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Darboux transformations and coherent interaction of the light pulse with two-level media

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Abstract

This work deals with the applications of the Darboux transformation method to study of the reduced Maxwell-Bloch system (RMB) and of the self-induced transparency (SIT) equations. Both system describe in some reasonable approximation the propagation of the ultrashort optical pulses in the two-level medium. Main result of the present work is a construction of the multisoliton solutions on the arbitrary background for RMB and SIT equations. Particular cases of those solutions are discussed in some details.

1. Introduction

Darboux transformation as a method of construction of infinite families of exactly solvable linear Schrödinger equations was proposed at the end of the last entury by Gaston Darboux (1882). The renewal of interest in the method and its essential generalization, applicable to partial differential and difference problem and numerous applications to soliton problems is due to the works of the first author (1979). For further developpments see Bobenko (1982), Salle (1982), Bobenko, Matveev, Salle (1982), Babich, Matveev, Salle (1985), Leble, Salle (1985), Rybin, Salle (1985). For an extension of the Darboux method to the multidimensional stationary problems and applications to supersymmetrical quantum mechanics see Borisov, Andrianov, Joffe (1984), (1986).

In applications to solitons theory generalized Darboux transformation method - which we shall call shortly DT method - is the simplest way to produce so called multisoliton solutions on the arbitrary background, rational solutions, solutions depending on the functional parameters for the K-P and 2-dimensional Toda-like equations, without requiring full implementation of the inverse problem method. In addition the obtained expressions can be helpful for describing the asymptotic behavior at large times since they enable one to take into account automatically the interaction of the soliton part of the solution with the continuous spectrum. Below we apply DT method to some well known in nonlinear optics completely integrable systems namely reduced Maxwell-Bloch equations and self-induced transparency equations. The reduced Maxwell-Bloch system is written

$$r_{1t} = -\omega r_2 \,, \quad r_{2t} = \omega r_1 + \frac{2\mu}{\hbar} E r_3 \,,$$

$$r_{3t} = -\frac{2\mu}{\hbar} E r_2 \,, \quad c E_x + E_t = -2\pi\mu \langle r_{1t} \rangle \,. \tag{1.1}$$

In (1.1) r_i are the following linear combinations of the matrix elements ρ_{ik}

of the density matrix:

$$r_1 = \rho_{21} + \rho_{12}\,,\ r_2 = i(\rho_{21} - \rho_{12})\,,\ r_3 = \rho_{22} - \rho_{11}\,,$$

n is the concentration of the atoms, μ-matrix element of the dipole-momentum operator. All the quantities entering in (1.1) are realvalued. f means the average with some density $g_1(\omega)$

$$\langle f \rangle = \int_0^\infty f(\omega) g_1(\omega)\, d\omega$$

Associated term in (1.1) takes into account so-called nonuniform broadening of the lines. Case $g_1(\omega) = \delta(\omega - \omega_0)$ corresponds to infinitely narrow line.

The described model was introduced by Eilbeck (1972) and discussed in the works of Eilbeck et al. (1973), Ablowitz et al. (1974), Bullough et al. (1979). In those works partuclarly multisolitons solutions and breather-like solutions were explicitly constructed.

Self induced transparency equations (SIT) are of the form

$$\mathcal{E}_\xi = \langle \rho \rangle \,, \qquad \rho_\tau + 2i\eta\rho = N\mathcal{E}$$

$$N_\tau = -\frac{1}{2}\left(\mathcal{E}^*\rho + \mathcal{E}\rho^*\right), \tag{1.2}$$

$$\rho = 2i\rho_{21}\exp\{ikx + i\omega_0 t\},\ \eta = \frac{\omega - \omega_0}{2\Omega}\,,$$

$$\xi = \frac{\Omega x}{c}\,,\ \tau = \Omega\left(t - \frac{x}{c}\right),\ \Omega^2 = \frac{2\pi n \mu^2 \omega_0}{\hbar}$$

being the complex enveloppe of the electric field, N and ρ depend on η

$\langle \rho \rangle$ stays for the integral:

$$\langle \rho \rangle = \int_{-\infty}^{\infty} \rho(x,t,\eta)\, g_2(\eta)\, d\eta \,.$$

In the case of infinitely narrow line – $g_2(\eta) = \delta(\eta)$ –, $\mathcal{E}$ real valued and $N = \cos\phi$, $\rho = \sin\phi$, the system (1.2) reduces to the sine-Gordon equation

$$\phi_{tt} - \phi_{xx} = \sin\phi$$

The detailed discussion of the physical origine of the SIT equations and the list of known exact solutions may be found in the book of Dodd et al. (1982).

