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by

J.A. Magpantay

Deutsches Elektronen-Synchrotron DESY, Hamburg

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Phase Space Formulation of Stochastic Quantization

José A. Magpantay [*+]

Deutsches Elektronen-Synchrotron DESY, Hamburg

Abstract

Stochastic quantization is formulated in terms of coordinates and momenta. In this formulation, we also have a white noise and Langevin equation for each momenta. We then derive the equivalent Fokker-Planck formulation and show that the distribution has the equilibrium value given by the Hamiltonian formulation of the path-integral, i.e. $\exp\{-\int dt[p\dot{q} - H(q,p)]\}$. Then we consider singular systems where the constraints depend on both q and p. By going to an enlarged phase space involving both bosonic and fermionic coordinates and momenta, we show that the Fokker-Planck distribution has the equilibrium value given by the Dirac-Fadeev path integral, i.e.,
$\delta(\phi^a)\,\delta(\chi^b)\,\det[\phi^a,\chi^b]\,\exp\{-\int dt\,[p\dot{q} - H(q,p)]\}$. Finally, we consider the perturbation expansion in this formulation.

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* Alexander von Humboldt Fellow

\+ On leave of absence from the University of the Philippines

1. Introduction

Ever since the stochastic quantization method was proposed by Parisi and Wu /1/, it had been applied to gauge theories /2/, large N field theories /3/, non-linear σ model /4/, lattice gauge theories /5/, quantization of fermions /6/, chiral anomalies /7/ and derivation of the tunnelling amplitude /8/. The fact that this method can treat these various physical systems proves that the stochastic method is a viable approach to quantum theory.

In this paper, we will attempt a different formulation of the stochastic approach. The problem we will try to solve is suggested by the path-integral quantization, which has both a Lagrangian and Hamiltonian formulation. The original Parisi-Wu formulation is based on a Lagrangian and by going through the Fokker-Planck formulation, it was shown to give the Lagrangian formulation of the path-integral. It is then natural to ask if there is a Hamiltonian version of stochastic quantization which will correspond to the Hamiltonian formulation of the path-integral. The answer is yes and we find that we can also assign a separate white noise and Langevin equation for each momenta. This makes the momenta at the same footing with the coordinates as far as stochastic dynamics is concerned. This seems to be natural because a particle in random motion will not only have its coordinates random but also its momenta because its path is nowhere differentiable.

The advantage of the Hamiltonian formulation over the usual Lagrangian formulation is in dealing with constrained systems where the constraints depend on both coordinates and momenta. In the Lagrangian formulation, a constraint $\phi(q, p=\frac{\partial L}{\partial \dot{q}})$ is actually a dynamical equation. The problem is how to take this into account in the Langevin dynamics. For some systems, like gauge theories, the presence of such constraints (in gauge theories, they take the form of Gauss' law) do not affect the straightforward way of solving the Langevin equation. This is due to the fact that the relevant operator in the vector field equation, i.e., $[\frac{\partial}{\partial\tau} - (\partial^2\delta_{\mu\nu} - \partial_\mu\partial_\nu)]$ is non-singular even though the operator $\partial^2\delta_{\mu\nu} - \partial_\mu\partial_\nu$ is singular (due to gauge-invariance). However, in general it is not known now to incorporate "constraints" $\phi(q, p = \frac{\partial L}{\partial \dot{q}}) = 0$ in the Lagrangian formulation of stochastic quantization.

If we formulate stochastic quantization in terms of the Hamiltonian, then the constraint $\phi(q,p) = 0$ is not a dynamical e quation. This can easily be incorporated in the Hamiltonian version of the Langevin dynamics by extending the work of Namiki et al. /9/ on constraints of the form $\phi(q)$. However, in this paper, we will choose a different avenue and we will also get the result required by the Hamiltonian formulation of the path-integral as discussed by Fadeev /10/.

As for the disadvantage of the Hamiltonian formulation, we find that the perturbation expansion is not as manageable as in the Parisi-Wu formulation because of the complicated nature of the τ dependent Green's function. However, general arguments based on the Fokker-Planck formulation showed that the perturbation expansion must be the same because once we integrate out the momenta in $P[q, p; \tau\to\infty]$, we will get $P[q; \tau\to\infty] \sim \exp\{-S_E[q]\}$.

We will arrange the paper as follows: In Section 2, the Langevin equations for both coordinates and momenta are presented. Then we follow the "canonical" procedure to derive the Fokker-Planck Hamiltonian corresponding to the Langevin equations. Then we solve for the zero mode of this Fokker-Planck Hamiltonian and this gives the equilibrium distribution which agrees with the Hamiltonian

formulation of the path-integral. As an example, we treat the problem with velocity dependent potential. Then in Section 3 we extend the formalism to include singular Lagrangians. Here we introduce fermionic coordinates and momenta so as to have Langevin equations with additive white noises only. We show that the equilibrium distribution function agrees with Fadeev's path-integral for singular systems. In Section 4 we discuss perturbation theory by treating the problem of the harmonic oscillator. Then we end with a discussion of what has been achieved in this paper in Section 5.

2. Non-Singular Systems

Consider a non-singular Lagrangian $L(q_i, \dot{q}_i)$. Correlation functions are given by

$$\langle q(t_1)\cdots q(t_n)\rangle = \int [dq]\, q(t_1)\cdots q(t_n) \exp\{-\int dt\, L_E(q,\dot{q})\}. \tag{2.1}$$

When we quantize the system stochastically, we introduce the Langevin equations

$$\frac{\partial q_i}{\partial \tau} = -\left.\frac{\delta S_E}{\delta q_i}\right|_{q_i(t,\tau)} + \eta_i(t,\tau)\,, \tag{2.2}$$

where we introduced an extra time τ and a white noise η_i for each coordinate q_i. The stochastic evolution of the system is determined by (2.2) which we basically solve for $q_\eta(t,\tau)$. Corresponding to this Langevin equation is an equivalent Fokker-Planck formulation defined by

$$\langle q_\eta(t_1,\tau)\cdots q_\eta(t_n,\tau)\rangle_\eta = \int [dq(t)]\, q(t_1)\cdots q(t_n)\, P[q(t);\tau]\,, \tag{2.3}$$

where $P[q(t);\tau]$ is the Fokker-Planck distribution. The distribution satisfies a Schroedinger type equation

$$-\frac{\partial P}{\partial \tau} = H_{FP}[q(t)]\, P \quad , \tag{2.4}$$

where (2.3) defines the form of the Fokker-Planck Hamiltonian. Comparing (2.3) and (2.1), we see that Euclidean quantum theory is recovered if $P[q;\tau\to\infty]$ is proportional to $e^{-S_E[q]}$. This is guaranteed if the only zero mode of H_{FP} is $e^{-S_E[q]}$ and H_{FP} is a positive semi-definite operator.

