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Improved Continuum Limit Lattice Action for Quarks

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ABSTRACT. Symanzik's improvement programme is pursued for Wilson fermions using the concept of "on shell" improvement. Twisted antiperiodic boundary conditions for lattice fermions are introduced. The energy values and wave functions of quarks in the twisted world are considered to one loop order perturbation theory. The coefficient $c(g_0^2)$ of the proposed O(a)-"on shell" -improved action is calculated to first nontrivial order by considering a suitable "on shell" quantity.

1. INTRODUCTION

In lattice gauge theory a gauge invariant ultraviolet cutoff is introduced through the finite lattice spacing 'a'. Consequently all physical amplitudes become cutoff-dependent.
From the work of Symanzik [1] one learns that for perturbative calculations the following expansion holds for lattice Feynman diagrams

$$D \underset{a\to 0}{\sim} a^{-\omega} \sum_{n=0}^{\infty} \sum_{m=0}^{l} c_{nm} a^n (\log a)^m \tag{1.1}$$

where $\omega \geq 0$ depends on the superficial degree of divergence of the diagram and l is the number of loops contained in it.
From this formula one sees that if after renormalization the lattice spacing 'a' is decreased then the corrections to the continuum limit terms only die off relatively slowly. It is therefore worthwhile thinking about how to improve the continuum limit behaviour of the lattice approximation.
To this end two procedures have been proposed. One is due to Wilson and is basically non-perturbative [2]. The other approach given by Symanzik uses weak coupling perturbational methods [3].
Symanzik's method employs the fact that a lattice action for a given continuum theory is not unique, i.e. various lattice actions yielding the same continuum limit exist. Given one lattice action for a certain continuum theory other lattice actions with the same continuum limit can be produced by adding operators obeying the symmetries of the theory and containing higher powers of the cutoff.
The main idea of Symanzik's improvement programme is to fine tune the coefficients of these operators in such a way as to eliminate the cutoff dependency up to a given order in the lattice spacing. The values of these coefficients can be determined perturbatively or experimentally via Monte Carlo simulations.
The action obtained by pursuing this programme is universal in the sense that all physical quantities calculated with it are supposed to be improved. Whether this is true in the framework of lattice gauge theory is however not yet clear. Symanzik himself was able to show however that this was indeed possible for the Φ^4-theory [1] and for the nonlinear σ-model [4].
For the nonlinear σ-model Monte Carlo simulations employing this action yielded encouraging results [5,6].
Despite of the conceptual problems still pending much effort was invested in extending Symanzik's programme to lattice gauge theory.
This was first pursued for pure Yang Mills theory in a paper by Weisz [7] dealing mainly with the general form of the action and tree level improvement.
He found that at most three next to nearest neighbour terms have to be added to the standard one plaquette Wilson action to perform $O(a^2)$ improvement and by considering "classical" improvement found for the tree level values of their coefficients

$$c_0^{(0)} = \frac{5}{3}, \qquad c_1^{(0)} = -\frac{1}{12}, \qquad c_2^{(0)} = c_3^{(0)} = 0 \tag{1.2}$$

where

$$c_i(g_0^2) = \sum_{\nu=0}^{\infty} g_0^{2\nu} c_i^{(\nu)}. \tag{1.3}$$

The next step was an attempt to calculate these coefficients to one loop order perturbation theory. This calculation was performed by Weisz and the author by considering the static quark-antiquark potential [8]. It turned out however that this quantity only yielded information on the linear combination of the three coefficients viz for the standard $SU(N)$ Wilson action

$$c_1^{(1)} - c_2^{(1)} - c_3^{(1)} =$$

$$-N(0.0012167(1)) + \frac{(2N^2-3)}{12N}(c_0^{(0)}(0.0305219) + c_1^{(0)}(1.1043780)). \quad (1.4)$$

The last steps to complete the treatment of Symanzik's improvement programme for pure Yang Mills theory were then made by Lüscher and Weisz. In a sequence of papers [9-12] they presented both theoretical contributions and numerical calculations.
As an important conceptual tool they introduced the concept of "on shell" improvement. Instead of demanding the improvement of all Greens functions this means improvement of only all "on shell" quantities, i.e. low lying (with momenta small compared to the cutoff) energy levels, S-matrix elements etc.
By studying the behaviour of the action under transformations that leave "on shell" quantities unchanged it is possible to distinguish between two kinds of operators. If the coefficient of an operator changes under such a spectrum conserving transformation of the action then its value has no effect on "on shell" quantities. Operators with this property are called redundant, their coefficients are arbitrary and can be chosen for convenience. This property is valid to all orders of perturbation theory.
It is therefore only the coefficients of operators that remain unchanged under spectrum conserving transformations that have to be determined perturbatively or via Monte Carlo.
In the case of pure Yang Mills theory this means that one of the coefficients was found to be redundant. It was then set equal to zero for convenience so that after a change of notation compared to [7,8] i.e. $c_2 \leftrightarrow c_3$ only the remaining two coefficients c_1 and c_2 had to be determined.
The main numerical result presented by Lüscher and Weisz is the calculation of these coefficients to first nontrivial order perturbation theory, verifying also the result given in [8] by means of a completely independent calculation.
For this calculation they developed several efficient and useful numerical techniques [12], which also have been employed for the work presented here.
One is a highly efficient integration routine for periodic functions ensuring exponential convergence by equidistant descretisation, which in addition also gives a reliable estimate for the error of the numerical integration.
The second important method is to compactify two space-dimensions by introducing twisted periodic boundary conditions thus providing an explicit infrared cutoff and giving complete control of the zero modes. The theory then has a mass gap and the lowest lying stable particles can be expressed by gauge invariant operators. It is therefore possible to consider gluon-gluon scattering and a wealth of "on shell" quantities for the calculation of the improvement coefficients becomes available.
By employing this concept it was possible to obtain the second relation needed for the determination of $c_1^{(1)}$ and $c_2^{(1)}$ by considering an appropriate gluon-three-point-function.
The next step in the application of Symanzik's improvement programme was the inclusion of lattice fermions.
This problem is relevant especially because of the well known problem of species doubling

which can only be rendered completely by the sacrifice of chiral symmetry. As this, at least in the framework of the fermion lattice action proposed by Wilson [13], is already performed by adding an operator containing a higher power of the cutoff it is quite suggestive to try to soften the breaking of chiral symmetry with an improved action. This could ultimately lead to a quicker regain of chiral symmetry when taking the continuum limit.
Although some early attempts [14-16] had been made in that direction no systematic treatment using especially the fruitful concept of "on shell" -improvement had been given.
The first steps in that direction were made by B. Sheikholeslami and the author [17].
In [17] the structure of the action suitable to perform improvement for Wilson lattice fermions was studied and tree level improvement was carried out.
In the work presented here the improvement coefficient emerging in the scheme of O(a)-"on shell" improvement is calculated to one loop order perturbation theory.
To this end the results obtained in [17] are summarized in Section 2.
The remainder of the paper is then organized as follows.
In Section 3 the notations used are given.
Section 4 deals with the extension of the concept of twisted boundary conditions introduced in [12] to lattice fermions.
In Section 5 the propagator and vertices emerging from the action quoted in Section 2 are elaborated for quarks with antiperiodic twisted boundary conditions.
In Section 6 the problem of the spectrum and stable particles is discussed.
The purpose of Section 7 is to find a suitable "on shell" quantity to yield the necessary improvement condition to one loop order. To this end a quantity is studied that yields the tree level improvement condition.
Section 8 deals with the fermion energy and the fermion wave functions to one loop order.
In Section 9 the definition of the quantity for which improvement will be demanded is given and the Feynman diagrams contributing to its calculation are depicted.
Section 10 deals with the integration routines used to perform the lattice integrals in the continuous directions.
In Section 11 the technical aspects of the calculation are discussed.
In Section 12 the results are summarized and discussed, the final result of the computation is presented and possibilities to extend the calculation to two loop order are proposed.
Further technical and conceptual aspects of the calculation are presented in the Appendices A,B.

2. "On Shell" Improvement for Wilson Fermions

Here the main results obtained by B. Sheikholeslami and the author are summarized because they are the basis for the work presented in the following Sections.
While in pure Yang Mills Theory only lattice artifacts with even powers of the cutoff appear (i.e. $O(a^2)$,$O(a^4)$ ... corrections to the continuum limit) in lattice QCD with Wilson fermions also odd powers of the cutoff emerge. As stressed in [17] for lattice fermions two different approaches of improvement are possible. In the minimal approach only the especially disturbing O(a) terms are intended to be cancelled. This concept will be called O(a) improvement in the following. More ambitious is the task to additionally remove the $O(a^2)$ corrections to the continuum limit which will be called $O(a^2)$ improvement. For both O(a) and $O(a^2)$ improvement the most general form of the lattice fermion action was derived. This amounts to the construction of an action containing (up to total derivatives) all possible gauge-invariant scalar operators of at most dimension five for O(a) and six for $O(a^2)$ improvement that are invariant under discrete rotations, parity and charge conjugation transformations.
For $O(a^2)$ improvement one ends up with 18 admissable operators. Considering the isospectral transformation of the action only five of them are found to be redundant. Although to lowest order classical improvement is compatible with "on shell" improvement and allows for a considerable simplification of the action the coefficients of the remaining non redundant operators of higher dimensions remain to be calculated in higher orders of perturbation theory. Especially the disturbing four fermion contact terms cannot be excluded. As moreover for the gluon sector the $O(a^2)$ improved action has to be used the approach of $O(a^2)$ improvement for Wilson fermions is impractable and must be abandoned.
If one however only pursues O(a) improvement the situation looks favourable. For the gluon sector the standard one plaquette Wilson action can be used, because it only produces $O(a^2)$ deviations from the continuum limit which are not intended to be cancelled. The action is drastically simplified and contains only two admissable operators of dimension five in addition to the lowest order operators of dimension less or equal to four.
Consideration of the isospectral transformation of the action shows that only one of them is non-redundant i.e. its coefficient has to be calculated perturbatively. The redundant operator can however be used to break chiral symmetry to lowest order and thus avoid species doubling. At first sight it is surprising that the term which removes the unwanted additional low lying energy states at the edges of the Brillouin zone gives no effects when considering a spectrum conserving transformation. It must be noted however that the concept of "on shell" improvement only applies for all momenta small compared to the cutoff i.e. not for the large lattice momenta where species doubling arises.
One ends up with the standard Wilson lattice fermion action plus one additional term which can still be formulated using only next neighbour interactions

$$S = \frac{a^4}{g_0^2} \sum_{\substack{x \text{ all} \\ \text{lattice sites}}} \frac{1}{2a}\Big\{-2[M_0 a + 4 - g_0^2 b_0(g_0^2)]\bar{\Psi}(x)\Psi(x)$$
$$+\sum_\mu \bar{\Psi}(x)\left[(r-\gamma_\mu)U_\mu(x)\Psi(x+\hat{\mu}) + (r+\gamma_\mu)U_\mu^+(x-\hat{\mu})\Psi(x-\hat{\mu})\right] \qquad (2.1)$$
$$-\frac{i}{2}c(g_0^2)\sum_{\mu,\nu}\bar{\Psi}(x)\sigma_{\mu\nu}P_{\mu\nu}(x)\Psi(x)\Big\}.$$

Where

$$P_{\mu\nu}(x) = -\tfrac{1}{4}\big[U_\nu(x)U_\mu(x+\hat{\nu})U_\nu^+(x+\hat{\mu})U_\mu^+(x)$$
$$+U_\nu^+(x-\hat{\nu})U_\mu^+(x-\hat{\mu}-\hat{\nu})U_\nu(x-\hat{\mu}-\hat{\nu})U_\mu(x-\hat{\mu})$$
$$-U_\nu(x)U_\mu^+(x-\hat{\mu}+\hat{\nu})U_\nu^+(x-\hat{\mu})U_\mu(x-\hat{\mu}) \qquad (2.2)$$
$$-U_\nu^+(x-\hat{\nu})U_\mu(x-\hat{\nu})U_\nu(x-\hat{\nu}+\hat{\mu})U_\mu^+(x)\big]$$

is a selfadjoint version of (2.9) in [17] and the shorthand notation $\bar{\Psi}(x)$,$\Psi(x)$ for the N_C row and column matrices of the fermions transforming under the fundamental representation of $SU(N_C)$ and $U_\mu(x)$ for the $SU(N_C)$ matrix of the parallel transporter has been used. Furthermore a mass-subtraction term $b_0(g_0^2)$ has made its appearance. As is well known and will be discussed in detail later, this term is necessary in order to remove unwanted $O(\frac{1}{a})$-effects in perturbation theory for fermions of zero bare mass.
The coefficient

$$c(g_0^2) = \sum_{\nu=0}^{\infty} c^{(\nu)} g_0^{2\nu} \qquad (2.3)$$

has to be calculated order by order in perturbation theory by imposing suitable "on shell" improvement conditions.
To lowest order [17] one has

$$c^{(0)} = r \qquad (2.4)$$

where r is the coefficient of the chirality breaking term introduced by Wilson.
As explained above "on shell" improvement cannot be used to determine suitable values of r because it only puts the constraint (2.4).
To this purpose the discussion of the energy momentum relation of the lattice fermions as given in [17] proves useful.
For improved actions these considerations have been first made by Lüscher for a free scalar field [18].
For the action (2.1) the following observations are made. Consider the free massless theory.
The one-particle energy states are related to the poles of the time Fourier transformed propagator. They are given by

$$E_i = \log z_i$$

where z_i are the poles inside the unit circle.
The main effect that arises is the appearance of "unphysical" energy states besides the "physical" ones which approximate the continuum energy momentum relation

$$E^2(\bar{p}) = \bar{p}^2 \qquad (\bar{p} = (p_1, p_2, p_3)).$$

For $r = 0$ there are two energy states, one real and one complex, whose real parts coincide. As a consequence of chiral symmetry there are additional zeroes of the energy at the edges of the Brillouin zone.
In the case $r \neq 0$ the "unphysical" second energy state has a small-$\bar{p}$ expansion of the form

$$E(\bar{p}) = \frac{1}{a}\log\frac{(r+1)}{(r-1)} + \frac{1}{4}a\frac{(\lambda^2+1)}{\lambda}\bar{p}^2 + O(a^3). \qquad (2.5)$$

One sees that for $r < 1$ the solution is complex but its real part is shifted to values above the "physical" energy values as r approaches 1.

For $r = 1$ this solution becomes infinite corresponding to a pole of the propagator at zero while for $r > 1$ $E(\vec{p})$ is real and for $r \gg 1$ again tends towards the "physical" energy momentum relation.
The important observation is that for $r = 1$ this unwanted state is completely removed, so this is the natural choice for the coefficient of the chirality breaking term in the action. Another merit of this choice of r is that in the hopping parameter expansion no backtracking is encountered. The calculation presented in the following is thus done for

$$r = 1 \quad \text{and} \quad c^{(0)} = 1 \tag{2.5}$$

3. Notations

Throughout this paper a four dimensional hypercubic lattice is considered with the lattice spacing 'a' set equal to one so lattice sites correspond to points in $\mathbb{Z}^4$. If necessary 'a' can always be reintroduced through dimensional analysis.
$\hat{\mu}$ denotes the unit vector in the μ-direction.
For the parallel transporter $U_\mu(x)$ the choice

$$U_\mu(x) = e^{g_0 A_\mu(x)}$$

with antihermitian gauge fields A_μ belonging to the fundamental representation of $SU(N)$ is made.
For the fermion sector a set of euclidean γ-matrices with the properties

$$\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}\mathbb{1}$$
$$\gamma_\mu^+ = \gamma_\mu$$
$$\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$$

is introduced.
When performing explicit calculations the following representation was chosen

$$\gamma_0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \qquad \gamma_j = i\begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix} \qquad j = 1,2,3$$

where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

For lattice momenta the common notations $\hat{k}_\mu = 2\sin\frac{k_\mu}{2}$ and

$$\hat{k}^2 = \sum_\mu \hat{k}_\mu^2$$
$$\hat{k}^4 = \sum_\mu \hat{k}_\mu^4$$

will often be used.

