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Covariant Derivative Expansion of the One-Loop Scalar Effective Action at Finite Temperature

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Abstract

We extend the covariant derivative expansion of the one-loop effective action for a general scalar theory to include temperature effects. This method is applied to the O(N+1)/O(N) non-linear σ model to get a few leading terms depending on zero, two, and four derivatives of the scalar fields at $T\neq 0$. Based on this calculation, the temperature dependence of the pion decay constant and of the pion mass and the associated critical temperature, above which the non-linear σ model breaks down, are identified. Some of the results in the O(N+1)/O(N) non-linear σ model are rederived starting from the O(N+1) linear σ model in the limit of large sigma particle mass.

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I. Introduction

Recently, there have been several papers /1-3/ in the derivative expansion of one-loop effective action. This expansion seems to be quite viable in describing low energy physics. The first leading term in this expansion, which is just the negative of effective potential /4/, has been used in studying the structure of spontaneous symmetry breaking. The other terms containing derivatives of external (background) fields, become relevant in various physical situations: Skyrmion physics /5/; the strongly interacting Higgs sector of the standard model; ordinary and supersymmetric non-linear σ models; supergravity models /6/ and (super)string models /7/.

Due to these interesting applications, much effort has been devoted to developing the derivative expansion in more and more general contexts. The first attempt to obtain terms containing derivatives is due to Iliopoulos et al. /8/. However, their method was not systematic and only the two derivative terms could be calculated. Some years later, several people invented a more systematic approach /1/ in the simple case where no internal symmetry is involved. More recently, Gaillard, Zuk, and Chan modified, independently this method in the general case where internal symmetries are involved, in a way that those internal symmetries are manifestly preserved in the expansion /2,3/.

The above discussion referred only to the zero temperature case, i.e. usual quantum field theory. However, temperature effects are important in studying the structure of the early universe and the phase structure of various theories. The temperature dependent effective potential, which corresponds to the first leading term in the derivative expansion, was studied long ago in connection with the phase structure of spontaneous symmetry breaking /9/ in linear models /F.1/. Binétruy and Gaillard /10/ extended this calculation for general theories which include non-linear theories and they applied it to the supergravity models in the early universe. Recently Moss et al. /11/ developed a systematic way to calculate derivative terms in studying stabilities of theories at finite temperature. However, they didn't mention internal symmetries and thus their method is limited to the case of linear models.

In this paper, I provide a systematic derivative expansion of the one-loop effective action in a manifestly covariant way at finite temperature in the case where an internal symmetry is involved. This systematic expansion can be performed due to the fact that the differential equations satisfied by the finite-temperature Green functions are the same as those at zero temperature, with only the boundary conditions for the time variable being changed. (At finite temperature periodic boundary conditions for imaginary time are what is relevant.) The method at zero temperature developed by Gaillard /2/ can be easily extended to the case at finite temperature; the formal structure is the same, but the integrations over the zeroth component of the internal momentum are replaced by appropriate discrete summations.

The formalism developed is applied to an O(N+1)/O(N) non-linear σ model, including an explicit symmetry breaking term. I calculate, in explicit form, the non-derivative terms, two-derivative terms and four derivative terms of the one loop effective action at finite temperature. Based on this calculation, the temperature dependence of the pion decay constant f is shown to be

$$f^2(T) = f^2(0) - \frac{N-1}{12} T^2 \tag{1}$$

after removing ultraviolet divergences by renormalization of the parameters in the model at zero temperature. From this formula, the critical temperature T_c, above which the non-linear σ model description breaks down, is determined to be $\sqrt{12/N-1}$ f. It may be helpful to compare this result with the case of the linear σ model. In the latter case, the temperature dependence is obtained from the analysis of the effective potential, while it is derived from the kinetic term (two derivative term) in the non-linear σ model case. Our result shows a different N dependence in the linear σ model case with the factor of N-1 being replaced by N+3. This reduction may be understood from the decoupling of the massive σ particle when its mass becomes much larger than the critical temperature.

This paper is organized as follows. In section 2, a general framework for the covariant derivative expansion of the one-loop scalar effective action is provided at T≠0. In section 3, we apply it to the O(N+1)/O(N) non-linear σ model as a specific example and we evaluate the zero-, two-, and four-derivative terms in explicit form. The temperature dependence of a few physically interesting parameters is also discussed here. In section 4, the O(N+1) linear model is studied and a part of the results found in section 3 is rederived in the limit $m_\sigma \to \infty$ The final section is devoted to some additional remarks and our conclusion.

II. General Theory

I will consider first an ordinary quantum field theory, i.e. at zero temperature. The finite temperature formulation can be easily obtained from the zero temperature formulation by imposing the periodic boundary condition on the imaginary part of the time /9/. The material contained in this part may be considered as a review of ref. /2/.

A most general theory of scalars ϕ_p (p=1,2,...,N) may be described by the action

$$S = \int d^4x \left\{ \frac{1}{2} Z^{pq} \partial_\mu \phi_p \partial^\mu \phi_q - V(\phi) \right\} \tag{2}$$

where Z^{pq} is the metric function and V is the potential function depending on the scalars ϕ_p. A specific model can be realized if we choose a specific form of these functions, Z^{pq} and V. The Christoffel symbols are defined in terms of the metric through the relation

$$\Gamma_p{}^{qr} = \Gamma_p{}^{rq} = (Z^{-1})_{ps} \Gamma^{srq} ,$$
$$\Gamma^{pqr} = \frac{1}{2} \left(\frac{\partial Z^{pr}}{\partial \phi_q} + \frac{\partial Z^{pq}}{\partial \phi_r} - \frac{\partial Z^{qr}}{\partial \phi_p} \right) , \tag{3}$$

which will prove useful in my subsequenct discussions.