Let us try to formulate stochastic quantization in terms of the Hamiltonian. The classical equations are now given by $\dot{q}_i = \frac{\partial H}{\partial p_i}$ and $\dot{p}_i = -\frac{\partial H}{\partial q_i}$. Let us consider the following Langevin equations:

$$\frac{\partial q_i}{\partial \tau} = \frac{\partial p_i}{\partial t} + \frac{\partial H}{\partial q_i} + \eta_i \quad , \tag{2.5a}$$

$$\frac{\partial p_i}{\partial \tau} = -\frac{\partial q_i}{\partial t} + \frac{\partial H}{\partial p_i} + \xi_i \quad . \tag{2.5b}$$

Now we also have introduced the white noise $\xi_i(t,\tau)$ for each momenta p_i. The ξ_i's have the same correlations as the η_i's. We only have to add the rule that

$$\langle \eta_i \cdots \eta_j \xi_k \cdots \xi_l \rangle = \langle \eta_i \cdots \eta_j \rangle \langle \xi_k \cdots \xi_l \rangle \, . \tag{2.6}$$

To see whether (2.5a,b) is the correct Langevin equation for the Hamiltonian

approach, let us go over to the Fokker-Planck formulation. This we do by considering the Wiener integral.

Let us consider the stochastic evolution of the system. Suppose it started at $\tau = 0$ with configuration $q_i^0(t)$ and at stochastic time τ with configuration $q_i(t)$. The corresponding momenta must then be consistent with the Langevin equation (2.5a,b). The transition probability for the system to evolve from initial to final configuration is given by

$$P = \int_{\text{end points}} d(\text{path}) \, [\text{weight of each path}] \tag{2.7}$$

The Langevin equations (2.5a,b) show that given a certain noise $\{\eta_i, \xi_i\}$, we have a particular path. The weight of each path is given by the Gaussian distribution of the noises, thus,

$$P = \int_{\text{end points}} [d\eta_i][d\xi_i] \exp\left\{-\frac{1}{4}\int dt\, d\tau \left[\eta_i^2 + \xi_i^2\right]\right\} \tag{2.8}$$

We can express (2.8) in terms of q and p by using the Langevin equations (2.5a,b). The result is

$$\begin{aligned} P = \int_{\text{end points}} [dq_i(t,\tau)][dp_i(t,\tau)]\, J\left(\frac{\eta, \xi}{q, p}\right) \\ \times \exp\left\{-\frac{1}{4}\int dt\, d\tau \left\langle \left[\frac{\partial q_i}{\partial \tau} - \left(\frac{\partial p_i}{\partial t} + \frac{\partial H}{\partial q_i}\right)\right]^2 \right.\right. \\ \left.\left. + \left[\frac{\partial p_i}{\partial \tau} - \left(-\frac{\partial q_i}{\partial t} + \frac{\partial H}{\partial p_i}\right)\right]^2 \right\rangle \right\} . \end{aligned} \tag{2.9}$$

Jacobian is given by

$$J = \det \begin{Bmatrix} \frac{\partial}{\partial \tau}\delta_{ij}\,\delta(t-t')\delta(\tau-\tau') - \frac{\partial^2 H}{\partial q_i(t,\tau)\partial q_j(t',\tau')} & -\frac{\partial}{\partial t}\delta_{ij}\,\delta(t-t')\delta(\tau-\tau') - \frac{\partial^2 H}{\partial q_i(t,\tau)\partial p_j(t',\tau')} \\ \frac{\partial}{\partial t}\delta_{ij}\,\delta(t-t')\delta(\tau-\tau') - \frac{\partial^2 H}{\partial p_i(t,\tau)\partial q_j(t',\tau')} & \frac{\partial}{\partial \tau}\delta_{ij}\,\delta(t-t')\delta(\tau-\tau') - \frac{\partial^2 H}{\partial p_i(t,\tau)\partial p_j(t',\tau')} \end{Bmatrix}$$

Factoring out $\frac{\partial}{\partial \tau}$, then expressing the determinant as $\exp.\,\mathrm{tr}\ln(\mathbb{1}+\mathfrak{X})$ and taking the τ trace will drop all the terms except the order $\mathfrak{X}$. This will then give

$$J = \exp.\left\{-\frac{1}{2}\int dt\,d\tau \left[\frac{\partial^2 H}{\partial p_i^2(t,\tau)} + \frac{\partial^2 H}{\partial q_i^2(t,\tau)}\right]\right\}, \tag{2.10}$$

where we have used the midpoint rule $\Theta(0) = 1/2$. Substituting in (2.8), we find the Fokker-Planck "Lagrangian" density

$$L_{FP} = \frac{1}{4}\int dt \left\{\left[\frac{\partial q_i}{\partial \tau} - \left(\frac{\partial p_i}{\partial t} + \frac{\partial H}{\partial q_i}\right)\right]^2 + \left[\frac{\partial p_i}{\partial \tau} - \left(-\frac{\partial q_i}{\partial t} + \frac{\partial H}{\partial p_i}\right)\right]^2 + 2\left[\frac{\partial^2 H}{\partial q_i^2} + \frac{\partial^2 H}{\partial p_i^2}\right]\right\}. \tag{2.11}$$