The application of the DT method gives a possibility to construct the multi-soliton solutions on the arbitrary background in a very simple way both for RMB and SIT equations and to isolate some new particular interesting cases of those solutions, which seems to be previously unknown.

2. Darboux transformations for Zakharov-Shabat spectral problem

Here we recall and put in the convenient form some general formulae related with Darboux transformations for the first-order linear systems. Those formulae were firstly obtained by Bobenko (1982) and Salle (1982) and used by Salle (1982) for solving NS equation. Somewhat later those results were rediscovered by Neugebauer and Meinel (1984). Consider the system

$$\psi_x = -i\lambda\psi + iq\varphi$$

$$\varphi_x = i\lambda\varphi + ir\psi \tag{2.1}$$

and the related equation for the fundamental matrix solution:

$$\Psi_x = J\Psi\Lambda + U\Psi,$$

$$J=\begin{pmatrix}-i & 0\\ 0 & i\end{pmatrix},\ \Lambda=\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix},\ U=\begin{pmatrix}0 & iq\\ ir & 0\end{pmatrix},\ \Psi=\begin{pmatrix}\psi_1 & \psi_2\\ \varphi_1 & \varphi_2\end{pmatrix}, \tag{2.2}$$

$\begin{pmatrix}\psi_k\\ \varphi_k\end{pmatrix}$, k = 1, 2 form a solution of (2.1) with $\lambda = \lambda_k$. The direct substitution proves that the equation (2.2) is covariant with respect to the transformation:

$$\Psi \to \Psi[1],\quad \Psi[1] = \Psi\Lambda - \Psi_1\Lambda_1\Psi_1^{-1}\Psi, \tag{2.3}$$

Ψ_1 being a fixed solution of (2.2) with $\Lambda = \Lambda_1$. The induced transformation of the matrix-potential U U[1] is given by the formula

$$U[1] = U + [J, \Psi_1\Lambda_1\Psi_1^{-1}]. \tag{2.4}$$

N-times repeated Darboux transformation is evidently of the form:

$$\Psi[N] = \Psi\Lambda^N + S_1\Psi\Lambda^{N-1} + \dots + S_{N-1}\Psi\Lambda + S_N\Psi, \tag{2.5}$$

$$U[N] = U + [S_1, J], \tag{2.6}$$

and the coefficients S_j may be defined by the conditions

$$\Psi[N]\Big|_{\Lambda=\Lambda_k,\ \Psi=\Psi_k} = 0,\quad k=1,\dots,N, \tag{2.7a}$$

Ψ_k being a fixed solution of the system (2.2) with $\Lambda = \Lambda_k$. The equations (2.7a) may be rewritten in the form

$$\begin{aligned} &S_1\Psi_1\Lambda_1^{N-1} + S_2\Psi_1\Lambda_1^{N-2} + \dots + S_N\Psi_1 = -\Lambda_1^N\Psi_1,\\ &\vdots\\ &S_1\Psi_N\Lambda_N^{N-1} + S_2\Psi_N\Lambda_N^{N-2} + \dots + S_N\Psi_N = -\Lambda_N^N\Psi_N. \end{aligned} \tag{2.7b}$$

(2.7b) represents the system of (2N) scalar linear equations which can be easily solved by the Kramer rule. In this way we get particularly for S_1

$$S_1 = -\frac{1}{\Delta}\begin{pmatrix}\Delta_{11} & \Delta_{12}\\ \Delta_{21} & \Delta_{22}\end{pmatrix}$$

where

$$\Delta = \det\left(\lambda_i^{N-1}\psi_i, \lambda_i^{N-1}\varphi_i, \lambda_i^{N-2}\psi_i, \lambda_i^{N-2}\varphi_i, \dots, \psi_i, \varphi_i\right),$$

$$\Delta_{11} = \det\left(\lambda_i^{N}\psi_i, \lambda_i^{N-1}\varphi_i, \lambda_i^{N-2}\psi_i, \lambda_i^{N-2}\varphi_i, \dots, \psi_i, \varphi_i\right)$$

$$\Delta_{22} = \det\left(\lambda_i^{N-1}\psi_i, \lambda_i^{N-1}\varphi_i, \lambda_i^{N-2}\psi_i, \lambda_i^{N-2}\varphi_i, \dots, \psi_i, \varphi_i\right)$$

$$\Delta_{12} = \det\left(\lambda_i^{N-1}\psi_i, \lambda_i^{N}\psi_i, \lambda_i^{N-2}\varphi_i, \dots, \psi_i, \varphi_i\right)$$

$$\Delta_{21} = \det\left(\lambda_i^{N}\varphi_i, \lambda_i^{N-1}\varphi_i, \lambda_i^{N-2}\psi_i, \lambda_i^{N-2}\varphi_i, \dots, \psi_i, \varphi_i\right), \tag{2.8}$$

