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QUANTUM CHAOS AND GEOMETRY

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Abstract

We present several exact relations between classical and quantum mechanics in a simple ergodic Hamiltonian system: a point particle sliding freely on a surface of constant negative curvature. The classical chaotic behaviour of the system is well understood, and is completely determined by the exponentially proliferating number of periodic geodesics on a compact Riemann surface with two or more handles. The Selberg trace formula leads to a striking duality relation between the quantum mechanical energy spectrum and the lengths of the classical periodic orbits. It constitutes a deep connection between quantum chaos and geometry.

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1. Introduction

While we are celebrating Professor W. Thirring's 60th birthday at this conference, we should also commemorate the 70th birthday of the theory of quantum chaos. Most of you are probably surprised about this latter anniversary, for this seems historically almost impossible, since quantum mechanics was not yet invented in 1917. There only existed the Bohr-Sommerfeld quantization condition. However, by a thorough study of these quantization conditions, Einstein /1/ realised the important role played by what we call today invariant tori on the energy-surface in phase space. ("Man hat sich den Phasenraum jeweilen in eine Anzahl »Trakte« gespalten zu denken, ..." /1/.) For systems which possess invariant tori, Einstein established the most general quantization conditions. But he then made the crucial remark that for ergodic systems, i.e. systems without invariant tori, the whole quantization method of Bohr and Sommerfeld fails. Until its rediscovery by Keller, Gutzwiller and others more than 40 years later, Einstein's paper was totally ignored.

In this talk I shall consider a prototype example of an ergodic system for which one can establish exact relations which are a substitute for the Bohr-Sommerfeld-Einstein quantization rules. These relations have recently been derived in /2/.

The classical dynamics of our prototype example is a Hamiltonian system of two degrees of freedom: a particle with mass m sliding freely on a surface of constant negative curvature. This model was introduced by Hadamard (1898), and is described by the Lagrangian $L(x,\dot{x}) = (m/2)(ds/dt)^2$, $ds^2 = g_{ij}dx^i dx^j$, where g_{ij} is the coordinate-dependent metric tensor of a compact Riemann surface M of genus $g \geq 2$. The energy E = L is the only constant of motion, and the dynamics is the geodesic flow on M, $ds = (2E/m)^{1/2}dt$. There are no invariant tori in phase space, the system has very sensitive dependence on initial conditions (Hadamard), and almost all orbits are dense (Artin /3/). The system has the Anosov property /4/: neighbouring trajectories diverge with time at the rate $\exp \omega t$, i.e. the trajectories are unstable, a typical property of classical chaos. From Jacobi's equation for the geodesic deviation one obtains for the Lyapunov exponent $\omega = (2E/m R^2)^{1/2}$, where $K = -1/R^2$ is the negative Gaussian curvature on M. Pesin's equality /5/, $h = \omega$, relates ω to the Kolmogorov-Sinai entropy h /6/, which in turn determines the exponential proliferation of the closed periodic geodesics on M: $\#\{\gamma : T(\gamma) \leq T\} \sim \exp(hT)/hT$, $T \to \infty$, where γ denotes a primitive periodic orbit on M, and $T(\gamma)$ its period. With $\ell(\gamma) = hT(\gamma)R$ = length of periodic orbit γ with energy E and period $T(\gamma)$, we obtain Huber's law /7/: $\nu(\ell) \equiv \#\{\gamma \in M : \ell(\gamma) \leq \ell\} \sim (R/\ell)\exp(\ell/R)$, $\ell \to \infty$. Thus the length spectrum $\{\ell(\gamma)\}$ on M shows an exponential proliferation of long periodic orbits.

The quantum mechanics of this model is governed by the Hamiltonian $H = (-\hbar^2/2mR^2)\Delta$, where Δ is the Laplacian on M (Laplace-Beltrami operator) $\Delta = \hat{g}^{-1/2}\partial_i(\hat{g}^{1/2} g^{ij}\partial_j)$, $\hat{g} = \det(g_{ij})$, g^{ij} = inverse of g_{ij}, x_i measured in units of R. This model was first studied by Gutzwiller /8/, who discovered the relation of his semiclassical trace formula /9/ to the rigorous Selberg trace formula /10/. The latter formula is the mathematical basis of our work /2/. It constitutes a very deep connection between quantum chaos and geometry.