Using the canonical procedure, we find the following Fokker-Planck Hamiltonian

$$H_{FP} = \int dt \left\{\Pi_{q_i}\left[\Pi_{q_i} + \left(\frac{\partial p_i}{\partial t} + \frac{\partial H}{\partial q_i}\right)\right] + \Pi_{p_i}\left[\Pi_{p_i} + \left(-\frac{\partial q_i}{\partial t} + \frac{\partial H}{\partial p_i}\right)\right] - \frac{1}{2}\left[\frac{\partial^2 H}{\partial q_i^2} + \frac{\partial^2 H}{\partial p_i^2}\right]\right\} \tag{2.12}$$

After operator ordering the cross-terms, we find the following Fokker-Planck Hamiltonian operator

$$\hat{H}_{FP} = \int dt \left\{\frac{\delta}{\delta q_i(t)}\left[\frac{\delta}{\delta q_i(t)} - \left(\frac{\partial p_i}{\partial t} + \frac{\partial H}{\partial q_i}\right)\right] + \frac{\delta}{\delta p_i(t)}\left[\frac{\delta}{\delta p_i(t)} - \left(-\frac{\partial q_i}{\partial t} + \frac{\partial H}{\partial p_i}\right)\right]\right\}, \tag{2.13}$$

where $\hat{\Pi}_{q_i} = -\frac{\delta}{\delta q_i}$, $\hat{\Pi}_{p_i} = -\frac{\delta}{\delta p_i}$ so that $[q_i, \hat{\Pi}_{q_j}] = [p_i, \hat{\Pi}_{p_j}] = \delta_{ij}$.

The zero mode of this operator is

$$\psi_0 \sim \exp\left\{-\int dt\left[p\dot{q} - H(q,p)\right]\right\}, \tag{2.14}$$

and thus the distribution function relaxes then to the result required by the Hamiltonian formulation of the path-integral /11/. The result is actually expected because $\int dt[p\dot{q} - H(q,p)]$ is just $S_E[q,p]$ and the restoring forces in (2.5a,b) are just $\frac{\delta S_E}{\delta q}$ and $\frac{\delta S_E}{\delta p}$, respectively. Even the Fokker-Planck Hamiltonian (2.13) and the Langevin equations (2.5a,b) can be generalized similar to the ones described by Sakita (see reference 4) so as to guarantee the semi-positive definiteness of H_{FP} insuring that ψ_0 given by (2.14) is the equilibrium distribution function.

To give an example, let us apply this formalism to the problem with velocity dependent potential. In terms of the Lagrangian formulation of stochastic quantization, the effective action with the so called Lee-Yang term was derived by a change of coordinates and using the Stratonovic calculus to transform a stochastic equation with additive white noise to a stochastic equation with multiplicative white noise /12/. In the Hamiltonian formulation, the solution is straightforward.

Consider the following (Euclideanized) Lagrangian

$$L = \frac{1}{2}\, \dot{q}_i A_{ij}(q)\, \dot{q}_j + V(q) \,, \tag{2.15}$$

where $A_{ij}(q)$ is a symmetric, non-singular matrix. The conjugate momenta and Hamiltonian are

$$p_i = A_{ij}\, \dot{q}_j \,, \tag{2.16a}$$

$$H = \frac{1}{2}\, p_i A^{-1}_{ij} p_j - V(q) \,. \tag{2.16b}$$

By (2.14), the equilibrium distribution function is

$$P[q,p;\tau \to \infty] \sim \exp\left\{ -\int dt \left[p_i \dot{q}_i - \frac{1}{2} p_i A^{-1}_{ij} p_j + V(q) \right] \right\} . \tag{2.17}$$

The purely coordinate equilibrium distribution function is derived by integrating out the momenta and this gives

$$P[q;\tau \to \infty] = \int [dp]\, P[q,p;\tau \to \infty]$$
$$= \exp\left\{ \frac{1}{2}\, \delta(0)\, \mathrm{tr} \ln A - \int dt \left[\frac{1}{2} \dot{q}_i A_{ij} \dot{q}_j + V(q) \right] \right\} , \tag{2.18}$$

3. The Singular Case

Suppose the Lagrangian $L(q_i, \dot{q}_i)$ with $i = 1, \ldots, N$ is singular and that after following Dirac's /13/ consistency iteration we arrive at a total number of M constraints $\phi^a(q,p) = 0$, $a = 1, \ldots, M < N$. Let us also assume that they are all first-class and we would make them second-class by adding a subsidiary condition $\chi^a(q,p) = 0$ for each first-class constraint.

To carry out the stochastic quantization program, one approach is to just deal with the independent coordinates and momenta. Let us call them $\{q^*_r, p^*_r\}$ where $r = 1, \ldots, N-M$. This means we have to solve for the dependent $\tilde{q}_a(q^*, p^*)$ and $\tilde{p}_a(q^*, p^*)$ from the constraints ϕ^a and χ^a. The Hamiltonian

$$H(q_i, p_i) = H(q^*_r, p^*_r; \tilde{q}_a(q^*, p^*), \tilde{p}_a(q^*, p^*)) = H^*(q^*, p^*) \,. \tag{3.1}$$

The Hamiltonian Langevin equations are those given by (2.5a,b) and the entire procedure described in Section 2 is valid. The equilibrium distribution function is

$$P[q^*, p^*; \tau \to \infty] \sim \exp\left\{ -\int dt \left[p^*_r \dot{q}^*_r - H^*(q^*, p^*) \right] \right\} . \tag{3.2}$$

Following Fadeev (see reference 10), we can then write

$$\int [dq^*][dp^*]\, P[q^*, p^*; \tau \to \infty] = \int [dq][dp]\, \delta(\phi^a)\, \delta(\chi^b)\, \det[\phi^a, \chi^b]$$
$$\times \exp\left\{ -\int dt \left[p_i \dot{q}_i - H(q_i, p_i) \right] \right\} , \tag{3.3}$$

and thus argue that the equilibrium limit of the distribution function for the full set of coordinate $\{q_i, p_i\}$ is given by

$$P[q_i, p_i; \tau\to\infty] \sim \delta(\phi^a)\,\delta(\chi^b)\,\det[\phi^a,\chi^b]\,\exp\left\{-\int dt\,[p_i\dot{q}_i - H]\right\} . \qquad (3.4)$$

However, this way of doing things does not enlighten us on how to incorporate constraints directly in the Langevin equations. So instead, we will work directly with the full set $\{q_i, p_i\}$. Our objective now is to write down the Hamiltonian Langevin equations that will give (3.4).

