derived from proposition 7 in the next section. Therefore it remains only to prove theorem 2, given validity of proposition 8.

The tree formula, theorem 1, can be written as

$$(P|\psi) = \sum_{\underline{x}} \sum_{\eta} \int ds\, f(\eta|s) \left[\int d\mu_w(\varphi)\right.$$

$$\prod_{(xy) \in T(\underline{x},\eta)} \left\{ v_{xy} \left(\frac{\partial}{\partial \varphi_x} \frac{\partial}{\partial \varphi_y} + \frac{\partial}{\partial w_{xy}} \right) \right\} e^{\delta e(P|w)} \prod_{x \in P} e^{-W(g[\varphi_x + \psi_x])} \Big|_{w = v[\underline{x},s]} \qquad (4.22)$$

We may expand, and resum after bounds are substituted, using

$$\sum_{\emptyset \subseteq S \subseteq T} D^S E^{T-S} = (D+E)^T \quad \text{if} \quad D^S = \prod_{\ell \in S} D(\ell) \quad \text{etc.} \qquad (4.23)$$

We bound the φ-derivatives of e^{-W} by using the Cauchy formula as before, and the v-derivatives of $e^{\delta e(P|v)} = Z_U(P|0)^{-1}$ by proposition 8. This bound is valid for general v , with a mass m that depends on v through requirement (2.21). Given that (2.21) holds for v , the corresponding interpolating propagator $w = v[\underline{x},s]$ obeys also

$$\sup_x \sum_y |w_{xy}| \leq (ma)^{-2} \qquad (4.24)$$

by its construction (2.17).

Inserting the bounds and resumming with the help of (4.23) we obtain the bound

$$|A(P|\psi)| \leq \sum_{\underline{x}} \sum_{\eta} \int ds\, f(\eta|s) \left(\prod_{i=1}^{n} d_\eta(i)!\right)(2b)^n \prod_{i=2}^{n} |v_{x_i x_{\eta(i)}}|$$

$$\cdot \left[\frac{g^2}{(1-\varepsilon)^2} + 32 b^2 g_1^2\right]^{n-1} .$$

We used the fact that

$$\prod_x d_{T(\underline{x},\eta)}(x)! = \prod_i d_\eta(i)!$$

since $T(\underline{x},\eta)$ consists of links $(\underline{x}(i), \underline{x}(\eta(i))) \equiv (x_i, x_{\eta(i)})$. From here on the procedure is the same as in the derivation of the bounds (4.17) on the unnormalized activities. If g satsfies the inequality stated as hypothesis of theorem 2 then $g_1^2 \leq \frac{7}{6} g^2$, and the bounds of theorem 2 follow. (Note that $b \geq 1$).

5. Consequences of Mayer Montroll equations

It remains to establish the bounds of proposition 8 for the derivatives of the inverse partition function $Z_u(X|0)^{-1}$ with respect to free propagators. They will be derived from the bounds of proposition 7 on activities.

One considers the "reduced correlation functions" defined by

$$\rho_\Lambda(X) = \frac{Z_u(\Lambda \setminus X|0)}{Z_u(\Lambda|0)} = \frac{\partial}{\partial A_u(X|0)} \ln Z_u(\Lambda|0) . \tag{5.1}$$

Λ may be an arbitrary subset of the infinitely extended lattice $(a\mathbb{Z})^\nu$. We will ultimately be interested in the special case

$$\rho_X(X) = Z_u(X|0)^{-1} . \tag{5.2}$$

The reduced correlation functions admit an expansion as a formal power series in activities [14]. As Fredenhagen and Marcu have shown [17], a form of this expansion that is particularly convenient for purposes of estimation is obtained by iterating the Mayer Montroll equations for ρ_Λ (see Appendix C). To state it we need some definitions.

Two polymers P_1 and P_2 were called compatible if they do not overlap. In this case we write $P_1 \sim P_2$. A collection $\mathcal{P}$ of polymers will be called admissible if $P \sim P'$ for all $P, P' \in \mathcal{P}$, $P \neq P'$. We write $K(X)$ for the set of all admissible collections of polymers $P \subseteq X$. Admissible $\mathcal{P}$ are partitions of some subset of X, therefore

$$K(X) = \bigcup_{Y \subseteq X} \Pi(Y) . \tag{5.3}$$

In particular, $K(X)$ contains the empty collection $\mathcal{P} = \phi$. Two admissible collections $\mathcal{P}_1$ and $\mathcal{P}_2$ are called compatible if $P_1 \in \mathcal{P}_1, P_2 \in \mathcal{P}_2$ implies $P_1 \sim P_2$. In this case we write $\mathcal{P}_1 \sim \mathcal{P}_2$. Finally one defines

$$Conn_X(\mathcal{P}) = \{ \mathcal{P}' \in K(X) \mid P' \not\sim \mathcal{P} \text{ for all } P' \in \mathcal{P}' \} . \tag{5.4}$$

Thus, $\mathcal{P}' \in Conn_X(\mathcal{P})$ consists of polymers that are incompatible with at least one polymer in $\mathcal{P}$.

