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E. Elizalde

II. Institut f. Theoretische Physik, Universität Hamburg

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Analytic solutions of the Klein-Gordon equation in a curved background

E. Elizalde[1]
II. Institut für Theoretische Physik der Universität Hamburg,
Luruper Chaussee 149, D-2000 Hamburg 50, Fed. Rep. Germany

Abstract

Several formal solutions of the Klein-Gordon equation in a curved background, given by power series expansions, are proven to render in fact convergent solutions in very extense domains. In particular, one family of solutions is analytic in the whole of space-time. For a complex domain of values of the frequency, the family of solutions corresponds to particles of real mass $m > 0$.

[1] Alexander von Humboldt Foundation Fellow. On leave from and permanent address: Facultat de Física, Universitat de Barcelona, Diagonal 647, 08028 Barcelona, Spain.

1. Introduction.

The factorizable solutions of the Klein-Gordon equation

$$\left(\nabla_\mu \nabla^\mu - m^2\right) \Phi = 0 \tag{1.1}$$

in a curved, Schwarzschild space-time

$$ds^2 = \left(1 - \frac{2GM}{r}\right) dt^2 - \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 - r^2 \left(d\theta^2 + \sin^2\theta \, d\varphi^2\right) \tag{1.2}$$

can be written as [1]

$$r^{-1} f_{\omega l}(r) Y_{lm}(\theta, \varphi) \exp(-i\omega t), \tag{1.3}$$

where the Y_{lm} are spherical harmonics and the radial functions $f_{\omega l}$ satisfy

$$\frac{d^2 f_{\omega l}(r)}{d r_*^2} + \left\{ \omega^2 - \left[m^2 + \frac{l(l+1)}{r^2} + \frac{2GM}{r^3} \right] \left(1 - \frac{2GM}{r}\right) \right\} f_{\omega l}(r) = 0, \tag{1.4}$$

r_* being the Regge-Wheeler coordinate

$$r_* = r + 2GM \ln \left| \frac{r}{2GM} - 1 \right| . \tag{1.5}$$

Notice our definition with absolute value which will allow us to use this coordinate for $r < 2GM$. In the asymptotic region r going to infinity, the solutions of (1.4) are

$$\exp(\mp i k r_*), \quad k = (\omega^2 - m^2)^{1/2}, \tag{1.6}$$

so that (1.3) reduces to

$$r^{-1} Y_{lm}(\theta, \varphi) \exp\left[-i(\omega t \pm k r_*)\right] . \tag{1.7}$$

A considerable number of attempts have been carried out in order to solve equation (1.4) and very much is already known about the general properties of the solutions [2], which, on the other hand, have proven to be very difficult to give explicitely. Normally one has to work without knowing them in terms of simple functions. The derivation of the Hawking effect [3] is a beautiful example of how to proceed without solving the exact equation [1,4].

It seems clear that any effort aiming at the derivation of regular solutions of eq. (1.4) in the most explicit way possible and valid over very large domains of space-time should be welcome. Concerning this last requeriment, it is apparent that the Schwarzschild radius $r_S \equiv 2GM$ builds a natural frontier for these domains (see, however, the first and third of our solutions) as the physical properties of the solutions in the region outside and in the region inside the hypersphere $r = r_S$ (usually called regions I and II, respectively) are different –and this in spite of the fact that $r = r_S$ is not a true singularity of the Schwarzschild metric. In our calculations we have divided the whole of space-time correspondingly. With respect to the first requeriment, we shall in some cases be able to produce solutions given by elementary functions. They will be analytic in the whole domain I or II in each case and some of them will even be valid in part of the complementary domain. The last ones may prove to be relevant for the study of gravitational collapse.

We shall actually analyze three different types of solutions to eq. (1.4), two of them valid in the region I, exterior to the Schwarzschild horizon and the other one valid in the interior region II. The precise derivation of the solutions was done elsewhere [5] and will not be repeated here. We shall concentrate in the proof of the convergence of each of the solutions, a task which involves very careful analysis of terms of different order, as we shall see.