Suppose we naively write down equations similar to (2.5a,b), only we separate out the $\{q^*_r, p^*_r\}$ and the $\{\tilde{q}_a, \tilde{p}_a\}$.

$$\frac{\partial q^*_r}{\partial \tau} = \frac{\partial p^*_r}{\partial t} + \frac{\partial H}{\partial q^*_r} + \eta^*_r , \qquad (3.5a)$$

$$\frac{\partial p^*_r}{\partial \tau} = -\frac{\partial q^*_r}{\partial t} + \frac{\partial H}{\partial p^*_r} + \xi^*_r , \qquad (3.5b)$$

$$\frac{\partial \tilde{q}_a}{\partial \tau} = \frac{\partial \tilde{p}_a}{\partial t} + \frac{\partial H}{\partial \tilde{q}_a} + \tilde{\eta}_a , \qquad (3.5c)$$

$$\frac{\partial \tilde{p}_a}{\partial \tau} = -\frac{\partial \tilde{q}_a}{\partial t} + \frac{\partial H}{\partial \tilde{p}_a} + \tilde{\xi}_a . \qquad (3.5d)$$

One immediately sees that they are inconsistent because of the constraints. The noises in (3.5a,b,c,d) are all supposed to be independent white noises. But

if we use $\phi^a = \chi^a = 0$ to solve for q_a and p_a as functions of q^* and p^* and substitute in (3.5c,d), we find a contradiction, i.e., the noises $\tilde{\eta}_a$ and $\tilde{\xi}_a$ are not independent of η^*_r and ξ^*_r.

Another way to see the inconsistency is by writing down the Wiener integral, a crucial ingredient of which is the Jacobian $J\left(\frac{\eta_i, \xi_i}{q_j, p_j}\right)$. This is zero because not all the q's and p's are independent and thus some columns in the Jacobian are not independent from the rest.

One way to resolve these problems is by generalizing the work of Namiki et al. to phase space. Here we give a short discussion of their work and point out the extension to phase space. They considered a Lagrangian with constraints $\phi^a(q) = 0$, i.e., only coordinate dependent constraints. The constraints then determine an N-M dimensional surface in the full q space (there are N coordinates and M constraints). They then write down the Langevin equation using the action

$$S' = \int (L + \lambda^a \phi^a)\, dt = S + \int \lambda^a \phi^a\, dt \quad . \qquad (3.6)$$

Imposing that the stochastic dynamics of the constraints be consistent, i.e., $\frac{d\phi^a}{d\tau} = 0$, yields an expression for the Lagrange multipliers λ^a . The Langevin equations then become

$$\frac{\partial q_i}{\partial \tau} = \left\{\delta_{ij} - \frac{\partial \phi^a}{\partial q_i}\left[\frac{\partial \phi^a}{\partial q_k}\frac{\partial \phi^b}{\partial q_k}\right]^{-1}\frac{\partial \phi^b}{\partial q_j}\right\}\left(-\frac{\delta S}{\delta q_j} + \eta_j\right) . \qquad (3.7)$$

They then constructed a new set of coordinates $\{Q^*_r\}$ and $\{\tilde{Q}^a\}$ where

$r = 1,\ldots,N-M$ and $a=1,\ldots,M$ such that Q_a do not evolve stochastically and $\{Q^*_r\}$ contains the full stochastic dynamics, only this time with a multiplicative white noise.

If we follow this procedure in phase space, we know that the constrained system only covers a 2(N-M) dimensional phase space. We can then consider the Hamiltonian

$$H' = H + \lambda^a \phi^a + \beta^a \chi^a \,, \tag{3.8}$$

and impose that the stochastic dynamics of the constraints be consistent, i.e., $\frac{d\phi^a}{d\tau} = \frac{d\chi^a}{d\tau} = 0$. This would enable us to solve for the multipliers λ^a and β^a. The problem is that the resulting Hamiltonian Langevin equations are too complicated and it is difficult to define new coordinates and momenta such that 2M are stochastically trivial and 2(N-M) contain the full stochastic dynamics with multiplicative white noise.

We instead propose to solve the problem in a different way. To illustrate the new procedure, let us do again the problem considered by Namiki et al. The classical theory is given by the Lagrangian $L(q_i,\dot q_i)$ with constraints $\phi^a(q_i)$ where $i = 1,\ldots,N$ and $a = 1,\ldots,M < N$. The number of independent coordinates is N-M. Let us designate these independent coordinates as q^*_r, $r=1,\ldots,N-M$ and the dependent coordinates $\tilde q_a(q^*_r)$ are solved from the constraints $\phi^a(q) = 0$. The classical dynamics of the independent coordinates q^*_r is given by the Lagrangian

$$L(q^*_r, \dot q^*_r; \tilde q_a(q^*), \dot{\tilde q}_a(q^*)) = L^*(q^*_r, \dot q^*_r) \,. \tag{3.9}$$

The classical equations of motion are

$$\frac{d}{dt}\left(\frac{\partial L^*}{\partial \dot q^*_r}\right) - \frac{\partial L^*}{\partial q^*_r} = 0 \quad , \; r=1,\ldots,N-M, \tag{3.10}$$

and using (3.9), we can express these in terms of L as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q^*_r}\right) - \frac{\partial L}{\partial q^*_r} + \frac{\partial \tilde q^a}{\partial q^*_r}\left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\tilde q}_a}\right) - \frac{\partial L}{\partial \tilde q_a}\right] = 0 \,. \tag{3.11}$$

We would now look for an equivalent system whose classical equations for the independent coordinates q^*_r give (3.11). It is this equivalent system that we would stochastically quantize and show that we can derive in a simpler way the result of Namiki et al.