Let us introduce the following notations for activities

$$A_u(P|0) = \delta_{1,n} + M_u(P)$$
$$M^{\mathcal{P}} = \prod_{P \in \mathcal{P}} M_u(P) . \tag{5.5}$$

In this notation, the relation (2.1a) between partition functions and activities reads

$$Z_u(X|0) = \sum_{\mathcal{P} \in K(X)} M^{\mathcal{P}} \tag{5.6}$$

M are called activities of a polymer system "with empty sites" because not all sites $x \in X$ need to be covered by a polymer $P \in \mathcal{P}$ if $\mathcal{P} \in K(X)$. The power series expansion of ρ_Λ reads

$$\rho_\Lambda(X) = \lim_{n \to \infty} \sum_{\substack{\mathcal{P}_1, \dots, \mathcal{P}_n \in K(X) \\ \mathcal{P}_i \in Conn(\mathcal{P}_{i-1}),\, i = 1 \dots n}} (-M)^{\mathcal{P}_1 + \dots + \mathcal{P}_n} \tag{a}$$

(5.7)

$$= \sum_{n \geq 0} \sum_{\substack{\mathcal{P}_1, \dots, \mathcal{P}_n \in K(X) \\ \phi \neq \mathcal{P}_i \in Conn(\mathcal{P}_{i-1}),\, i = 1 \dots n}} (-M)^{\mathcal{P}_1 + \dots + \mathcal{P}_n} \tag{b}$$

where $\mathcal{P}_0$ is the collection of monomers in X,

$$\mathcal{P}_0 = \{ \{x\} \mid x \in X \} . \tag{c}$$

Our activities $M_u(P)$ will depend on free propagators $v_\ell = v_{xy}$ associated with pairs $l = (xy)$ of points in P. If T is a possibly disconnected tree graph consisting of links l we write

$$\partial_T = \prod_{\ell \in T} \frac{\partial}{\partial v_\ell}$$

as before. Our desired bounds will follow from

Lemma 9. Assume that for some $b_1 \geq 0$ and all possibly disconnected trees T

$$\sum_{P \not\sim \mathcal{P}} (2e^{b_1})^{|P|} \cdot |\partial_T M_u(P)| \leq g_2^{\,t} (e^{b_1} - 1) \, \|\mathcal{P}\| \, \varkappa(T) \tag{5.8a}$$

where $\|\mathcal{P}\|$ = total no. of points in $\sum_{P \in \mathcal{P}} P$, t = no. of links in T, $g_2 > 0$, and $\varkappa$ satisfies

$$\varkappa(T_1 + T_2) \geq \varkappa(T_1)\varkappa(T_2) \quad , \quad \varkappa(\phi) = 1 \tag{5.8b}$$

for disjoint possibly disconnected trees T_1, T_2. Then

$$|\partial_T \rho_\Lambda(X)| \le g_2^t \varkappa(T) e^{b_1|X|} \tag{5.9}$$

for all Λ.

Proof: We follow the procedure of Fredenhagen and Marcu [17], adding to it as needed to handle the derivatives. First one establishes

Lemma 10. Under the hypothesis of lemma 9

$$\sum_{\mathcal{P}' \nsim \mathcal{P}} (2e^{b_1})^{\|\mathcal{P}'\|} |\partial_T M^{\mathcal{P}'}| \le g_2^t \varkappa(T) e^{b_1 \|\mathcal{P}\|} .$$

Proof of lemma 10: By abuse of notation we write $T \subset P$ if $\ell = (x,y) \in T$ implies $x \in P$ and $y \in P$. Let $\mathcal{P}' = \{P_1, \dots, P_n\}$ be a set of compatible polymers P_i. Since $M_u(P)$ depends on v_ℓ only for $\ell = (x,y) \subset P$ it follows that either $\partial_T M^{\mathcal{P}'} = 0$ or there exists a uniquely specified partition of T into (possibly empty) subset $T_i \subset P_i$. In this case

$$\partial_T M^{\mathcal{P}'} = \prod_i \partial_{T_i} M_u(P_i) .$$

We write $\mathcal{P}' \nsim \mathcal{P}$ if $P' \in \mathcal{P}'$ implies $P' \nsim \mathcal{P}$ (i.e. P' is incompatible with some $P \in \mathcal{P}$).

Then

$$\sum_{\mathcal{P}' \nsim \mathcal{P}} (2e^{b_1})^{\|\mathcal{P}'\|} |\partial_T M^{\mathcal{P}'}| = \sum_{n \ge 0} \frac{1}{n!} {\sum_{\substack{P_1, \dots, P_n \\ P_i \nsim \mathcal{P}}}}' \prod_{i=1}^{n} (2e^{b_1})^{|P_i|} \partial_{T_i} M_u(P_i) . \tag{5.10}$$

Summation over $\{P_i\}$ is restricted to polymers P_i that are mutually compatible and such that the above mentioned partition of T into T_i exists. Now we bound the sums over $P_n, \dots, P_1$ following Fredenhagen [28].

The first polymer P_1 must intersect the support of $\mathcal{P}$, a set with $\|\mathcal{P}\|$ sites. Applying inequality (5.8a) will produce a factor $g_2^{t_1}(e^{b_1}-1)\|\mathcal{P}\|\varkappa(T_1)$ when one sums over P_1 later on. Polymer P_2 must not intersect P_1 and must therefore be compatible with $\mathcal{P} \setminus P_1$ whose support has at most $\|\mathcal{P}\|-1$ sites. Summing over such P_2 with the help of inequality (5.8a) will produce a factor as before, with in place of $\|\mathcal{P}\|$. Doing the summations in this way, beginning with P_n, produces the bound

$$(5.10) \le \varkappa(T) g_2^t \sum_{n \ge 0} \binom{\|\mathcal{P}\|}{n} (e^{b_1}-1)^n = \varkappa(T) g_2^t e^{b_1 \|\mathcal{P}\|} .$$

We have used that $\varkappa(T_1) \cdots \varkappa(T_n) \le \varkappa(T)$ by hypothesis (5.8b), and $g_2^{t_1} \cdots g_2^{t_n} = g_2^t$ because $T_1, \dots, T_n$ is a partition of T. This proves lemma 10.