4. Fermions with antiperiodic twisted Boundary Conditions on a Lattice with two compact Dimensions

The concept of twisted periodic boundary conditions for gluons has been discussed in detail in [12]. Here the modifications necessary to formulate this concept for fermion fields are elaborated.
As in [12] consider a lattice with finite size of L lattice spacings in the x_1- and x_2-directions. Winding once around the torus in these compact directions the gluon fields U_μ transform like

$$U_\mu(x + L\hat{\nu}) = \Omega_\nu U_\mu(x)\Omega_\nu^{-1} \qquad \nu = 1,2. \tag{4.1}$$

In this expression the Ω_ν are constant $SU(N)$-matrices with the following properties

$$\begin{aligned} \Omega_1\Omega_2 &= z\Omega_2\Omega_1 \qquad & z &= e^{i\frac{2\pi}{N}} \\ \Omega_\nu^N &= (-1)^{N-1}\mathbb{1} \qquad & \nu &= 1,2. \end{aligned} \tag{4.2}$$

The corresponding gauge group G_Ω consists of all fields $\Lambda(x) \epsilon SU(N_C)$ with

$$\Lambda(x + L\hat{\nu}) = \Omega_\nu \Lambda(x)\Omega_\nu^{-1} \tag{4.3}$$

and $U_\mu(x)$ transforms according to

$$U_\mu(x) \to \Lambda(x)U_\mu(x)\Lambda^{-1}(x + \hat{\mu}). \tag{4.4}$$

To formulate twisted boundary conditions consistently for quark fields one introduces in addition to the colour group $SU(N_C)$ a so called "smell group" $SU(N_S)$ with $N_S = N_C = N$. This idea was first developed by Parisi [19] and its practical use has been pointed out to the author by Lüscher. Omitting spinor and flavour indices the quark fields are then $N_C \times N_S = N \times N$-matrices $\Psi(x) = \Psi_{cs}(x)$ with e.g.

$$\sum_{c,s} \bar{\Psi}_{sc}(x)\Psi_{cs}(x) = Tr(\bar{\Psi}(x)\Psi(x)).$$

Now an analogous condition to (4.1) can be formulated

$$\Psi(x + L\hat{\nu}) = \Omega_\nu \Psi(x)\Omega_\nu^{-1}\epsilon^{i\frac{\pi}{N}} \qquad \nu = 1,2. \tag{4.5}$$

The additional factor $\epsilon^{i\frac{\pi}{N}}$ guarantees the antiperiodicity thus suppressing even multiples of the minimal momenta (especially the zero modes) because of

$$\Psi(x + NL\hat{\nu}) = -\Psi(x) \qquad \nu = 1,2. \tag{4.6}$$

The gauge transformations of $\Psi(x)$ and $\bar{\Psi}(x)$ under elements of G_Ω are

$$\Psi(x) \to \Lambda(x)\Psi(x) \tag{4.7}$$
$$\bar{\Psi}(x) \to \bar{\Psi}(x)\Lambda^{-1}(x) \tag{4.8}$$

A basis of plane waves is given by

$$\Gamma_p e^{ipx}. \tag{4.9}$$

Antiperiodicity puts the following constraint on the momenta

$$\Gamma_p = -e^{ip_i LN}\Gamma_p \qquad i = 1,2. \tag{4.10}$$

Thus

$$p_i = \frac{\pi}{LN}(2\nu_i + 1) \qquad \nu_1, \nu_2 \epsilon \mathbf{Z}. \tag{4.11}$$

The Γ_p are unique up to a phase. As in the gluon case it is chosen to give

$$\Gamma_p = z^{\frac{1}{2}(\nu_1+\nu_2)(\nu_1+\nu_2-1)}\Omega_1^{-\nu_2}\Omega_2^{\nu_1}. \tag{4.12}$$

Define

$$\int\!\!\!\!\!\!\sum_p = (L^2N)^{-1}\sum_{p_\perp}\iint \frac{dp_0}{2\pi}\frac{dp_3}{2\pi} \tag{4.13}$$

with

$$p = (p_0, p_\perp, p_3), \qquad -\pi \le p_0, p_3 \le \pi$$
$$p_\perp = (p_1, p_2) = \frac{\pi}{LN}(2\nu_1 + 1, 2\nu_2 + 1), \qquad \nu_1, \nu_2 \epsilon [0, LN-1] \cap \mathbf{Z}.$$

Then the relations

$$\Psi(x) = \int\!\!\!\!\!\!\sum_p e^{ipx}\Gamma_p\tilde{\Psi}(p) \tag{4.14}$$

and

$$\tilde{\Psi}(p) = \sum_x e^{-ipx} Tr\{\Gamma_p^+\Psi(x)\} \tag{4.15}$$

for the inverse hold.
Apart from spinor and flavour indices $\tilde{\Psi}(p)$ is a scalar quantity because the colour and smell group structure is absorbed in the Γ_p.
The analogous formulae for $\bar{\Psi}$ are

$$\bar{\Psi}(x) = \int\!\!\!\!\!\!\sum_p e^{-ipx}\Gamma_p^{(+}\tilde{\bar{\Psi}}(p) \tag{4.16}$$

$$\tilde{\bar{\Psi}}(p) = \sum_x e^{ipx} Tr\{\Gamma_p\bar{\Psi}(x)\}. \tag{4.17}$$

For quark momenta p, p'[1]

$$p_\perp = \frac{\pi}{LN}(2\nu_1 + 1, 2\nu_2 + 1) \qquad \nu_i \epsilon \mathbf{Z}$$
$$p'_\perp = \frac{\pi}{LN}(2\nu'_1 + 1, 2\nu'_2 + 1) \qquad \nu'_i \epsilon \mathbf{Z}$$

as for gluon momenta the symmetric and antisymmetric products

$$(p, p') = \nu_1\nu'_1 + \nu_2\nu'_2 + (\nu_1 + \nu_2)(\nu'_1 + \nu'_2) \tag{4.18}$$
$$\langle p, p'\rangle = \nu_1\nu'_2 - \nu_2\nu'_1 \tag{4.19}$$

are introduced.
Using these expressions one defines

$$z(p, p') = z^{\frac{1}{2}((p,p')-(p,p'))}. \tag{4.20}$$

If k, k' denote gluon momenta [12] with

$$k_\perp = \frac{2\pi}{LN}(\mu_1, \mu_2), \qquad k'_\perp = \frac{2\pi}{LN}(\mu'_1, \mu'_2) \qquad \mu_i, \mu'_i \epsilon \mathbf{Z}$$

one finds for the products of momenta occurring in Feynman diagrams

$$\begin{aligned} \Gamma_k\Gamma_{k'} &= z(k, k')\Gamma_{k+k'} \\ \Gamma_k\Gamma_p &= z(k, p)\Gamma_{k+p} && (k + p \text{ is a fermion momentum}) \\ \Gamma_p^+\Gamma_{p'} &= z^{-\frac{1}{2}(p,p')}z(p, p')^{-1}\Gamma_{p'-p} && (p' - p \text{ is a gluon momentum}). \end{aligned} \tag{4.21}$$

5. Propagator and Vertices of the O(a)-"On Shell" -Improved Lattice Action for Quarks with antiperiodic twisted Boundary Conditions

Using the Ansatz (2.1) with $r = c = 1$ and taking into account the twisted boundary conditions for the quark and gluon fields the quark propagator and the quark-gluon vertices are given by

$$g_o^2\tilde{S} =$$
$$\int\!\!\!\!\!\!\sum_p \tilde{\bar{\Psi}}(p)(S_F^{-1}(p) - g_0^2 b_0(g_0^2))\tilde{\Psi}(p) + \sum_{r=1}^{\infty}\frac{1}{r!}g_o^r \int\!\!\!\!\!\!\sum_{p_1}\int\!\!\!\!\!\!\sum_{p_2}\int\!\!\!\!\!\!\sum_{k_1}\cdots\int\!\!\!\!\!\!\sum_{k_r}\sum_{\mu_1}\cdots\sum_{\mu_r}\tilde{A}_{\mu_1}(k_1)\ldots\tilde{A}_{\mu_r}(k_r)$$
$$\cdot\,\delta(p_1, p_2 + \sum_{i=1}^{r}k_i)\tilde{\bar{\Psi}}(p_1)V_r^{FG}(p_1, p_2; k_1, \mu_1; \ldots; k_r, \mu_r)\tilde{\Psi}(p_2). \tag{5.1}$$

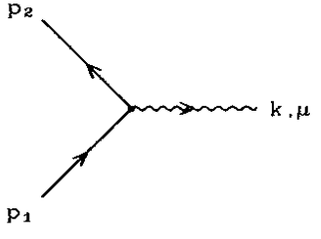
[1] quark momenta are always denoted by p, p_i etc. whereas for gluon momenta k, k_i etc. are used.

One finds for the inverse propagator

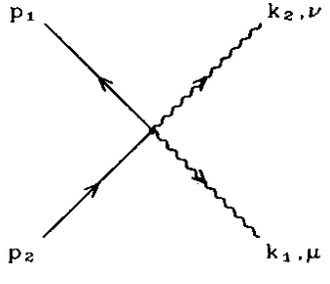
$$S_F^{-1}(p) = M_0 + \sum_\mu \left(i\gamma_\mu \sin p_\mu + 2\sin^2 \frac{p_\mu}{2} \right). \tag{5.2}$$

and the quark-gluon vertices

$$V_1^{FG}(p_1, p_2; k, \mu) = z(k, p_2) \left\{ -\gamma_\mu \cos \tfrac{(p_1+p_2)_\mu}{2} + i \sin \tfrac{(p_1+p_2)_\mu}{2} - \tfrac{c(g_o^2)}{2} \cos \tfrac{k_\mu}{2} \sum_\nu \sigma_{\mu\nu} \sin k_\nu \right\}, \tag{5.3}$$

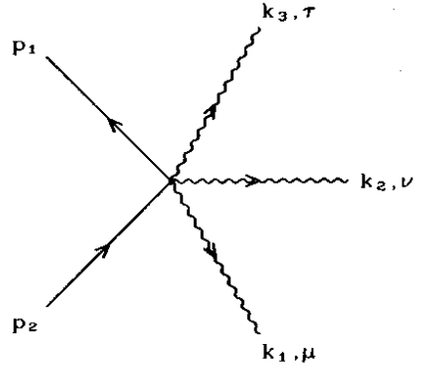


$$\begin{aligned} V_2^{FG}(p_1, p_2; k_1, \mu; k_2, \nu) = & \\ & -\tfrac{1}{2} z(p_1 - p_2, p_2) \\ & \cdot \Big\{ \delta_{\mu\nu} \Big[\big[z(k_1, k_2) + z(k_2, k_1) \big] \big[-\cos \tfrac{(p_1-p_2)_\mu}{2} + i\gamma_\mu \sin \tfrac{(p_1+p_2)_\mu}{2} \big] \\ & - \tfrac{i}{2} c(g_o^2) \big[z(k_1, k_2) - z(k_2, k_1) \big] \sin \tfrac{(p_1-p_2)_\mu}{2} \sum_\rho \sigma_{\mu\rho} (\sin k_{1\rho} - \sin k_{2\rho}) \Big] \\ & + i\sigma_{\mu\nu} c(g_o^2) \big[z(k_1, k_2) - z(k_2, k_1) \big] \\ & \cdot \Big[2 \cos \tfrac{(p_1-p_2)_\mu}{2} \cos \tfrac{k_{2\mu}}{2} \cos \tfrac{(p_1-p_2)_\nu}{2} \cos \tfrac{k_{1\nu}}{2} - \cos \tfrac{k_{1\mu}}{2} \cos \tfrac{k_{2\nu}}{2} \Big] \Big\} \end{aligned} \tag{5.4}$$



and

$$\begin{aligned} \Gamma_3^{FG}(p_1, p_2; k_1, \mu; k_2, \nu; k_3, \sigma) & \\ = -\tfrac{1}{2} z(p_1 - p_2, p_2) & \Big[\big[z(k_1, k_2 + k_3) z(k_2, k_3) + z(k_3, k_2) z(k_2 + k_3, k_1) \big] \\ \cdot \Big\{ \delta_{\mu\nu} \delta_{\mu\sigma} & \Big(\tfrac{1}{3} \big(\gamma_\mu \cos \tfrac{(p_1+p_2)_\mu}{2} - i \sin \tfrac{(p_1+p_2)_\mu}{2} \big) \\ + c(g_o^2) \cos \tfrac{(p_1-p_2)_\mu}{2} & \sum_\rho \sigma_{\mu\rho} \big(\tfrac{1}{6} \sin (p_1 - p_2)_\rho - \cos \tfrac{(p_1-p_2)_\rho}{2} \sin \tfrac{k_{2\rho}}{2} \cos \tfrac{(k_1-k_3)_\rho}{2} \big) \Big) \\ + \sigma_{\mu\nu} c(g_o^2) & \Big[2\delta_{\nu\sigma} \sin \tfrac{k_{1\nu}}{2} \cos \tfrac{(p_1-p_2)_\nu}{2} \cos \tfrac{(p_1-p_2)_\mu}{2} \cos \tfrac{(k_2+k_3)_\mu}{2} \\ & + \delta_{\nu\sigma} \sin \tfrac{(k_2-k_3)_\nu}{2} \cos \big(\tfrac{k_1}{2} + k_2 \big)_\mu \\ & + \delta_{\mu\sigma} \sin \tfrac{(k_1+k_3+2k_2)_\mu}{2} \cos \tfrac{(p_1-p_2)_\nu}{2} \cos \tfrac{(k_1-k_3)_\nu}{2} \Big] \Big\} \\ & + 2 \text{ cycl. Perms.} \Big]. \end{aligned} \tag{5.5}$$



For $\Gamma_1^{FG}(p_1, p_2; k, \mu)$ the Slavnov Taylor identity holds:

$$\hat{k}_\mu \Gamma_1^{FG}(p_1, p_2; k, \mu) = iz(k, p_2) \left\{ S_F^{-1}(p_1) - S_F^{-1}(p_2) \right\}. \tag{5.6}$$

6. Spectrum and stable Particles

Calculated in e.g. the Coulomb gauge the fermion two point function is expected to have poles corresponding to the existence of massive charged unconfined fermions in the theory. The LSZ-construction can be applied and S-matrix elements calculated in perturbation theory.
Consider momentum configurations $p = (\omega, \vec{p})$ with $\vec{p} = (p_1, p_2, p_3) = (p_\perp, p_3)$.
Near the pole $\omega = iE(\vec{p})$ the fermion two point function has the form

$$g_2^F(\vec{p}) = \frac{Z(\vec{p})}{\omega^2 + E^2(\vec{p})} + O(1) \tag{6.1}$$

where $E(\vec{p})$ and $Z(\vec{p})$ have the power series expansion[1]

$$E(\vec{p}) = \sum_{\nu=0}^{\infty} g_0^{2\nu} E^{(\nu)}(\vec{p}) \tag{6.2}$$

$$Z(\vec{p}) = \sum_{\nu=0}^{\infty} g_0^{2\nu} Z^{(\nu)}(\vec{p}) \tag{6.3}$$

To lowest order $E^{(0)}(\vec{p})$ is given by the location of the pole in the free fermion propagator

$$S_F^{-1}(\omega, \vec{p}) \mid_{\omega = iE^{(0)}(\vec{p})} = 0. \tag{6.4}$$

Thus

$$-\sinh^2 E^{(0)}(\vec{p}) + \sum_{i=1}^{3} \sin^2 p_i + (M_0 - 2\sinh^2 \frac{E^{(0)}(\vec{p})}{2} + \frac{\hat{\vec{p}}^2}{2})^2 = 0. \tag{6.5}$$

and hence

$$\begin{aligned} 4\sinh^2 \frac{E^{(0)}(\vec{p})}{2} &= \frac{M_0^2 + (1 + M_0)\hat{\vec{p}}^2 + \frac{1}{2}((\hat{\vec{p}}^2)^2 - \hat{p}^4)}{1 + M_0 + \frac{\hat{\vec{p}}^2}{2}} \\ &= \hat{\vec{p}}^2 + M_0^2(1 - M_0) + O(a^2). \end{aligned} \tag{6.6}$$

[1] The expansion for the energy $E(\vec{p})$ is given in terms of the bare coupling g_0 but of course this quantity when reexpressed as a function of the renormalized coupling has a finite limit when 'a' goes to zero.