$i = 1, \dots, 2N,$

$$\det(a_i, b_i, \dots, f_i) \stackrel{\det}{=} \det\begin{pmatrix} a_1, & b_1, & \dots & f_1\\ a_2, & b_2, & & f_2\\ \vdots & \vdots & & \vdots\end{pmatrix}.$$

From (2.6), (2.8) we have the following expressions for $q[N]$, $r[N]$:

$$\begin{cases} q[N] = q + 2\dfrac{\Delta_{12}}{\Delta},\\[2mm] r[N] = r - 2\dfrac{\Delta_{21}}{\Delta}. \end{cases} \tag{2.9}$$

3. Multisoliton solutions on an arbitrary background for RMB system

For simplify the final expressions we consider RMB system in ξ, τ variables:

$$r_{1\xi} = -\omega r_2 , \quad r_{2\xi} = \omega r_1 + \mathcal{E} r_3 ,$$
$$r_{3\xi} = -\mathcal{E} r_2 , \quad \mathcal{E}_\tau = -\langle r_{1\xi} \rangle . \qquad (3.1)$$

Equations (3.1) may be considered as a condition of compatibility of the following linear system:

$$\Psi_\xi = J \Psi \Lambda + U_1 \Psi , \qquad (3.2)$$

$$\Psi_\tau = \langle M_1 \Psi P_1 + M_2 \Psi P_2 \rangle , \qquad (3.3)$$

U_1 is obtained from U under the reduction constraint

$$q = r = \frac{\mathcal{E}}{2} , \qquad (3.4)$$

$$M_1 = \frac{\omega}{8} \begin{pmatrix} i r_3 , & r_2 + i r_1 \\ -r_2 + i r_1 , & -i r_3 \end{pmatrix} , \quad M_2 = \frac{\omega}{8} \begin{pmatrix} i r_3 , & r_2 - i r_1 \\ -r_2 - i r_1 , & -i r_3 \end{pmatrix} , \qquad (3.5)$$

$$P_1 = \begin{pmatrix} \frac{1}{\lambda_1 - \frac{\omega}{2}} , & 0 \\ 0 , & \frac{1}{\lambda_2 - \frac{\omega}{2}} \end{pmatrix} , \quad P_2 = \begin{pmatrix} \frac{1}{\lambda_1 + \frac{\omega}{2}} , & 0 \\ 0 , & \frac{1}{\lambda_2 + \frac{\omega}{2}} \end{pmatrix} . \qquad (3.6)$$

The equation (3.2) is covariant with respect to Darboux transformation (2.3) - (2.5) under the complementary restrictions on Ψ and Λ : $\lambda_2 = -\lambda_1$, $\psi_2 = \varphi_1$, $\varphi_2 = \psi_1$ in (2.3) and $\lambda_{2k} = -\lambda_{2k-1}$, $\psi_{2k} = \varphi_{2k-1}$, $\varphi_{2k} = \psi_{2k-1}$ in the formulae for the N-times iterated Darboux transformation. Consequently the formulae (2.9) make it possible to find $\mathcal{E}[N]$. For obtain $r_k[N]$ it is nesserary to check the covariance of the equation (3.3) with respect to the same Darboux-transformation. The proof of this covariance is based on the use of the identities:

$$\Lambda P_1 = P_1 \Lambda = I + \frac{\omega}{2} P_1 ,$$
$$\Lambda P_2 = P_2 \Lambda = I - \frac{\omega}{2} P_2 . \qquad (3.7)$$

The induced transformation of tme $M_{1,2}$-matrices is described by the formulae

$$M_1[1] = \left(\frac{\omega}{2} - \sigma \right) M_1 \left(\frac{\omega}{2} - \sigma \right)^{-1} ,$$

$$M_2[1] = \left(\frac{\omega}{2} + \sigma \right) M_2 \left(\frac{\omega}{2} + \sigma \right)^{-1} ,$$

$$\sigma = \Psi_1 \Lambda_1 \Psi_1^{-1} . \qquad (3.8)$$

For N-times repeated D-transformation the corresponding expressions are:

$$M_1[N] = \sigma_1 M_1 \sigma_1^{-1} ,$$
$$M_2[N] = \sigma_2 M_2 \sigma_2^{-1} , \qquad (3.9)$$

$$\sigma_1 = \sum_{j=0}^{N} \left(\frac{\omega}{2}\right)^{N-j} S_j ,$$

$$\sigma_2 = \sum_{j=0}^{N} \left(-\frac{\omega}{2}\right)^{N-j} S_j , \tag{3.10}$$