Now we return to the proof of lemma 9. Consider the auxiliary quantities ($n = 0, 1, 2, \dots$, t = no. of links in T)

$$\sigma_{n,T}(\mathcal{P}_0) = \sum_{\substack{P_1 \dots P_n \in K(\Lambda) \\ P_i \in \mathrm{Conn}(\mathcal{P}_{i-1}),\, i=1\dots n}} e^{b_1 \|\mathcal{P}_n\|} |\partial_T M^{P_1 + \dots + P_n}| \varkappa(T)^{-1} g_2^{-t} \tag{a}$$

$$\hspace{20em} (5.11)$$

$$= \sum_{m=0}^{n} \sum_{\substack{P_1 \dots P_n \in K(\Lambda) \\ \phi \ne P_i \in \mathrm{Conn}(\mathcal{P}_{i-1}),\, i=1\dots m}} |\partial_T M^{P_1 + \dots + P_m}| \varkappa(T)^{-1} g_2^{-t} f_{mn} \tag{b}$$

with $f_{nn} = \exp b_1 \|\mathcal{P}_n\|$ and $f_{mn} = 1$ for $m \ne n$. $M^\phi = 1$, $\varkappa(\phi) = 1$, therefore

$$\sigma_{0,T}(\mathcal{P}_0) = \begin{cases} e^{b_1 \|\mathcal{P}_0\|} & \text{if } T = \phi \\ 0 & \text{otherwise} \end{cases} \tag{c}$$

Suppose that $\sup_T \sigma_{n,T}(\mathcal{P}_0)$ can be shown to be a nonincreasing function of n. Then it follows that the series (5.7b) for $\rho_\Lambda(X)$ and its derivatives is absolutely convergent (and represents the unique solution of the Mayer Montroll equations). Moreover, since evidently $|\partial_T \rho_\Lambda(\mathcal{P}_0)| \le \varkappa(T) g_2^t \sigma_{\infty,T}(\mathcal{P}_0)$ it follows that

$$|\partial_T \rho_\Lambda(X)| \le \varkappa(T) g_2^t \sup_{T'} \sigma_{0,T'}(\mathcal{P}_0) = \varkappa(T) e^{b_1 \|\mathcal{P}_0\|} g_2^t .$$

Since $\|\mathcal{P}_0\| = |X|$ by definition (5.7b), lemma 9 will follow. Thus it remains to prove that $\sup_T \sigma_{n,T}(\mathcal{P}_0)$ is a nonincreasing function of n.

By definition

$$\sigma_{n,T}(\mathcal{P}_0) = \varkappa(T)^{-1} \sum_{T = T_1 + T_2} \sum_{P_1, \dots, P_{n-1}} |\partial_{T_1} M^{P_1 + \dots + P_{n-1}}| \Big\{ \sum_{P_n \in \mathrm{Conn}(\mathcal{P}_{n-1})} e^{b_1 \|\mathcal{P}_n\|} \partial_{T_2} M^{P_n} \Big\} g_2^{-t} .$$

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*) We adhere to the convention that empty products are always 1.

Let us discuss how many sets T_2 can make a nonvanishing contribution. Let us write $\ell \in \mathcal{P}$ if $\ell \subset \mathcal{P}$ for some polymer $P \in \mathcal{P}$, and similarly for points x. T_2 must consist of links $\ell \in \mathcal{P}_n$, otherwise $\partial_{T_2} M^{\mathcal{P}_n} = 0$. Similarly, T_1 must consist of links $\ell \in \bigcup_{k<n} \mathcal{P}_k$. Let $R \subseteq T$ consist of those links ℓ in T which meet both requirements, i.e.

$$\ell \in \mathcal{P}_n \qquad \text{and} \qquad \ell \in \bigcup_{k<n} \mathcal{P}_k .$$

A partition $T = T_1 + T_2$ is specified by an assignment of every $\ell \in R$ to either T_1 or T_2. Suppose that R contains N links. Then there are 2^N such assignments. Since R is a possibility disconnected tree, it must have at least N + 1 vertices x. All of them are in $\mathcal{P}_n$ (i.e. $x \in \mathcal{P}_n$), therefore

$$N < \| \mathcal{P}_n \| .$$

In conclusion, there are fewer than $2^{\|\mathcal{P}_n\|}$ nonvanishing terms in the sum over partitions $T = T_1 + T_2$. Thus

$$\sigma_{n,T}(\mathcal{P}_0) \le \varkappa(T)^{-1} \sup_{T = T_1 + T_2} \sum_{\mathcal{P}_1, \dots, \mathcal{P}_{n-1}} | \partial_{T_1} M^{\mathcal{P}_1 + \dots + \mathcal{P}_{n-1}} |$$

$$\Big\{ \sum_{\mathcal{P}_n \in \mathrm{Conn}(\mathcal{P}_{n-1})} | 2 e^{b_1} |^{\|\mathcal{P}_n\|} \partial_{T_2} M^{\mathcal{P}_n} \Big\} g_2^{-t} .$$

We apply lemma 10 to bound the expression in

$$\{ \} \le g_2^{t_2} \varkappa(T_2) e^{b_1 \|\mathcal{P}_{n-1}\|}$$

where t_2 = no. of links in T_2. Since $\varkappa(T)^{-1}\varkappa(T_2) \le \varkappa(T_1)^{-1}$ by hypothesis (5.8b) it follows that

$$\sigma_{n,T}(\mathcal{P}_0) \le \sup_{T_1} \varkappa(T_1)^{-1} g_2^{-t_1} \sum_{\mathcal{P}_1 \dots \mathcal{P}_{n-1}} | \partial_{T_1} M^{\mathcal{P}_1 + \dots + \mathcal{P}_{n-1}} | e^{b_1 \|\mathcal{P}_{n-1}\|}$$

$$= \sup_{T_1} \sigma_{n-1, T_1}(\mathcal{P}_0) .$$

This establishes that $\sup_T \sigma_{n,T}(\mathcal{P}_0)$ does not increase in n, completing the proof of lemma 10.