With the help of the notion of "normal coordinates" /12/, the one-loop effective action, in the presence of external background field ϕ, can be written as /2/

$$\Gamma = \frac{i}{2} \int d^4x \; \mathrm{tr} \log \left\{ (D_\mu D^\mu + R + U)_x \, \delta^4(x-y) \right\} \Big|_{y=x}$$
$$- \frac{i}{2} \int d^4x \; \mathrm{tr} \log \left\{ \partial_\mu \partial^\mu \, \delta^4(x-y) \right\} \Big|_{y=x} . \tag{4}$$

In the above expression, the last term is the field-independent normalization constant, which assumes the absence of unphysical field-independent quartic divergences. For convenience we will neglect it for a while. We have used the covariant derivative D_μ defined as

$$D_\mu^{(x)} = \partial_\mu^{(x)} + \gamma_\mu(x) \tag{5}$$

with the matrix valued connection function

$$\gamma_{\mu\, p}{}^{q} = \partial_\mu \phi_r \, \Gamma_p{}^{qr} . \tag{6}$$

The mass matrix U, which depends on the background field ϕ, is determined by

$$U_p{}^q = (Z^{-1})_{pr} \left(\frac{\partial^2 V}{\partial \phi_q \partial \phi_s} - \Gamma_s{}^{rq} \frac{\partial V}{\partial \phi_s} \right). \tag{7}$$

The other matrix R in Eq. (6) is related to the curvature tensor $R_p{}^{qrs}$ in the following way,

$$R_p{}^q = \partial_\mu \phi_r \, \partial^\mu \phi_s \, R_p{}^{rsq} \tag{8}$$

with

$$R_p{}^{qrs} = \frac{\partial}{\partial \phi_s} \Gamma_p{}^{qr} + \Gamma_p{}^{ts} \Gamma_t{}^{qr} - \frac{\partial}{\partial \phi_r} \Gamma_p{}^{qs} - \Gamma_p{}^{tr} \Gamma_t{}^{qs}$$
$$= - R_p{}^{qsr} . \tag{9}$$

It is convenient to introduce the Fourier variable p_μ, which is nothing but the internal momentum variable, by writing the delta function as

$$\delta^4(x-y) = \int \frac{d^4p}{(2\pi)^4} \, e^{i\,p(x-y)} \tag{10}$$

Inserting this identity into equation (5) and rearranging gives the following expression for Γ:

$$\Gamma = \frac{i}{2} \int d^4x \int \frac{d^4p}{(2\pi)^4} \; \mathrm{tr} \log \left\{ -(i p_\mu + \gamma_\mu (x - i \tfrac{\partial}{\partial p}))^2 + H(x - i \tfrac{\partial}{\partial p}) \right\} \tag{11}$$

with $H = U + R$.

The above is the result given in ref. /2/ at zero temperature. Now let us impose the periodic boundary conditions on the imaginary part of the time variable. Correspondingly, in the imaginary time formulation /9,10/, the zeroth component of the momentum variable is replaced by the discrete values and the integrations over it should be replaced by discrete summations. After this modification, the finite temperature effective action can be written as

$$\Gamma_\beta = \frac{i}{2} \int d^4x \int_\beta \frac{d^4p}{(2\pi)^4} \, \mathrm{tr} \log \left\{ -\left(i p_\mu - Y_\mu (x - i\tfrac{\partial}{\partial p}) \right)^2 + H (x - i\tfrac{\partial}{\partial p}) \right\} \tag{12}$$

with

$$p^0 = \frac{2\pi n}{-i\beta} \quad (n = 0, \pm 1, \pm 2, \cdots) \tag{13}$$

and

$$\int_\beta \frac{d^4p}{(2\pi)^4} = \frac{1}{-i\beta} \sum_n \int \frac{d^3p}{(2\pi)^3} \tag{14}$$

In the remaining part of this paper, I will restrict myself to the case where the background fields ϕ are time independent. This is because the time-derivate of time dependent fields can get large values, in the high temperature limit in which we are interested /11/.

The desired derivative expansion follows by using the simple Taylor expansion

$$\begin{aligned} F(x - i\tfrac{\partial}{\partial p}) &= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \partial_{i_1} \cdots \partial_{i_n} F \frac{\partial^n}{\partial p_{i_1} \cdots \partial p_{i_n}} \\ &= e^{-i \partial \cdot \frac{\partial}{\partial p}} F(x) \, e^{i \partial \cdot \frac{\partial}{\partial p}} \end{aligned} \tag{15}$$

with the abbreviation $a \cdot b = -a^1 b^1 - a^2 b^2 - a^3 b^3$. However, it is more convenient to follow /2/, in order to get a manifest covariant expression. Let's introduce the identity

$$1 = e^{-i D \cdot \frac{\partial}{\partial p}} e^{i \partial \cdot \frac{\partial}{\partial p}} e^{-i \partial \cdot \frac{\partial}{\partial p}} e^{i D \cdot \frac{\partial}{\partial p}} \tag{16}$$

and insert it into the formula (12), using the cyclic property of trace

$$\mathrm{Tr}(ABC) = \mathrm{Tr}(BCA). \tag{17}$$

In the rearrangement of the effective action, the following identities are useful.

$$\begin{aligned} & e^{-i D \cdot \frac{\partial}{\partial p}} e^{i \partial \cdot \frac{\partial}{\partial p}} \left(i p_j - Y_j (x - i\tfrac{\partial}{\partial p}) \right) e^{-i \partial \cdot \frac{\partial}{\partial p}} e^{i D \cdot \frac{\partial}{\partial p}} \\ &= e^{-i D \cdot \frac{\partial}{\partial p}} \left(i p_j - D_j^{(x)} \right) e^{i D \cdot \frac{\partial}{\partial p}} \\ &= i p_j + i \tilde{G}_{kj} \frac{\partial}{\partial p_k} \end{aligned} \tag{18}$$

where

$$\tilde{G}^{kj} = \sum_{n=0}^{\infty} (-i)^n \frac{n+1}{(n+2)!} [D_{i_1}, [D_{i_2}, \cdots [D_{i_n}, G^{kj}] \cdots]] \frac{\partial^n}{\partial p_{i_1} \cdots \partial p_{i_n}} \tag{19}$$

and

$$G^{kj}{}_p{}^{\beta} = [D^k, D^j]_p{}^{\beta} = \partial^k\phi_r\, \partial^j\phi_s\, R_p{}^{\beta s r}. \tag{20}$$