S_j are defined by (2.7), S_o = I. The sums in (3.10) may be converted into more simple and convenient expressions:

$$\sigma_1 = -\left(\frac{\omega}{2}\right)^{-\frac{(N-1)(N-2)}{2}} \frac{1}{\Delta} \begin{pmatrix} \delta^{(11)}, & \delta^{(12)} \\ \delta^{(21)}, & \delta^{(22)} \end{pmatrix} \tag{3.11}$$

Δ is defined by (2.8) and $\delta^{(ik)}$ are:

$$\delta^{(ik)} = \det\left(\delta_{mn}^{(ik)}\right), \quad 1 \le m, n \le 2N,$$

$$\delta_{mn}^{(11)} = \begin{cases} \left[\lambda_n^N - \left(\frac{\omega}{2}\right)^N\right]\psi_n , & m=1 \\ \lambda_n^{N-k}\varphi_n , & m=2k,\ k=1,\dots,N \\ \lambda_n^{N-j}\left[\left(\frac{\omega}{2}\right)^{j-1} - \lambda_n^{j-1}\right]\psi_n, & m=2j-1,\ j=2,3,\dots,N. \end{cases}$$

$$\delta_{mn}^{(12)} = \begin{cases} \lambda_n^{N-k}\psi_n , & m=2k-1,\ k=1,\dots,N \\ \lambda_n^{N}\psi_n , & m=2 \\ \lambda_n^{N-j}\left[\left(\frac{\omega}{2}\right)^{j-1} - \lambda_n^{j-1}\right]\varphi_n, & m=2j,\ j=2,\dots,N. \end{cases}$$

$$\delta_{mn}^{(21)} = \begin{cases} \lambda_n^{N}\varphi_n , & m=1 \\ \lambda_n^{N-k}\varphi_n , & m=2k,\ k=1,\dots,N \\ \lambda_n^{N-j}\left[\left(\frac{\omega}{2}\right)^{j-1} - \lambda_n^{j-1}\right]\psi_n, & m=2j-1,\ j=2,3,\dots,N. \end{cases}$$

$$\delta_{mn}^{(22)} = \begin{cases} \lambda_n^{N-k}\psi_n , & m=2k-1,\ k=1,2,\dots,N \\ \left[\lambda_n^{N} - \left(\frac{\omega}{2}\right)^N\right]\varphi_n , & m=2 \\ \lambda_n^{N-j}\left[\left(\frac{\omega}{2}\right)^{j-1} - \lambda_n^{j-1}\right]\varphi_n, & m=2j,\ j=2,\dots,N. \end{cases}$$

In (3.11) the restriction $\lambda_{2k} = \lambda_{2k-1}$, $\psi_{2k} = \varphi_{2k-1}$, $\varphi_{2k} = \psi_{2k-1}$, $k = 1, \dots, N$ must be imposed. The formulae for σ_2 are obtained from σ_1 by the change $w \mapsto -w$.

<u>Remark</u>. Another dressing formulae for $r_1[N]$ may be deduced from the equations (3.1):

$$r_1[N] = \frac{1}{\omega}\left\{ r_{2\xi}[N] - \mathcal{E}[N]\, r_3[N] \right\}.$$

Introducing M by the formula

$$M = M_1 + M_2 = \frac{\omega}{4}\begin{pmatrix} ir_3 & r_2 \\ -r_2 & -ir_3 \end{pmatrix}, \tag{3.12}$$

we have

$$M[1] = M - \sigma_\tau\,, \quad M[N] = M + S_{1\tau}\,,$$

S_1 being defined by (2.8). Particularly for 1-time Darboux dressing we have

$$\mathcal{E}[1] = \mathcal{E} - 8\,\frac{\lambda_1 \psi_1 \varphi_1}{\psi_1^2 - \varphi_1^2}\,, \tag{3.13}$$

$$r_2[1] = r_2 - \frac{8\lambda_1}{\omega}\left(\frac{\psi_1 \varphi_1}{\psi_1^2 - \varphi_1^2}\right)_\tau\,, \tag{3.14}$$

$$r_3[1] = r_3 + \frac{4i\lambda_1}{\omega}\left(\frac{\psi_1^2 + \varphi_1^2}{\psi_1^2 - \varphi_1^2}\right)_\tau\,. \tag{3.15}$$