Finally we should apply lemma 10 to prove proposition 8. We set

$$\varkappa(T) = \prod_{x \in \Lambda} d_T(x)! . \tag{5.11}$$

This satisfies hypothesis (5.8b) of lemma 9. We set

$$g_2 / 4 g_1^2 = \varepsilon^{-1}$$

with g_1 as in proposition 7, and define μ_T by

$$| \partial_T M_U(P) | = \varkappa(T) g_2^t \mu_T(P) .$$

Throughout the following argument, t = no. of links in T, $n = |P| =$ no. of points in P. Hypothesis (5.8a) of lemma 9 will be satisfied for a certain b_1 if

$$B(\xi) = \sup_x \frac{1}{\xi} \Big[1 + \sum_{P: x \in P} (2\xi)^{|P|} \mu_T(P) \Big] \le 1$$

for $\xi = e^{b_1}$ and all T. Proposition 7 tells us that

$$\mu_T(P) = 0 \quad \textit{unless} \quad T \subset P , \quad \textit{hence} \quad t \le n-1 ,$$

and

$$\mu_\emptyset(\{x\}) \le g_3 b \ ; \quad \sum_{\substack{P \\ x \in P, |P| = n}} | \mu_T(P) | \le b^n (c g_3)^{n-1-t} \varepsilon^t$$

with

$$g_3 = (g/ma)^2 , \quad c = 16 e^2 .$$

Thus if $T \neq \emptyset$

$$B(\xi) \le \xi^{-1} \Big[1 + \sum_{n \ge t+1} (2b\xi)^n (c g_3)^{n-1-t} \varepsilon^t \Big]$$

$$= \xi^{-1} \Big[1 + (2b\xi)^{1+t} (1 - 2 c g_3 b \xi)^{-1} \varepsilon^t \Big] .$$

This is ≤ 1 for $\xi = 2$ since $c g_3 b \le \frac{1}{24}$ by assumption and if we choose $4 b^2 \varepsilon < \frac{1}{4}$. In the case t = 0, we must use the improved bound (4.6) on the monomer activity to see that $B(\xi) \le 1$. Since $g^2/m^2a^2 \le b^{-2}/384$ we have $c g_3 b \le \frac{1}{24}$, therefore $B(\xi) \le 1$, and the assertion of proposition 8 follows from lemma 9.

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Appendix A Gaussian measures and generalized truncated expectation values.

Let us write $\langle\ \rangle_v$ for expectation value with respect to the normalized Gaussian measure $d\mu_v(\varphi)$ for a free field φ with propagator v. Given $X \subseteq \Lambda$ set

$$v_X(x,y) = \begin{cases} v(x,y) & \text{if } x,y \in X \\ 0 & \text{otherwise} \end{cases} . \tag{A.1}$$

A Gaussian measure $d\mu_v(\varphi)$ remains well defined when its covariance matrix has zero eigenvalues. The simplest example is the Dirac δ-meaure $d\mu_0(\xi)$ on $\mathbb{R}$. v_X is positive semidefinite if v is, and the measure $d\mu_{v_X}(\varphi)$ is supported on fields φ with $\varphi(x)=0$ for $x \notin X$. Moreover, if $f(\varphi)$ depends only on $\varphi(x)$ for $x \in X$ then

$$\langle f(\varphi)\rangle_v = \langle f(\varphi)\rangle_{v_X} \equiv \int d\mu_{v_X}(\varphi) f(\varphi) . \tag{A.2}$$

If X has n points $x_1, \ldots, x_n$ then (A.2) is effectively an n-dimensional integral, with integration variables $\varphi(x_1) \ldots \varphi(x_n)$. Let us give the explicit formula for convenience, expressed in terms of dimensionless variables (in ν dimensions)

$$\varphi_i = a^d \varphi(x_i) \quad , \quad v_{ij} = a^{2d} v(x_i, x_j) \quad , \quad d = \frac{\nu-2}{2} .$$

It reads

$$\langle f \rangle_v = (\det 2\pi v)^{-\frac{1}{2}} \int d\varphi_1 \ldots d\varphi_n \, f(\varphi_1 \ldots \varphi_n) \exp\left[-\tfrac{1}{2}\sum_{i,j} \varphi_i (v^{-1})_{ij} \varphi_j\right] .$$

v^{-1} is the inverse of the nxn matrix $v = (v_{ij})$.

All these statements are readily verified with the help of the formula for the Fourier transform of a Gaussian measure

$$\int d\mu_v(\varphi) e^{i(q,\varphi)} = e^{-\frac{1}{2}(q,vq)} . \tag{A.4}$$

Let us now return to the consideration of truncated expectation values. We use the costumary semicolon notation for them

$$\langle f_1(\varphi); f_2(\varphi); \cdots; f_n(\varphi)\rangle_v \equiv \langle \prod_{i=1}^{n} [f_i(\varphi);] \rangle_v . \tag{A.5}$$

Given a collection of (integrable) functions f_0, f_1, ..., the truncated expectation values are determined by the defining relation

$$\langle \prod_{i\in I} f_i(\varphi)\rangle_v = \sum_{I=\Sigma J} \prod_J \langle \prod_{j\in J} [f_j(\varphi);] \rangle_v . \tag{A.6}$$

Summation is over partitions of the finite index set $I \subset \mathbb{N}$ into nonempty subsets. If f_i are polynomials in φ, then Wicks theorem expresses the truncated expectation values as finite sums of connected Feynman diagrams.

The relation (A.6) is of the same form as the relation (2.1a) between partition functions and activities of a polymer system, with ordinary expectation values playing the role of partition functions, and truncated expectation values the role of activities. Generalization to index sets I other than sets of natural numbers is obvious.