The constrainted system span an N-M coordinate space. If we consider as variables all the N coordinates q_i and the M multipliers λ^a that go with every constraint ϕ^a, it necessarily means that we have to introduce 2M fermionic coordinates because they have negative dimensions /14/. Thus, we have to consider an (N+M) bosonic plus a 2M fermionic coordinate space such that the classical dynamics yield (3.11). The Lagrangian for such a system is given by

$$L' = L + \lambda^a \phi^a + \bar C^a \underline{X}^{ab} C^b \,, \tag{3.12}$$

where λ^a are the multipliers (bosonic), $\overline{X}^{ab}$ is purely a function of q and C^a and $\bar C^a$ are fermionic coordinates. The equations of motion derived from (3.12) are

$$\bar{c}^a \chi^{ab} = \chi^{ab} c^b = 0 , \tag{3.13a}$$

$$\phi^a = 0 , \tag{3.13b}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\tilde{q}}_a}\right) - \frac{\partial L}{\partial \tilde{q}_a} - \lambda^b \frac{\partial \phi^b}{\partial \tilde{q}^a} - \bar{c}^b \frac{\partial \chi^{bc}}{\partial \tilde{q}^a} c^c = 0 , \tag{3.13c}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^*_r}\right) - \frac{\partial L}{\partial q^*_r} - \lambda^b \frac{\partial \phi^b}{\partial q^*_r} - \bar{c}^b \frac{\partial \chi^{bc}}{\partial q^*_r} c^c = 0 . \tag{3.13d}$$

Using the classical equation (3.13a), we can drop the last terms of (3.13c,d) because $\bar{c}^a \chi^{ab} c^b = 0$ implies that $\frac{\partial}{\partial q_i}\left(\bar{c}^a \chi^{ab} c^b\right) = \bar{c}^a \frac{\partial \chi^{ab}}{\partial q_i} c^b = 0$. From (3.13c), we can solve for the multiplier λ^a,

$$\lambda^a = Y^{-1ab}\left\{ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\tilde{q}}_b}\right) - \frac{\partial L}{\partial \tilde{q}_b} \right\} \tag{3.14}$$

where $Y^{ab} = \frac{\partial \phi^a}{\partial \tilde{q}_b}$ and we assume this matrix to be non-singular. Substituting (3.14) in (3.13d) and using $\frac{\partial \phi^a}{\partial q^*_r} = -\frac{\partial \phi^a}{\partial \tilde{q}_b}\frac{\partial \tilde{q}_b}{\partial q^*_r}$, we get precisely (3.11). Thus, (3.12) is a reasonable starting point for doing the stochastic quantization procedure.

The Langevin equations are

$$\frac{\partial q_i}{\partial \tau} = -\left(\frac{\delta S}{\delta q_i} + \lambda^a \frac{\partial \phi^a}{\partial q_i} + \bar{c}^a \chi^{ab} c^b\right) + \eta_i , \tag{3.15a}$$

$$\frac{\partial \lambda^a}{\partial \tau} = -\phi^a + \eta^a , \tag{3.15b}$$

$$\frac{\partial c^a}{\partial \tau} = -\chi^{ab} c^b + \rho^a , \tag{3.15c}$$

$$\frac{\partial \bar{c}^a}{\partial \tau} = -\bar{c}^b \chi^{ba} + \bar{\rho}^a , \tag{3.15d}$$

where $S = \int L(q,\dot{q})\,dt$. Note that the Langevin equations have purely additive white noises. Following the same procedure as in Section 2, we find that the corresponding Fokker-Planck Hamiltonian is

$$\begin{aligned} \hat{H}_{FP} = \int dt \Big\{ & \frac{\delta}{\delta q_i(t)}\left[\frac{\delta}{\delta q_i(t)} - \frac{\delta}{\delta q_i}\left(S + \int dt\, \lambda^a \phi^a + \bar{c}^a \chi^{ab} c^b\right)\right] \\ & + \frac{\delta}{\delta \lambda^a(t)}\left[\frac{\delta}{\delta \lambda^a(t)} - \phi^a\right] + \frac{\delta}{\delta \bar{c}^a}\left[\frac{\delta}{\delta c^a} - \bar{c}^b \chi^{ba}\right] \\ & + \frac{\delta}{\delta c^a(t)}\left[\frac{\delta}{\delta \bar{c}^a} - \chi^{ab} c^b\right] \Big\} . \end{aligned} \tag{3.16}$$

The zero mode of this operator is

$$\Psi_0 \sim \exp.\left\{-\int dt\left[L + \lambda^a \phi^a + \bar{c}^a \chi^{ab} c^b\right]\right\} ,$$

which is also the equilibrium distribution for $P[q,\lambda,c,\bar{c};\tau]$. Integrating out λ^a, c and $\bar{c}$, we will get the equilibrium distribution $P[q;\tau\to\infty]$ and this is just

$$P[q;\, t\to\infty] \sim \delta(\phi^a)\, \det \Sigma^{ab} \exp\left\{-\int dt\, L(q,\dot q)\right\} . \tag{3.17}$$

This is precisely the result given by Namiki et al. if we put

$$\Sigma^{ab} = \left(\frac{\partial \phi^a}{\partial q_i}\,\frac{\partial \phi^b}{\partial q_i}\right)^{1/2}$$

If the constraints are of the form $\phi^a(q,p)=0$, we cannot use the Lagrangian formulation because as stated already, this becomes a dynamical equation $\phi^a(q,\frac{\partial L}{\partial \dot q})=0$. What we need to do then is to carry out the same procedure in the Hamiltonian formulation. We will first introduce the simplification used also by Fadeev by taking $\chi^a=\tilde p_a=0$. The Hamiltonian of the independent coordinates and momenta q^* and p^* is .

$$H(q_i,p_i)=H(q_r^*,p_r^*,\tilde q_a(q^*,p^*),\tilde p_a=0)=H^*(q_r^*,p_r^*) . \tag{3.18}$$

From this we derive the equations of motion

$$\frac{dq_r^*}{dt}=\frac{\partial H^*}{\partial p_r^*}=[q_r^*,H]-[q_r^*,\phi^a][\phi^a,\tilde p_b]^{-1}[\tilde p_b,H] , \tag{3.19a}$$

$$\frac{dp_r^*}{dt}=-\frac{\partial H^*}{\partial q_r^*}=[p_r^*,H]-[p_r^*,\phi^a][\phi^a,\tilde p_b]^{-1}[\tilde p_b,H] . \tag{3.19b}$$

The right side of (3.19a,b) are the modified Poisson brackets of q^* , p^* with H and to derive it we used the fact that $d\phi^a=0$ to get an expression for $\frac{\partial \tilde q_a}{\partial q_r^*}$ and $\frac{\partial \tilde q_a}{\partial p_r^*}$.