4. <u>Particular solutions of the RMB system</u>

Simplest solution may be obtained if we put $r_2 = 0$; $r_1, \mathcal{E}, r_3$ be the constants related by the condition

$$\omega_1 r_1 + \mathcal{E} r_3 = 0 \tag{4.1}$$

Then for $\lambda_1 = \frac{i\mathcal{E}}{2}$ we have

$$\psi_1 = \eta + \frac{1}{\mathcal{E}}\,, \quad \varphi_1 = i\left(\eta - \frac{1}{\mathcal{E}}\right),$$

$$\eta = \xi - \xi_0 + \left\langle \frac{\omega r_3}{\mathcal{E}^2 + \omega^2} \right\rangle t\,. \tag{4.2}$$

In this case (3.13) - (3.15) describe rational soliton of the RMB system. In the case $\lambda \neq \frac{i\mathcal{E}}{2}$ we obtain the solution depending on η,

$$\eta = \xi - \xi_0 - \left\langle \frac{\omega r_3}{4\lambda_1^2 - \omega^2} \right\rangle t\,, \tag{4.3}$$

$$\begin{pmatrix} \psi_1 \\ \varphi_1 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{2(\nu+\lambda)}{\mathcal{E}} \end{pmatrix} c_1 e^{i\nu\eta} + \begin{pmatrix} 1 \\ \frac{2(\lambda-\nu)}{\mathcal{E}} \end{pmatrix} c_2 e^{-i\nu\eta}\,, \tag{4.4}$$

$\nu = \sqrt{\lambda^2 + \frac{\mathcal{E}^2}{4}}$, the branch of the square root may be chosen in arbitrary way. For the case $\mathrm{Re}\,\nu = 0$, we obtain the well known ch-soliton. In the case $\mathrm{Im}\,\nu = 0$ we have some elementary periodic solution. Formulae for the multiple DT produces a large family of solutions of the RMB system in assumtion that the solutions of associated starting linear system are of the form (4.2) - (4.4). By the same method we can easy obtain the following solution of the RMB system:

$$r_1 = f(\omega)\cos k\xi \quad , \quad \langle f(\omega)\rangle = 0 ,$$

$$r_2 = \frac{k f(\omega)}{\omega}\sin k\xi \, , \quad r_3 = \sqrt{c - \frac{f^2}{2}\left(1-\frac{k^2}{\omega^2}\right)\cos 2k\xi}$$

$$\mathcal{E} = -\omega f(\omega)\left(1-\frac{k^2}{\omega^2}\right)\frac{\cos k\xi}{\sqrt{c - \frac{f^2(\omega)}{2}\left(1-\frac{k^2}{\omega^2}\cos k\xi\right)}} .$$

5. Multisoliton solutions on arbitrary background for SIT equations

The equations (1.2) admit the following zero-curvature representation:

$$\psi_t = J\psi\Lambda + U_2\psi , \tag{5.1}$$

$$\psi_x = \langle M_3 \psi P\rangle , \tag{5.2}$$

U_2 is obtained from U by imposing the reduction constraint $q = r^* = i\frac{\mathcal{E}}{2}$,

$$M_3 = -\frac{i}{4}\begin{pmatrix} N, \rho \\ \rho^*, -N\end{pmatrix} , \quad P = \begin{pmatrix} \frac{1}{\eta-\lambda_1} & 0 \\ 0 & \frac{1}{\eta-\lambda_2}\end{pmatrix} .$$

Covariance of the equation (5.2) with respect to Darboux transformation (2.3) may be easily proved by use of the identity

$$P\Lambda = \Lambda P = -I + \eta P . \tag{5.3}$$

The induced law of transformation of the coefficient M reads:

$$M_3[1] = M_3 + \sigma_x = (\eta - \sigma) M_3 (\eta - \sigma)^{-1} , \tag{5.4}$$

$$\sigma = \Psi_1 \Lambda \Psi_1^{-1} \, , \quad M_3[N] = M_3 - S_{1x} ,$$

S_1 defined by the formula (2.8) The reduction constraints on Λ and Ψ take the form :

$$\lambda_{2k} = \lambda^*_{2k-1} \quad , \quad \varphi_{2k} = \psi^*_{2k-1} \quad , \quad \psi_{2k} = -\varphi^*_{2k-1} ,$$

$$k = 1, \dots, N \quad , \quad \text{i.e}$$

$$\Lambda_k = \begin{pmatrix} \lambda_k & 0 \\ 0 & \lambda_k^* \end{pmatrix} , \quad \Psi_k = \begin{pmatrix} \psi_k , & -\varphi_k^* \\ \varphi_k , & \psi_k^* \end{pmatrix} . \tag{5.5}$$

- In (5.5) we have redefined the indices $\lambda_{2k-1} \to \lambda_k , \psi_{2k-1} \to \psi_k$.