Suppose now that the functions $f_i(\varphi)$ depend only on $\varphi(x)$ for $x \in X_i$. Assume that X_i are disjoint for $i \in I$ and set

$$v_I \equiv v_{X_I} \quad , \quad X_I = \sum_{i\in I} X_i \tag{A.7}$$

(Σ = union of disjoint sets). Then we may use eq. (A.2) to write

$$\begin{aligned} \langle \prod_{i\in I} f_i(\varphi)\rangle_v &= \langle \prod_{i\in I} f_i(\varphi)\rangle_{v_I} \\ &= \sum_{I=\Sigma J} \prod_J \langle \prod_{j\in J} [f_j(\varphi);] \rangle_{v_J} . \end{aligned} \tag{A.8}$$

We will introduce a notion of generalized, or renormalized, truncated expectation values. They have a generalization of (A.8) as their defining relation.

We consider functions F_i which depend not only on the field φ which is to be integrated over, but also on the propagator v, and possibly on other parameters Ψ

$$F_i(\varphi | v, \Psi)$$

Suppose that the φ-dependence of F_i obeys the conditions stated before (A.7) for f_i, and define generalized expectation values

$$\langle\!\langle \prod_{i\in I} F_i \rangle\!\rangle_{v_I}(\Psi) \equiv \langle \prod_{i\in I} F_i(\varphi | v_I, \Psi)\rangle_{v_I} . \tag{A.9}$$

The generalization consists in the fact that v_I determines not only the measure used to compute the expectation value, but is also substituted as a variable in F_i. Generalized truncated expectation values are introduced by the defining relation

$$\langle\!\langle \prod_{i\in I} F_i \rangle\!\rangle_{v_I}(\Psi) = \sum_{I=\Sigma J} \prod_J \left(\langle\!\langle \prod_{j\in J} [F_i;] \rangle\!\rangle_{v_J}(\Psi) \right) . \tag{A.10}$$

We are interested in the following special situation:

$$F_i(\varphi|v,\psi) = f_i(\varphi,\psi)e^{\delta e_i(v)} \tag{A.11}$$

and $\delta e_i(v)$ are uniquely determined by renormalization conditions of the form

$$\langle \prod_{i\in J}[F_i;]\rangle_{v_J}(\psi=0) = r_J\,\delta_{1,n} \tag{A.12}$$

n = no. of points in J. In this case we speak of a "renormalized" truncated expectation value of f_i's. (More precisely, the vacuum energy is renormalized. One may impose further renormalization conditions [9] but we will not expand on this here.) According to their defining relation (2.10) and the renormalization condition (2.11b), our Mayer amplitudes $\mathcal{M}$ are renormalized truncated expectation values as defined here

$$\langle\!\langle \prod_{x\in P}[F_x;]\rangle\!\rangle_{v_P}(\psi) = \delta_{1,n} + n!\,a^{\nu n}\mathcal{M}(x_1\ldots x_n|\psi) \tag{A.13}$$

for $P = (x_1, \ldots, x_n)$, with

$$F_x(\varphi|v,\psi) = \exp\left[\delta e_x(v) - W(g[\varphi_x+\psi_x])\right] \tag{A.14}$$

for interaction V of the form $V(\varphi(x)) = a^{-\nu}W(g\varphi_x)$.

The tree formula, theorem 1, is really a formula for generalized truncated expectation values.

Appendix B <u>Representations of Mayer amplitudes using superpropagators</u>

Let us first consider the unnormalized partition functions and corresponding activities

$$Z_u(X|\psi) = \int d\mu_v(\varphi)\prod_{x\in X}e^{-W(g[\varphi_x+\psi_x])} \tag{B.1}$$

$$= \sum_{\mathcal{P}\in\Pi(X)}\prod_{P\in\mathcal{P}}A_u(P|\psi)\,. \tag{B.2}$$

The unnormalized Mayer amplitudes (1.8) are related to A_u in the same way (2.10) as $\mathcal{M}$ are related to A.

Insert the Fourier representation

$$\tilde{F}_x(k) = (2\pi)^{-1}\int d\xi\,\exp[-W(g\xi)-ik\xi]\,.$$

Formula (A.4) for the Fourier transform of a Gaussian measure yields a representation of Z_u as partition function of an unbounded spin system [20]

$$Z_u(X|\psi) = \int\prod_{x\in X}\left[dq_x\,e^{iq_x\psi_x}\tilde{F}_x(q_x)\right]\exp\Big(-\tfrac{1}{2}\sum_{x,y}q_x v_{xy}q_y\Big)$$

$$= \int\prod_{x\in X}\mathcal{D}q_x\prod_{(x,y)\in X^*}\left[1+f_{xy}(q)\right] \tag{B.3}$$

with

$$\mathcal{D}q_x = dq_x\,e^{iq_x\psi_x}\tilde{F}_x(q_x)e^{-\frac{1}{2}q_x^2v_{xx}} \equiv dq_x\,\hat{F}_x(q_x)e^{iq_x\psi_x} \tag{a}$$

$$f_{xy}(q) = -1+\exp[-q_xv_{xy}q_y] \tag{B.4 b}$$

X^* is the set of unordered pairs of distinct points in X. Let us note that the Fourier transformation in the field which we have performed is not of the kind used in duality trasnformations. Nevertheless one may now proceed as in standard high temperature expansions [1]. One expands in products of f's. The terms in this sum are labelled by graphs. They are decomposed into connected graphs G_P on vertex P. The q-integral factorizes and one obtains a representation of the form (B.2) with

$$A_u(P|\psi) = \sum_{G_P}\int\prod_{x\in P}\left[dq_x\,\hat{F}(q_x)e^{iq_x\psi_x}\right]\prod_{(x,y)\in G_P}f_{xy}(q)\,. \tag{B.5}$$

Summation is over all sets G_P of pairs of distinct points in P such that the following graph is connected: Its vertices are the points $x \in P$ and its links are the pairs $(xy) \in G_P$ (connected Mayer graphs). Eq. (B.5) is the promised representation in terms of superpropagators $f_{xy}(q)$ given by eq. (B.4b) and non-polynomial vertex function $F(q_x)$, given by eq. (B.4).