By analogy with the previous case, we look for an equivalent system whose Hamiltonian dynamics will reproduce (3.19a,b). This Hamiltonian is given by

$$H'=H(q_i,p_i)+\lambda^a\phi^a+\Pi^a\tilde p_a+\bar c_\alpha \Sigma^{\alpha\beta}c_\beta , \tag{3.20}$$

where $\{\lambda^a\}$ is a set of bosonic coordinates, $\{\Pi^a\}$ is its conjugate momenta, $\{c_\alpha,\ \alpha=1,\dots,2M\}$ is a set of fermionic coordinates and $\{\bar c_\alpha\}$ is its conjugate momenta and the matrix $\Sigma^{\alpha\beta}$ by analogy with (3.17) is given by

$$\Sigma^{\alpha\beta}=\left[\begin{array}{c|c}[\phi,\phi]=0 & [\phi^a,\tilde p_b] \\ \hline [\tilde p_b,\phi_a] & [\tilde p_a,\tilde p_b]=0\end{array}\right]^{1/2} .$$

The rational for extending the phase space to 2(N+M) bosonic phase space plus 4M fermionic phase space is due to the fact that the constrained phase space is only 2(N-M) dimensional. But the more important support for doing this is if we can show that we can recover the classical dynamics of the independent degrees of freedom as given by (3.19a,b) and we can derive the quantum theory.

The equations of motion for the independent degrees of freedom $\{q_r^*,p_r^*\}$ and the dependent degrees of freedom $\{\tilde q_a, \tilde p_a\}$ are

$$\dot q_r^*=\frac{\partial H}{\partial p_r^*}+\lambda^a\frac{\partial \phi^a}{\partial p_r^*} , \tag{3.21a}$$

$$\dot p_r^*=-\frac{\partial H}{\partial q_r^*}-\lambda^a\frac{\partial \phi^a}{\partial q_r^*} , \tag{3.21b}$$

$$\dot{\tilde{q}}_a = \frac{\partial H}{\partial \tilde{p}_a} + \lambda^b \frac{\partial \phi^b}{\partial \tilde{p}_a} + \Pi^a , \tag{3.21c}$$

$$\dot{\tilde{p}}_a = -\frac{\partial H}{\partial \tilde{q}_a} - \lambda^b \frac{\partial \phi^b}{\partial \tilde{q}^a} = 0 . \tag{3.21d}$$

In deriving above, we used the equations of motion for $\{c_\alpha, \bar{c}_\alpha\}$ and $\{\lambda^a, \Pi^a\}$. From (3.21d), we get $\lambda^a = -[\phi^a, \tilde{p}_b][H, \tilde{p}_b]$ and substituting this in (3.21a,b), we get the results given by (3.19a,b). Equation (3.21c) will solve for the other multiplier Π^a which is not relevant for q^*_r, p^*_r .

The Hamiltonian Langevin equations based on (3.20) are

$$\frac{\partial q_i}{\partial \tau} = \frac{\partial p_i}{\partial t} + \frac{\partial H}{\partial q_i} + \lambda^a \frac{\partial \phi^a}{\partial q_i} + \bar{c}_\alpha \frac{\partial X^{\alpha\beta}}{\partial q_i} c_\beta + \eta_i , \tag{3.22a}$$

$$\frac{\partial p_i}{\partial \tau} = -\frac{\partial q_i}{\partial t} + \frac{\partial H}{\partial p_i} + \lambda^a \frac{\partial \phi^a}{\partial p_i} + \Pi^a \delta^{ai} + \bar{c}_\alpha \frac{\partial X^{\alpha\beta}}{\partial p_i} c_\beta + \xi_i , \tag{3.22b}$$

$$\frac{\partial \lambda^a}{\partial \tau} = \phi^a + \rho^a , \tag{3.22c}$$

$$\frac{\partial \Pi^a}{\partial \tau} = \tilde{p}_a + \gamma^a , \tag{3.22d}$$

$$\frac{\partial c_\alpha}{\partial \tau} = X^{\alpha\beta} c_\beta + \eta_\alpha , \tag{3.22e}$$

$$\frac{\partial \bar{c}_\alpha}{\partial \tau} = \bar{c}_\beta X^{\beta\alpha} + \bar{\eta}_\alpha . \tag{3.22f}$$

The corresponding Fokker-Planck Hamiltonian is

$$\begin{aligned} \hat{H}_{FP} = \int dt \Big\{ & \frac{\delta}{\delta q_i(t)} \Big[\frac{\delta}{\delta q_i} - \Big(\frac{\partial p_i}{\partial t} + \frac{\partial H}{\partial q_i} + \lambda^a \frac{\partial \phi^a}{\partial q_i} + \bar{c}_\alpha \frac{\partial X^{\alpha\beta}}{\partial q_i} c_\beta \Big) \Big] \\ & + \frac{\delta}{\delta p_i(t)} \Big[\frac{\delta}{\delta p_i} - \Big(-\frac{\partial q_i}{\partial t} + \frac{\partial H}{\partial p_i} + \lambda^a \frac{\partial \phi^a}{\partial p_i} + \Pi^a \delta^{ai} + \bar{c}_\alpha \frac{\partial X^{\alpha\beta}}{\partial p_i} c_\beta \Big) \Big] \\ & + \frac{\delta}{\delta \lambda^a(t)} \Big[\frac{\delta}{\delta \lambda^a} - \phi^a \Big] + \frac{\delta}{\delta \Pi^a} \Big[\frac{\delta}{\delta \Pi^a} - \tilde{p}_a \Big] \\ & + \frac{\delta}{\delta \bar{c}_\alpha} \Big[\frac{\delta}{\delta c_\alpha} - \bar{c}_\beta X^{\beta\alpha} \Big] + \frac{\delta}{\delta c_\alpha} \Big[\frac{\delta}{\delta \bar{c}_\alpha} - X^{\alpha\beta} c_\beta \Big] \Big\} . \end{aligned} \tag{3.23}$$