With

$$M_R = M_0(1 - \frac{M_0}{2}) \tag{6.7}$$

this yields

$$E^{(0)}(\vec{p})^2 = \vec{p}^2 + M_R^2 + O(a^4) = p_\perp^2 + p_3^2 + M_R^2 + O(a^2), \tag{6.8}$$

i.e. up to $O(a^2)$ corrections the action (5.1) describes a continuum theory with renormalized mass M_R.
Furthermore

$$\begin{aligned} &E^{(0)}(\vec{p}) \geq \sqrt{(M_R^2 + \hat{p}_\perp^2)} = \sqrt{(M_R^2 + m^2((2\nu_1 + 1)^2 + (2\nu_2 + 1)^2))} \\ &\text{where} \qquad p_\perp = m(2\nu_1 + 1, 2\nu_2 + 1), \qquad \nu_1, \nu_2 \epsilon \mathbf{Z}, \qquad m = \frac{\pi}{LN} \end{aligned} \tag{6.9}$$

The introduction of twisted antiperiodic boudary conditions has led to the fact that even for zero bare mass the theory has a mass gap.
For the lowest lying energy levels X and Y one finds

$$\begin{aligned} p_\perp^{X\,2} &= 2m^2 \qquad (\nu_1^X, \nu_2^X = 0, -1) \\ p_\perp^{Y\,2} &= 10m^2 \qquad (\nu_1^Y = 0, -1 \wedge \nu_2^Y = 1, -3) \vee (1 \leftrightarrow 2) \end{aligned} \tag{6.10}$$

For the gluon sector qualitatively the same picture holds true. Although here, differing from [12], the "unimproved" gluon propagator

$$D_{\mu\nu}(k) = -\frac{1}{2} \chi_k z(k, k) [\delta_{\mu\nu} \hat{k}^2 + (\alpha - 1) \hat{k}_\mu \hat{k}_\nu] \frac{1}{(\hat{k}^2)^2} \tag{6.11}$$

with a general covariant gauge fixing is used, the location of its poles is still to lowest order given by

$$k_0 = \pm \sqrt{k_\perp^2 + k_3^2}. \tag{6.12}$$

χ_k has been introduced by Lüscher and Weisz [12] and performs the supression of the zero momentum modes for the gluons:

$$\chi_k = \begin{cases} 0 & \text{if } k_\perp = 0 \text{ (mod N)} \\ 1 & \text{otherwise.} \end{cases} \tag{6.13}$$

Taking into account (6.13) the lowest lying massive particles in the gluon sector are A- and B-mesons with

$$k_\perp^{A\,2} = 4m^2 \tag{6.14}$$

$$k_\perp^{B\,2} = 8m^2 \tag{6.15}$$

corresponding to minimal energies of

$$E^{(0)}(k_\perp^A, 0) = 2m \tag{6.16}$$

$$E^{(0)}(k_\perp^B, 0) = 2\sqrt{2}m. \tag{6.17}$$

To lowest order in 'a' X and Y quarks correspond to stable particles for they cannot decay into another quark and a gluon because

$$E^{(0)}(p_\perp^{X,Y}, 0)^2 < (E^{(0)}(p_\perp^X, 0) + E^{(0)}(k_\perp^A, 0))^2 = (\sqrt{(M_R^2 + 2m^2)} + 2m)^2. \tag{6.18}$$

At weak coupling this property can be expected to persist.
Whereas in pure Yang Mills theory both A and B mesons are stable particles one finds that this only holds true for the lowest lying A mesons when fermions are included.
While the lowest lying gluon states can be realized by gauge invariant operators i.e. Wilson loops winding around the world in the compact directions, this cannot be accomplished for the stable quark states.
For the considerations of tree level "on shell" improvement made in the next Section one temporarily considers the general form of the action given in (2.1) where the coefficient of the Wilson term has not yet been fixed, but has still the generic value r.
Then one would have

$$E^{(0)}(\bar{p})^2 = \bar{p}^2 + 2m^2 + M_0^2(1 - rM_0) + O(a^2) \tag{6.19}$$

instead of (6.8) i.e. $M_R = M_0(1 - \frac{r}{2}M_0)$.
The propagator then has a pole at $\omega = \pm iE^{(0)}(\bar{p})$ with the residue $Z^{(0)}(\bar{p})$.
It follows that

$$\frac{\partial}{\partial\omega^2}\left[\omega^2 + (E^{(0)}(\bar{p})^2 - M_R^2) + (M_0 + \frac{r}{2}(\omega^2 + E^{(0)}(\bar{p})^2 - M_R^2)^2)\right]\Big|_{\omega^2 = -E^{(0)2}} + O(a^2)$$
$$= 1 + r(M_0 - \frac{r}{2}M_R^2) = 1 + rM_R + O(a^2). \tag{6.20}$$

Therefore

$$Z^{(0)}(\bar{p})\,|_{\omega = iE^{(0)}(\bar{p})} = -\frac{1}{(1 + rM_R)}(-\gamma_0 \sinh E^{(0)}(\bar{p}) + i\sum_{\nu=1}^{3}\gamma_\nu \sin p_\nu - M_R) + O(a^2)$$
$$= \left[\sinh E^{(0)}(\bar{p}) + M_R\right]^{-1} u_\alpha^{(0)}(\bar{p})\bar{u}_\alpha^{(0)}(\bar{p}) + O(a^2) \tag{6.21}$$

with

$$u_\alpha^{(0)}(p) = -(1 + rM_R)^{-\frac{1}{2}}(-\gamma_0 \sinh E^{(0)}(\bar{p}) + i\sum_{\nu=1}^{3}\gamma_\nu \sin p_\nu - M_R)u_\alpha^0$$

$$\bar{u}_\alpha^{(0)}(p) = -(1 + rM_R)^{-\frac{1}{2}}\bar{u}_\alpha^0(-\gamma_0 \sinh E^{(0)}(\bar{p}) + i\sum_{\nu=1}^{3}\gamma_3 \sin p_\nu - M_R) \tag{6.22}$$

$$(u_\alpha^0)_\beta = \delta_{\alpha\beta} \qquad \bar{u}_\alpha^0 = {}^T u_\alpha^0 \gamma_0 \qquad \alpha = 1, 2$$

where α denotes the two possible polarisations of the quarks.
For later use, a modified Gordan identity is introduced. Let a_μ be an arbitrary vector then

$$2M_R \bar{u}_\alpha^{(0)}(\bar{p}) \not{a}\, u_\beta^{(0)}(\bar{p}') =$$
$$\bar{u}_\alpha^{(0)}(\bar{p})\{a_0(\sinh E^{(0)}(\bar{p}) - \sinh E^{(0)}(\bar{p}')) - i\sum_{\nu=1}^{3}a_\nu(\sin p_\nu - \sin p'_\nu))$$
$$- a_\mu(\sigma_{\mu 0} i(\sinh E(\bar{p}) - \sinh E(\bar{p}')) - \sum_{\nu=1}^{3}\sigma_{\mu\nu}(\sin p_\nu - \sin p'_\nu))\}u_\beta^{(0)}(\bar{p}'). \tag{6.23}$$

The desired identities are obtained by differentiation with respect to a_μ.
Furthermore one has

$$\bar{u}_\alpha^{(0)}(p)u_\beta^{(0)}(p) = 2(1 - rM_R)^{-1}M_R(\sinh E^{(0)}(\bar{p}) + M_R)\delta_{\alpha\beta}. \tag{6.24}$$

7. "On Shell"-Improvement and lowest Order

In [17] the fact that "classical" improvement is compatible with "on shell" improvement was employed to deduce the relation (2.4) for the tree level "on shell" improved action.
In order to perform the calculation of $c^{(1)}$ an appropriate "on shell" quantity has to be found for which one loop improvement will be demanded.
To this end one first considers "on shell" quantities to lowest order. If they yield the tree level improvement condition (2.4) they are suitable to derive $c^{(1)}$.
In the case of pure Yang Mills theory the mass of the lowest lying gluons and an appropriate gluon three point function derived from a gluon S-matrix element were chosen to deduce the necessary two independent relations needed to determine the two improvement coefficients emerging in this scheme to one loop order.
The following argument shows that the quantities derived from the fermion two point function i.e. the fermion mass renormalisation and the fermion wave function renormalisation cannot be used to calculate $c^{(1)}$.
For the fermion self-energy $\Sigma(p)$ and the three point function $\Gamma_3(p_1,p_2;k,\mu)$ the Ward and Slavnov-Taylor identities hold to all orders of perturbation theory. Let

$$\Sigma(p) = \sum_{\nu=1}^{\infty} g_0^{2\nu}\Sigma^{(\nu)}(p) \tag{7.1}$$

$$\Gamma_3(p_1,p_2;k,\mu) = g_0 \sum_{\nu=0}^{\infty} g_0^{2\nu}\Gamma_3^{(\nu)}(p_1,p_2;k,\mu) \tag{7.2}$$

then the following relations are valid

$$-i\frac{\partial}{\partial p_\mu}\Sigma^{(\nu)}(p) = \Gamma_3^{(\nu)}(p,p;0,\mu) \tag{7.3}$$

and

$$iz(k,p_2)\left(\Sigma^{(\nu)}(p_2) - \Sigma^{(\nu)}(p_1)\right) = \hat{k}_\mu\Gamma_3^{(\nu)}(p_1,p_2;k,\mu). \tag{7.4}$$

In all orders of perturbation theory the contribution of $c^{(\nu)}$ to $\Gamma_3^{(\nu)}(p_1,p_2;k,\mu)$ reads

$$\Gamma_{3c}^{(\nu)}(p_1,p_2;k,\mu) = -\frac{c^{(\nu)}}{2}z(k,p_2)\cos\frac{k_\mu}{2}\sum_\rho \sigma_{\mu\rho}\sin k_\rho. \tag{7.5}$$

Since

$$\Gamma_{3c}^{(\nu)}(p,p;0,\mu) = 0 \tag{7.6}$$

and

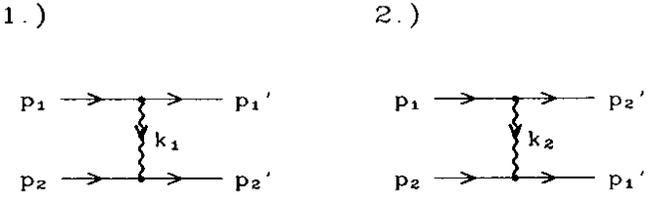
$$\begin{aligned}&\hat{k}_\mu\Gamma_{3c}^{(\nu)}(p_1,p_2;k,\mu)\\ &= -\frac{1}{2}z(k,p_2)\sum_{\mu\rho}\sigma_{\mu\rho}\sin k_\mu \sin k_\rho = 0\end{aligned} \tag{7.7}$$

the following conclusions can be drawn.

Since the contribution of $c(g_0^2)$ to any difference of fermion self-energies

$$\Sigma^{(\nu)}(p_1) - \Sigma^{(\nu)}(p_2)$$

vanishes for generic p_1,p_2, $c(g_0^2)$ can only contribute a constant to the fermion self energy. As $\Sigma(p)$ is an analytic function of p this constant vanishes.
Hence to the ν'th order of perturbation theory $c^{(\nu)}$ gives neither a contribution to the fermion mass renormalization nor to the fermion wave function renormalization. As both quantities cannot be used for a perturbative calculation of $c(g_0^2)$ one has to turn to scattering processes between stable fermion and gluon states.
Consider quark-quark-scattering.
The lowest order diagrams are



In order to suppress the contribution of one diagram the following "on shell" momenta of the lowest lying excitations are chosen

$$\begin{aligned} p_1 &= (iE, m, \quad m, \quad is) & p_1' &= (iE, m, -m, -is)\\ p_2 &= (iE, m, -m, -is) & p_2' &= (iE, m, \quad m, \quad is)\end{aligned} \tag{7.8}$$

i.e.

$$\begin{aligned} p_1 = p_2', \qquad & p_1' = p_2\\ k_1 = p_1 - p_1' = p_2' - p_2 &= (0,0,2m,2is)\\ k_2 = p_1 - p_2' = p_1' - p_2 &= (0,0,0,0).\end{aligned} \tag{7.9}$$

As pointed out in the previous Section due to the boundary conditions quarks and gluons carrying these minimal momenta in the $p_\perp$, $k_\perp$ are stable, so for these particles scattering amplitudes are meaningful quantities.
As intended diagram 2 doesn't contribute because the gluon propagator vanishes. s is chosen such as to make k_1 "on shell" namely

$$s = m + O(a^2).$$

The scattering amplitude is then to lowest order

$$S^{(0)} = T_\mu^{(0)} D_{\mu\nu}(k_1) T_\nu^{(0)\prime} \tag{7.10}$$

with

$$T_\mu^{(0)} = \bar{u}_\alpha^{(0)}(\bar{p}_1) V_1^{FG}(p_1, p_1'; k_1, \mu) u_\beta^{(0)}(\bar{p}_1'). \tag{7.11}$$

T_ν' is obtained from T_μ by replacing

$$p_1 \longrightarrow p_2, \qquad p_1' \longrightarrow p_2', \qquad \mu \longrightarrow \nu, \qquad k_1 \longrightarrow -k_1. \tag{7.12}$$

Using the generalized Gordon identity (6.23) one finds ($c^{(0)} = c$)

$$T_\mu^{(0)} = z(k_1, p_1') \bar{u}_\alpha^{(0)}(\bar{p}_1) [-\gamma_\mu (1 + cM_R) - \tfrac{i}{2}(c - r)(p_1 + p_1')_\mu + O(a^2)] u_\beta^{(0)}(\bar{p}_1'). \tag{7.13}$$

Due to the Slavnov Taylor identity (5.6) only the $\delta_{\mu\nu}$- part of the propagator contributes to the scattering amplitude.
The small 'a' expansion of S then gives

$$\begin{aligned} S^{(0)} = & -\tfrac{1}{2} \chi_{k_1} \frac{z(k_1, k_1)}{\hat{k}_1^2} \Big\{ \bar{u}_\alpha^{(0)}(\bar{p}_1) \big[-\gamma_\mu (1 + cM_R) - \tfrac{i}{2}(c - r)(p_1 + p_1')_\mu - O(a^2) \big] u_\beta^{(0)}(\bar{p}_1') \\ & \cdot \bar{u}_\gamma^{(0)}(\bar{p}_1') \big[-\gamma_\mu (1 + cM_R) - \tfrac{i}{2}(c - r)(p_1 + p_1')_\mu + O(a^2) \big] u_\delta^{(0)}(\bar{p}_1) \Big\} \\ = & -\tfrac{1}{2} \chi_{k_1} \frac{z(k_1, k_1)}{\hat{k}_1^2} \big\{ \bar{u}_\alpha^{(0)}(\bar{p}_1) \gamma_\mu u_\beta^{(0)}(\bar{p}_1') \bar{u}_\gamma^{(0)}(\bar{p}_1') \gamma_\mu u_\delta^{(0)}(\bar{p}_1) (1 + cM_R)^2 \\ & - 2iM_R \bar{u}_\alpha^{(0)}(\bar{p}_1) u_\beta^{(0)}(\bar{p}_1') \bar{u}_\gamma^{(0)}(\bar{p}_1') u_\delta^{(0)}(\bar{p}_1) (c - r)(1 + cM_R) + O(a^2) \big\}. \end{aligned} \tag{7.14}$$

If $S^{(0)}(1), u_{\alpha,1}^{(0)}(\bar{p}_1)$ etc. denote the corresponding infinite volume quantities one finds

$$\begin{aligned} S^{(0)} = & -\tfrac{1}{2} \chi_{k_1} \frac{z^{-1}}{\hat{k}_1^2} \frac{1}{(1 + rM_R)^2} \big\{ \bar{u}_{\alpha,1}^{(0)}(\bar{p}_1) \gamma_\mu u_{\beta,1}^{(0)}(\bar{p}_1') \bar{u}_{\gamma,1}^{(0)}(\bar{p}_1') \gamma_\mu u_{\delta,1}^{(0)}(\bar{p}_1) (1 + cM_R)^2 \\ & - 2iM_R \bar{u}_{\alpha,1}^{(0)}(\bar{p}_1) u_{\beta,1}^{(0)}(\bar{p}_1') \bar{u}_{\gamma,1}^{(0)}(\bar{p}_1') u_{\delta,1}^{(0)}(\bar{p}_1) (c - r)(1 + cM_R) + O(a^2) \big\} \\ = & \frac{1}{(1 + rM_R)^2} \big\{ (1 + cM_R)^2 S^{(0)}(1) \\ & + \frac{z^{-1} \chi_{k_1}}{\hat{k}_1^2} iM_R (c - r)(1 + cM_R) \bar{u}_{\alpha,1}^{(0)}(\bar{p}_1) u_{\beta,1}^{(0)}(\bar{p}_1') \bar{u}_{\gamma,1}^{(0)}(\bar{p}_1') u_{\delta,1}^{(0)}(\bar{p}_1) \big\}. \end{aligned} \tag{7.15}$$

Thus for $M_R \neq 0$ S yields the tree level improvement condition (2.4).

