DEUTSCHES ELEKTRONEN-SYNCHROTRON DESY

DESY 87-100
August 1987

PROBABILISTIC INTERPRETATION AND THE QUANTUM THEORY OF MEASUREMENT

by

Z. Maki

Deutsches Elektronen-Synchrotron DESY, Hamburg

and

Research Institute f. Fundamental Physics, Kyoto University

ISSN 0418-9833

NOTKESTRASSE 85 · 2 HAMBURG 52

DESY
Bibliothek
Notkestrasse 85
2 Hamburg 52
Germany

DESY 87-100
August 1987

ISSN 0418-9833

PROBABILISTIC INTERPRETATION AND THE QUANTUM THEORY OF MEASUREMENT

by

Ziro MAKI

Deutsches Elektronen-Synchrotron DESY, Hamburg

and

Research Institute for Fundamental Physics,
Kyoto University, Kyoto *

Abstract

The logical foundation of the probabilistic interpretation of quantum-mechanical states is re-examined in view of Fukuda's new theory of measurement. We suggest that the probabilistic interpretation could be viewed as a natural consequence of the reduction of states upon measurement, rather than an a priori ansatz contained in the "external observation" framework of quantum mechanics.

---

* Permanent address

1. Introduction

Recently much attentions have been paid to the apparent limitation of the "external observation" formalism of quantum mechanics, since this kind of formalism does not look applicable to any isolated and closed quantum system, for example the whole universe, involving the measuring apparatus and observers inside. To get rid of this difficulty many physicists [1] have been tempted to adopt the "many-worlds interpretation" of quantum mechanics suggested by Everett, III some years ago [2].

Although this latter scheme of quantum theory, known as the "relative state" formulation, poses a possible closed theory based exclusively on the superposition principle and the causal-unitary timedevelopment of the quantum states, there are still some difficulties with conceiving infinite multiple of the whole world. On the other hand it has long been an open question [3] whether or not the reduction of wave-packet could well be described in some way as a physical process of measurement and not as an axiomatic proposition.

In this connection, a new approach to the theory of measurement, developed recently by Fukuda [4] looks particularly appealing to us. As it will be outlined in the next section he treated the motion of measuring apparatus in the large number limit of a many-body quantum mechanical system and proved that the Hilbert space for the states of macroscopic detector undergoes a sort of phase-transition, converting itself into a set of a large number of disconnected subspaces in an extremely short period of time.

If one takes this view-point for granted, one may then ask whether or not the probabilistic interpretation of the state should really be regarded as one of the starting hypothesis for the quantum theory.

The point is that now it is no longer necessary to treat the measurement in terms of the "external observation" formalism.

The present note addresses some points concerning this issue, which may suggest a new way of axiomatization of quantum mechanics. In Section 3, the probabilistic interpretation of the state, i.e. Born's ansatz [5], is re-examined not as starting hypothesis but as a notion which acquires meaning connected only with the measurement through the macroscopic apparatus. The last section of the paper will be devoted to discuss of these results.

2. Macroscopic variables in the measuring process

As a preparation to the later discussion we recapitulate briefly the essential aspects of the theory of measurement developed by Fukuda [4].

The measuring apparatus, as a macroscopic system, consists of, in general, an infinitely large number ($N \to \infty$) of degrees of freedom. The relevant quantities which, somehow, record the results of measurement are the Class I intensive variables[F1], which are obtained by averaging the local variables over a macroscopic region. These variables remain finite in the limit $V \to \infty$ with N/V fixed. The volume V denotes the spatial extension of the relevant part of the measuring apparatus; this volume could be very small but it is assumed to be infinitely larger than the size of atoms or nuclei. Now it is evident that such a macroscopic variable thus defined obeys the individual c-number equation of motion without any quantum fluctuations. But how does this come about in terms of the large number limit of quantum mechanics? This is just the problem which Fukuda treated successfully by using the method of functional integration,

F1) We follow Fukuda's terminology here. The intensive variables not belonging to the Class I are called Class II. On the contrary, the extensive variables cannot exist in the limit $V \to \infty$ since they are infinite operators. This section does not contain any new result beyond Fukuda's treatment, except for a minor change of notations. The interested reader can obtain further details in the original paper [4].

thereby clarifying the structure of the Hilbert space for any macroscopic system.