The zero mode of this operator is given by

$$\Psi_0 \sim \exp\Big\{ -\int dt \big[p_i \dot{q}_i - H(q,p) - \lambda^a \phi^a - \Pi^a \tilde{p}^a - \bar{c}_\alpha X^{\alpha\beta} c_\beta \big] \Big\} ,$$

which is also the equilibrium distribution in the extended phase space. Integrating out λ^a, Π^a, c_α and $\bar{c}_\alpha$, we will get the equilibrium distribution for q_i, p_i, i.e.,

$$\begin{aligned} P[q, p; \tau \to \infty] &= \int [d\lambda^a][d\Pi^a][dc_\alpha][d\bar{c}_\alpha] \; \Psi_0 \\ &= \delta(\phi^a)\, \delta(\tilde{p}^b) \det[\phi^a, \tilde{p}^b] \exp\Big\{ -\int dt [p\dot{q} - H(q,p)] \Big\} , \end{aligned} \tag{3.24}$$

and this is the result given by Fadeev.

Before we leave this section, two comments are in order. First, the choice of

(3.12) and (3.20) are actually hinted by the path-integral results. The fermionic terms are needed to express the determinants which are quantum effects. Thus, these fermionic terms are expected not to affect the classical equations of motion as was explicitly verified in the calculations.

Second, the treatment here is not covariant unlike the Fradkin-Vilkovisky formulation /15/. Doing the stochastic quantization of the fully covariant Fradkin-Vilkovsky formulation is a lot more complicated and it is not obvious (at least to the author) how it will give the Dirac-Fadeev result.

4. Perturbation Theory

Let us consider the harmonic oscillator. The Euclideanized Hamiltonian is

$$H = \frac{1}{2} p^2 - \frac{1}{2} q^2 \tag{4.1}$$

The Langevin equations are

$$\frac{\partial q}{\partial \tau} = \frac{\partial p}{\partial t} - q + \eta , \tag{4.2a}$$

$$\frac{\partial p}{\partial \tau} = -\frac{\partial q}{\partial t} + p + \xi . \tag{4.2b}$$

The structure of these equations is like that of a 2-dimensional Dirac equation with an external source. We will need then the Green's function for the first-order operator

$$\mathcal{D} = \mathbb{1}\,\partial_\tau + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \partial_t + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \tag{4.3}$$

defined in the region $-\infty \leq t \leq \infty$ and $0 \leq \tau \leq \infty$, and subject to the condition $g(t-t';\tau-\tau')$ equals zero for $\tau < \tau'$. The solution is (see Appendix)

$$g(t-t';\tau-\tau') = \left(\frac{1}{2\pi}\right)\int_{-\infty}^{\infty} dk\, e^{-\epsilon k^2}\Big\{\cosh\left[(1+k^2)^{1/2}(\tau-\tau')\right] - \frac{1}{(1+k^2)^{1/2}}\begin{pmatrix} 1 & -ik \\ ik & -1 \end{pmatrix}\sinh\left[(1+k^2)^{1/2}(\tau-\tau')\right]\Big\} e^{ik(t-t')}\,\Theta(\tau-\tau') , \tag{4.4}$$

where $e^{-\epsilon k^2}$ is a convergence factor and ϵ will be put to zero at the end of calculations.

The coordinate solution of (4.2a,b) is

$$q_\eta(t,\tau) = \int_0^\tau d\tau_1 \int_{-\infty}^{\infty} dt_1 \left(\frac{1}{2\pi}\right)\int_{-\infty}^{\infty} dk\, e^{-\epsilon k^2}\Big\{\Big\langle\cosh\left[(1+k^2)^{1/2}(\tau-\tau_1)\right] - \frac{1}{(1+k^2)^{1/2}}\sinh\left[(1+k^2)^{1/2}(\tau-\tau_1)\right]\Big\rangle\,\eta(t_1,\tau_1) + \frac{ik}{(1+k^2)^{1/2}}\sinh\left[(1+k^2)^{1/2}(\tau-\tau_1)\right]\xi(t_1,\tau_1)\Big\}\, e^{ik(t-t_1)} . \tag{4.5}$$

From (4.4), we can calculate $\langle q_\eta(t,\tau)\, q_\eta(t',\tau)\rangle_\eta$ and this is also shown in the Appendix and we find that it gives the Euclidean result

$$G_E(t-t') \sim e^{-|t-t'|} \quad .$$

When we add an interaction term $V(q) \sim \mathcal{O}(q^3)$ and higher, the Langevin equation becomes

$$\mathcal{D}\begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} -\frac{\partial V}{\partial q} \\ 0 \end{pmatrix} + \begin{pmatrix} \eta \\ \xi \end{pmatrix} . \tag{4.6}$$

The iteration of (4.5) with the Green's function (4.3) is more complicated than in the Parisi-Wu Lagrangian formulation. We will not verify here whether this Hamiltonian formulation can reproduce the usual perturbative expansion. It is sufficient to note that the Fokker-Planck formulation which gives the equilibrium distribution $e^{-\int [p\dot{q} - H(q,p)]dt}$ and upon integration of p gives $e^{-\int L_E dt}$ suggests that the usual perturbative expansion can be reproduced.