2. The relation between classical and quantum mechanics in a chaotic system

A compact Riemann surface M of genus $g \geq 2$ can be identified with U/Γ, the action of a Fuchsian group Γ on the upper half-plane $\mathcal{U} = \{z = x + iy : x \in R, y > 0\}$ endowed with the (conformal) Poincaré metric $ds^2 = y^{-2}(dx^2+dy^2)$. This is the classical model for hyperbolic geometry of constant negative curvature, $K = -1$. Γ is a discrete subgroup of $PSL(2,R) = SL(2,R)/\{\pm 1\}$, the group of Möbius transformations. From the Gauß-Bonnet theorem we infer $K \cdot A = 2\pi\,\chi = 4\pi(1-g)$, where A denotes the area of M and χ its Euler characteristic, i.e. $A = 4\pi(g-1)$. In the Poincaré metric the Laplacian on M is given by $\Delta = y^2(\partial^2/\partial x^2 + \partial^2/\partial y^2)$, and the Schrödinger equation reads $-\Delta\psi_n = E_n\psi_n$. (For a surface of arbitrary constant negative curvature, $K = -1/R^2$, the energy eigenvalues scale as $E_n = (\hbar^2/2mR^2)\lambda_n$, where λ_n is independent of $\hbar$, m and R. In the following we set $\hbar = 2m = R = 1$). The wavefunctions on M have to satisfy periodic boundary conditions which are realized as follows: one considers a fundamental region $F \subset U$ for the group Γ, i.e. a connected subset of U whose images under Γ are a tiling of U. For genus g, F has the form of a hyperbolic polygon of 4g sides. If the sides of F are identified in pairs according to the action of Γ, we have a realization of M. For the wavefunctions, the boundary condition implies $\psi_n(\gamma z) = \psi_n(z)\ \forall \gamma \in \Gamma$ (automorphic functions), $\psi_n(z) \in L_2(F)$, where the integration measure in F is $dA = dxdy/y^2$. Mathematically, the problem is now reduced to harmonic analysis of homogeneous spaces and discontinuous groups /10-13/. The spectrum of $H = -\Delta$ on M is discrete and real, $0 = E_0 < E_1 \leq E_2 \leq \ldots$, where the zero mode ($E_0 = 0$) belongs to a constant wavefunction. One has Weyl's law /14/: $\#\{E_n \leq E\} \sim (A/4\pi)E$ asymptotically. (For a comprehensive review of the "chaos on the pseudosphere", see /15/.)

The basic relation of spectral geometry is the Selberg trace formula on M /10-12/

$$\sum_{n=0}^{\infty} h(p_n) = \frac{A}{2\pi}\int_0^{\infty} dp\, p \tanh \pi p\, h(p) + \sum_{\{\gamma\}}\sum_{n=1}^{\infty} \frac{l(\gamma)\, g(n l(\gamma))}{2\sinh \frac{n l(\gamma)}{2}} \tag{1}$$

which is the non-commutative analogue of the classical Poisson summation formula. Here all series and the integral converge absolutely under the following conditions on the function h(p): i) $h(-p) = h(p)$, ii) h(p) is holomorphic in a strip $|\operatorname{Im} p| \leq \frac{1}{2}+\varepsilon$, $\varepsilon > 0$, iii) $|h(p)| \leq a(1+|p|^2)^{-2-\varepsilon}$, $a > 0$. The function g(x) is the Fourier transform of h(p)

$$g(x) = \frac{1}{\pi}\int_0^{\infty} dp \cos px\, h(p). \tag{2}$$

On the left-hand side of (1) the sum runs over the eigenvalues of H parametrized by the momentum p: $E_n = \frac{1}{4} + p_n^2$ with $p_n \geq 0$ for $E_n \geq \frac{1}{4}$, and p_n purely imaginary for $0 \leq E_n < \frac{1}{4}$. The first term on the right-hand side of (1) is the "zero length contribution" (free motion on U), and the last term is a sum over the length spectrum of M (primitive conjugacy classes in Γ).

The trace formula (1) is the only known exact substitute for the Bohr-Sommerfeld-Einstein quantization rules for a chaotic system. It establishes a striking duality relation between the quantum mechanical energy spectrum and the lengths of the classical closed periodic orbits.