The arguments proceed as follows. Let $\zeta(\underset{\sim}{x})$ be a quantized field which, with its canonical conjugate, represent the infinitely many degrees of freedom of the macroscopic system under consideration. Then the state prepared at the time t_0, $|\Phi_{t_0}\rangle$ is written as a wave functional $\langle \zeta | \Phi_{t_0} \rangle \equiv \Phi_{t_0}[\zeta]$, and the subsequent time development of the state is prescribed by the kernel K as

$$\Phi_t[\zeta] = \int K_{t,t_0}(\zeta, \zeta_0)\, \Phi_{t_0}[\zeta_0]\, [d\zeta_0] \,, \tag{1}$$

where, as is well-known, K is computed by the method of functional path-integration:

$$K \sim \int [d\zeta'] \exp\left(\frac{i}{\hbar}\int_{t_0}^{t} L\,[\zeta(t'), \dot{\zeta}(t')]\, dt'\right) \tag{2}$$

with $\dot{\zeta}(t) \equiv d\zeta/dt$. The space coordinate $\underset{\sim}{x}$ in $\zeta(\underset{\sim}{x}, t)$ is omitted for brevity. The functional integration here is performed over all possible forms of the c-number field $\zeta(\underset{\sim}{x}, t')$ with $t_0 < t' < t$, with two boundary functions $\zeta(\underset{\sim}{x}, t) \equiv \zeta$ and $\zeta(\underset{\sim}{x}, t_0) \equiv \zeta_0$ taking fixed values, respectively. Given the Lagrangian of the system, L, our next task is to rewrite the kernel (2) as a sum of contributions from the actions, each of which now being given in terms of suitably chosen collective variables rather than of original canonical field variables. Since we are dealing with a macroscopic system, the total action obtained by the integration $\int d[\zeta']$ must be proportional to the volume V, so that the predominant contribution to the action density comes solely from the Class I intensive opertors, denoted by $A_i[\zeta(t)]$ $(i = 1,2,\ldots)$.[F2) In other words, A_i's are the set of relevant collective variables, all possible values of which could contribute to the kernel. However, for the same reason that, in the limit of classical action, the classical path of motion of a particle determined by the Euler-Lagrange equation saturates the contribution to the whole action, the effective action density in our case is uniquely determined once we evaluate it in terms of the classical solution $a_i(t)$ of A_i from its equation of motion[F3), under the given boundary conditions. More precisely, for each A_i there exist a number of different c-number solutions corresponding to mutually different types of equations of motion. We differentiate labelling them by the letter r, say $a^{(r)}(\zeta, \zeta_0)$. The set of $a^{(r)}$, $\{a^{(r)}(\zeta,\zeta_0)\}$, thus exhausts all the macroscopic properties of the system. Since the equation of motions are deterministic, no fluctuation occurs in any of their solutions. The state functional $\Phi_t[\zeta]$ is now written in a form

$$\Phi_t[\zeta] = \sum_{(r)} \int [d\zeta_0]\, \Phi^{(r)}[\zeta,\zeta_0] \tag{3}$$

with

$$\Phi^{(r)}[\zeta,\zeta_0] \equiv C_{\zeta,\zeta_0}[a^{(r)}(\zeta,\zeta_0)] \exp\left\{\frac{i}{\hbar} V \Gamma^{(r)}[a^{(r)}(\zeta,\zeta_0)]\right\}. \tag{4}$$

F2) Operators A_i are certain functionals of $\zeta(x,t)$ and its canonical conjugate, the latter being rewritten here as a function of $\zeta(t)$ and $\dot{\zeta}(t)$.

F3) In general, this kind of equation of motion is derived from the requirement that the effective action $\Gamma[a(t)]$ should be stationary under the variation of $a \to \delta a$, $\delta\Gamma = 0$.

where $\Gamma^{(r)}$ is obviously the effective action density responsible for the corresponding solution $a^{(r)}(\zeta,\zeta_0)$, and C is an amplitude associated with the respective action.

Eqs. (3) and (4) mean that, in the limit $V \to \infty$, each term in (3) gets an infinite phase and the phase difference of any pair of terms is, in general, also infinite. Thus the phase correlation among the pair of terms in (3) vanish within a short time Δt even if one prepares at $t = t_0$ the state $\Phi_{t_0}[\zeta_0]$ with a definite phase. Δt is estimated to be of the order of magnitude 1/V or equivalently 1/N. It is also easily understood that there exist no finite operators which have non-vanishing matrix element between any pair of terms in the expansion (3); this implies that the Hilbert spaces spanned by the set of states $\{\Phi^{(r)}[\zeta,\zeta_0]\}$ are completely disjoint of each other.