Using the notation

$$\hat{H} = e^{-i D\cdot\frac{\partial}{\partial p}}\, e^{i\partial\cdot\frac{\partial}{\partial p}}\, H(x - i\tfrac{\partial}{\partial p})\, e^{-i\partial\cdot\frac{\partial}{\partial p}}\, e^{iD\cdot\frac{\partial}{\partial p}}$$

$$= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} [D_{i_1}, [D_{i_2}, \cdots [D_{i_n}, H]\cdots]] \frac{\partial^n}{\partial p_{i_1}\cdots\partial p_{i_n}}, \tag{21}$$

the effective action Γ_β can now be cast into the form

$$\Gamma_\beta = \frac{i}{2}\int d^4x \int_\beta \frac{d^4p}{(2\pi)^4}\, \mathrm{tr}\,\log\left\{ (p_\mu + \eta_\mu{}^i\, \tilde{G}_{ji} \frac{\partial}{\partial p_j})^2 - \hat{H} \right\} \tag{22}$$

with $\eta^{0i} = 0$ and $\eta^{ij} = -\delta^{ij}$. This expression has a manifestly covariant form.

Eq. (22) is ultraviolet divergent. In order to isolate these infinities we regularize it by making two subtractions /F.2/, so that

$$\Gamma_{reg}(\beta) = \left(\Gamma_\beta - \Gamma_\beta\Big|_{H\to H+\Lambda^2}\right) - \left(\Gamma_\beta\Big|_{H\to H+\Lambda^2} - \Gamma_\beta\Big|_{H\to H+2\Lambda^2}\right). \tag{23}$$

These subtractions do not change the temperature dependent part, since it does not involve any ultraviolet divergence. (One can check that the temperature dependent part in the subtraction terms vanishes in the limit $\Lambda\to\infty$. See Eq. (29) below.) One can write the regularized finite temperature effective action $\Gamma_{reg}(\beta)$ using a Feynman parameter α as follows

$$\Gamma_{reg}(\beta) = -\frac{i}{2}\Lambda^2\int_0^1 d\alpha \int d^4x \int_\beta \frac{d^4p}{(2\pi)^4}\, \mathrm{tr}\,\Big\{ \big[-p^2 + H + \alpha\Lambda^2 + 2p_i\tilde{G}_j{}^i\frac{\partial}{\partial p_j} - \tilde{G}_{ij}\tilde{G}^{ij} + (\hat{H} - H)\big]^{-1} - [\alpha\to\alpha+1]^{-1}\Big\}. \tag{24}$$

This quantity can be expanded, taking advantage of the identity

$$\frac{1}{A+B} = \frac{1}{A} - \frac{1}{A}B\frac{1}{A} + \frac{1}{A}B\frac{1}{A}B\frac{1}{A} - \cdots \tag{25}$$

with the identification $A = -p^2 + H + \alpha\Lambda^2$. In each term in the above expansion, the coordinate dependent part and the momentum dependent part are factorized. The explicit evaluation of the momentum integral gives us the wanted derivative expansion. However, this integration is not so easy in general due to the non-commutativity $[\partial_\mu U, U] \neq 0$. Thus we postpone its more complete evaluation until we consider a specific model. Here we give only the contribution corresponding to the first term in the right hand side of (25). One finds:

$$-\frac{i}{2}\Lambda^2\int_0^1 d\alpha\int d^4x\int_\beta \frac{d^4p}{(2\pi)^4}\,\mathrm{tr}\left\{\frac{1}{-p^2+H+\alpha\Lambda^2} - \frac{1}{-p^2+H+(\alpha+1)\Lambda^2}\right\}$$

$$= -\frac{1}{2}\int\frac{d^4x}{(4\pi)^2}\,\mathrm{tr}\left(2\Lambda^2 H\log 2 + \frac{1}{2}H^2\log 2H/\Lambda^2 - \frac{3}{4}H^2\right)$$

$$-\frac{1}{2\pi^2\beta^4}\int d^4x\,\mathrm{tr}\left\{J(\beta^2 H) - 2J(\beta^2(H+\Lambda^2)) + J(\beta^2(H+2\Lambda^2))\right\} \tag{26}$$

with

$$J(a) = \int_0^\infty dy\, y^2 \log\left(1 - e^{-(y^2+a^2)^{1/2}}\right), \tag{27}$$

upto field independent constants. This integral J(a) has the following asymptotic forms

$$J(a) \underset{a\to 0}{=} -\frac{\pi^4}{45} + \frac{\pi^2}{12} a - \frac{\pi}{6} a^{3/2} - \frac{1}{32} a^2 \log a + \frac{C}{32} a^2 + O(a^{5/2}) \tag{28}$$

with $C = \frac{3}{2} + 2\log 4\pi - 2\gamma_E$ (γ_E; Euler constant), and

$$J(a) \underset{a\to\infty}{=} O(e^{-a}) . \tag{29}$$

Note that the last two terms in (26) drop out in the limit $\Lambda \to \infty$ (with fixed β). Equations (24) and (25) will be the basic formulas, which we shall use in the following sections.

III. O(N+1)/O(N) non-linear σ model

The standard form of the action for the O(N+1)/O(N) non-linear σ model is, at the tree level,

$$S = \int d^4x \left\{ Z^{pq} \partial_\mu \pi_p \partial^\mu \pi_q + \varepsilon \sqrt{f^2 - \pi^2} \right\}, \text{ for } p,q = 1,2,\cdots,N, \tag{30}$$

where f is the pion decay constant and the constant parameter ε is related to the mass of pions. The metric function Z^{pq} is given by the formula

$$Z^{pq} = \delta_{pq} + \frac{\pi_p \pi_q}{f^2 - \pi^2} . \tag{31}$$

With the above metric, one easily computes

$$\begin{aligned} \Gamma_p{}^{qr} &= Z^{qr} \pi_p / f^2, \\ R_p{}^{qrs} &= (Z^{qr} \delta_{ps} - Z^{qs} \delta_{pr}) / f^2, \\ U_p{}^{q} &= \varepsilon\, \delta_{pq} \sqrt{f^2 - \pi^2} / f^2 \equiv \delta_{pq}\, u . \end{aligned} \tag{32}$$