5. Discussion

In this paper, we have constructed the Hamiltonian approach to stochastic quantization. We have written down the Langevin equations for both coordinates and momenta. Then we derived the corresponding Fokker-Planck formulation and showed that the distribution function relaxes to the value given by the Hamiltonian formulation of the path integral. Then we showed how singular Lagrangians can be treated by adding extra degrees of freedom. This trick gives a *simpler* stochastic quantization because the Langevin equations have purely additive white noises. Finally, we discussed the perturbation expansion. Although it is not transparent nor was it shown here that we can recover the usual perturbation series, general arguments relying on the Fokker-Planck formulation showed that

the perturbation expansion must be the same. After all if we integrate out the momentum in the equilibrium distribution function $P[q,p;s\to\infty]$, we will get the distribution $P[q,s\to\infty] \sim e^{-S[q]}$. To conclude, this work has shown the Hamiltonian or phase space formulation of stochastic quantization.

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Appendix

We will be solving for the Green's function of the operator defined in (4.3). Let us write

$$g(t-t';\mathfrak{z}-\mathfrak{z}') = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, \tilde{g}(k,\mathfrak{z}-\mathfrak{z}')\, e^{ik(t-t')} \qquad \text{(A.1)}$$

This gives the equation for $\tilde{g}(k,\mathfrak{z}-\mathfrak{z}')$

$$\left[\partial_{\mathfrak{z}} + \begin{pmatrix} 1 & -ik \\ ik & -1 \end{pmatrix}\right]\tilde{g} = \delta(\mathfrak{z}-\mathfrak{z}'), \qquad \text{(A.2)}$$

and we find that

$$\tilde{g}(k,\mathfrak{z}-\mathfrak{z}') = \begin{cases} B\, e^{-\begin{pmatrix} 1 & -ik \\ ik & -1 \end{pmatrix}(\mathfrak{z}-\mathfrak{z}')}, & \mathfrak{z} > \mathfrak{z}' \\ C\, e^{-\begin{pmatrix} 1 & -ik \\ ik & -1 \end{pmatrix}(\mathfrak{z}-\mathfrak{z}')}, & \mathfrak{z} < \mathfrak{z}'. \end{cases} \qquad \text{(A.3)}$$

Integrating (A.2) in a small region centered at $\mathfrak{z}$, and noting that g is zero for $\mathfrak{z} < \mathfrak{z}'$, we find B = 1 and C = 0. Thus, we have

$$g(t-t';\mathfrak{z}-\mathfrak{z}') = \left(\frac{1}{2\pi}\right)\int_{-\infty}^{\infty} dk\; e^{-\begin{pmatrix} 1 & -ik \\ ik & -1 \end{pmatrix}(\mathfrak{z}-\mathfrak{z}')}\, e^{ik(t-t')}\; \Theta(\mathfrak{z}-\mathfrak{z}'). \qquad \text{(A.4)}$$

Define A = $\begin{pmatrix} 1 & -ik \\ ik & -1 \end{pmatrix}(\mathfrak{z}-\mathfrak{z}')$. Expanding e^{-A} and noting that

$$A^{2n} = (1+k^2)^n \cdot (\mathfrak{z}-\mathfrak{z}')^{2n}, \qquad \text{(A.5a)}$$

$$A^{2n+1} = (1+k^2)^n \begin{pmatrix} 1 & -ik \\ ik & -1 \end{pmatrix}(\mathfrak{z}-\mathfrak{z}')^{2n+1}, \qquad \text{(A.5b)}$$

we find the expression (4.5). The convergence factor $e^{-\epsilon k^2}$ with $\epsilon \to 0$ at the end of calculations is inserted to make the k integrals well-defined.

The expression for q(t,$\mathfrak{z}$) is now given by (4.5). Evaluating $\langle g_\eta(t,\mathfrak{z}) g_\eta(t',\mathfrak{z})\rangle_\eta$ we find the expression

$$\langle g_\eta(t,\mathfrak{z}) g_\eta(t',\mathfrak{z})\rangle_\eta = \left(\frac{1}{\pi}\right)\int_0^{\mathfrak{z}} d\mathfrak{z}_1 \int_{-\infty}^{\infty} dk\, e^{-\epsilon k^2}\Big\{\cosh^2 y + \sinh^2 y - \frac{2}{(1+k^2)^{1/2}}\sinh y \cosh y\Big\} e^{ik(t-t')}, \qquad \text{(A.6)}$$

where y = $(1+k^2)^{1/2}(\mathfrak{z}-\mathfrak{z}_1)$. Integrating out $\mathfrak{z}_1$, we find

$$\langle g_\eta(t,\mathfrak{z}) g_\eta(t',\mathfrak{z})\rangle_\eta = I_1 + I_2 + I_3, \qquad \text{(A.7)}$$

where

$$I_1 = \left(\frac{1}{2\pi}\right)\int_{-\infty}^{\infty} dk\, e^{-\epsilon k^2}\frac{e^{ik(t-t')}}{k^2+1} = \frac{1}{2}e^{-|t-t'|}, \qquad \text{(A.8a)}$$

$$I_2 = -\left(\frac{1}{4\pi}\right)\int_{-\infty}^{\infty} dk\, e^{-\epsilon k^2}\frac{\cosh[2(1+k^2)^{1/2}\mathfrak{z}]}{k^2+1}e^{ik(t-t')} = -\frac{1}{4}e^{-|t-t'|}, \qquad \text{(A.8b)}$$

$$I_3 = \frac{1}{4\pi}\int_{-d}^{d} dk\, e^{-\epsilon k^2} \frac{\sinh[2(1+k^2)^{1/2}\tau]}{(1+k^2)^{1/2}} e^{ik(t-t')}$$

$$= \frac{1}{2}\frac{\partial I_3}{\partial \tau} = 0 . \qquad \text{(A.8c)}$$

Note that in getting (A.8b), we need not take the $\tau \to \infty$ limit only the $\epsilon \to 0$. The presence of the convergence factor $e^{-\epsilon k^2}$ which makes the k integrals well-defined also enable the system attain equilibrium at finite τ. Equation (A.7) then gives the Euclidean result.

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