The last, but the most important observation by Fukuda is the disappearance of the Hamiltonian which would govern the time evolution of the whole system. A simple reason for this is that the total Hamiltonian is an extensive operator and thus infinite. The unitary development of the state is only operative within each Hilbert space characterized by (r), and the state vectors in each space develope themselves controlled by the quantum mechanics concerning the Class II intensive variables.

To summarize, the time development of the macroscopic system is describable in a quantum mechanical basis; to each c-number value of Class I intensive variable a Hilbert space is associated, and these spaces belonging to the different species of solution $a^{(r)}(\zeta,\zeta_0)$ are completely disjoint; the development of the state (Eqs. (3) and (4)) is not unitary nor time-reversal invariant - it is a sort of phase-transitions undergone within a very short time interval $\Delta t \sim 1/V$ (or 1/N).

3. Measuring process and the probabilistic interpretation

The measuring process is a change of the state of a system composed of a 'to-be-measured' system S and a macroscopic detector M, under the interaction between these two.

We start with the usual notion of the state vector $|\psi\rangle$ and linear operators corresponding to physical observables, but without any kind of assumptions concerning the expected values of observables, or equivalently, the probabilistic interpretation of state vectors (Born's ansatz).

The usual assumption of 'good measurement' may of course be understood here. - It states that "If a system is in the state $|\lambda_k\rangle$, the eigenstate of an observable Λ associated to its eigenvalue λ_k, then the result of the measurement of Λ always gives its value λ_k."

Clearly the above statement, too, does not involve any probabilistic conception.

Now we introduce the statistical ensemble and define the 'state of ensemble' through the operators

$$\left.\begin{array}{ll} \text{(a)} & U = |\psi\rangle\langle\psi| \text{ , or} \\ \text{(b)} & U' = \sum_m \omega_m |\psi_m\rangle\langle\psi_m| \text{, with } \sum_m \omega_m = 1 . \end{array}\right\} \quad (5)$$

The former operator (a) represents the state of ensemble in which every system lies on one and the same state $|\psi\rangle$, and this state of ensemble is called 'pure state'. On the contrary, the latter (b) represents the state of ensemble in which there exists M systems in the state $|\psi_m\rangle$ among the total N systems, ω_m being the ratio M/N provided that both M and N are quite large numbers. This state of ensemble is called as 'mixture' (Gemisch).

All these have been quite well-known as parts of the definition of statistical operators. But, in order that U or U' are statistical operators in von Neumann's sense [6], we need to add the 'expected value hypothesis', that is

$$\langle\Lambda\rangle_{EV} = \begin{cases} T_r\, U\Lambda & \text{(pure state)} \\ T_r\, U'\Lambda & \text{(mixture),} \end{cases} \quad (6)$$

where $\langle\Lambda\rangle_{EV}$ is the expected value of the observable Λ.

In the following, however, we will not impose the above condition on our U or U'. Therefore our U (or U') are still not statistical operators but the equivalent substitute for the concept of state, extended to ensembles.[F4] The relation (6) will be derived later on. One should also notice that the 'statistical' element entered in the definition (b) of U' has nothing to do with the 'statistical interpretation' of quantum mechanics, but concerns only the classic calculus of probabilities. We now proceed to discuss the measuring process.

In the presence of the macroscopic detectors M, the state of M, $|\Phi\rangle$, receives a change due to the interaction of M with the object S. To make the measurement (of the operator Λ of S) successful, we should select an appropriate Class-I variable of M, which couples effectively to the operator Λ one wishes to measure. This is actually an abstract criterion to arrange a suitable apparatus for a relevant measurement. Provided this has been already done, we start with similar equations to Eqs. (1) and (2), but replace them by more general ones which correspond to the state of S+M, $|\Psi_t\rangle$. That is to say, we work with $\langle\zeta,\cdot\cdot|\Psi_t\rangle \equiv \Psi_t[\zeta,\cdot\cdot]$, $L^{S+M}[\zeta,\dot{\zeta};\cdot\cdot]$ and $\int[d\zeta'][d(\cdot\cdot)]$ instead of the old ones. Here $(\cdot\cdot)$ abbreviates the other variables of the object.