To handle the Mayer amplitudes proper we will need to consider more general graphs, analogous to renormalized Feynman graphs which involve insertions that are themselves given by graphs. Let us decompose the vacuum energy counter terms

$$\delta e(X|v) = \sum_{Y \subseteq X} \delta\varepsilon(Y|v) \ . \tag{B.6}$$

This decomposition is unique. Since $Z(X|\psi) = e^{\delta e(X|v)} Z_u(X|\psi)$, we obtain in place of eq. (B.3).

$$Z(X|\psi) = \int \prod_{x \in X} \tilde{D}q_x \prod_{\substack{Y \subseteq X \\ |Y| \geq 2}} \left[f_Y(q) + 1 \right]$$

with

$$f_Y(q) = -1 + \exp\left[\delta\varepsilon(Y|v) - q_x v_{xy} q_y\right] \quad \text{if } Y = (x,y) \tag{a}$$

and

$$f_Y(q) = e^{\delta\varepsilon(Y|v)} - 1 \qquad \text{if } |Y| > 2 \tag{B.7b}$$

$$\tilde{D}q_x = Dq_x \, e^{\delta\varepsilon(\{x\}|v)} \ . \tag{c}$$

Now we can expand in products of f's as before, and label the terms in the sum by graphs. A Mayer graph G_P on vertex set P shall be a collection of distinct subsets $Y \subseteq P$ with $\geq$ 2 elements such that the following graph is connected: Its vertices are the points of P, and two of them are linked if they are both in one of the sets $Y \in G_P$. Decomposing graphs into connected ones and using factorization of q-integrals, one finds that Z admits a polymer representation

$$Z(X|\psi) = \sum_{\mathcal{P} \in \Pi(X)} \prod_{P} A(P|\psi) \tag{B.8}$$

with activities

$$A(P|\psi) = \sum_{G_P} \int \prod_{x \in P} \left[dq_x \hat{F}(q_x) e^{iq_x\psi_x + \delta\varepsilon(\{x\}|v)} \right] \prod_{Y \in G_P} f_Y(q) \ . \tag{B.9}$$

Summation is over all Mayer graphs on vertex set P as were just defined. From this formula one could compute the activities $A(P|\psi)$ if one knew the counter terms $\delta\varepsilon(Y|v)$ which enter into the f's.

What we want, however, is a formula from which we can compute the activities $A(P|\psi)$ and the counter terms $\delta\varepsilon(P|v)$ simultaneously, in order of increasing number n of points in P. To obtain it, we introduce an auxiliary, "partially normalized" partition function

$$Z_p(X|\psi) = Z(X|\psi) e^{-\delta\varepsilon(X|v)} = Z_u(X|\psi) \exp \sum_{Y \subseteq X,\, Y \neq X} \delta\varepsilon(Y|v) \ .$$

To compute it one needs $\delta\varepsilon(Y|v)$ only for proper subsets of X. Consider now a fixed set X. Evidently we obtain $Z_p(X|\psi)$ from expression (B.8) if we replace $\delta\varepsilon(X|v)$ by 0 for the chosen set X. This affects only $f_X(q)$ among all f_Y with $Y \subseteq X$, rendering it 0. Therefore only the term $A(X|\psi)$ in the sum (B.1) is changed which is associated with the trivial partition $\mathcal{P} = \{X\}$. In formula (B.9) for $A(X|\psi)$, all the terms in the sum over G_X are unchanged, except those with $X \in G_X$. They become zero. Thus

$$Z(X|\psi) - Z_p(X|\psi) = \left(e^{\delta\varepsilon(X|v)} - 1\right) Z_p(X|\psi) = \sum_{G_P:\, X \in G_P} \int (\cdots) \ .$$

Therefore

$$A(X|\psi) = \left(e^{\delta\varepsilon(X|v)} - 1\right) Z_p(X|\psi) + A_p(X|\psi) \tag{a}$$

$$A_p(X|\psi) = \sum_{G_X:\, X \notin G_X} \int \prod_{x \in X} dq_x \hat{F}_x(q_x) e^{iq_x\psi_x + \delta\varepsilon(\{x\}|v)} \prod_{Y \in G_X} f_Y(q) \ . \tag{B.9 b}$$

Suppose that the counter terms $\delta\varepsilon(Y|v)$ for proper subsets Y of X are already known. Then the only dependence on an unknown counter term in formula (B.9a,b) is in the factor $e^{\delta\varepsilon(X|v)} - 1$. Imposing the normalization condition

$$A(X|0) = 0 \qquad \text{for } |X| \geq 2$$

will therefore permit determination of both $\delta\varepsilon(X|v)$ and $A(X|\psi)$. In the case of a monomer, eq. (B.9) simplifies to

$$A(\{x\}|\psi) = \left(e^{\delta\varepsilon(\{x\}|v)} - 1\right) Z_u(\{x\}|\psi) \tag{B.9c}$$

The quantities f_Y and F were defined in eqs. (B.7) and (B.4a). Eq. (B.9) is the generalization of the superpropagator representation (B.5) of the unnormalized activities. The Mayer amplitudes $\mathcal{M}$ are related to activities $A(X|\Psi)$ by eqs. (2.10), viz.

$$A(X|\Psi) = \delta_{1,n} + n!\, a^{\nu n} \mathcal{M}(x_1 \ldots x_n|\Psi) \quad \text{if} \quad X = \{x_1 \ldots x_n\} .$$

Appendix C <u>Mayer Montroll equations</u> [14, 17]

We use the notations of section 5. For $\mathcal{P} \in \Pi(X)$ reduced correlation functions are defined by

$$\rho_\Lambda(\mathcal{P}) = Z_u(\Lambda|0)^{-1} \sum_{\mathcal{P}' \in \Pi(\Lambda):\, \mathcal{P}' \sim \mathcal{P}} M^{\mathcal{P}'} .$$