In this O(N+1)/O(N) non-linear σ model, the momentum integrals in (24), which is the basic equation in the derivative expansion, may be easily evaluated. We separate the one-loop finite temperature effective action $\Gamma_{reg}(\beta)$ into

$$\Gamma_{reg}(\beta) = \Gamma_{reg}^{(0)}(\beta) + \Gamma_{reg}^{(2)}(\beta) + \Gamma_{reg}^{(4)}(\beta) + \cdots \tag{33}$$

according to the power of derivatives, with the superscripts denoting the corresponding number of derivatives. Note that there is no odd power term, since they are forbidden because of parity. The first term in (33), i.e. the non-derivative term may be obtained from equation (26) with the substitution H $\to$ U. The result, in the high temperature limit, after subtracting the field independent constants, is given by

$$\begin{aligned} \Gamma_{reg}^{(0)}(\beta) = &- \frac{N}{2} \int \frac{d^4x}{(4\pi)^2} \left(2u\, \Lambda^2 \log 2 + \frac{1}{2} u^2 \log 2u/\Lambda^2 - \frac{3}{4} u^2 \right) \\ &+ N \int d^4x \left(-\frac{1}{24\beta^2} u + \frac{1}{12\pi\beta} u^{3/2} + O(\log \beta) \right) . \end{aligned} \tag{34}$$

It is nothing but the negative of the temperature dependent effective potential in the one loop approximation. Note that there is a relation between the quadratic divergence in a zero temperature quantity and the T^2-dependence as noted in /9/.

A part of the two derivative term may be also found from equation (26) with the substitution H → U+R, since R has two derivatives already. The other part of $\Gamma^{(2)}_{reg}(\rho)$ is obtained from the corresponding term in (24,25) as

$$-\frac{1}{2}\int d^4x \ \mathrm{tr}\left\{ [D_i,[D_j,U]]\,(4 I_4^{ij} + \eta^{ij} I_3) + 4[D_i,U][D_j,U]\, I_5^{ij} \right\} \tag{35}$$

where

$$I_m = i\Lambda^2 \int_0^1 d\alpha \int_\rho \frac{d^4p}{(2\pi)^4} \left([-p^2+\alpha\Lambda^2+u]^{-m} - [\alpha \to \alpha+1]^{-m} \right)$$

and

$$I_m^{ij} = i\Lambda^2 \int_0^1 d\alpha \int_\rho \frac{d^4p}{(2\pi)^4}\, p^i p^j \left([-p^2+\alpha\Lambda^2+u]^{-m} - [\alpha \to \alpha+1]^{-m} \right) \tag{36}$$

Using the formulas

$$\mathrm{tr}\, R = (N-1)\, Z^{\rho\delta}\, \partial_\mu \pi_\rho\, \partial^\mu \pi_\delta \ ,$$

$$[D_\mu, U]_\rho{}^\delta = \delta_{\rho\delta}\, \partial_\mu u \tag{37}$$

and the asymptotic form of the integrals I_3, I_4^{ij} (I_5^{ij} can be derived from these formulas by successive differentiation with u.)

$$I_3 = \frac{1}{2}\frac{1}{(4\pi)^2} \log 2u/\Lambda^2 - \frac{1}{2}\frac{1}{8\pi\beta} u^{-1/2} + O(\log\beta),$$

$$I_4^{ij} = -\frac{1}{12}\frac{1}{(4\pi)^2}\, 2\eta^{ij} \log 2u/\Lambda^2 + \frac{1}{12}\frac{1}{8\pi\beta}\, \eta^{ij} u^{-1/2} + O(\log\beta), \tag{38}$$

$\Gamma^{(2)}_{reg}(\rho)$ may be cast into the form

$$\Gamma^{(2)}_{reg}(\beta) = -\frac{1}{2}\int \frac{d^4x}{(4\pi)^2} \left\{ \frac{N-1}{f^2} Z^{\rho\delta}\, \partial_i\pi_\rho \partial^i\pi_\delta \,(2\Lambda^2\log 2 + u\log(2u/e\Lambda^2)) - \frac{N}{12}\, \partial_i u\, \partial^i u/u \right\}$$
$$+ \int d^4x \left\{ \frac{N-1}{f^2} Z^{\rho\delta} \partial_i\pi_\rho\partial^i\pi_\delta \left(-\frac{1}{24\beta^2} + \frac{\sqrt{u}}{8\pi\beta}\right) + \frac{1}{48}\frac{N}{8\pi\beta}\, \partial_i u\, \partial^i u/\sqrt{u} \right\}$$
$$+ O(\log\beta). \tag{39}$$

Once again we can see the relation between quadratic divergences at zero temperature and T^2-dependent term, even for the case of derivate terms. Note that the zero temperature contribution in the first line is the same as the result given in equation (3.8) of reference /2/.

Before proceeding with the calculations further, I want to discuss the instability of this theory triggered by temperature effects. For this purpose, it is enough to consider only the two-derivative terms in zeroth order in ε and the non-derivative terms in the leading order of ε regarding ε and the derivative of the scalar fields as small quantities. Adding the tree level action and the one-loop results, (38) and (39) gives us

$$S + \Gamma_{reg}(\rho) = \int d^4x \left\{ \frac{1}{2}(\delta_{\rho\delta} + \theta_\rho\theta_\delta/(1-\theta^2))\, \partial_i\theta_\rho\partial^i\theta_\delta \times \right.$$
$$\left. \left(f^2 - \frac{N-1}{(4\pi)^2} 2\Lambda^2\log 2 - \frac{N-1}{12} T^2\right) + \varepsilon f\sqrt{1-\theta^2}\left(1 - \frac{N}{(4\pi)^2}\frac{\Lambda^2}{f^2}\log 2 - \frac{1}{24}\frac{T^2}{f^2}\right)\right\}$$
$$+ O(\varepsilon^2, \varepsilon\partial^2, \partial^4) + O(\log\beta) \tag{40}$$

with the rescaled fields $\theta_\rho = \pi_\rho/f$. The theory becomes renormalizable in this limited form. The ultraviolet divergences depending on the cutoff Λ can be absorbed into the renormalization of the two parameters f and ε as

$$f_R = f\left(1 - \frac{N-1}{(4\pi)^2}\, 2\frac{\Lambda^2}{f^2}\log 2\right)^{1/2},$$

$$\varepsilon_R f_R = \varepsilon f\left(1 - \frac{N}{(4\pi)^2}\frac{\Lambda^2}{f^2}\log 2\right). \tag{41}$$