The simplest dressing formulae for the solutions of the SIT equation corresponding to 1-time Darboux transformation is of the form:

$$\mathcal{E}[1] = \mathcal{E} + \frac{4i(\lambda_1 - \lambda_1^*)\psi_1\varphi_1^*}{|\psi_1|^2 + |\varphi_1|^2} , \tag{5.6}$$

$$\mathcal{N}[1]=\mathcal{N}+4i\left(\frac{\lambda_1|\psi_1|^2+\lambda_1^*|\varphi_1|^2}{|\psi_1|^2+|\varphi_1|^2}\right)_x, \tag{5.7}$$

$$\rho[1]=\rho+4i(\lambda_1-\lambda_1^*)\left(\frac{\psi_1\varphi_1^*}{|\psi_1|^2+|\varphi_1|^2}\right)_x. \tag{5.8}$$

(5.7) provides a conservation of the real valuedness of N because we can rewrite (5.7) as follows:

$$\mathcal{N}[1]=\mathcal{N}-4\ \frac{\lambda_1-\lambda_1^*}{2i}\left(\frac{|\psi_1|^2-|\varphi_1|^2}{|\psi_1|^2+|\varphi_1|^2}\right)_x \tag{5.9}$$

The physical derivation of the SIT equation is based on the assumption of the absence of nonpure states. This fact is taken into account by the condition $\mathcal{N}^2+|\rho|^2=1$. In terms of M_3 this condition reads $M_3^2=-\frac{1}{16}\,I$. Those restrictions are conserved by the action of the Darboux transformation.

We can however forget the origine of the SIT equations and consider them as some equations for the density matrix omitting the condition $\mathcal{N}^2+|\rho|^2=1$, i.e. admiting the existence of nonpure states.

Associated solutions are of the interesting mathematical structure and probably may be interpreted in a reasonable way from the experimental view-point. That's why we discuss below in a few words some of those solutions.

6. Some particular solutions of the SIT equations

Taking starting solution of the system (1.2) in the form $\rho=\mathcal{E}=0,\ \mathcal{N}=\pm 1$ we get for the solutions of the system (5.1 - 5.2):

$$\psi=C_1\exp\{-i\lambda t-i\varkappa x\},$$

$$\varphi=C_2\exp\{i\lambda t+i\varkappa x\},$$

$$\varkappa=\pm\frac{1}{4}\left\langle\frac{1}{\eta-\lambda}\right\rangle, \tag{6.1}$$

$$\lambda=\lambda_R+i\lambda_I,\qquad \varkappa=\varkappa_R+i\varkappa_I$$

The dressed solution of the SIT system takes the form

$$\mathcal{E}[1]=-8\lambda_1\frac{\exp\{-2i\lambda_R t-2i\varkappa_R x+\alpha_0\}}{\operatorname{ch}\{2\lambda_1 t+2\varkappa_I x+\beta_0\}},$$

$$\mathcal{N}[1]=\pm 1\mp\frac{8\lambda_I\varkappa_I}{\operatorname{ch}^2(2\lambda_I t+2\varkappa_I x+\beta_0)}, \tag{6.2}$$

The construction of "background multisolitons" from the exact formulae for $\mathcal{E}[N]$, $\rho[N]$ and the starting solutions of the form (6.1) with different values of λ is straightforward.

If we try to look at the solutions of the same system without the constraint $N^2+|\rho|^2=1$, i.e. nonpure states should also be admited, it would be possible to take starting solution in the form $\mathcal{E}=\rho=0$, $N=N(x)$, or even in more general form:

$$\mathcal{E}=\mathcal{E}_0\exp\left\{i\omega t-i\int_{x_0}^{x}\left\langle\frac{N}{\omega+2\eta}\right\rangle dx\right\},$$

$$\rho=-\frac{iN\mathcal{E}_0}{\omega+2\eta}\exp\left\{i\omega t-i\int_{x_0}^{x}\left\langle\frac{N}{\omega+2\eta}\right\rangle dx\right\},$$
$$N=N(x,\eta), \tag{6.3}$$

N being an arbitrary function on x and η . In this case the constraint $N^2+|\rho|^2=1$ is violated:

$$N^2+|\rho|^2=N^2\left(1+\frac{\mathcal{E}_0^2}{(\omega+2\eta)^2}\right).$$

Introducing the notation

$$\theta=i\omega t-i\int_{x_0}^{x}\left\langle\frac{N}{\omega+2\eta}\right\rangle dx,$$

we find φ and ψ :

$$\psi=e^{\frac{\theta}{2}}u(z),\quad \varphi=e^{-\frac{\theta}{2}}v(z),\quad z=t+\frac{1}{4}\int_{x_0}^{x}\left\langle\frac{N}{(\frac{\omega}{2}+\eta)(\eta-\lambda)}\right\rangle dx, \tag{6.4}$$

$$\begin{cases} u_z=-i\left(\lambda+\frac{\omega}{2}\right)u-\frac{\mathcal{E}_0}{2}v \\ v_z=\frac{\mathcal{E}_0^*}{2}u+i\left(\lambda+\frac{\omega}{2}\right)u \end{cases} \tag{6.5}$$