To illustrate the physical significance of (1), we calculate /2/ the trace of the regularized resolvent of H on M, $\mathrm{Tr}(E-H)^{-1}_{reg}$, $(E = s(1-s))$

$$\frac{1}{E} + \sum_{n=1}^{\infty}\left[\frac{1}{E-E_n} + \frac{1}{E_n}\right] = \gamma_\Delta + 2(g-1)\psi(s) - \frac{1}{2s-1}\frac{Z'(s)}{Z(s)}. \tag{3}$$

Here the sum over the classical periodic orbits has been expressed in terms of the Selberg zeta function on M

$$Z(s) \equiv \prod_{\{\gamma\}}\prod_{n=0}^{\infty}\left[1 - e^{-(s+n)l(\gamma)}\right]. \tag{4}$$

$\psi(s)$ is the digamma function, and γ_Δ denotes the generalized Euler constant of the Laplacian on M ($\gamma = 0.5722\ldots$ = Euler's constant)

$$\gamma_\Delta \equiv 2(g-1)\gamma - 1 + \frac{1}{2}\frac{Z''(1)}{Z'(1)}. \tag{5}$$

Notice that the zero mode had to be treated separately (infrared problem), and that the sum over the eigenvalues cannot be broken up, otherwise convergence is lost. (H^{-1} is not of trace class; ultraviolet problem. The relation given in /8/, /11/ is wrong.) The sum rule (3) extends meromorphically to all $s \in \mathbb{C}$, and we infer that Z(s) is an entire function of s of order 2 with "trivial" zeros at $s = 1-k$, $k \in N_0$. Apart from a finite number of zeros on the real line between 0 and 1 (corresponding to eigenvalues $E_n \leq \frac{1}{4}$), the "non-trivial" zeros of Z(s) are located at $s = \frac{1}{2} \pm i p_n$ (corresponding to $E_n > \frac{1}{4}$), i.e. on the critical line $\text{Re}\, s = \frac{1}{2}$. $Z'(s)/Z(s)$ has a Laurent expansion near s = 1 /2/ with a simple pole at s = 1 with residue 1. From this we can deduce the asymptotic behaviour of the length spectrum $\{\ell(\gamma)\}$ on M, i.e. Huber's law. The latter ensures the convergence of (4) for $\text{Re}\, s > 1$.

From (3) we obtain for $E > \frac{1}{4}$ $(s=\frac{1}{2}+\varepsilon-ip,\ p=\sqrt{E-\frac{1}{4}},\ \varepsilon>0)$ the spectral density

$$d(E) \equiv \sum_{n=0}^{\infty} \delta(E-E_n) = \frac{A}{4\pi} \tanh \pi p + \frac{1}{2\pi p} \lim_{\varepsilon \to 0} \text{Im} \left\{ i \frac{Z'(\frac{1}{2}+\varepsilon-ip)}{Z(\frac{1}{2}+\varepsilon-ip)} \right\}. \tag{6}$$

The first term (zero length contribution) gives the improved Weyl's law for the spectral staircase

$$\langle N(E) \rangle \equiv \int_{1/4}^{E} dE' \frac{A}{4\pi} \tanh\left(\pi \sqrt{E'-\tfrac{1}{4}}\right)$$

$$= \frac{A}{4\pi}\left(E - \frac{1}{3}\right) + \frac{A}{2\pi^2}\sqrt{E}\, e^{-2\pi\sqrt{E}} + O\left(\frac{e^{-2\pi\sqrt{E}}}{\sqrt{E}}\right). \tag{7}$$

Unfortunately, the contribution from the periodic orbits in (6) requires an analytic continuation of Z(s) to the line $\text{Re}\, s = \frac{1}{2}$, $\text{Im}\, s < 0$, which at present we do not know how to perform. To get an explicit relation, we define a smeared spectral density with a real smearing parameter $\sigma > 1/2$

$$\bar{d}(E) \equiv \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{\sigma}{(E-E_n)^2 + \sigma^2}. \tag{8}$$

(For a similar procedure, employed some time ago in nuclear physics and QCD, see /16/.) We then obtain from (3) and (4) $(E > \frac{1}{4},\ \sigma > \frac{1}{2})$

$$\bar{d}(E) = -\frac{A}{2\pi^2} \text{Im}\, \psi\left(\tfrac{1}{2}+\sigma-ip\right) + \frac{1}{4\pi(p^2+\sigma^2)} \sum_{\{\gamma\}} \sum_{n=1}^{\infty} \frac{\ell(\gamma)\, e^{-\sigma n \ell(\gamma)}}{\sinh \frac{n\ell(\gamma)}{2}} \left[p \cos(pn\ell(\gamma)) + \sigma \sin(pn\ell(\gamma)) \right]. \tag{9}$$