In the case of zero bare mass $M_R = M_0 = 0$ a slightly different momentum configuration has to be chosen:

$$\begin{aligned} p_1 &= m(\ \ i, 1, \ \ 1, \ \ i) & p_1' &= m(\ \ i, 1, -1, -i) \\ p_2 &= m(-i, 1, -1, -i) & p_2' &= m(-i, 1, \ \ 1, \ \ i) \end{aligned} \tag{7.16}$$

and

$$\begin{aligned} p_1 - p_1' = 2m(i, 1, 0, 0), \qquad & p_2 + p_2' = 2m(-i, 1, 0, 0) \\ k_1 = p_1 - p_1' = p_2' - p_2 &= 2m(0, 0, 1, i) \\ k_2 = p_1 - p_2' = p_1' - p_2 &= 2m(i, 0, 0, 0). \end{aligned} \tag{7.17}$$

Again diagram 2 doesn't contribute and one finds

$$T_\mu^{(0)} = z(k_1, p_1') \bar{u}_\alpha^{(0)}(\bar{p}_1) [-\gamma_\mu - \tfrac{i}{2}(c - r)(p_1 + p_1')_\mu + O(a^2)] u_\beta^{(0)}(\bar{p}_1') \tag{7.18}$$

$$T_\nu^{(0)\prime} = z(-k_1, p_2') \bar{u}_\delta^{(0)}(\bar{p}_2) [-\gamma_\nu - \tfrac{i}{2}(c - r)(p_2 + p_2')_\nu + O(a^2)] u_\gamma^{(0)}(\bar{p}_2'). \tag{7.19}$$

The scattering amplitude to lowest order is

$$S^{(0)} = -\tfrac{1}{2} \chi_{k_1} \frac{z(k_1, k_1)}{\hat{k}_1^2} T_\mu^{(0)} T_\mu^{(0)\prime} \tag{7.20}$$

Now set

$$\begin{aligned} T_\mu^{(0)} &= T_\mu^{(0)}(1) + T_\mu^{(0)}(a) + O(a^2) \\ T_\mu^{(0)\prime} &= T_\mu^{(0)\prime}(1) - T_\mu^{(0)\prime}(a) - O(a^2), \end{aligned} \tag{7.21}$$

where the term in brackets indicates the O(1) and O(a) contributions. Then

$$T_\mu^{(0)}(a) = T_\mu^{(0)\prime}(a) = 0, \qquad \mu = 2, 3 \tag{7.22}$$

and

$$S^{(0)} = -\tfrac{1}{2} \chi_{k_1} z(k_1, k_1) \frac{1}{\hat{k}_1^2} \Big[T_\mu^{(0)}(1) T_\mu^{(0)\prime}(1) + \sum_{\nu=0}^{1} \big(T_\nu^{(0)}(1) T_\nu^{(0)\prime}(a) + T_\nu^{(0)}(a) T_\nu^{(0)\prime}(1) \big) + O(a^2) \Big]. \tag{7.23}$$

The first order corrections to the continuum limit therefore originate from the term

$$\sum_{\nu=0}^{1} \big(T_\nu^{(0)}(1) T_\nu^{(0)\prime}(a) + T_\nu^{(0)}(a) T_\nu^{(0)\prime}(1) \big) \tag{7.24}$$

For definite polarisations in the in- and out-states, i.e. $\alpha = \beta = \delta = \gamma = 1$ one finds

$$\begin{aligned} T_0^{(0)}(1) &= -2zm^2(1-i) & T_0^{(0)\prime}(1) &= -2m^2(1+i) \\ T_0^{(0)}(a) &= 2izm^3(c-r) & T_0^{(0)\prime}(a) &= 2im^3(c-r) \end{aligned} \tag{7.25}$$

yielding

$$T_0^{(0)}(1)T_0^{(0)\prime}(a) + T_0^{(0)}(a)T_0^{(0)\prime}(1) = -8izm^5(c-r) \tag{7.26}$$

and

$$\begin{aligned} T_1^{(0)}(1) &= 2zm^2(1+i) & T_1^{(0)\prime}(1) &= 2m^2(1-i) \\ T_1^{(0)}(a) &= 2zm^3(c-r) & T_1^{(0)\prime}(a) &= -2m^3(c-r) \end{aligned} \tag{7.27}$$

giving

$$T_1^{(0)}(1)T_1^{(0)\prime}(a) + T_1^{(0)}(a)T_1^{(0)\prime}(1) = -8izm^5(c-r). \tag{7.28}$$

So to lowest order both polarisations in the intermediate gluon state yield the same contribution to the scattering amplitude.
Thus for the momentum configuration (7.16) (7.17) consideration of S yields once again the tree level improvement condition.

8. Energy and Wave Functions to first nontrivial Order

From now on only quarks with zero bare mass will be considered. The constant r in the action (2.1) is set equal to 1 again which according to Section 2 is the natural choice for this coefficient. Thus the value of the improvement coefficient $c(g_0^2)$ to lowest order becomes $c = 1$.
In Appendix A the following formulae for $E^{(1)}(\bar{p})$ and $Z^{(1)}(\bar{p})$ are derived

$$E^{(1)}(\bar{p})Z^{(0)}(\bar{p}) = -\frac{1}{2E^{(0)}(\bar{p})} Z^{(0)}(\bar{p})\Sigma^{(1)}(p)\,|_{p_0=iE^{(0)}(\bar{p})} Z^{(0)}(\bar{p}) \tag{8.1}$$

$$\begin{aligned} Z^{(1)}(\bar{p}) = {} & \frac{1}{2iE^{(0)}(\bar{p})} Z^{(0)}(\bar{p})\Big(\frac{\partial}{\partial p_0}\Sigma^{(1)}(p)\Big)\,|_{p_0=iE^{(0)}(\bar{p})} Z^{(0)}(\bar{p}) \\ & + \frac{1}{2iE^{(0)}(\bar{p})}\big[N_1(\bar{p})\Sigma^{(1)}(p)\,|_{p_0=iE^{(0)}(\bar{p})} Z^{(0)}(\bar{p}) + Z^{(0)}(\bar{p})\Sigma^{(1)}(p)\,|_{p_0=iE^{(0)}(\bar{p})} N_1(\bar{p})\big] + O(a^3). \end{aligned} \tag{8.2}$$

According to Appendix B to one loop order the fermion self energy has the form

$$\Sigma^{(1)}(p)\,|_{p_0=iE^{(0)}(\bar{p})} = i(\gamma_\mu \sin p_\mu)\,|_{p_0=iE^{(0)}(\bar{p})}\Big(\frac{P_1}{4} + \frac{a-1}{2}(P_{2\mu} - 2P_2 + P(\bar{p}) + P_{3\mu})\Big) + i\not{a}(\bar{p}) + b(\bar{p}) \tag{8.3}$$

where $a_\mu(\bar{p})$ and $b(\bar{p})$ are gauge invariant contributions.
One finds

$$b(\bar{p}) = b_0^{(0)} + b_1^{(0)} m^2 + O(a^3), \tag{8.4}$$

where

$$\begin{aligned} b_0^{(0)} &= -\frac{N^2-1}{2N} \quad 0.2025565(4) \\ b_1^{(0)} &= \frac{N^2-1}{2N} \quad 0.0093304(8) \end{aligned} \tag{8.5}$$

and $b_0^{(0)}$. $b_1^{(0)}$ are universal for X and Y quarks with momenta of

$$m(\pm i, \pm 1, \pm 1, \pm i) \tag{8.6}$$

and

$$m(\pm i, \pm 1, \pm 3, \pm 3i) \tag{8.7}$$

respectively.
Therefore as already indicated in (2.1) a counterterm of the form

$$-g_0^2 b_0(g_0^2)\bar{\Psi}(x)\Psi(x) \qquad \text{with} \quad b_0(0) = b_0^{(0)}$$

has to be incorporated in the action in order to cancel the $O(\frac{1}{a})$ effect arising from $b_0^{(0)}$. With the relations

$$\begin{aligned} & Z^{(0)}(\bar{p})Z^{(0)}(\bar{p}) = O(a^2) \\ & Z^{(0)}(\bar{p})\gamma_0 Z^{(0)}(\bar{p}) = -2\sinh E^{(0)}(\bar{p})Z^{(0)}(\bar{p}) + O(a^2) \\ & Z^{(0)}(\bar{p})\gamma_\mu Z^{(0)}(\bar{p}) = 2i\sin p_\mu Z^{(0)}(\bar{p}) - O(a^2) \qquad \mu = 1,2,3 \end{aligned} \tag{8.8}$$

one finds

$$E^{(1)}(\bar{p}) = \frac{1}{E^{(0)}(\bar{p})}(\sum \sin p_\mu a_\mu(\bar{p}))|_{p_0=iE^{(0)}(\bar{p})} + O(a^2) \tag{8.9}$$

Using the numerical results quoted in Appendix B one finds for X and Y quarks with the above momenta

$$E^{(1)}(\bar{p}) = -\frac{N^2-1}{2N}(0.0093304(8))E^{(0)}(\bar{p}) + O(a^2) \tag{8.10}$$

where use has been made of the remarkable relation obtained there i.e.

$$b_1^{(0)} m^2 = (\sum a_\mu(\bar{p}) \sin p_\mu)|_{p_0=iE^{(0)}(\bar{p})} \tag{8.11}$$

holding for the action given in (2.1).
In Appendix B it is furthermore shown that

$$\frac{1}{2iE^{(0)}(\bar{p})} Z^{(0)}(\bar{p})(\frac{\partial}{\partial p_0}\Sigma^{(1)}(p))|_{p_0=iE^{(0)}(\bar{p})} Z^{(0)}(\bar{p}) = C(\bar{p})Z^{(0)}(\bar{p}) + O(a^2) \tag{8.12}$$

where $C(\bar{p})$ is O(1) and O($\log m^2$) but has no O(a) contributions.
Inserting this into (8.2) yields

$$Z^{(1)}(\bar{p}) = C(\bar{p})Z^{(0)}(\bar{p}) - \frac{1}{E^{(0)}(\bar{p})}(\sum_{\nu=0}^{4} \mathrm{M}_\nu B_\nu) + O(a^2) \tag{8.13}$$

where

$$\begin{aligned}
\mathrm{M}_0 &= i(-\cos p_0 \sin p_0 b_1^{(0)} + \sin p_0 \sum \sin p_\mu a_\mu(\bar{p})) + O(a^2) \\
\mathrm{M}_1 &= \cos p_0(\sum \sin p_\mu a_\mu(\bar{p})) + \sin^2 p_0 b_1^{(0)} + O(a^2) \\
\mathrm{M}_2 &= \cos p_0(a_0(\bar{p}) \sin p_1 - \sin p_0 a_1(\bar{p})) + \sin p_0 \sin p_1 b_1^{(0)}) + O(a^2) \\
\mathrm{M}_3 &= \mathrm{M}_2(1 \to 2) \\
\mathrm{M}_4 &= \mathrm{M}_2(1 \to 3).
\end{aligned} \tag{8.14}$$

Due to the relation (8.11) one has for the O(a) improved action $\mathrm{M}_0 = O(a^2)$ and

$$Z^{(1)}(\bar{p}) = C(\bar{p})Z^{(0)}(\bar{p}) - \frac{1}{2E^{(0)}(\bar{p})}(\gamma_0(i\not{a}(\bar{p}))Z^{(0)}(\bar{p}) + Z^{(0)}(\bar{p})(i\not{a}(\bar{p}))\gamma_0) + O(a^2). \tag{8.15}$$

Together with the fact that $C(\bar{p})$ and $a_\mu(\bar{p})$ are O(1) with O(a^2) corrections (see Appendix B) this shows that $Z^{(1)}(\bar{p})$ has no O(a) corrections.
It has been checked that relation (8.11) is a merit of O(a) improvement because for the Wilson action one finds[1]

$$\begin{aligned}
b_0^{(0),W} &= -\frac{N^2-1}{2N}(0.3257151(7)) \\
b_1^{(0),W}(\bar{p}) &= 0.02920(4) \qquad N = 2
\end{aligned} \tag{8.16}$$

[1] The numerical values quoted here have been derived only for X quarks with momenta given by (8.6).

whereas

$$(\sum a_\mu^W(\bar{p}) \sin p_\mu)|_{p_0=iE^{(0)}(\bar{p})} = 0.00617(12) \qquad N = 2. \tag{8.17}$$

The result for $b_0^{(0),W}$ given in (8.16) agrees with the one obtained by Stehr and Weisz in [20].
The values of the hopping parameter K of the equivalent zero quark mass theory is then

$$K = \frac{1}{8}((1 - g_0^2 \frac{b_0^{(0)}}{4}) + O(g_0^4)) \tag{8.18}$$

i.e.

$$K = \frac{1}{8}(1 + 0.0506391(1)\frac{N^2-1}{2N} g_0^2 + O(g_0^4)) \tag{8.19}$$

for the O(a)-improved lattice action and

$$K^W = \frac{1}{8}(1 + 0.0814288(2)\frac{N^2-1}{2N} g_0^2 + O(g_0^4)) \tag{8.20}$$

for the Wilson action.
Now the form of the wave functions to one loop order is to be determined. In analogy to (6.21) one defines

$$\sinh E(\bar{p}) Z(\bar{p}) = u_\alpha(\bar{p})\bar{u}_\alpha(\bar{p}). \tag{8.21}$$

Expanding up to first order yields

$$Z^{(1)}(\bar{p}) = -\frac{\sinh E^{(1)}(\bar{p})}{\sinh E^{(0)}(\bar{p})} Z^{(0)}(\bar{p}) + \frac{1}{\sinh E^{(0)}(\bar{p})}(u_\alpha^{(0)}(\bar{p})\bar{u}_\alpha^{(1)}(\bar{p}) + u_\alpha^{(1)}(\bar{p})\bar{u}_\alpha^{(0)}(\bar{p})) + O(a^2). \tag{8.22}$$