Notice that we renormalize our theory at T=0. Now the temperature dependent quantities in the kinetic term serve to define a temperature dependent decay constant $f(T)$ /F.3/. One has, after reintroducing the pion fields $\pi_\rho \equiv f(T)\,\theta_\rho$

$$S + \Gamma(\beta) = \int d^4x \left\{ \frac{1}{2} Z^{\rho\delta}(f(T))\, \partial_i \pi_\rho \partial^i \pi_\delta + \varepsilon_R f_R \sqrt{1 - \pi^2/f^2(T)} \left(1 - \frac{I^2}{f_R^2} \frac{N}{24} \right) \right\} + O(\varepsilon^2, \varepsilon\partial^2, \partial^4, \log\beta) \tag{42}$$

with

$$f^2(T) = f_R^2 - \frac{N-1}{12} T^2 \tag{43}$$

From the above expression, one can easily read off the temperature dependent mass of pions

$$m_\pi^2(T) = m_\pi^2 \left(1 + \frac{N-2}{24} \frac{I^2}{f_R^2} \right), \quad m_\pi^2 \equiv \varepsilon_R / f_R \tag{44}$$

It increases as the temperature increases and it vanishes in the exact symmetry limit, i.e. $\varepsilon_R \to 0$ whether the temperature is zero or not.

The critical temperature, where f(T) vanishes, is determined from Equation (43) to be $\sqrt{12/N-1}\; f_R$. Above this temperature the theory becomes unstable. This number is different from the result obtained in the O(N+1) linear model associated with the O(N+1)/O(N) non linear model. This difference, however, may be understood in the limit of very large sigma particle mass. This is explicitly shown in the next section.

Before doing so, let us go back to the evaluation of further derivative terms. There are several contributions to the four derivative term $\Gamma^{(4)}_{reg}(\beta)$ in equation (26). After some algebra and integration by parts, one finds

$$\begin{aligned}
\Gamma^{(4)}_{reg}(\beta) = \int d^4x \, \mathrm{tr} \Big\{ & \frac{1}{2} R^2 I_3 - \frac{1}{2} G_{ij} G_k{}^j I_4^{ik} + R\,(-\partial_i\partial_j I_4^{ij} \\
& - \partial_i\partial_j u \,[\, 8 I_5^{ij} + \eta^{ij} I_4 \,] - 4\, \partial_j u\, \partial_i I_5^{ij} + \partial_i\partial_j u\, [\, 10 I_6^{ij} + 2\eta^{ij} I_5 \,]) \\
& - \partial_i\partial_j\partial_k\partial_l u \,(8 I_6^{ijkl} + 6 \eta^{ij} I_5^{kl} + \tfrac{1}{2} \eta^{ij}\eta^{kl} I_4) \\
& + \partial_i\partial_j u \; \partial_k\partial_l u \,(8 I_7^{ijkl} + 4 \eta^{ij} I_6^{kl} + \tfrac{1}{2} \eta^{ij}\eta^{kl} I_5) \\
& - \partial_i\partial_j\partial_k u \;\; \partial_l u \,(16 I_7^{ijkl} + 8 \eta^{ij} I_6^{kl}) \\
& - \partial_i u \;\, \partial_j u \;\, \partial_k\partial_l u \,(14 I_8^{ijkl} - 12 \eta^{ij} I_7^{kl} + 2 \eta^{ij}\eta^{kl} I_6) \\
& + \partial_i u \;\, \partial_j u \;\, \partial_k u \;\, \partial_l u \,(144 I_9^{ijkl} + 48 \eta^{ij} I_8^{kl} + 4 \eta^{ij}\eta^{kl} I_7) \Big\}
\end{aligned} \tag{44}$$

with

$$I_m^{ijkl} = i \Lambda^2 \int_0^1 d\alpha \int_\beta \frac{d^4p}{(2\pi)^4} \left\{ \frac{p^i p^j p^k p^l}{(-p^2 + \alpha\Lambda^2 + u)^m} - \frac{p^i p^j p^k p^l}{(\alpha \to \alpha+1)^m} \right\}. \tag{45}$$

Integrating over the momentum variables, Equation (44) may be cast into the form

$$\begin{aligned}
\Gamma^{(4)}_{reg}(\beta) = \int \frac{d^4x}{(4\pi)^2} \Big\{ & \mathrm{tr}\,(R^2 + \tfrac{1}{6} G_{ij} G^{ij}) \tfrac{1}{4} \log 2u/\Lambda^2 \\
& + \frac{N-1}{f^2} Z^{\rho\delta} \partial_i \pi_\rho \partial^i \pi_\delta \,(\tfrac{1}{12} \partial_i \partial^i u/u - \tfrac{1}{24} \partial_i u \partial^i u / u^2) \\
& + N\,(-\tfrac{1}{240} (\partial_k \partial^k u)^2/u^2 + \tfrac{1}{180} \partial_i u \partial^i u \, \partial_j \partial^j u / u^3 - \tfrac{1}{480} (\partial_i u \partial^i u)^2/u^4) \Big\} \\
+ \int \frac{d^4x}{8\pi\beta} \frac{1}{\sqrt{u}} \Big\{ & -\tfrac{1}{4}\, \mathrm{tr}\,(R^2 + \tfrac{1}{6} G_{ij} G^{ij}) + \frac{N-1}{f^2} Z^{\rho\delta} \partial_i \pi_\rho \partial^i \pi_\delta \,(\tfrac{1}{24} \partial_i \partial^i u/u \\
& \qquad - \tfrac{1}{32} \partial_i u \partial^i u/u^2) \\
& + N (-\tfrac{1}{320} (\partial_i \partial^i u)^2/u^2 + \tfrac{5}{384} \partial_i u \partial^i u \, \partial_j \partial^j u / u^3 + \tfrac{77}{3072} (\partial_i u \partial^i u)^2/u^4) \Big\},
\end{aligned} \tag{46}$$