F4) Our U (or U') obey the time evolution equation $i\hbar\partial U/\partial t$ = HU-UH as usual.

At the time t_o, before the measurement, the state $|\Psi_t\rangle$ is a product of two states $|\psi_{t_o}\rangle$ and $|\Phi_{t_o}\rangle$, corresponding to the object and the macroscopic detector, respectively:

$$|\Psi_{t_o}\rangle = |\psi_{t_o}\rangle |\Phi_{t_o}\rangle. \tag{7}$$

We have assumed here that both are in their pure state. The state of ensemble compromising both S and M is, accordingly, expressed by an operator

$$U^{S+M} = |\psi\rangle\langle\psi| \otimes |\Phi\rangle\langle\Phi| \tag{8}$$

at $t = t_o$. This definition is unique since at least one of both is assumed to be in the pure state, as was proved by von Neumann. More specifically, we assume the object was in the state

$$|\psi_{t_o}\rangle = \sum_k c_k |\lambda_k\rangle \tag{9}$$

and hereafter all the state vectors, we assume , to be normalized, e.g. $\langle\psi|\psi\rangle = 1$, $\langle\lambda_k|\lambda_{k'}\rangle = \delta_{kk'}$ etc.

After a very short passage of time $t_o \to t_o + \Delta t$, Δt being $O(1/V)$ as was remarked previously, we can reasonably suppose that the state $|\Psi_{\Delta t}\rangle$ still keep the product form of Eq. (7) (and so does $U^{S+M}_{\Delta t}$) approximately, and suppress the object Lagrangian from $L(\zeta,\dot{\zeta};\cdots)$ retaining only the interaction part of S with M:

$$L(\zeta,\dot{\zeta};\cdots) = L^M(\zeta,\dot{\zeta}) + L^{int}(\zeta,\dot{\zeta};\Lambda), \tag{10}$$

where the interaction part is introduced through the observable

, the function of dynamical variables of the object. The time variation of the coefficients in (9) is to be estimated perturbatively. But, for sufficiently small Δt the variation will be non-appreciable, or the measurement would lose its physical meaning.

In this situation, it is easy to write

$$\Psi_{\Delta t}[\zeta,\cdots]$$
$$= \sum_k c_k(t)|\lambda_k\rangle \int [d\zeta'][d\zeta_o] \exp\left\{\frac{i}{\hbar}\int_{t_o}^{t_o+\Delta t} dt' \left(L^M + L^{int}(\zeta';\lambda_k)\right)\right\} \Phi_{t_o}[\zeta_o]. \tag{11}$$

This is again Fukuda's formula, and the second factor in r.h.s. of Eq. (11), denoted by $\Phi_{\Delta t}[\zeta;\lambda_k]$, would also have the similar form as Eqs. (3) and (4). Since the effective action densities derived from the original Lagrangian are different corresponding to different values λ_k ; the obtained classical variables for the relevant Class I operators vary with different λ_k's; and just this fact enables us to record the results of measurement in macroscopic terms. Evidently, any pair of Hilbert spaces $\{\Phi_{\Delta t}[\zeta;\lambda_k]\}$ and $\{\Phi_{\Delta t}[\zeta;\lambda_{k'}]\}$ have no phase correlation and are completely disconnected as far as $k \neq k'$ F5).

A state of the ensemble of the system S+M, after measurement, is expressed as before by

$$U^{S+M}_{\Delta t} = |\Psi_{\Delta t}\rangle\langle\Psi_{\Delta t}|. \tag{12}$$

F5) Each space $\{\Phi[\zeta;\lambda_k]\}$ is, in general, decomposed further into a large number of disconnected subspaces because everywhere we have infinite phase differences in the limit $V \to \infty$.

The problem is to define similar operator referring only to the system S. We postulate that this can be done by taking the 'Trace' with respect to the states of M. Then

$$U^S_{\Delta t} = \sum_n \langle \Phi_n | \Psi_{\Delta t} \rangle \langle \Psi_{\Delta t} | \Phi_n \rangle, \qquad (13)$$

where $|\Phi_n\rangle$ is an arbitrarily chosen complete-orth-normal vector in the Hilbert space of M. Substituting the expression (11) into Eq. (13) we have