With the Ansatz

$$\begin{aligned}
u^{(1)}(\bar{p}) &= M(\bar{p})u^{(0)}(\bar{p}) \\
\bar{u}^{(1)}(\bar{p}) &= \bar{u}^{(0)}(\bar{p})\gamma_0 M(\bar{p})^+ \gamma_0
\end{aligned} \tag{8.23}$$

this yields

$$Z^{(1)}(\bar{p}) = -\frac{\sinh E^{(1)}(\bar{p})}{\sinh E^{(0)}(\bar{p})} Z^{(0)}(\bar{p}) + (M(\bar{p})Z^{(0)}(\bar{p}) + Z^{(0)}(\bar{p})\gamma_0 M(\bar{p})^+ \gamma_0 + O(a^2). \tag{8.24}$$

From this and (8.15) one deduces a solution

$$M(\bar{p}) = \frac{1}{2}\left[\frac{\sinh E^{(1)}(\bar{p})}{\sinh E^{(0)}(\bar{p})} + C(\bar{p})\right]\mathbb{1} - \frac{i}{2E^{(0)}(\bar{p})}\gamma_0 \not{a}(\bar{p}) + O(a^2). \tag{8.25}$$

Abbreviating the term proportional to $\mathbb{1}$ by $A(\bar{p})$ one therefore finds

$$\begin{aligned}
u^{(1)}(\bar{p}) &= A(p)u^{(0)}(\bar{p}) - \frac{i}{2E^{(0)}(\bar{p})}\gamma_0 \not{a}(\bar{p})u^{(0)}(\bar{p}) + O(a^2) \\
\bar{u}^{(1)}(\bar{p}) &= A(\bar{p})\bar{u}^{(0)}(\bar{p}) - \frac{i}{2E^{(0)}(\bar{p})}\bar{u}^{(0)}(\bar{p})\not{a}(\bar{p})\gamma_0 + O(a^2).
\end{aligned} \tag{8.26}$$

So as $Z^{(1)}(\bar{p})$ the fermion wave functions to one loop order perturbation theory have no O(a) corrections. This property could already be expected from the considerations in Section 6

where it was argued that tree level improvement already accounts for the O(a) improvement of "on shell" quantities depending only on the fermion two point function to one loop order. Next the O(a) effects in the three point function arising from the wave functions to one loop order will be studied. Consider

$$g_2^{-1}(\bar{p})u(\bar{p}) \tag{8.27}$$

up to first order. One then has

$$\begin{aligned}
&(S_F^{-1}(\bar{p}) - \Sigma^{(1)}(p))u(\bar{p}) = \\
&\qquad \big(S_F^{-1}(\bar{p}) - i\not{a}(\bar{p}) - b_1\big)\big(u^{(0)}(\bar{p}) + A(\bar{p})u^{(0)}(\bar{p}) - \frac{i}{2E^{(0)}(\bar{p})}\gamma_0 \not{a}(\bar{p})u^{(0)}(\bar{p})\big) \\
&= \frac{i}{2E^{(0)}(\bar{p})}u^{(0)}(\bar{p})\gamma_0\not{a}(\bar{p})u^{(0)}(\bar{p}) - \sinh E^{(1)}(\bar{p})\gamma_0 u^{(0)}(\bar{p}) - i\not{a}(\bar{p})u^{(0)}(\bar{p}) - b_1 u^{(0)}(\bar{p}) + O(a^2) \\
&\qquad = -\frac{i}{2E^{(0)}(\bar{p})}\gamma_0 u^{(0)}(\bar{p})\not{a}(\bar{p})u^{(0)}(\bar{p}) + i\not{a}(\bar{p})u^{(0)}(\bar{p}) \\
&\qquad - \sinh E^{(1)}(\bar{p})\gamma_0 u^{(0)}(\bar{p}) - i\not{a}(\bar{p})u^{(0)}(\bar{p}) - b_1 u^{(0)}(\bar{p}) + O(m^4) \\
&\qquad\qquad = -b_1 u^{(0)}(\bar{p}) + O(a^2)
\end{aligned} \tag{8.28}$$

where

$$S_F^{-1}(\bar{p}) = -u^{(0)}(\bar{p}) - \sinh E^{(1)}(\bar{p})\gamma_0 + O(a^2) \tag{8.29}$$

has been used.
Analogously one finds

$$\bar{u}(\bar{p})g_2^{-1}\bar{p}) = -\bar{u}^{(0)}(\bar{p})b_1(\bar{p}) + O(g_0^4) + O(a^2). \tag{8.30}$$

With this relation a generalized version of the Gordon-Identity (6.23) can be derived. From

$$\bar{u}(\bar{p})((g_2^{-1}(\bar{p}) + b_1\not{p} + \not{p}'(g_2^{-1}(\bar{p}') + b_1))u(\bar{p}') = O(a^2) + O(g_0^4) \tag{8.31}$$

it follows that to leading order

$$\bar{u}(\bar{p})(-i((p+p')_\mu - (a(\bar{p}) + a(\bar{p}'))_\mu) + \sigma_{\mu\nu}((p-p')_\nu - (a(\bar{p}) - a(\bar{p}'))_\nu)u(\bar{p}') = 0. \tag{8.32}$$

Using this one obtains the following relation for the O(a)-effects arising from $\Gamma_3^{(0)}(p,p';k,1)$[1]

$$\begin{aligned}
&\bar{u}(\bar{p})(i\frac{(p+p')_1}{2} + \frac{1}{2}(\sigma_{01}(p-p')_0 - \sigma_{12}(p-p')_2 - \sigma_{13}(p-p')_3))u(\bar{p}') \\
&\qquad = \bar{u}(\bar{p})(i\frac{(a+a')_1}{2} + \frac{1}{2}(\sigma_{01}(a-a')_0 - \sigma_{12}(a-a')_2 - \sigma_{13}(a-a')_3))u(\bar{p}')
\end{aligned} \tag{8.33}$$

where the abbreviations $a_\mu = a_\mu(\bar{p})$, $a'_\mu = a_\mu(\bar{p}')$ $\mu = 0..3$ have been introduced.
As the terms in the bracket in the second expression are already $O(g_0^2)$ the wave functions multiplying them can be taken to lowest order.
One therefore has

$$\bar{u}_1^{(0)}(\bar{p})(\sum_{\nu=0}^{15} \mathrm{M}_\nu B_\nu)u_1^{(0)}(\bar{p}'). \tag{8.34}$$

[1] By a slight abuse of notation here the $O(g_0^2)$ effects arising from the contribution of $E^{(1)}(\bar{p}).E^{(1)}(\bar{p}')$ are incorporated in $\Gamma_3^{(0)}(p,p';k,1)$.

For X quarks with momenta given by (7.16) the relations

$$a'_1 = a_1 \qquad a'_2 = -a_2 \qquad a'_3 = -a_3 = a_1 \tag{8.35}$$

hold i.e.

$$\begin{aligned}
\mathrm{M}_0 &= ia_1 \\
\mathrm{M}_1 - \mathrm{M}_8 &= 0 \\
\mathrm{M}_9 &= -a_1. \\
\mathrm{M}_{10} &= -a_0 \\
\mathrm{M}_{11} - \mathrm{M}_{15} &= 0
\end{aligned} \tag{8.36}$$

For the O(a) contributions from the fermion wave functions to one loop order one therefore finds using the notations introduced in Section 7

$$T_1^{(1)}(a)^{WF} = -2m^2(ia_0(\bar{p}_1) + a_1(\bar{p}_1))(1+i) = -m^3 b_1^{(0)}(1+i) \tag{8.37}$$

whereas

$$T_1^{(1)\prime}(a)^{WF} = 2m^2(-ia_0(\bar{p}_2) + a_1(\bar{p}_2))(1-i) = m^3 b_1^{(0)}(1-i) \tag{8.38}$$

Taking the combination contributing to $S^{(1)}$ (see (7.28))

$$T_1^{(0)}(1)T_1^{(1)\prime}(a)^{WF} + T_1^{(1)}(a)^{WF}T_1^{(0)\prime}(1) = 2zm^5 b_1^{(0)}((1+i)(1-i) - (1+i)(1-i)) = 0 \tag{8.39}$$

one observes that the O(a) contributions from $u^{(1)}(\bar{p})$ cancel exactly in the 1-channel of gluon polarisations.

9. Improvement to One Loop Order Perturbation Theory

As pointed out in Section 7 the 1-channel of the S-matrix element considered there independently yields the tree level improvement condition $c^{(0)} = 1$. Due to the factorisation properties of the S-matrix this can be expected to persist, so the quantity for which O(a) improvement will be demanded to one loop order is in accordance with [12] chosen proportional to the residue of the pole of the S-matrix element

$$\begin{aligned}\lambda &= \frac{1}{m^4} z(-k,p_1')T_1T_1' \\ &= \frac{1}{m^4} z(-k,p_1')Z(k)^G \bar{u}_1(\bar{p}_1)\Gamma_3(\tilde{p}_1,\tilde{p}_1';\tilde{k},1)u_1(\bar{p}_1') \\ &\quad \cdot \bar{u}_1(\bar{p}_2)\Gamma_3(\tilde{p}_2,\tilde{p}_2';-\tilde{k},1)u_1(\bar{p}_2') \end{aligned} \tag{9.1}$$

where

$$\begin{aligned} p_1 &= m(i,1,1,i), \qquad p_1' = m(i,1,-1,-i), \qquad k = p - p' = 2m(0,0,1,i) \\ p_2 &= m(-i,1,-1,-i), \qquad p_2' = m(-i,1,1,i) \\ \text{and} \qquad \tilde{p} &= (p_0 + \sum_{\nu=1}^{\infty} g_0^{2\nu} E^{(\nu)}(\bar{p}),\bar{p}) \quad \text{etc.} \end{aligned} \tag{9.2}$$

whereas $Z(k)^G$ is the gluon wave function renormalization. For all quantities a perturbative expansion of the form $\lambda = \sum_{\nu=0}^{\infty} g_0^{2\nu}\lambda^{(\nu)}$, $T_\mu = g_0 \sum_{\nu=0}^{\infty} g_0^{2\nu} T_\mu^{(\nu)}$ etc. is assumed.
Using the notations introduced in Section 7 one finds up to one loop order

$$\begin{aligned}&\lambda^{(0)} + g_0^2\lambda^{(1)} = \\ &\frac{1}{m^4} z(-k,p_1')(T_1^{(0)}T_1^{(0)\prime}(Z^{(0)}(k)^G + g_0^2 Z^{(1)}(k)^G) + g_0^2(T_1^{(1)}T_1^{(0)\prime} + T_1^{(0)}T_1^{(1)\prime})Z^{(0)}(k)^G). \end{aligned} \tag{9.3}$$

Now the O(a)-corrections to $\lambda^{(1)}$ have to be eliminated by properly adjusting the g_0^2-part $c^{(1)}$ of the constant $c(g_0^2)$.
Since

$$\begin{aligned} Z^{(0)}(k)^G &= 1 + O(a^2) \\ Z^{(1)}(k)^G &= O(1) + O(a^2) \end{aligned} \tag{9.4}$$

and

$$T_1^{(0)}(a) = T_1^{(0)\prime}(a) = 0$$

due to tree level improvement one finds

$$\lambda^{(1)}(a) = \frac{1}{m^4} z(-k,p_1')(T_1^{(1)}(a)T_1^{(0)\prime}(1) - T_1^{(0)}(1)T_1^{(1)\prime}(a)). \tag{9.5}$$

In the previous chapter it was shown that the contributions to $\lambda^{(1)}(a)$ stemming from the fermion wave functions to one loop order and the insertion of $E^{(1)}(\bar{p})$ etc. into $\Gamma_3^{(0)}$ cancel. If $T_1^{(1)}(a)^{VF}$ and $T_1^{(1)\prime}(a)^{VF}$ denote the O(a) contributions to

$$\bar{u}_1^{(0)}(\bar{p}_1)\Gamma_3^{(1)}(p_1,p_1';k,1)u_1^{(0)}(\bar{p}_1') \qquad \text{and} \qquad \bar{u}_1^{(0)}(\bar{p}_2)\Gamma_3^{(1)}(p_2,p_2';-k,1)u_1^{(0)}(\bar{p}_2')$$

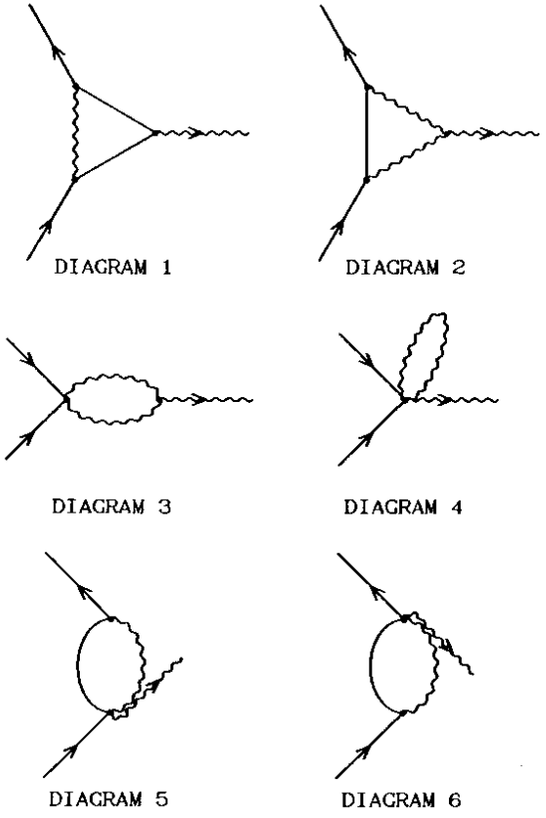
respectively one finally finds

$$\lambda^{(1)}(a) = \frac{1}{m^4} z(-k,p_1')(T_1^{(1)}(a)^{VF}T_1^{(0)\prime}(1) + T_1^{(0)}(1)T_1^{(1)\prime}(a)^{VF}). \tag{9.6}$$

The contribution of $c^{(1)}$ to $T_1^{(1)}(a)^{VF}$ and $T_1^{(1)\prime}(a)^{VF}$ can be obtained from the tree level computations of the last chapter

$$T_1^{(1)}(a)^{VF,c^{(1)}} = 2zm^3c^{(1)} \qquad T_1^{(1)\prime}(a)^{VF,c^{(1)}} = -2m^3c^{(1)} \tag{9.7}$$

whereas the following six diagrams contribute to to $T_1^{(1)}(a)^{VF}$ and $T_1^{(1)\prime}(a)^{VF}$



10. Computation of Feynman Integrals on a Lattice with twisted Boundary Conditions

The computation of one-loop Feynman integrals on a lattice with twisted boundary conditions amounts to the evaluation of

$$I(f) = \int_{p'} f(p')$$

for inner fermion lines or

$$I(f) = \int_{k'} f(k')$$

for inner gluon lines.
After properly taking into account the Clebsch-Gordan coefficients arising in the twisted world and the functions $\chi_{k'}$ suppressing the zero modes of gluon propagators the momentum sums over the two compactified dimensions can be performed by computer for a given lattice size L in these directions. This means that in these directions fermion momenta take the values

$$(2\nu + 1)m \qquad \nu = 1..LN$$

while gluon momenta are given by

$$2m\nu \qquad \nu = 1..LN.$$

The remaining two-dimensional integrals over the Brillouin zones of the continuous directions 0 and 3 are treated via one dimensional subintegrals using the exponentially convergent integration procedure described in [12]. To this end the integrand is summed over 2^n equidistant points

$$-\pi + \frac{2\pi}{n}\nu \qquad \nu = 1..2^n.$$

If the integrand has been correctly substituted so that no poles on the real axis occur these sums differ from the exact value of the integral only by a term

$$Ae^{-B2^n}, \qquad \text{B real.}$$

This is of course a benefit of the mass gap created by the introduction of the twisted boundary conditions for both quarks and gluons.
The summation is first done for $n = 1, 2, 3$. From the first three sums the constants A and B and hence an estimate for the error are determined. If it is greater than the requested absolute error for a subintegral, then n is increased by one i.e. the number of integration points doubled and another estimate is won.
To this end it of course suffices to determine only 2^{n-1} new values of the integrand. This is done up to a maximum of $n = 7$ (128 integration points).
Another benefit of this method is that the trigonometric functions occurring in the specific Feynman-integral can be tabulated beforehand, so that this time consuming calculation has to be done only once for the allowed maximum of 2^n equidistant points of integration.
After the integration procedure has terminated the result can be further improved by correcting it with the value of the estimated deviation from the exact result.
All this leads to an extremely efficient integration routine which allows for results of high

accuracy. With a maximum of 128 points in each continuous direction a precision of 12 digits can be gained without difficulty provided the integrand has been properly substituted.
The required computer time is kept sufficiently small so that even the lengthy integrands usually emerging in computations with improved actions can be performed.
Moreover a reliable estimate of the error can be won which is one of the major features of this procedure compared to other numerical integration routines.
It can be assumed [12] that the results obtained this way have an expansion of the form

$$I(f) = \sum_{\nu=0}^{\infty} (a_\nu + b_\nu \log((ma)^2))(ma)^{\delta+\nu} \tag{10.1}$$

where the value of δ depends on the superficial degree of divergence of f.
With 'a' set equal to one this results in a power series expansion of $I(f)$ in terms of m. To extract the values of the first coefficients a_ν and b_ν, $I(f)$ is evaluated for a whole range of lattice sizes L i.e. different values of $m = \frac{\pi}{LN}$.
For the calculation of the gauge invariant part of λ this was done for even values of L ranging up to 36 for $N = 2$ and 30 for $N = 3$ respectively.
The results obtained that way where then fitted with a behaviour of the form 10.1 and the values of a_ν and b_ν for small ν extracted.
It was found that in the integrals occurring in this computation the coefficients with odd ν vanish.
Although through the necessary fit procedure a significant loss in precision was encountered it was still possible to extract the values of a_0 and b_0 up to four digits.