Given $X \subsetneq \Lambda$ we set $\mathcal{P}_0$ = collection of monomers $x \in X$, $\rho_\Lambda(X) = \rho_\Lambda(\mathcal{P}_0)$. The reduced correlation functions satisfy the Mayer Montroll equations

$$\begin{aligned} \rho_\Lambda(\mathcal{P}) &= \sum_{\mathcal{P}' \in \mathrm{Conn}_\Lambda(\mathcal{P})} (-M)^{\mathcal{P}'} \rho_\Lambda(\mathcal{P}') \\ &= 1 + \sum_{\phi \neq \mathcal{P}' \in \mathrm{Conn}_\Lambda(\mathcal{P})} (-M)^{\mathcal{P}'} \rho_\Lambda(\mathcal{P}') . \end{aligned} \tag{C.1}$$

Let us reproduce the proof [17]. Abbreviate $Z = Z_u(\Lambda|0)$ and write $\|\mathcal{P}\|$ for the total number of sites in all the $P \in \mathcal{P}$.

$$Z\rho_\Lambda(\mathcal{P}) = \sum_{\mathcal{P}' \sim \mathcal{P}} M^{\mathcal{P}'} = \sum_{\mathcal{P}'} (1-1)^{\mathcal{P}'_{\mathcal{P}}} M^{\mathcal{P}'} = \sum_{\mathcal{P}'} \sum_{\mathcal{P}'' \subset \mathcal{P}'_{\mathcal{P}}} (-1)^{\|\mathcal{P}''\|} M^{\mathcal{P}'}$$

where $\mathcal{P}'_{\mathcal{P}} = \{P' \in \mathcal{P}' ; P' \not\sim \mathcal{P}\}$. The restriction $\mathcal{P}'' \subset \mathcal{P}'_{\mathcal{P}}$ means $\mathcal{P}'' \subset \mathcal{P}'$. $\mathcal{P}'' \in \mathrm{Conn}_\Lambda(\mathcal{P})$. Write $\mathcal{P}' = \mathcal{P}'' + \mathcal{P}'_1$. Then $\mathcal{P}'_1 \sim \mathcal{P}''$ because P in $\mathcal{P}'$ are compatible. So

$$\text{above} = \sum_{\mathcal{P}'' \in \mathrm{Conn}_\Lambda(\mathcal{P})} (-1)^{\|\mathcal{P}''\|} \sum_{\mathcal{P}'_1 \sim \mathcal{P}''} M^{\mathcal{P}'_1 + \mathcal{P}''} = \sum_{\mathcal{P}'' \in \mathrm{Conn}_\Lambda(\mathcal{P})} (-M)^{\mathcal{P}''} Z\rho_\Lambda(\mathcal{P}'') \quad \text{q.e.d.}$$

Iterating eq. (C.1) yields the expansion (5.7).

Appendix D Connected Greens functions

The expansions of section 2 for the full disconnected Greens functions will determine expansions for the connected Greens functions in sums of products of Mayer amplitudes, multiplied by combinatorial coefficients. These combinatorial coefficients can be found as follows.

$$G_c(x_1 \dots x_n) = \frac{\delta^n}{\delta\Psi(x_1)\cdots\delta\Psi(x_n)} \ln Z(\Lambda|\Psi)\Big|_{\Psi=0} \tag{D.1}$$

Set again

$$A(P|\Psi) = \delta_{1,n} + M(P|\Psi), \qquad n = \text{no. of points in } P.$$

It is well known that lnZ admits expansion in a formal power series in M's, as follows [1, 14, 16]. Given X, a cluster $\mathbb{Q}$ on X is a nonempty collection of not necessarily distinct polymers with the property that the following graph $\gamma(\mathbb{Q})$ is connected: Draw a vertex for every polymers P in $\mathbb{Q}$ and a line joining P to P' if P intersects P'. It is costumary to write

$$\mathbb{Q} = (P_1^{n_1}, \dots, P_k^{n_k}) \tag{D.2a}$$

if $\mathbb{Q}$ contains k distinct polymers with multiplicities n_i. The expansion formula reads

$$\ln Z(\Lambda|\Psi) = \sum_{\mathbb{Q}} a(\mathbb{Q}) \prod_{P\in\mathbb{Q}} M(P|\Psi). \tag{D.2b}$$

The sum runs over all clusters on Λ, and the combinatorial coefficients

$$a(\mathbb{Q}) = \sum_{C\subseteq\gamma(\mathbb{Q})} (-1)^{\ell(C)} \Big/ \prod_i n_i! \,. \tag{D.2c}$$

Summation is over all connected subgraphs C of the graph $\gamma(\mathbb{Q})$ that was mentioned above, and $\ell(C)$ is the number of lines in C. A convenient alternative formula for $a(\mathbb{Q})$ can also be extracted from the Fredenhagen Marcu [17] expansion (5.1) for the reduced correlation functions $\rho_\Lambda(X) = \frac{\partial}{\partial M(X)} \ln Z(X)$

Let supp $\mathbb{Q}$ be the disjoint union of $P\in\mathbb{Q}$ - i.e. a point $x\in\Lambda$ is contained in supp$\mathbb{Q}$ with a certain multiplicity n(x). It equals the number of polymers $P\in\mathbb{Q}$ with contain x. Now define, for X = finite family of not necessarily distinct elements x_i of Λ

$$\tilde{M}(X|\Psi) = \sum_{\mathbb{Q}:\,\mathrm{supp}\,\mathbb{Q}=X} a(\mathbb{Q}) \prod_{P\in\mathbb{Q}} M(P|\Psi)$$

$$n!\, a^{n\nu} \tilde{\mathcal{K}}(x_1 \dots x_n|\Psi) = \tilde{M}(X|\Psi) \cdot \prod_{\substack{\text{distinct}\\ x\in X}} n(x)! \tag{D.2d}$$

for $X = \{x_1, \dots, x_n\}$. Then

$$\ln Z(\Lambda|\Psi) = \sum_X \tilde{M}(X|\Psi) = \sum_{n\geq 1} \int_{x_1} \dots \int_{x_n} \tilde{\mathcal{K}}(x_1 \dots x_n|\Psi) \tag{D.2e}$$

as a formal power series. Differentiating at $\Psi = 0$, only a finite number of terms survives on a finite lattice because of the normalization condition (2.11b) which implies

$$\tilde{\mathcal{K}}(x_1 \dots x_n|0) = 0 \,. \tag{D.2f}$$

As a result, expansion (1.5) for connected Greens functions is obtained.