with
$$\mathrm{tr}\, R^2 = \frac{1}{f^4}(N-2)\left(Z^{pq}\partial_i\pi_p\partial^i\pi_q\right)^2 + \frac{1}{f^4} Z^{pq}Z^{rs}(\partial_i\pi_p\partial_j\pi_r)(\partial^i\pi_q\partial^j\pi_s) ,$$
$$\mathrm{tr}\, G_{ij}G^{ij} = \frac{1}{f^4}\partial_i\pi_p\partial_j\pi_q\partial^i\pi_r\partial^j\pi_s\left(Z^{ps}Z^{qr} - Z^{pr}Z^{qs}\right). \quad (47)$$

The first term is the well-known one-loop logarithmically divergent term in the non-linear σ model. It is expressed in geometrical quantities so that it is invariant under the redefinition of the pion fields. This is due to our manifestly covariant formalism. Otherwise, we would have further logarithmically divergent terms which has no geometrical meaning. (Even if these terms can always be removed by a suitable redefinition of pion fields /12,13/.) Note also that equation (46) becomes the result derived in /11/, in the limit of trivial symmetry, i.e. R=0.

The expression (46) has some infrared divergences in the limit of exact symmetry, $\varepsilon \to 0$. The first term diverges logarithmically and the temperature dependent terms diverge as $\varepsilon^{-1/2}$. These infrared divergences are expected since the power of derivatives, in the derivative expansion, should be accompanied with the corresponding power of the pion mass /14/, which vanishes in the exact symmetry limit, $\varepsilon \to 0$. Generally, based on the power counting argument, one can easily find that the n-derivative terms behave as $\varepsilon^{(4-n)/2}$ while the corresponding temperature dependent terms behave as $\varepsilon^{(3-n)/2}$ in the limit $\varepsilon \to 0$. Therefore one needs to retain a finite (but still small) ε . This is not unphysical since, after all, the real pion mass is not strictly zero.

IV. O(N+1) linear σ model and decoupling

The classical action for the O(N+1) linear σ model is given by

$$S = \int d^4x \left\{ \frac{1}{2}\partial_\mu\phi_\alpha\partial^\mu\phi_\alpha + \varepsilon\sigma - \frac{1}{4}\lambda\left(\phi^2 - f^2\right)^2 \right\} , \quad (48)$$

where $\phi^2 = \sum_\alpha \phi_\alpha^2$, $\phi_0 \equiv \sigma$, $\phi_p \equiv \pi_p$ for $\alpha = 0,1,\dots,N$ and $p=1,2\dots,N$. This action corresponds to the general action (2) with the metric $Z^{\alpha\beta} = \delta_{\alpha\beta}$ and $V = \frac{1}{4}\lambda(\phi^2 - f^2)^2 - \varepsilon\sigma$. It is easy to see that the connection field in this linear model is simply zero and the mass matrix U is determined as

$$U_\alpha{}^\beta = \frac{\partial^2 V}{\partial\phi_\alpha\partial\phi_\beta} = \lambda\left(\delta_{\alpha\beta}(\phi^2 - f^2) + 2\phi_\alpha\phi_\beta\right) \quad (49)$$

This mass matrix U has two different eigenvalues: N are degenerate and have the value

$$m^2 = \lambda(\phi^2 - f^2) \quad (50)$$

while the other has the distinguished value

$$m_0^2 = \lambda(3\phi^2 - f^2) \quad (51)$$

The former eigenvalues are related to the mass of pions, while the latter is related to the mass of the σ particle. Both of them are dependent on the external background fields.

There is a relation between the O(N+1)/O(N) non-linear model and the O(N+1) linear model. If we take the limit $\lambda \to \infty$ in the action (48), the value of the σ field is fixed at the value $\sigma^2 = f^2 - \pi^2$. Substituting this fixed value in the action (48), the non-linear σ model described by (3) is easily derived. When λ is finite but very large, the mass of the σ particle is also very large and thus the σ field should be decoupled from the low energy sector. At zero temperature, there is a formal proof /15/ that the effective action of a non-linear model is obtained from the effective action of the corresponding linear model through the decoupling of heavy particles, in all orders. This is also verified by calculating some terms in the derivative expansion of the one-loop effective action in ref. /2,13/. The relation between linear and non-linear models, through decoupling, seems also to be true even at finite temperature, provided one freezes out the degree of freedom corresponding to heavy particles, if their masses are sufficiently larger than the temperature in question.

In the following, we shall verify this statement by rederiving the zero- and two-derivative terms $\Gamma^{(0)}_{reg}(\rho)$ and $\Gamma^{(2)}_{reg}(\rho)$ from the finite temperature effective action for the O(N+1) linear model. For this purpose, it is convenient to introduce the following change of variables

$$\sigma = \rho \cos(\theta/f) \;, \quad \pi_p = \rho\,(\theta_p/\theta)\sin(\theta/f) \tag{52}$$

with

$$\theta = \Big(\sum_p \theta_p^2\Big)^{1/2} \;, \quad \rho^2 = \phi^2 = \sigma^2 + \sum_p \pi_p^2 \;. \tag{53}$$

In the limit $\lambda \to \infty$ in the action (48), only ρ is fixed while θ moves freely since the potential term carrying λ only involves the ρ field. With a finite but sufficiently large λ, the ρ field may be fixed by minimizing /2,13/ the action (48);

$$\begin{aligned} 0 &= \frac{\partial\sigma}{\partial\rho}\frac{\delta S}{\delta\sigma} + \frac{\partial\phi_p}{\partial\rho}\frac{\delta S}{\delta\phi_p} \\ &= \lambda\rho(\rho^2 - f^2) - \frac{\varepsilon\sigma}{\rho} + \frac{1}{\rho}\phi_\alpha\partial^2\phi_\alpha \end{aligned} \tag{54}$$