11. Evaluation of the Feynman Diagrams contributing to the Vertex Function to One Loop Order Perturbation Theory

Here the procedure of obtaining the integrands is explained which were afterwards treated numerically using the methods described in the previous Section.
The contribution of each diagram will be abbreviated by

$$\bar{u}_1^{(0)}(\bar{p})(D\mu)u_1^{(0)}(\bar{p}') \qquad \mu = 1..6 \tag{11.1}$$

where $D\mu$ is obtained by combining the appropriate vertices and propagators.
Before the contraction with the polarisation vectors $\bar{u}_1^{(0)}, u_1^{(0)}$ the expressions for the integrands resulting from the various Feynman diagrams contributing to the vertex function have an expansion

$$\sum_{\nu=0}^{15}\left(\int N_\nu \frac{C_{CG}\lambda}{\mathrm{Den}^{D\mu}}\right)B_\nu \tag{11.2}$$

where $\mathrm{Den}^{D\mu}$ is the denominator produced by the propagators of the internal lines, C_{CG} is the Clebsch Gordan coefficient of the diagram, λ is the combination of the $\lambda_{k'}$ of all internal gluon lines and B_ν denotes the basis spanning the sixteen dimensional space of Dirac spinors

$$\begin{array}{llll} B_0 = \mathbb{1}, & B_1 = \gamma_0, & B_2 = \gamma_1, & B_3 = \gamma_2 \\ B_4 = \gamma_3, & B_5 = \gamma_5, & B_6 = \sigma_{01}, & B_7 = \sigma_{02} \\ B_8 = \sigma_{03}, & B_9 = \sigma_{12}, & B_{10} = \sigma_{13}, & B_{11} = \sigma_{23} \\ B_{12} = \gamma_5\gamma_0, & B_{13} = \gamma_5\gamma_1, & B_{14} = \gamma_5\gamma_2. & B_{15} = \gamma_5\gamma_3. \end{array} \tag{11.3}$$

This expansion was worked out analytically for each diagram.
Intermediate steps of the calculation were checked using the algebraic formula manipulation program REDUCE.
The final results from the analytical calculation which were coded into the computer by hand were then once again checked to prevent any errors occurring in the calculation or in the procedure of coding them. This was done numerically by inserting definite values for the internal momenta and comparing the results with expressions for the numerator of the specific diagram produced again with REDUCE by directly combining the appropriate vertices and propagators entering the diagram.
After this check the results were then used to produce the numerator of the Feynman integrals for the various N_ν's.
The vertex function has the form

$$\bar{u}_1^{(0)}(\bar{p})\Gamma^{(1)}(p,p';k,1)u_1^{(0)}(\bar{p}') = \bar{u}_1^{(0)}\Big(\sum_{\nu=0}^{15} O_\nu\Big)B_\nu\Big)u_1^{(0)}. \tag{11.4}$$

As for the tree level case the continuum limit terms are localised in the O_ν with $\nu = 1..4, 12..15$ while the O(a)-corrections that one is interested in are given by the O_ν with $\nu = 0, 5..11$. Moreover it is found that the O_ν with $\nu = 0, 5$ only contribute $\mathrm{O}(\mathrm{a}^3)$ corrections to the continuum limit. As the final multiplication with the polarisation vectors $\bar{u}_1^{(0)}, u_1^{(0)}$ projects on the O_ν with $\nu = 0, 1, 9, 15$ it suffices to calculate for every diagram the contribution to O_9 and determine its first order coefficients a_0,b_0 as explained in the previous Section.

It is thus an interesting result of the calculation that the spinor structure of the three point function under consideration allows for an analytical separation of the O(a) corrections from the leading order contribution which therefore doesn't have to be calculated with high numerical precision. Furthermore the tedious task to separate the first order corrections from the leading terms by means of a fit procedure is avoided which would have necessarily led to a substantial loss in numerical precision.
Moreover it was found that similar to the tree level the O(a) corrections to $T_1^{(1)}$ and $T_1^{(1)\prime}$ are the same up to a factor of $-z$. Therefore only the contributions to $T_1^{(1)}(a)$ had to be evaluated numerically to high precision. In the following Sections the results emerging for the gauge invariant and gauge variant contributions to $T_1^{(1)}(a)$ of each diagram are given for $N = 2$ and $N = 3$.
As explained above this amounts to the determination of the $\mathrm{O}(m)$ and $\mathrm{O}(m\log m^2)$ contributions a_0, b_0 of each diagram to the coefficient O_9 for the external momenta

$$p = m(i,1,1,i) \qquad p' = m(i,1,-1,-1) \qquad k = 2m(0,0,1,i). \tag{11.5}$$

For all one loop diagrams a_0 and b_0 have an expansion in the gauge fixing parameter $\alpha - 1$ of the form

$$a_0 = \sum_{\nu=0}^{2} a_0^{(\alpha-1)^\nu}(\alpha-1)^\nu \tag{11.6}$$

and

$$b_0 = \sum_{\nu=0}^{2} b_0^{(\alpha-1)^\nu}(\alpha-1)^\nu \tag{11.7}$$

respectively.
The coefficients $a_0^{(\alpha-1)^0}$, $b_0^{(\alpha-1)^0}$ will be called the gauge invariant contribution whereas the remaining terms will be referred to as the gauge variant contributions to a_0, b_0.
On the results the following checks can be made.
Because of the general covariant gauge fixing, gauge invariance can explicitly be checked. Furthermore the logarithmic contributions to $c^{(1)}$ must cancel. Moreover logarithmic contributions to leading order must vanish, which means that the sum of the coefficients b_0 have to vanish. These features provide an excellent consistency check especially on the combinatorial factors attached to each diagram.

12. Conclusions

Collecting the contributions of all diagrams[1] one finds

N=2				
Diagram	a_0^{GI}	b_0^{GI}	$a_0^{GV,(\alpha-1)}$	$b_0^{GV,(\alpha-1)}$
1	0.03864(2)	0.0063301(8)	0.0096833(1)	-
2	−0.01630(1)	−0.0189979(2)	−0.017895(5)	−0.0031652(8)
3	0.01512(1)	0.0189979(2)	0.017899(3)	0.0031661(8)
4	0.34860013(1)	-	0.06132780(1)	-
5	−0.03903(2)	−0.0031652(8)	−0.035505569(1)	-
6	−0.03903(2)	−0.0031652(8)	−0.035505569(1)	-
sum	0.30800(8)	−0.0000003(28)	0.000004(8)	0.0000009(16)

(12.1)

and

N=3		
Diagram	a_0^{GI}	b_0^{GI}
1	$0.05787(4) - i\ 0.0766861(1)$	0.004212(4)
2	$-0.03268(4) + i\ 0.0329050(8)$	−0.028500(4)
3	$0.03087(1) - i\ 0.0329050(8)$	0.028496(2)
4	0.59478907(1)	
5	$-0.05953(2) + i\ 0.0383431(1)$	−0.002108(1)
6	$-0.05953(2) + i\ 0.0383431(1)$	−0.002108(1)
sum	0.53179(14)	−0.000008(12)

N=3		
Diagram	$a_0^{GV,(\alpha-1)}$	$b_0^{GV,(\alpha-1)}$
1	$0.0064555(1) - i\ 0.0221846(1)$	-
2	−0.02819(1)	−0.004747(1)
3	0.02820(1)	0.004747(1)
4	0.08392226(4)	-
5	$-0.0451889(1) + i\ 0.0110912(1)$	-
6	$-0.0451889(1) + i\ 0.0110912(1)$	-
sum	$0.00001(2) - i\ 0.0000002(3)$	-

(12.2)

The result has all desired features. It is explicitly gauge invariant and both the logarithmic and complex contributions of individual diagrams cancel.

[1] A more detailed discussion of the evaluation of the individual diagrams is given in the author's thesis.

This cumulated presentation of the results for the individual diagrams displays the following features.

There is a natural grouping of diagrams if one takes into consideration the various cancellations taking place.

The first group of closely related diagrams are diagrams 2 and 3. Among them the gauge variant parts, the leading logarithms of the gauge invariant parts as well as the complex parts of a_0^{GI} for $N = 3$ cancel.

The second group of diagrams are diagrams 1,5 and 6. Among them the leading logarithms of the gauge invariant parts and the complex parts of a_0^{GI}, a_0^{GV} cancel. To perform also the cancellation of the real part of the gauge variant part however the contribution of diagram 4 is needed.

Taking into consideration the remarks made in Section 11 on the dependency of $T_1^{(1)}(a)^{VF}$ and $T_1^{(1)\prime}(a)^{VF}$ one eventually finds for the value of $\lambda^{(1)}(a)$ defined in Section 9

$$\lambda^{(1)}(a) = 4m \begin{cases} 0.308000(8) - 2c^{(1)} & N = 2 \\ 0.53179(14) - 2c^{(1)} & N = 3. \end{cases} \tag{12.3}$$

This gives the desired value of $c^{(1)}$ as

$$c^{(1)} = \begin{cases} 0.15400(4) & N = 2 \\ 0.26590(7) & N = 3. \end{cases} \tag{12.4}$$

Although the cancellations described above already provide a decent check on the calculation and all steps of the analytical work involved have been verified by computer, one would like to have a double check on the obtained result.

The natural way to do this would be a totally independent calculation. Here a compromise method was chosen.

As mentioned in Section 6 in addition to the lowest lying X quarks there are also the stable Y quarks. $\lambda^{(1)}(a)$ was therefore calculated again using a different momentum configuration involving these Y quarks, i.e. for external momenta

$$\begin{aligned} p_1 &= m(\ \ i,1,3,3i) & p_1' &= m(\ \ i,1,1,\ i) \\ p_2 &= m(-i,1,1,\ i) & p_2' &= m(-i,1,3,3i) \end{aligned} \tag{12.5}$$

and

$$\begin{aligned} p_1 + p_1' &= 2m(i,1,2,2i), \qquad p_2 + p_2' = 2m(-i,1,2,2i) \\ k_1 &= p_1 - p_1' = p_2' - p_2 = 2m(0,0,1,i) \\ k_2 &= p_1 - p_2' = p_1' - p_2 = 2m(i,0,0,0). \end{aligned} \tag{12.6}$$

instead of (7.16), (7.17).

For the reasons given below this calculation was however performed only for $N = 2$. Along the same lines as in Section 8 for the X quarks one finds that the contribution from the wave functions $u^{(1)}(\bar{p})$ and from the insertion of $E^{(1)}(\bar{p})$ into $\Gamma_3^{(0)}$ vanish within the error bars. The explicit calculation given in the Appendix yields a numerical contribution to $\lambda^{(1)}(a)$ of

$$\lambda^{(1)}(a)^{WF} = m \quad 0.00060(540). \tag{12.7}$$

Comparing the result (12.7) with the contributions to $\lambda^{(1)}$ given in (12.1), (12.8) shows that this contribution can indeed only be excluded within the error bars.

The observation that $T_1^{(1)}(a)^{VF}$ and $T_1^{(1)\prime}(a)^{VF}$ differ only by a factor of $-z$ persists, so again only the diagrams contributing to $T_1^{(1)}(a)^{VF}$ have to be evaluated to high numerical precision.
Since the Y mass is rather close to the threshold for production of a X quark and a A meson it is much more delicate to find the right substitution of the integrands. This leads to a slower rate of overall convergence compared to the case considered before. Moreover because of the considerable loss in symmetry in the external fermion lines the integrands become more complicated which too increases the amount of computer time needed for their numerical evaluation.
The recalculation of $T_1^{(1)}(a)^{VF}$ was therefore performed only for $N = 2$ and up to lower values of L and thus to lower numerical precision. The results are summarized in the following table

N=2				
Diagram	a_0^{GI}	b_0^{GI}	$a_0^{GV,(a-1)}$	$b_0^{GV,(a-1)}$
1	0.06316(8)	0.006348(15)	0.0096833(1)	-
2	−0.01629(4)	−0.018997(1)	−0.017889(8)	−0.0031648(8)
3	0.01512(1)	0.0189981(2)	0.017899(3)	0.0031661(8)
4	0.34860013(1)	-	0.06132780(1)	-
5	−0.04602(4)	−0.0031662(5)	−0.035505569(1)	-
6	−0.05687(8)	−0.003167(1)	−0.035505569(1)	-
sum	0.30770(25)	0.000015(18)	0.000010(11)	0.000013(16)

(12.8)

Comparing (12.8) with (12.1) the following observations can be made.
All diagrams that depend only on gluon momenta don't change their values. This is true for most of the gauge dependent parts and for the gauge independent parts of diagram 3 and 4. The degeneracy of the gauge independent part of diagram 5 and 6 is lifted. The grouping of diagrams described above however persists. Although the numerical precision of the second calculation is worse it nevertheless turns out that within the error bars the same result is found as for the original configuration of external momenta. The result obtained for the value of the improvement coefficient to one loop order perturbation theory is therefore believed to be trustworthy.
As already mentioned in the discussion of diagram 4 this diagram has all features of the total result.
Moreover this diagram yields the main contribution to the coefficient $c^{(1)}$. The sum of all other diagrams only amounts to a 10 % correction to its contribution.
So altogether this diagram gives a 10 % precise estimate of the final result at the price of only 2 % the effort. If one would wish to get a notion of the value of $c(g_0^2)$ to two loop order, one would therefore suggest to first evaluate only the diagrams containing closed gluon loops. The effort needed to do this could be expected to be comparable to the work presented here and one could hope to find a good estimate for the order of magnitude of higher order corrections.

As a last point the influence of the artificial "smell" group $SU(N_S)$ introduced in Section 4 on the value of $c^{(1)}$ obtained in this Section has to be studied.
To this end it suffices to note that in all diagrams contributing to the one loop calculation of $\lambda^{(1)}(a)$ no closed fermion loops appear, so no dependence on N_S occurs.