References

1. E. Seiler: Gauge theories as a problem in constructive field theory and statistical mechanics. Lecture Notes in Physics 159, Springer Heidelberg 1982 and references given there

   D. Ruelle: Statistical mechanics. Benjamin, London 1969

2. J. Glimm, A. Jaffe: Quantum physics, a functional integral point of view. Springer Heidelberg 1981

3. J. Glimm, A. Jaffe, T. Spencer: The Wightman exioms and particle structure in the $P(\varphi)_2$-quantum field model. Ann. Math. Phys. 100 (1974) 585

   D. Brydges: A rigorous approach to Debye screening in dilute classical Coulomb systems. Commun. Math. Phys. 58 (1978) 313

   G. Gallavotti: On the Ultraviolet stability in statistical mechanics and field theory. Ann. Math. Pura ed. Applicata CXX (1979) 1

4. D. Brydges, P. Federbush: A new form of the Mayer expansion in statistical mechanics. J. Math. Phys. 19 (1978) 2064

5. C. Itzykson, J.B. Zuber: Quantum field theory. Mc Graw Hill New York 1980, or other text books on quantum field theory.

6. D. Brydges: A short course on cluster expansions. Les Houches lectures 1984

7. H. Lehmann, in: Nonpolynomial Lagrangeans, Renormalization and Gravity. Abdus Salam (Ed.), CoralGables Conference 1971, Gordon and Breach, New York 1971

8. ref. 9, section 4

9. G. Mack, A. Pordt: Convergent perturbation expansions for Euclidean quantum field theory. Commun. Math. Phys. 97 (1985) 267

10. G. Battle, A. Federbush: A phase cell cluster expansion for Euclidean field theories. Ann. Phys. 142 (1982) 95

11. G. Mack: Exact renormalization group as a scheme for calculations. DESY 85-111 (Oct 1985), short version in Proc. 14th ICGTMP, Seoul, Korea (1985), Y.M. Cho (ed.), World Scientific, Singapore 1986

12. G. Battle, A. Federbush: A phase cell cluster expansion for a hierarchical ϕ_3^4-model. Commun. Math. Phys. 88 (1983) 263

    G. Battle: Federbush mode decomposition of ϕ_2^4. Reports on Math. Phys.
    - An elementary expansion of the Yukawa_2 field theory without vacuum energy counter terms. preprint

    A. Pordt: (unpublished)

13. G. Battle: An intrinsic cluster expansion for statistical mechanics. preprint Texas A & M, College Station

14. Ch. Gruber, A. Kunz: General properties of polymer systems. Commun. Math. Phys. 22 (1971) 133

15. see e.g. ref. 9, Appendix A

16. G. Gallavotti: A Martin Löf, S. Miracle Solé: in Statistical mechanics and math. problems, A. Lenard (ed). Lecture Notes in Physics 20, Springer Heidelberg 1973

17. K. Fredenhagen, M. Marcu: Charged states in Z_2 gauge theories. Commun. Math. Phys. 92 (1983) 81, Appendix

18. ref. 2, section 2.4

19. K. Gawedzki, A. Kupiainen: Renormalization group for a critical lattice model. Commun. Math. Phys. 88 (1983) 77
    - Block spin renormalization group for dipole gas and $(\nabla\phi)^4$. Ann. Phys. 147 (1983) 198
    - Massless ϕ_4^4 theory. Rigorous control of a renormalizable asymptotically free model. Commun. Math. Phys. 97 (1985) 267

20. C. Cammarota: Decay of correlations for infinite range interactions in unbounded spin systems. Commun. Math. Phys. 85 (1982) 517

21. M. Göpfert , G. Mack: Iterated Mayer expansions for classical gases at low temperatures. Commun. Math. Phys. 81 (1981) 97

    H. Kunz: Analyticity and clustering properties of unbounded spin systems. Commun. Math. Phys. 59 (1987) 53

    J. Bricmont, J.L. Lebowitz, C.E. Pfister: Low temperature expansion for continuous spin Ising models. Commun. Math. Phys. 78 (1980) 117

22. G. Battle, P. Federbush, R. Robinson: Tree graphs and quasi-bounded spin systems. preprint

23. G. Battle: A new combinatorial estimate for cluster expansions. Commun. Math. Phys. 94 (1984) 133

G. Battle, P. Federbush: A note on cluster expansions, tree graph identities, extra 1/N! factors!!! Lett. Math. Phys. 8 (1984) 55, and ref. 22

24. M. Göpfert, G. Mack: Proof of confinement of static quarks in 3-dimensional U(1) lattice gauge theory for all values of the coupling constant. Commun. Math. Phys. 82 (1982) 545

25. V.N.P. Anghel, L.O. Tătaru: Cluster expansions for unbounded spin systems. Preprint Univ. of Craiova, Romania

E. Speer: Combinatoric identities for cluster functions. Rutgers University preprint

26. ref. 2, section 9.2

27. ref. 2, theorem 6.3.3

28. K. Fredenhagen: private communication