This equation can be solved approximately for large λ. From that solution we have

$$\begin{aligned} m^2 &= \mu + \frac{1}{f^2}\phi_\alpha\partial^2\phi_\alpha + O(\tfrac{1}{\lambda}) + O(\partial^4) \\ &= \mu + \frac{1}{f^2} Z^{p\beta}\partial_\mu\pi_p\partial^\mu\pi_\beta + O(\tfrac{1}{\lambda}) + O(\partial^4), \\ m_0^2 &= 2\lambda f^2 + O(1) + O(\partial^2). \end{aligned} \tag{55}$$

The non-derivative term in the effective action (24) for the O(N+1) linear model may be written as

$$\Gamma'_{reg}(\rho) = \frac{1}{2}\int d^4x\,\{ N\, I_1(m^2) + I_1(m_0^2) \} \tag{56}$$

with

$$I_1(m^2) = i\Lambda^2\int_0^1 d\alpha \int_\beta \frac{d^4p}{(2\pi)^4}\{ (-p^2+m^2+\alpha\Lambda^2)^{-1} - (\alpha\to\alpha+1)^{-1} \} . \tag{57}$$

The last function has the following asymptotic forms

$$\begin{aligned} I_1(m^2) \longrightarrow & -\frac{1}{(4\pi)^2}\big(2\Lambda^2 m^2 \log 2 + \tfrac{1}{2}m^4\log(2m^2/\Lambda^2) - \tfrac{3}{4}m^4\big) \\ & - \frac{1}{24\beta^2} + \frac{1}{8\pi\beta}\sqrt{m^2} + O(\log\beta)\,, \quad \text{for } m^2 << T^2, \Lambda^2 , \\ \longrightarrow & -\frac{1}{(4\pi)^2}\,2\Lambda^4\log m^2/\Lambda^2 + O(e^{-\beta(m^2)^{1/2}})\,, \quad \text{for } m^2 >> T^2, \Lambda^2 . \end{aligned} \tag{58}$$

Now it is easy to see that the second term in (56) vanishes for sufficiently large λ . After substituting the formulas for m^2, equation (56) becomes

$$\Gamma'_{reg}(\beta) = -\frac{N}{2}\int\frac{d^4x}{(4\pi)^2}\Big\{ 2\Lambda^2 u \log 2 + \frac{1}{2}u^2\log 2u/\Lambda^2 - \frac{3}{4}u^2 + \frac{1}{f^2}Z^{\rho\gamma}\partial_i\pi_\rho\partial^i\pi_\gamma\left(2\Lambda^2\log 2 + u\log 2u/\Lambda^2\right)\Big\}$$
$$+ N\int d^4x\,\Big\{-\frac{1}{24\beta^2}u + \frac{1}{12\pi\beta}u\sqrt{u} + \frac{1}{f^2}Z^{\rho\gamma}\partial_i\pi_\rho\partial^i\pi_\gamma\left(-\frac{1}{24\beta^2}+\frac{\sqrt{u}}{8\pi\beta}\right)\Big\},$$
$$+ O\left(\tfrac{1}{\lambda}\right) \tag{59}$$

in the high temperature limit, which should still be valid since the pions are light compared with T_c.

To evaluate the two-derivative terms it is convenient to introduce the projection matrices P_π and P_σ;

$$[P_\pi]_\alpha{}^\beta = \delta_{\alpha\beta} - \phi_\alpha\phi_\beta/\phi^2,$$
$$[P_\sigma]_\alpha{}^\beta = \phi_\alpha\phi_\beta/\phi^2, \tag{60}$$

which satisfy

$$P_\pi^2 = P_\pi,\quad P_\sigma^2 = P_\sigma,\quad P_\pi P_\sigma = P_\sigma P_\pi = 0. \tag{61}$$

After separating the mass matrix U into

$$U = m_0^2 P_\sigma + m^2 P_\pi, \tag{62}$$

one finds the following terms which contain two derivatives.

$$\Gamma''_{reg}(\beta) = -\frac{1}{2}\int d^4x\ \mathrm{tr}\Big\{-2\,I_4^{ij}(m^2)(\partial_i\partial_j U P_\pi) + 4\,I_5^{ij}(m^2)(\partial_i U P_\pi\partial_j U P_\pi) + \frac{4}{2\lambda f^2}I_4^{ij}(m^2)(\partial_i U P_\sigma\partial_j U P_\pi)$$
$$-\frac{4}{(2\lambda f^2)^2}\left(I_3^{ij}(m^2) + 2u\,I_4^{ij}(m^2)\right)(\partial_i U P_\sigma\partial_j U P_\pi)\Big\} + O(\lambda^{-1}). \tag{63}$$

In deriving this expression, we have used the following expansion in λ^{-1},

$$\frac{1}{-p^2+m_0^2+\alpha\Lambda^2} = \frac{1}{2\lambda f^2} - \frac{1}{(2\lambda f^2)^2}(-p^2+\alpha\Lambda^2+3u) + O(\tfrac{1}{\lambda^3}). \tag{64}$$

After inserting the asymptotic forms of the I integrals and making use of the following quantities

$$\mathrm{tr}(\partial_i\partial_j U P_\pi) = N\,\partial_i\partial_j u + 4\lambda Z^{\rho\gamma}\partial_i\pi_\rho\partial_j\pi_\gamma + O(\tfrac{1}{\lambda}),$$
$$\mathrm{tr}(\partial_i U P_\pi\partial_j U P_\pi) = N\,\partial_i u\,\partial_j u + O(\tfrac{1}{\lambda}),$$
$$\mathrm{tr}(\partial_i U P_\pi\partial_j U P_\sigma) = 4\lambda^2(f^2+\tfrac{u}{\lambda})Z^{\rho\gamma}\partial_i\pi_\rho\partial_j\pi_\gamma + O(1). \tag{65}$$

one can rewrite the formula (63) as

$$\Gamma''_{reg}(\beta) = \frac{1}{2}\int\frac{d^4x}{(4\pi)^2}\Big\{\frac{1}{f^2}Z^{\rho\gamma}\partial_i\pi_\rho\partial^i\pi_\gamma\left(2\Lambda^2\log 2 + u\log 2u/e\Lambda^2\right) + \frac{N}{12}\partial_i u\,\partial^i u/u\Big\}$$
$$+\int d^4x\,\Big\{Z^{\rho\gamma}\partial_i\pi_\rho\partial^i\pi_\gamma\left(\frac{1}{24\beta^2}-\frac{1}{8\pi\beta}\sqrt{u}\right) + \frac{1}{48}\frac{1}{8\pi\beta}\partial_i u\,\partial^i u/\sqrt{u}\Big\}$$
$$+ O\left(\tfrac{1}{\lambda}, \log\beta\right). \tag{66}$$