Acknowledgement
I would like to thank Dr. P. Weisz for numerous discussions and Prof. Dr. M. Lüscher for various important suggestions.

APPENDIX A

The purpose of this Appendix is to derive formulae for the first order contributions $E^{(1)}(\bar{p})$ to the fermion energy and $Z^{(1)}(\bar{p})$ to the fermion wave function renormalization.
To this end the pole structure of the fermion two point function is studied in more detail.
At the pole the fermion two-point function has the expansion

$$g_2(p) = \frac{Z(\bar{p})}{\omega^2 + E(\bar{p})^2} + O(1). \tag{A.1}$$

To lowest order (A.1) yields

$$S_F(p) = \frac{Z^{(0)}(\bar{p})}{\omega^2 + E^{(0)}(\bar{p})^2} + O(1). \tag{A.2}$$

In order to determine the O(1)-term in (A.2) one sets

$$S_F(p) = \frac{N_F(p)}{D_F(p)}. \tag{A.3}$$

$N_F(p)$ and $D_F(p)$ have the following expansions

$$N_F(p) = N_0(\bar{p}) + (\omega - iE^{(0)}(\bar{p}))N_1(\bar{p}) - ... \tag{A.4}$$
$$D_F(p) = D_0(\bar{p})(\omega^2 + E^{(0)}(\bar{p})^2) + D_1(\bar{p})(\omega^2 + E^{(0)}(\bar{p})^2)^2 + ... \tag{A.5}$$

with

$$N_0(\bar{p}) = D_0(\bar{p})Z^{(0)}(\bar{p}) = \gamma_0 \sinh E^{(0)}(\bar{p}) - i\sum_{\nu=1}^{3} \sin p_\nu + 2(-\sinh^2 \frac{E^{(0)}(\bar{p})}{2} + \sum_{\nu=1}^{3} \sin^2 \frac{p_\nu}{2}) \tag{A.6}$$

$$N_1(\bar{p}) = \frac{\partial}{\partial\omega} N_F(p) \mid_{\omega=iE^{(0)}(\bar{p})} = -i\gamma_0 \cosh E^{(0)}(\bar{p}) + \sinh E^{(0)}(\bar{p}) \tag{A.7}$$

$$D_0(\bar{p}) = \frac{1}{2iE^{(0)}(\bar{p})}(\frac{\partial}{\partial\omega} D_F(p)) \mid_{\omega=iE^{(0)}(\bar{p})} = \frac{\sinh 2E^{(0)}(\bar{p})}{2E^{(0)}(\bar{p})} + O(a^4) = 1 - \frac{2}{3}E^{(0)}(\bar{p})^2 + O(a^4) \tag{A.8}$$

$$D_1(\bar{p}) = \frac{1}{8E^{(0)}(\bar{p})^2}(2D_0(p) - (\frac{\partial^2}{\partial\omega^2} D_F(p)) \mid_{\omega=iE^{(0)}(\bar{p})}) = \frac{1}{8E^{(0)}(\bar{p})^2}(2 - \frac{4}{3}E^{(0)}(\bar{p})^2 - 2\cosh 2E^{(0)}(\bar{p}) - 2\sinh^2 E^{(0)}(p)) + O(a^2) = -\frac{1}{12} + O(a^2). \tag{A.9}$$

One therefore has

$$S_F(p) = \frac{Z^{(0)}(\bar{p})}{\omega^2 + E^{(0)}(\bar{p})^2} + \frac{1}{2iE^{(0)}(\bar{p})}N_1(\bar{p}) - Z^{(0)}(\bar{p})\frac{D_1(\bar{p})}{D_0(\bar{p})} + O(a^2) + O(\omega - E^{(0)}(\bar{p})) = \frac{Z^{(0)}(\bar{p})}{(\omega - iE^{(0)}(\bar{p}))(2iE^{(0)}(\bar{p}))} + \frac{N_1(\bar{p})}{2iE^{(0)}(\bar{p})}) + Z^{(0)}(\bar{p})(\frac{1}{12} + \frac{1}{4E^{(0)}(\bar{p})^2}) + O(a^2) + O(\omega - E^{(0)}(\bar{p})). \tag{A.10}$$

To one loop order the relation

$$g_2(p) = S_F(p) + S_F(p)\Sigma^{(1)}(p)(p)S_F(p) \tag{A.11}$$

holds.
By inserting (A.1) and (A.10) one finds

$$\frac{Z^{(0)}(\bar{p})}{\omega^2 + E^{(0)}(\bar{p})^2} + \frac{Z^{(1)}(\bar{p})}{\omega^2 + E^{(0)}(\bar{p})^2} - \frac{2E^{(0)}(\bar{p})E^{(1)}(\bar{p})}{(\omega^2 + E^{(0)}(\bar{p})^2)^2} + O(1) = \frac{Z^{(0)}(\bar{p})}{\omega^2 + E^{(0)}(\bar{p})^2} + \frac{1}{(\omega^2 + E^{(0)}(\bar{p})^2)^2} Z^{(0)}(\bar{p})\Sigma^{(1)}(p)(p)Z^{(0)}(\bar{p}) + \frac{1}{6(\omega^2 + E^{(0)}(\bar{p})^2)} Z^{(0)}(\bar{p})\Sigma^{(1)}(p)(p)Z^{(0)}(\bar{p}) - \frac{1}{(\omega^2 - E^{(0)}(\bar{p})^2)2iE^{(0)}(\bar{p})}[N_1(\bar{p})\Sigma^{(1)}(p)Z^{(0)}(\bar{p}) + Z^{(0)}(\bar{p})\Sigma^{(1)}(p)N_1(\bar{p})] + O(1). \tag{A.12}$$

From this one deduces the following relation for $E^{(1)}(\bar{p})$

$$E^{(1)}(\bar{p})Z^{(0)}(\bar{p}) = -\frac{1}{2E^{(0)}(\bar{p})} Z^{(0)}(\bar{p})\Sigma^{(1)}(p) \mid_{\omega=iE^{(0)}(\bar{p})} Z^{(0)}(\bar{p}). \tag{A.13}$$

At the pole one finds for the residue $Z^{(1)}(\bar{p})$

$$\frac{Z(\bar{p})}{2E(\bar{p})} = \frac{Z^{(0)}(\bar{p})}{2E^{(0)}(\bar{p})} + \frac{Z^{(1)}(\bar{p})}{2E^{(0)}(\bar{p})} - \frac{Z^{(0)}(\bar{p})}{2E^{(0)}(\bar{p})^2} E^{(1)}(\bar{p}) - O(g_0^4) + \frac{Z^{(0)}(\bar{p})}{2E^{(0)}(\bar{p})} - \frac{1}{4iE^{(0)}(\bar{p})^2} Z^{(0)}(\bar{p})(\frac{\partial}{\partial\omega}\Sigma^{(1)}(p)) \mid_{\omega=iE^{(0)}(\bar{p})} Z^{(0)}(\bar{p}) - \frac{1}{E^{(0)}(\bar{p})} RZ^{(0)}(\bar{p})\Sigma^{(1)}(p) \mid_{\omega=iE^{(0)}(\bar{p})} Z^{(0)}(\bar{p}) - \frac{1}{4iE^{(0)}(\bar{p})^2} N_1(\bar{p})\Sigma^{(1)}(p) \mid_{\omega=iE^{(0)}(\bar{p})} Z^{(0)}(\bar{p}) - Z^{(0)}(\bar{p})\Sigma^{(1)}(p) \mid_{\omega=iE^{(0)}(\bar{p})} N_1(\bar{p})] + O(a^2) \tag{A.14}$$

where the abbreviation

$$R = \frac{1}{12} - \frac{1}{4E^{(0)}(p)^2}$$

has been introduced.
So finally one finds for $Z^{(1)}(\bar{p})$

$$Z^{(1)}(\bar{p}) = \frac{1}{2iE^{(0)}(\bar{p})} Z^{(0)}(\bar{p}) \left(\frac{\partial}{\partial\omega}\Sigma^{(1)}(p)\right)|_{\omega=iE^{(0)}(\bar{p})} Z^{(0)}(\bar{p}) + \frac{1}{2iE^{(0)}(\bar{p})}[N_1(\bar{p})\Sigma^{(1)}(p)|_{\omega=iE^{(0)}(\bar{p})} Z^{(0)}(\bar{p}) + Z^{(0)}(\bar{p})\Sigma^{(1)}(p)|_{\omega=iE^{(0)}(\bar{p})} N_1(\bar{p})] + O(a^3). \quad \text{(A.15)}$$

APPENDIX B

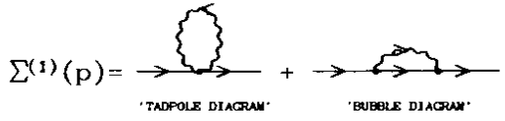
Here the fermion self energy is calculated to one loop order.
For

$$\Sigma^{(1)}(p)|_{p_0=iE^{(0)}(\bar{p})}$$

numerical values for the momentum configurations studied are derived.
For

$$\frac{\partial}{\partial p_0}(\Sigma^{(1)}(p))|_{p_0=iE^{(0)}(\bar{p})}$$

only a qualitative discussion is given, because for this quantity it suffices to know that it does not yield O(a)-corrections to $Z^{(1)}(\bar{p})$. Diagramatically one has



First the contribution of the tadpole diagram is treated.
The gauge variant contribution yields

$$\begin{aligned}\Sigma^{(1)}(p)^{TP,GV} &= -\frac{(\alpha-1)}{4}\int_{k'}\frac{\lambda_{k'}z(k',k')}{(\hat{k}'^2)^2}\hat{k}'_\mu\hat{k}'_\nu V_2^{FG}(p,p;k',\mu;-k',\nu)\\ &= \frac{(\alpha-1)}{4}\int_{k'}\frac{\lambda_{k'}\hat{k}'^2_\mu}{(\hat{k}'^2)^2}(i\gamma_\mu\sin p_\mu-\cos p_\mu) \qquad \text{(B.1)}\\ &= \frac{(\alpha-1)}{4}(i\gamma_\mu\sin p_\mu-\cos p_\mu)P_{2\mu}\end{aligned}$$

where

$$P_{2\mu} = \int_{k'}\frac{\lambda_{k'}\hat{k}'^2_\mu}{(\hat{k}'^2)^2} \qquad \mu = 0\ldots 3. \qquad \text{(B.2)}$$

Symmetry requires

$$P_{20} = P_{23} \quad \text{and} \quad P_{21} = P_{22}. \qquad \text{(B.3)}$$

Now[1]

$$P_{20} = \begin{cases} 0.0581000218(6) - 0.0111778(1)m^2 + O(a^4) & N=2 \\ 0.10328893(1) - 0.026576(5)m^2 + O(a^4) & N=3 \end{cases} \qquad \text{(B.4)}$$

[1] The leading terms in the small 'a' expansion of these and other gluon integrals in the twisted world can be obtained from the well known values [21] of the corresponding integrals in the continuum by multiplication with a factor $\frac{N}{N^2-1}$.

and

$$P_{21} = \begin{cases} 0.0581000210(6)+O(a^4) & N=2 \\ 0.10328892(1)+O(a^4) & N=3. \end{cases} \quad \text{(B.5)}$$

The gauge invariant part of the tadpole diagram yields

$$\Sigma^{(1)}(p)^{TP,GI} = \frac{1}{4}(\sum_\mu i\gamma_\mu \sin p_\mu - \cos p_\mu)P_1 \quad \text{(B.6)}$$

where

$$P_1 = \int_{k'} \frac{\chi_{k'}}{\hat{k}'^2}. \quad \text{(B.7)}$$

Numerically one finds

$$P_1 = \begin{cases} 0.232400086(1)-0.023552(4)m^2+O(a^4) & N=2 \\ 0.4131557(1)-0.053149(5)m^2+O(a^4) & N=3. \end{cases} \quad \text{(B.8)}$$

The contribution of the bubble diagram is more involved. The gauge variant part however can be considerably simplified using the Slavnov Taylor identity (5.6)

$$\begin{aligned}\Sigma^{(1)}(p)^{BD,GV} &= \\ &-\frac{(\alpha-1)}{2}\int_{k'} \frac{\chi_{k'} z(k',k')}{(\hat{k}'^2)^2}\hat{k}'_\mu \hat{k}'_\nu V_1^{FG}(p,p-k';k',\mu)S_F(p-k')V_1^{FG}(p-k',p;-k',\nu) \\ &= -\frac{(\alpha-1)}{2}\int_{k'} \frac{\chi_{k'}}{(\hat{k}'^2)^2}(2S_F^{-1}(p) - S_F^{-1}(p)S_F(p-k')S_F^{-1}(p) - S_F^{-1}(p-k')). \quad \text{(B.9)}\end{aligned}$$

First the second term of the integrand is studied. Numerical evaluation yields

$$\frac{1}{2}\Big(\int \frac{\chi_{k'}}{(\hat{k}'^2)^2} S_F^{-1}(p)S_F(p-k')S_F^{-1}(p)\Big)\Big|_{p_0=iE^{(0)}(\bar{p})} = i(\gamma_\mu \sin p_\mu)\Big|_{p_0=iE^{(0)}(\bar{p})} P(\bar{p}) + O(a^3) \quad \text{(B.10)}$$

where

$$P(\bar{p}) = \begin{cases} 0.0465464(5)-0.00949686(3)\log m^2+O(a^4) & N=2 \\ 0.0048071(3)-0.01688686(3)\log m^2+O(a^4) & N=3 \end{cases} \quad \text{(B.11)}$$

for X quarks with momenta

$$m(\pm i, \pm 1, \pm 1, \pm i) \quad \text{(B.12)}$$

and

$$P(\bar{p}) = \begin{cases} 0.01077(1)-0.00949686(3)\log m^2+O(a^4) & N=2 \end{cases} \quad \text{(B.13)}$$

for Y quarks with momenta

$$m(\pm i, \pm 1, \pm 3, +3i). \quad \text{(B.14)}$$

Altogether one finds for the gauge variant contribution of the bubble diagram

$$\begin{aligned}&\Sigma^{(1)}(p)^{BD,GV}\Big|_{p_0=iE^{(0)}(\bar{p})} = \\ &-\frac{\alpha-1}{2} i\gamma_\mu \sin p_\mu (2P_2 - P(\bar{p}) - P_{3\mu}) - \frac{1}{2}\cos p_\mu P_{2\mu} - \sin p_\mu P_{4\mu}\Big|_{p_0=iE^{(0)}(\bar{p})} + O(a^2) \quad \text{(B.15)}\end{aligned}$$

with

$$P_2 = \int_{k'} \frac{\chi_{k'}}{(\hat{k}'^2)^2}$$

$$P_{3\mu} = \int_{k'} \frac{\chi_{k'} \cos k'_\mu}{(\hat{k}'^2)^2} \qquad \mu = 0\ldots 3, \qquad P_{30} = P_{33}, \qquad P_{31} = P_{32} \quad \text{(B.16)}$$

$$P_{4\mu} = \int_{k'} \frac{\chi_{k'} \sin k'_\mu}{(\hat{k}'^2)^2} \qquad \mu = 0\ldots 3, \qquad P_{40} = P_{43}, \qquad P_{41} = P_{42}.$$