$$U^S_{\Delta t} = \sum_n \sum_k |c_k|^2 |\lambda_k\rangle\langle\lambda_k| \otimes \langle \Phi_n | \Phi_{\Delta t}[\zeta;\lambda_k]\rangle \langle \Phi_{\Delta t}[\zeta;\lambda_k] | \Phi_n\rangle$$
$$+ \sum_n \sum_{k \neq k'} c_k c^*_{k'} |\lambda_k\rangle\langle\lambda_{k'}| \otimes \langle \Phi_n | \Phi_{\Delta t}[\zeta;\lambda_k]\rangle \langle \Phi_{\Delta t}[\zeta;\lambda_{k'}] | \Phi_n\rangle, \qquad (14)$$

where we have omitted some summation symbols such as $\int[d\zeta']$, $\sum_{\alpha_k}$ (α_k belongs to the k-th Hilbert space) etc. to simplify the notations. It is now immediately clear that the second term in r.h.s. of Eq. (14) vanishes in the limit $V \to \infty$, because of the disjointness of the space of $\{\Phi_{\Delta t}[\zeta]\}$. However, with respect to the first term, some minor portion of $\sum_n$ may survive for each λ_k. This would give certain quantum correction coming from the Class-II operators, which would be negligibly small in any measurement in terms of classical variables. Thus, discarding such a small correction we arrive at the formula

$$U^S_{\Delta t} = \sum_k |c_k|^2 |\lambda_k\rangle\langle\lambda_k|. \qquad (15)$$

This implies that the ensemble state of S, originally in the pure state $|\psi\rangle\langle\psi|$, converted itself to the mixture state (15).

Comparing with the definition of the mixture state (b) given earlier, we are forced to interprete $|c_k|^2$ as the probability w_k for the state $|\lambda_k\rangle$ to occur in the mixture, namely[F6)]

$$|c_k|^2 = w_k. \qquad (16)$$

From (16), the operators U and U' are shown to become von Neumann's statistical operators (or density matrices), so that the relations in (6) have been proved a posteriori.

4. Remarks

(4-1) No secret exists anywhere in our deduction. A new recipe is to regard U and U' defined in Eq. (5) simply as mathematical expressions for the 'state of ensemble' just like $|\psi\rangle$ for the 'state of a system'; thereby attaching no statistical meaning to them. The path-integral formula does not involve in itself any probabilistic notion, since it consists solely of the multiplication law of amplitudes.

(4-2) It is desirable to formulate the measuring process on a more abstract ground. To this end we have only to establish a precise definition of a 'good detector' in terms of quantum-mechanical language. Such a 'good detector' would have to involve, in general, an infinite number of degrees of freedom. A macroscopic detector is a possible example of the good detector as was shown by Fukuda.

(4-3) Different discussions of a macroscopic detector have been given by Machida and Namiki [7)]. Their method of achieving the state reduction is, however, not satisfactory for they introduced

---

F6) We simply set $C_k(t) = C_k$ in the spirit of our approximate treatment.

averaging functions in <u>ad hoc</u> manner, not derivable from the quantum theoretical basis.

(4-4) "Many-worlds interpretation" does not look any longer necessary for the quantum mechanics to treat the whole universe, provided it does contain an infinite number of degrees of freedom. The quantum state of the universe could be known through the accumulated results of 'measurements' performed by dividing the universe into subsystems in various ways, such that in one side of pairs of subsystems there contains a good detector.

Acknowledgement

The author is grateful to Professor R.D. Peccei for warm hospitality during his stay at DESY, without which this work would have never been completed.

References

1) B.S. DeWitt and N. Graham, "The many-worlds interpretation of quantum mechanics" (Princeton 1973).
J.B. Hartle and S.W. Hawking, Phys. Rev. D28 (1983), 2960.
S.W. Hawking, in "Intersection between Elementary Particle Physics and Cosmology Vol. I" (ed. T. Piran and S. Weinberg, World Scientific 1986), 72.

2) H. Everett III, Rev. Mod. Phys. 29 (1957), 454.
J.A. Wheeler, ibid. p. 463.

3) See for example: "Foundation of Quantum Mechanics, Proceedings of the Int'l School of Physics "Enrico Fermi" Course IL (ed. B. d'Espagnat (Academic Press, New York 1971)).

4) R. Fukuda, Phys. Rev. A 35 (1987), 8.

5) M. Born, Z. Physik 37 (1926).

6) J. von Neumann, "Mathematische Grundlagen der Quantenmechanik" (Springer, 1932).

7) S. Machida and M. Namiki, Prog. Theor. Phys. 63 (1980), 1457; 1833.