(The first term has a simple expression for $\sigma = 0, \frac{1}{2}, 1, \dots$.) Eq. (9) is an exact representation of the smeared spectral density as a sum over all the periodic orbits of the classical system. The last term of (9) can be rewritten in the suggestive form $(p \gg \sigma)$

$$\bar{d}_{osc}(E) = \sum_{\{\gamma\}} \sum_{n=-\infty}^{\infty}{}' \bar{A}_n\, e^{i \bar{S}_n}$$

$$\bar{A}_n = \frac{p}{8\pi(p^2+\sigma^2)} \frac{\ell(\gamma)\, e^{-\sigma |n| \ell(\gamma)}}{\sinh \frac{|n|\ell(\gamma)}{2}}, \quad \bar{S}_n = n\left(p\ell(\gamma) - \frac{\sigma}{\ln|p|}\right). \tag{10}$$

The number n counts the multiple traversals, where $n < 0$ corresponds to traversals backwards in time. The amplitudes $\bar{A}_n$ decrease exponentially with $\ell(\gamma)$ which is typical for a chaotic system (in contrast to an integrable one, where one has a power-law). This exponential decrease is crucial for the finiteness of (10), because it compensates the exponential proliferation of very long orbits according to Huber's law. Notice that this compensation breaks down for $\sigma \leq \frac{1}{2}$! To our knowledge, this is the first time that an exact periodic orbit sum for a chaotic system has been derived and for which the abscissa of convergence is exactly known. The beautiful semiclassical periodic orbit sums discussed recently (see e.g. Berry /17/) correspond to the limit $\sigma \to 0$ in (9) or (10), and therefore are in general not expected to be convergent.

3. Can one hear the shape of a compact Riemann surface?

This is a variation of the famous question posed by M. Kac /18/. To answer it, we need the trace of the heat kernel on M. One finds /2/

$$\text{Tr}\, e^{-\frac{t}{\hbar}H} = \sum_{n=0}^{\infty} e^{-\frac{E_n}{\hbar}t} = AR\left(\frac{m}{2\pi\hbar t}\right)^{3/2} \int_0^{\infty} dx \frac{x}{\sinh\left(\frac{x}{2R}\right)} e^{-\frac{m}{2\hbar t}x^2 - \frac{\Delta V}{\hbar}t}$$

$$+ \frac{1}{2}\left(\frac{m}{2\pi\hbar t}\right)^{1/2} \sum_{\{\gamma\}} \sum_{n=1}^{\infty} \frac{\ell(\gamma)}{\sinh\left(\frac{n\ell(\gamma)}{2R}\right)} e^{-\frac{m}{2\hbar t}(n\ell(\gamma))^2 - \frac{\Delta V}{\hbar}t} \tag{11}$$

where $\Delta V \equiv \hbar^2/8mR^2$ is a quantum correction which naturally arises also in an exact path integral treatment /19/ of the free motion on the Poincaré upper half-plane. Since the closed orbit contribution in (11) vanishes exponentially for $t \to 0+$, the small-t behaviour is completely determined by the "zero length term", and is explicitly given by the asymptotic expansion as $t \to 0$

$$\sum_{n=0}^{\infty} e^{-\frac{E_n}{\hbar}t} = \frac{mAR^2}{2\pi\hbar t} \sum_{n=0}^{N} b_n \left(\frac{\hbar t}{2mR^2}\right)^n + O(t^N)$$

$$b_0 = 1,\ b_n = \frac{(-1)^n}{2^{2n} n!}\left[1 + 2\sum_{k=1}^{n} \binom{n}{k} (2^{2k-1}-1)|B_{2k}|\right],\ n \in N \tag{12}$$

where B_{2k} are the Bernoulli numbers. Thus one can hear the area and the Euler characteristic of M (see also /20/).

4. Summary

As illustrative examples, some exact relations between the quantum mechanical energy spectrum and the lengths of the classical periodic orbits have been presented for a simple chaotic Hamiltonian system. (More relations can be found in /2/.) Relations as eq. (3) are of a non-perturbative nature and can be compared with e.g. the well-known Källén-Lehmann representation in quantum field theory. The "closed-orbit sums" (as eq. (9)) instead have the character of a perturbation expansion ("loop expansion") and are in general only convergent in a finite region. (Compare e.g. with the QCD-example discussed by Poggio, Quinn and Weinberg /16/.) In this talk we have concentrated on the energy spectrum. For a discussion of the wavefunctions we refer to the work of Pignataro and Wightman /21/.

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