Numerical evaluation of the integrals yields

$$P_2 = \begin{cases} 0.03505688(2)-0.009498859(2)\log m^2+0.0027937(5)m^2+O(a^4) & N=2 \\ 0.0591581(1)-\ 0.01688686(3)\log m^2+\ 0.006640(4)m^2+O(a^4) & N=3, \end{cases} \quad \text{(B.17)}$$

$$P_{30} = \begin{cases} 0.0060068654(4)-0.009498859(2)\log m^2+0.0027937(5)m^2+O(a^4) & N=2 \\ 0.00751354(7)-\ 0.01688686(3)\log m^2+\ 0.006640(4)m^2+O(a^4) & N=3 \end{cases} \quad \text{(B.18)}$$

$$P_{31} = P_{30} + O(a^4) \quad \text{(B.19)}$$

and

$$P_{40} = P_{41} = 0 \quad \text{(B.20)}$$

Using (B.18),(B.19) (B.15) can be furthermore simplified

$$\begin{aligned}&\Sigma^{(1)}(p)\Big|_{p_0=iE^{(0)}(\bar{p})}^{BD,GV} = \\ &-\frac{\alpha-1}{2}[i\gamma_\mu \sin p_\mu (2P_2 - P(\bar{p}) - P_{3\mu}) - \frac{1}{2}\cos p_\mu P_{2\mu}]\Big|_{p_0=iE^{(0)}(\bar{p})} + O(a^2) \quad \text{(B.21)}\end{aligned}$$

so that altogether the gauge variant contribution to $\Sigma^{(1)}(p)$ reads

$$\Sigma^{(1)}(p)^{GV}\Big|_{p_0=iE^{(0)}(\bar{p})} = \frac{\alpha-1}{2} i(\gamma_\mu \sin p_\mu)\Big|_{p_0=iE^{(0)}(\bar{p})}[P_{2\mu} - 2P_2 + P(\bar{p}) + P_{3\mu}] \quad \text{(B.22)}$$

The gauge invariant part of the bubble diagram yields

$$\Sigma^{(1)}(p)^{BD,GI} = -\frac{1}{2}\int_{k'} \frac{\chi_{k'}(i\gamma_\mu A_\mu - B)}{\hat{k}'^2(\sum_\rho \sin^2(p-k')_\rho - 4(\sum_\rho \sin^2 \frac{(p-k')_\rho}{2})^2)} \quad \text{(B.23)}$$

with

$$\begin{aligned}B &= (2\sum_\mu \cos(2p-k')_\mu - \frac{1}{2}\cos^2\frac{k'_\mu}{2}\sum_{\rho\neq\mu}\sin^2 k'_\rho)\sum_\nu \sin^2\frac{(p-k')_\nu}{2} \\ &-\sum_\mu \sin(2p-k')_\mu \sin(p-k')_\mu - \frac{1}{2}\sum_\mu(\cos p_\mu + \cos(p-k')_\mu)\sum_{\nu\neq\mu}\sin(p-k')_\nu \sin k'_\nu \quad \text{(B.24)}\end{aligned}$$

and

$$A_\mu = 2\sin(p-k')_\mu(1+\sin^2(p-\frac{k'}{2})_\mu)$$
$$-(2\sin(2p-k')_\mu - \sin k'_\mu \sum_{\rho\neq\mu}(\cos p_\rho + \cos(p-k')_\rho)\sum_\nu \sin^2\frac{(p-k')_\nu}{2}$$
$$+\frac{1}{2}\sin k'_\mu \sum_\nu \sin(p-k')_\nu(\sin p_\nu + \sin(p-k')_\nu)$$
$$-\frac{1}{2}(\cos p_\mu + \cos(p-k')_\mu)\sum_\nu \sin(p-k')_\nu \sin k'_\nu$$
$$-\frac{1}{2}\sin k'_\mu \sum_\nu \cos^2\frac{k'_\nu}{2}\sum_{\rho\neq\nu}\sin k'_\rho \sin(p-k')_\rho$$
$$+\frac{1}{2}\cos^2\frac{k'_\mu}{2}\sum_\nu \sin k'_\nu(\sin(p-k')_\nu \sin k'_\mu - \sin(p-k')_\mu \sin k'_\nu)$$
$$+\frac{1}{4}\sin(p-k')_\mu \sum_\nu \cos^2\frac{k'_\nu}{2}\sum_{\rho\neq\nu}\sin^2 k'_\rho \qquad \mu = 0\ldots 3. \quad \text{(B.25)}$$

Combining this result with the contribution from the tadpole diagram one gets

$$\Sigma^{(1)}(p)^{GI}|_{p_0=iE^{(0)}(\bar{p})} = i\frac{P_1}{4}(\gamma_\mu \sin p_\mu)|_{p_0=iE^{(0)}(\bar{p})} + ia\not{q}(\bar{p}) + b(\bar{p}), \quad \text{(B.26)}$$

where the contribution of

$$-\frac{P_1}{4}(\sum_\mu \cos p_\mu)|_{p_0=iE^{(0)}(\bar{p})}$$

from the tadpole has been absorbed into the constant $b(\bar{p})$.
Evaluating the integrals gives the following remarkable property. The numerical results for the leading orders of $b(\bar{p})$ are independent of the particular momentum configuration under consideration.
One finds

$$b(\bar{p}) = b_0^{(0)} + b_1^{(0)}m^2 + O(m^4), \quad \text{(B.27)}$$

where

$$b_0^{(0)} = -\frac{N^2-1}{2N} \quad 0.2025565(4)$$
$$b_1^{(0)} = \frac{N^2-1}{2N} \quad 0.0093304(8) \quad \text{(B.28)}$$

both for X quarks with momenta given by (B.12) and for Y quarks with momenta given by (B.14). The numerical evaluation of the $a_\mu(\bar{p})$ yields different values for X and Y quarks, i.e.

$$\frac{a_0(\bar{p})}{\sinh E^{(0)}(\bar{p})} = \begin{cases} 0.01001(2) - 0.004749(1)\log m^2 + O(a^2) & N=2 \\ 0.01602(10) - 0.008444(5)\log m^2 + O(a^2) & N=3 \end{cases} \quad \text{(B.29)}$$

$$\frac{a_1(\bar{p})}{\sin p_1} = \begin{cases} 0.00650(1) - 0.004749(1)\log m^2 + O(a^2) & N=2 \\ 0.00961(8) - 0.008444(5)\log m^2 + O(a^2) & N=3 \end{cases} \quad \text{(B.30)}$$

$$\frac{a_2(\bar{p})}{\sin p_2} = \frac{a_1(\bar{p})}{\sin p_1}$$
$$\frac{a_3(\bar{p})}{\sin p_3} = \frac{a_0(\bar{p})}{\sinh E^{(0)}(\bar{p})} \quad \text{(B.31)}$$

for X quarks with the above momenta.
For Y quarks only the case $N=2$ was considered

$$\frac{a_0(\bar{p})}{\sinh E^{(0)}(\bar{p})} = 0.01502(4) - 0.004745(5)\log m^2 + O(a^2) \qquad N=2 \quad \text{(B.32)}$$

$$\frac{a_1(\bar{p})}{\sin p_1} = 0.01819(4) - 0.004740(10)\log m^2 + O(a^2) \qquad N=2 \quad \text{(B.33)}$$

$$\frac{a_2(\bar{p})}{\sin p_2} = 0.01392(6) - 0.004747(5)\log m^2 + O(a^2) \qquad N=2 \quad \text{(B.34)}$$

$$\frac{a_3(\bar{p})}{\sin p_3} = 0.01500(2) - 0.004752(3)\log m^2 + O(a^2) \qquad N=2 \quad \text{(B.35)}$$

Nevertheless both for X and Y quarks the general relation

$$(\sin p_\mu a_\mu(\bar{p}))|_{p_0=iE^{(0)}(\bar{p})} = b_1^{(0)}m^2 \quad \text{(B.36)}$$

holds which is the reason why both the $O(\frac{1}{a})$ and O(a) effects in $Z^{(1)}(\bar{p})$ and $u^{(1)}(\bar{p})$ can be cancelled by a universal subtraction of

$$g_0^2 b_0 \bar{\Psi}(x)\Psi(x).$$

In Appendix A it was shown that the contributions of

$$(\frac{\partial}{\partial p_0}\Sigma^{(1)}(p))|_{p_0=iE^{(0)}(\bar{p})}$$

to $Z^{(1)}(\bar{p})$ have the form

$$\frac{1}{2iE^{(0)}(\bar{p})}Z^{(0)}(\bar{p})(\frac{\partial}{\partial p_0}\Sigma^{(1)}(p))|_{p_0=iE^{(0)}(\bar{p})}Z^{(0)}(\bar{p}), \quad \text{(B.37)}$$

so only these combinations will be considered. Moreover the relations (8.8) will frequently be used.
The contribution of the tadpole diagram is again simple. It reads

$$\frac{1}{2iE^{(0)}(\bar{p})}Z^{(0)}(\bar{p})(\frac{\partial}{\partial p_0}\Sigma^{(1)}(p)^{TP}|_{p_0=iE^{(0)}(\bar{p})}Z^{(0)}(\bar{p}) = -\frac{1}{4}Z^{(0)}(\bar{p})(P_1 + (\alpha-1)P_{20}) + O(a^2) \quad \text{(B.38)}$$

i.e. no O(a) corrections appear.
For the gauge variant contribution of the bubble diagram one finds

$$\frac{\partial}{\partial p_0}\Sigma^{(1)}(p)^{BD,GV} = -\frac{(\alpha-1)}{2}\int_{k'}\frac{\lambda_{k'}}{(\hat{k}'^2)^2}\Big\{2(i\gamma_0\cos p_0 - \sin p_0)$$
$$-(i\gamma_0\cos p_0 + \sin p_0)S_F(p-k')S_F^{-1}(p) - S_F^{-1}(p)S_F(p-k')(i\gamma_0\cos p_0 + \sin p_0)$$
$$-S_F^{-1}(p)(\frac{\partial}{\partial p_0}S_F(p-k'))S_F^{-1}(p) - (i\gamma_0\cos(p-k')_0 + \sin(p-k')_0)\Big\}. \quad \text{(B.39)}$$

From this one deduces that

$$\frac{1}{2iE^{(0)}(\bar{p})}Z^{(0)}(\bar{p})(\frac{\partial}{\partial p_0}\Sigma^{(1)}(p)^{BD,GV})|_{p_0=iE^{(0)}(\bar{p})}Z^{(0)}(\bar{p})$$
$$=(\alpha-1)Z^{(0)}(\bar{p})(P_2-\frac{P_{30}}{2})+O(a^2) \quad \text{(B.40)}$$

and again no O(a) corrections arise.
Finally the gauge invariant part of the bubble diagram is treated. It yields

$$\frac{\partial}{\partial p_0}\Sigma^{(1)}(p)^{BD,GI}=-\frac{1}{2}\int_{k'}\frac{\chi_{k'}}{\hat{k}'^2}\Big\{\frac{i\gamma_\mu C_\mu+D}{(\sum_\rho \sin^2(p-k')_\rho+4(\sum_\rho \sin^2\frac{(p-k')_\rho}{2})^2)}$$
$$-\frac{2\sin(p-k')_0(\cos(p-k')_0+\sum_\nu \sin^2\frac{(p-k')_\nu}{2})(i\gamma_\mu A_\mu+B)}{(\sum_\rho \sin^2(p-k')_\rho+4(\sum_\rho \sin^2\frac{(p-k')_\rho}{2})^2)^2}\Big\} \quad \text{(B.41)}$$

where B, A_μ are given by (B.24),(B.25) while

$$D=\sin(p-k')_0(\sum_\mu(\cos^2(p-\frac{k'}{2})_\mu-\sin^2(p-\frac{k'}{2})_\mu)-2(\cos^2(p-\frac{k'}{2})_0-\sin^2(p-\frac{k'}{2})_0)$$
$$-4\sin(2p-k')_0\sum_\mu \sin^2\frac{(p-k')_\mu}{2}-\sin(2p-k')_0\cos(p-k')_0$$
$$-\sin(p-k')_0\cos\frac{k'_0}{2}\sum_{\mu\neq 0}\sin(p-k')_\mu\sin k'_\mu+\cos(p-k')_0\sin k'_0\sum_{\mu\neq 0}\cos(p-k')_\mu\cos\frac{k'_\mu}{2}$$
$$-\frac{1}{4}\sin(p-k')_0\sum_\mu\cos^2\frac{k'_\mu}{2}\sum_{\nu\neq\mu}\sin^2 k'_\nu, \quad \text{(B.42)}$$

$$C_0=\sin(2p-k')_0\sin(p-k')_0-2\cos(p-k')_0(1+\sin^2(p-\frac{k'}{2})_0)$$
$$-4(\cos^2(p-\frac{k'}{2})_0-\sin^2(p-\frac{k'}{2})_0)\sum_\mu\sin^2\frac{(p-k')_\mu}{2}$$
$$-\sin(p-k')_0\sin k'_0(\sum_{\nu\neq 0}\cos(p-\frac{k'}{2})_\nu\cos\frac{k'_\nu}{2}-\cos(p-\frac{k'}{2})_0\cos\frac{k'_0}{2})$$
$$-\cos(p-\frac{k'}{2})_0\cos\frac{k'_0}{2}\sum_\nu\sin(p-k')_\nu\sin k'_\nu-\frac{1}{2}\sin^2 k'_0\cos(p-k')_0\sum_{\nu\neq 0}\cos^2\frac{k'_\nu}{2}$$
$$-\frac{1}{2}\cos^2\frac{k'_0}{2}\cos(p-k')_0\sum_{\nu\neq 0}\sin^2 k'_\nu-\frac{1}{4}\cos(p-k')_0\sum_\nu\cos^2\frac{k'_\nu}{2}\sum_{\rho\neq\nu}\sin^2 k'_\rho \quad \text{(B.43)}$$

and

$$C_\mu=-\sin(p-k')_0(\sin(2p-k')_\mu+\frac{1}{2}\sin k'_\mu\sum_{\nu\neq\mu}(\cos p_\nu+\cos(p-k')_\nu)$$
$$+\sin k'_\mu(\sin p_0+\sin(p-k')_0)\sum_\nu\sin^2\frac{(p-k')_\nu}{2}+\frac{1}{2}\sin k'_\mu(\sin(2p-k')_0+\sin 2(p-k')_0)$$
$$-\frac{1}{2}(\sin p_\mu+\sin(p-k')_\mu\sin k'_0\cos(p-k')_0$$
$$-\frac{1}{2}\sin k'_\mu\sin k'_0\cos(p-k')_0\sum_{\nu\neq 0,\mu}\cos^2\frac{k'_\nu}{2} \qquad \mu=1,2,3. \quad \text{(B.44)}$$

Thus

$$\frac{1}{2iE^{(0)}(\bar{p})}Z^{(0)}(\bar{p})(\frac{\partial}{\partial p_0}\Sigma^{(1)}(p)^{BD,GV})|_{p_0=iE^{(0)}(\bar{p})}Z^{(0)}(\bar{p})$$
$$=\frac{1}{2E^{(0)}(\bar{p})}\int_{k'}\frac{\chi_{k'}}{\hat{k}'^2}\Big\{\frac{c_\mu\sin p_\mu}{(\sum_\rho \sin^2(p-k')_\rho+4(\sum_\rho \sin^2\frac{(p-k')_\rho}{2})^2)}$$
$$-\frac{2\sin(p-k')_0(\cos(p-k')_0+\sum_\nu \sin^2\frac{(p-k')_\nu}{2})(a_\mu\sin p_\mu)}{(\sum_\rho \sin^2(p-k')_\rho+4(\sum_\rho \sin^2\frac{(p-k')_\rho}{2})^2)^2}\Big\}\Big|_{p_0=iE^{(0)}(\bar{p})}Z^{(0)}(\bar{p})$$
$$+O(a^2). \quad \text{(B.45)}$$

For the momentum configurations of interest one has

$$\frac{1}{2iE^{(0)}(\bar{p})}Z^{(0)}(\bar{p})(\frac{\partial}{\partial p_0}\Sigma^{(1)}(p)^{BD,GV})|_{p_0=iE^{(0)}(\bar{p})}Z^{(0)}(\bar{p})=C(\bar{p})Z^{(0)}(\bar{p})+O(a^2) \quad \text{(B.46)}$$

where

$$C(\bar{p})=O(1)+O(\log m^2) \quad \text{(B.47)}$$

so again no O(a) corrections appear.

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