Last system defining u and v has constant coefficients and its integration is trivial. Particularly in the case $\mathcal{E}_0=0$, we have solutions of SIT equations partly similar to considered above but including a functional parameter. In the case of infinitely narrow line $g(\eta)=\delta(\eta)$ we obtain for $N[1]$ the following representation:

$$N[1]=N\left\{1-\frac{2\lambda_I}{|\lambda|^2 ch^2\left(2\lambda_I t+\frac{\lambda_I}{2|\lambda|^2}\int_{x_0}^{x}N(x)dx+\beta\right)}\right\}. \tag{6.6}$$

Taking in (6.6) $x_0=\beta=0$,

$$N(x)=\begin{cases} c_1, & x>0 \\ c_2, & x<0 \end{cases}$$

we obtain

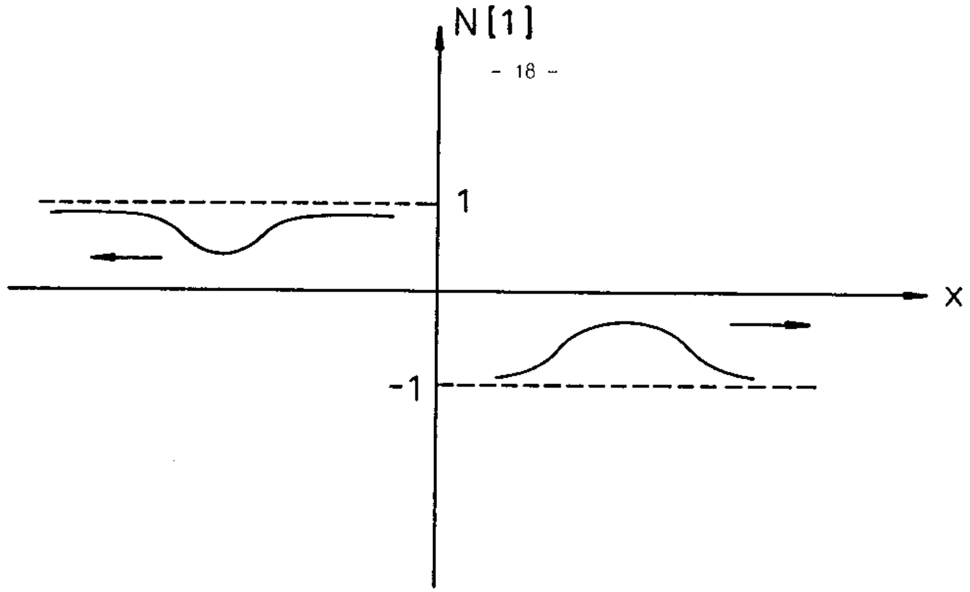
$$N[1]=\begin{cases} c_1\left(1-\frac{2\lambda_I}{|\lambda|^2 ch^2\left(2\lambda_I t+\frac{\lambda_I}{2|\lambda|^2}c_1 x\right)}\right), & x>0 \\ c_2\left(1-\frac{2\lambda_I}{|\lambda|^2 ch^2\left(2\lambda_I t+\frac{\lambda_I}{2|\lambda|^2}c_2 x\right)}\right), & x<0 \end{cases}$$

Let now $c_1=-1$, $c_2=1$, then at $t\to\infty$

$$N[1]\sim\begin{cases} -1 & x>0 \\ 1 & x>0 \end{cases}$$

Such behaviour does not contradict with the constraint $N^2+|\rho|^2=1$.

At $t\to-\infty$ we have two solitons propagating in the opposite directions :



So in the case $c_1 = 1$, $c_2 = -1$ we have a solution describing annihilation of solitons. The step structure of the N function is not essential: we can consider in the same spirit the case of smooth function N with the same asymptotic behaviour at infinities and arrive to the same conclusions. It is not difficult to consider in a same spirit the generalization for the case $g \neq \delta(\eta)$

Case $\mathcal{E}_0 \neq 0$ is also very interesing but would not be discussed here by the reason of limited volume of the paper.