Adding Eqs. (59) and (66) gives us precisely the same formulas for the zero-and two-derivative terms derived the non-linear σ model up to the terms of $O(\lambda^4)$

V. Concluding Remarks

In this paper, I have extended the zero temperature formulation of the covariant derivative expansion so that it is applicable at finite temperature. As a specific example, a few leading terms of the one-loop effective action for the O(N+1)/O(N) non-linear σ model have been explicitly calculated in a manifestly covariant way, at $T \neq 0$. Based on the analysis of the two-derivative terms and the effecting potential, the temperature dependences of the pion decay constant and the pion mass are derived and the critical temperature is also identified.

The most interesting case may be the N=3 case, since O(4)/O(3) is equivalent to SU(2)xSU(2)/SU(2) which corresponds to the chiral symmetry of QCD. The phase structure of this chiral symmetry is of high physical interest. However, it is very difficult to consider it directly in QCD, except for lattice simulations. Therefore it seems sensible to consider the same problem in the corresponding σ model, which is believed to describe the low energy sector of QCD. The O(N+1) linear σ model was investigated in /9/ and a critical temperature T_c' is derived from an analysis of the finite temperature effective potential. Our value of the critical temperature T_c is larger than that in ref. /9/ by the factor of $\sqrt{3}$, when N=3. This can be traced to the decreased N-factor in equation (40). This reduction in the effective N factor is explained starting from the O(N+1) linear σ model, as a result of the decoupling of the σ particle, which becomes very massive in the large coupling limit. Of course, the increasing of a factor $\sqrt{3}$ is valid only in the one loop approximation. Nevertheless, it might still be true that the slight increasement of the critical temperature really obtains

even beyond the one loop approximation.

There is another case where our analysis is relevant. It is now widely accepted that the Higgs sector of standard electroweak theory may be described by the SU(2)xSU(2)/SU(2) non-linear σ model /16/, if there is no light Higgs particle. Using our argument, one should expect that the critical temperature increases when the self coupling of Higgs particles is sufficiently large. Thus it could well be that $T_c = \sqrt{6}\,(\sqrt{2}\,G_F)^{-1/2}$ rather than $T_c = \sqrt{2}\,(\sqrt{2}\,G_F)^{-1/2}$, as might be expected from a straightforward analysis of the Higgs potential.

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Footnotes

/F.1/ If the curvature tensor (see Eq. (9)) on the scalar field manifold vanishes, we call that model a linear model; otherwise we call it a non-linear model.

/F.2/ Even after two subtractions, there may remain quartic divergences. However, they are cancelled by the normalization constant in (4).

/F.3/ In this derivation, we follow the loop-expansion philosophy, i.e. products of one-loop quantities are ignored. The temperature dependent term is also regarded as a one-loop quantity.

References

/1/ L.-H. Chan, Phys. Rev. Lett. 54 (1985) 1222;
O. Cheyette, Phys. Rev. Lett. 55 (1985) 2394;
I.J.R. Aitchison and C.M. Fraser, Phys. Rev. D31 (1985) 2605;
J.A. Zuk, Phys. Rev. D32 (1985) 2653

/2/ M.K. Gaillard, Nucl. Phys. B268 (1986) 669

/3/ J.A. Zuk, Phys. Rev. D33 (1986) 3645;
L.-H. Chan, Phys. Rev. Lett. 57 (1986) 1199

/4/ K. Symanzik, Commun. Math. Phys. 16 (1970) 48;
S. Coleman and E. Weinberg, Phys. Rev. D7 (1973) 1888

/5/ I.J.R. Aitchison and C.M. Fraser, in "Solitons in Nuclear and Elementary Particle Physics" ed. by A. Chodos, E. Hadjimichael and C. Tze (World Scientific, Singapore, 1984)

/6/ P. Binétruy and M.K. Gaillard, Phys. Lett. 168B (1986) 347

/7/ M.K. Gaillard, Berkeley preprint LBL-21465, UCB-PTH-86/10 (1986)

/8/ J. Iliopoulos, C. Itzykson and A. Martin, Rev. Mod. Phys. 47 (1975) 165

/9/ L. Dolan and R. Jackiw, Phys. Rev. D9 (1974) 3320;
S. Weinberg, Phys. Rev. D9 (1974) 3357;
D.A. Kirzhnits and A.D. Linde, Phys. Lett. 42B (1972) 471

/10/ P. Binétruy and M.K. Gaillard, Phys. Rev. D32 (1985) 931

/11/ I.G. Moss, D.J. Toms and W.A. Wright, New Castle preprint NCL 85-TP7 (1985)

/12/ J. Honerkamp, Nucl. Phys. B36 (1972) 130;
L. Tataru, Phys. Rev. D12 (1975) 3351;
D.G. Boulware and L.S. Brown, Ann. of Phys. (N.Y.) 138 (1982) 392

/13/ T. Appelquist and C. Bernard, Phys. Rev. D23 (1981) 425

/14/ I.J.R. Aitchison and C.M. Frazer, Phys. Rev. D32 (1985) 2190

/15/ C. Lee and P.Y. Pac, Nucl. Phys. B177 (1981) 87

/16/ T. Appelquist and C. Bernard, Phys. Rev. D22 (1980) 200