The paper is structured as follows. In Section 2 the asymptotic behaviour of the coefficients of the series expansion valid for $r > r_S = 2GM$ (the Schwarzschild radius) is analyzed. The existence of exact solutions to the recurrence equation satisfied by the coefficients is proven. In Section 3 these solutions are found. Attention is focussed on the precise initial conditions leading to exact and to very approximate solutions. The results of a numerical analysis carried out for different values of the parameters is also given here. In Sections 4 and 5 the same study is repeated for the other two series expansions considered, one of them valid for big r and the other for small r, respectively. Finally, Section 6 is devoted to discussions and conclusions.

2. Analysis of the recursion formula corresponding to the case $r > r_S$.

The most interesting of the series expansions we are going to analyze is the one about $r = r_S$. In principle it is valid for $r > r_S$ but it can be easily extended to $r_S/2 \leq r < r_S$. By doing the change of variables

$$\rho = 1 - \frac{2GM}{r} \tag{2.1}$$

equation (1.4) transforms into

$$\frac{d^2 f_{\omega l}}{d r_*^2} + \left\{ \omega^2 - \left(m^2 + \frac{l(l+1)+1}{4G^2M^2} \right)\rho + \frac{2l(l+1)+3}{4G^2M^2}\rho^2 - \frac{l(l+1)+3}{4G^2M^2}\rho^3 + \frac{\rho^4}{4G^2M^2} \right\} f_{\omega l} = 0 . \tag{2.2}$$

As was proven in [5], a solution of (2.2) is given by

$$f_{\omega l} = \alpha \exp\left[i(\omega r_* + g(\rho)) \right] , \quad g(\rho) = \sum_{s=1}^{\infty} c_s \rho^s , \tag{2.3}$$

where the coefficients are completely determined in terms of the parameters m, M, ω, and l as follows

$$\begin{aligned}
c_1 &= -\frac{b_1}{2\bar\omega - i} , \qquad c_2 = c_1 - \frac{c_1^2 + 2ic_1 - b_2}{4(\bar\omega - i)} , \\
c_3 &= \frac{2}{9 - 4\bar\omega^2}\Big\{ \frac{4}{3}\bar\omega c_1(c_1 - c_2) + \frac{2}{3}\bar\omega(4c_2 - c_1) + 10c_2 - 6c_1 \\
&\quad - \frac{\bar\omega}{3} b_3 + i\Big[2c_1(c_1 - c_2) + \bar\omega\Big(3c_1 - \frac{8}{3}c_2\Big) - \frac{b_3}{2} \Big] \Big\} , \\
c_4 &= \frac{1}{2(4 - \bar\omega^2)}\Big\{ -\bar\omega\Big(\frac{3}{2}c_1c_3 + c_2^2 - 4c_1c_2 + \frac{3}{2}c_1^2\Big) + \bar\omega^2(3c_3 - c_2) \\
&\quad + 21c_3 - 18c_2 + 5c_1 + \frac{\bar\omega}{4} b_4 + i\Big[-3c_1c_3 - 2c_2^2 + 8c_1c_2 \\
&\quad - 3c_1^2 - \bar\omega\Big(\frac{9}{2}c_3 - 7c_2 + \frac{5}{2}c_1\Big) + \frac{1}{2} \Big] \Big\} ,
\end{aligned} \tag{2.4}$$

where

$$\begin{aligned}
\bar\omega &= 2GM\omega , & \bar m &= 2GMm , \\
b_1 &= \bar m^2 + \bar l + 1 , & \bar l &= l(l+1) , \\
b_2 &= 2\bar l + 3 , & b_3 &= \bar l + 3 .
\end{aligned} \tag{2.5}$$