It seems natural to say a few words about the behaviour of "normalized" solutions on the periodic background. The simplest periodic solution of SIT equations is of the form

$$\mathcal{E} = A \exp\{ikx + i\omega t\},$$

$$\rho = \frac{iA^2 H(\eta)}{(\omega+2\eta)^2 + A^2} \exp\{ikx + i\omega t\},$$

$$N = -\frac{(\omega+2\eta) A H(\eta)}{(\omega+2\eta)^2 + A^2}, \tag{6.8}$$

where

$$k = \left\langle \frac{A H(\eta)}{(\omega+2\eta)^2 + A^2} \right\rangle. \tag{6.9}$$

Normalization condition $|\rho|^2 + N^2 = 1$ produces the restriction

$$H^2(\eta) = 1 + \frac{(\omega+2\eta)^2}{A^2}$$

which may be rewritten in terms of k:

$$k = \left\langle \frac{1}{\sqrt{1+(\omega+2\eta)^2}} \right\rangle$$

Starting solutions of the associated linear system are given by

$$\psi = u \exp\left\{\frac{1}{2}(ikx + i\omega t)\right\},$$

$$\varphi = v \exp\left\{-\frac{1}{2}(ikx + i\omega t)\right\},$$

$$u_z = -i\left(\lambda + \frac{\omega}{2}\right)u - \frac{A}{2}v$$

$$v_z = \frac{A}{2}u + i\left(\lambda + \frac{\omega}{2}\right)u,$$

$$z = t - \frac{A}{2}\left\langle \frac{H}{((\omega+2\eta)^2 + A^2)(\eta - \lambda)} \right\rangle x = t - cx. \tag{5.12}$$

In the case of multiple zero eigenvalue DT of the periodic background solution leads to the formula

$$\mathcal{E}[1] = Ae^{ikx + i\omega t}\left\{1 - \frac{\frac{2}{A} i c_I x - \frac{4}{A^2}}{|t - cx|^2 + \frac{1}{A^2}}\right\}. \tag{6.10}$$

- Similar expressions for ρ and N are omitted for economy of place - .

Solutions of the similar form were firstly discovered for the nonlinear Schrödinger equation - see Its, Salle, Rybin 1987 for the detailed discussion of the spectral nature of such solutions and different applications - . Characteristic feature of the solution (6.10) is that at fixed t it represents some localized formation on the periodic background disappearing it the large time limites.

Higher order - multiple - DT for the starting periodic solution produces a variety of solutions describing the interactions between the "solitons" on the periodic background and the (6.10)-like solutions. The study of those more complicated solutions shall be presented in a separated paper.

We can treat in a same manner nonnormalized solutions with a periodic - starting - background solution.

We also can derive from the formulae of this work some perturbative results in spirit of the work Rybin, Salle (1985). We shall return to this question in the fortcoming publication.

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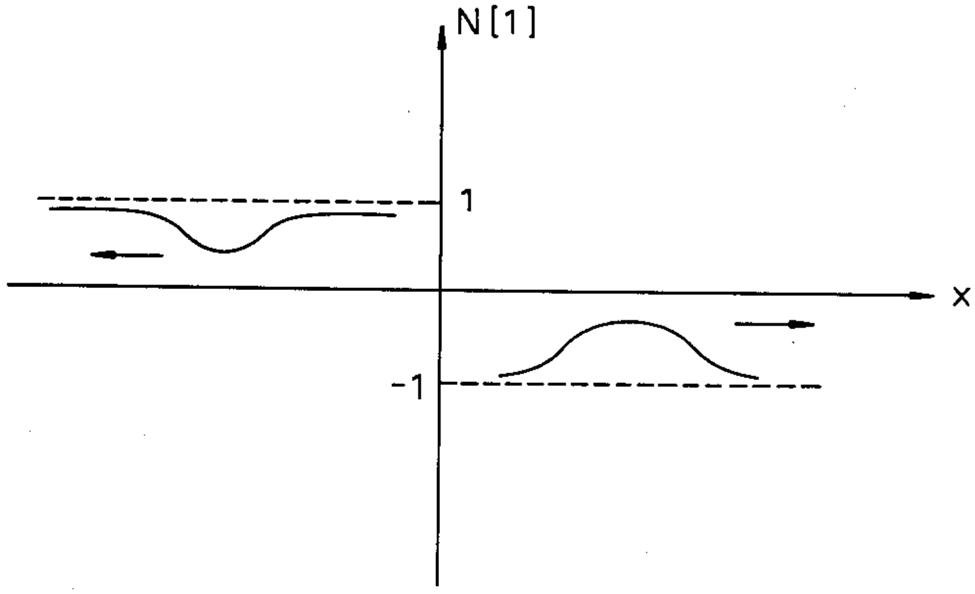
N [1]
1
x
-1