For $s \geq 5$ the general coefficient of the series solution (2.3) is given by the recurrence equation

$$
\begin{aligned}
c_s = \frac{2}{s(s^2+4\bar\omega^2)} \Big\{ & 2\bar\omega \textstyle\sum_s + 2\bar\omega^2 [2(s-1)c_{s-1} - (s-2)c_{s-2}] \\
& + s[(s-1)(2s-1)c_{s-1} - 3(s-1)(s-2)c_{s-2} + (s-3)(2s-3)c_{s-3} \\
& - \tfrac{1}{2}(s-2)(s-4)c_{s-4}] + i\,[s \textstyle\sum_s + \bar\omega\big(-2(s-1)^2 c_{s-1} \\
& + (s-2)(5s-6)c_{s-2} - 2(s-3)(2s-3)c_{s-3} + (s-2)(s-4)c_{s-4}\big)]\Big\}, \quad s \geqslant 5,
\end{aligned}
\tag{2.6}
$$

being

$$
\begin{aligned}
\textstyle\sum_s = & -\frac{1}{2}\sum_{j=1}^{s-1} j(s-j)c_j c_{s-j} + 2\sum_{j=1}^{s-2} j(s-j-1)c_j c_{s-j-1} \\
& -3\sum_{j=1}^{s-3} j(s-j-2)c_j c_{s-j-2} + 2\sum_{j=1}^{s-4} j(s-j-3)c_j c_{s-j-3} \\
& -\frac{1}{2}\sum_{j=1}^{s-5} j(s-j-4)c_j c_{s-j-4}\,.
\end{aligned}
\tag{2.7}
$$

Notice that some modifications have been introduced with respect to ref. 5 with the aim of rendering the expression more compact. We can go further and write (2.7) in an equivalent form more adequate for the manipulations to come, namely

$$
\begin{aligned}
\textstyle\sum_s = & -\sum_{j=1}^{[\frac{s}{2}]} j(s-j)c_j c_{s-j} + 4\sum_{j=1}^{[\frac{s-1}{2}]} j(s-j-1)c_j c_{s-j-1} - 6\sum_{j=1}^{[\frac{s-2}{2}]} j(s-j-2)c_j c_{s-j-2} \\
& + 4\sum_{j=1}^{[\frac{s-3}{2}]} j(s-j-3)c_j c_{s-j-3} - \sum_{j=1}^{[\frac{s-4}{2}]} j(s-j-4)c_j c_{s-j-4} \\
& + \left(\frac{s+1}{2} - \left[\frac{s}{2}\right]\right)\left(\frac{s}{2}\right)^2 c_{s/2}^2 - 4\left(\frac{s}{2} - \left[\frac{s-1}{2}\right]\right)\left(\frac{s-1}{2}\right)^2 c_{\frac{s-1}{2}}^2 \\
& + 6\left(\frac{s-1}{2} - \left[\frac{s-2}{2}\right]\right)\left(\frac{s-2}{2}\right)^2 c_{\frac{s-2}{2}}^2 - 4\left(\frac{s-2}{2} - \left[\frac{s-3}{2}\right]\right)\left(\frac{s-3}{2}\right)^2 c_{\frac{s-3}{2}}^2 \\
& + \left(\frac{s-3}{2} - \left[\frac{s-4}{2}\right]\right)\left(\frac{s-4}{2}\right)^2 c_{\frac{s-4}{2}}^2 \,,
\end{aligned}
\tag{2.8}
$$

where the square bracket means "integer value of". For s even, s=2p, this expression can be written as

$$
\begin{aligned}
\textstyle\sum_{2p} = \sum_{j=1}^{p-2} & \{ j c_j [-(s-j)c_{s-j} + 4(s-j-1)c_{s-j-1} - 6(s-j-2)c_{s-j-2} \\
& + 4(s-j-3)c_{s-j-3} - (s-j-4)c_{s-j-4}] \} - \tfrac{1}{2}p^2 c_p^2 - 3(p-1)^2 c_{p-1}^2 \\
& + \tfrac{1}{2}(p-2)^2 c_{p-2}^2 - (p^2-1)c_{p+1}c_{p-1} + 4p(p-1)c_p c_{p-1}\,, \\
& \qquad s = 2p\,,
\end{aligned}
\tag{2.9a}
$$

while for s odd, s=2p+1, one sees that

$$
\begin{aligned}
\textstyle\sum_{2p+1} = \sum_{j=1}^{p-1} & \{ j c_j [-(s-j)c_{s-j} + 4(s-j-1)c_{s-j-1} \\
& - 6(s-j-2)c_{s-j-2} + 4(s-j-3)c_{s-j-3} - (s-j-4)c_{s-j-4}] \} \\
& + 2p^2 c_p^2 - 2(p-1)^2 c_{p-1}^2 - p(p+1)c_p c_{p+1} \\
& + (p-1)(p-2)c_{p-1}c_{p-2}\,, \qquad s = 2p+1\,.
\end{aligned}
\tag{2.9b}
$$

Let us now proceed to the analysis of the behaviour of c_s as s tends to infinity. In this Section, under the hypothesis that c_s is analytical as a function of 1/s, as $s \to \infty$, we shall demonstrate that the series $\sum_{s=1}^{\infty} c_s \rho^s$ has radius of convergence $\rho_g = 1$, i.e. that the sum $g(\rho)$ exists for any value $\rho < 1$. To this end, let us call

$$
x_s = \frac{c_{s-1}}{c_s}\,.
\tag{2.10}
$$

x_s will be an analytical function of 1/s for s big enough

$$
x_s = \alpha + \frac{\beta}{s} + \frac{\gamma}{s^2} + \frac{\delta}{s^3} + \frac{\varepsilon}{s^4} + O(s^{-5})\,.
\tag{2.11}
$$

Eqs. (2.6) and (2.9) are then immediately rewritten as

$$
\begin{aligned}
c_s = \frac{2}{s(s^2+4\bar\omega^2)} \Big\{ & 2\bar\omega \textstyle\sum_s + 2\bar\omega^2 c_s [2(s-1)x_s - (s-2)x_s x_{s-1}] \\
& + s c_s [(s-1)(2s-1)x_s - 3(s-1)(s-2)x_s x_{s-1} + (s-3)(2s-3)x_s x_{s-1} x_{s-2} \\
& - \tfrac{1}{2}(s-2)(s-4)x_s x_{s-1} x_{s-2} x_{s-3}] + i\,[s \textstyle\sum_s \\
& + \bar\omega c_s \big(-2(s-1)^2 x_s + (s-2)(5s-6)x_s x_{s-1} \\
& - 2(s-3)(2s-3)x_s x_{s-1} x_{s-2} + (s-2)(s-4)x_s x_{s-1} x_{s-2} x_{s-3}\big)]\Big\}\,,
\end{aligned}
\tag{2.12}
$$

and

$$
\begin{aligned}
\textstyle\sum_s = \sum_{j=1}^{p-2} S_j(s) + c_p^2 & [-\tfrac{1}{2}p^2 - 3(p-1)^2 x_p^2 + \tfrac{1}{2}(p-2)^2 x_p^2 x_{p-1}^2 \\
& -(p^2-1)\frac{x_p}{x_{p+1}} + 4p(p-1)x_p]\,, \quad s = 2p\,,
\end{aligned}
\tag{2.13a}
$$

$$\Sigma_s = \sum_{j=1}^{p-1} S_j(s) + c_p^2\Big[2p^2 - 2(p-1)^2 x_p^2 - \frac{p(p+1)}{x_{p+1}} + (p-1)(p-2)x_p x_{p-1}\Big], \qquad s = 2p+1, \tag{2.13b}$$

where

$$\begin{aligned} S_j(s) = j\, c_j c_{s-j} \big[& -(s-j) + 4(s-j-1)x_{s-j} - 6(s-j-2)x_{s-j}x_{s-j-1} \\ & + 4(s-j-3)x_{s-j}x_{s-j-1}x_{s-j-2} - (s-j-4)x_{s-j}x_{s-j-1} \\ & \times x_{s-j-2}x_{s-j-3}\big]. \end{aligned} \tag{2.14}$$

At this point, the different products of series of the type x_s (2.11) are to be substituted in eqs. (2.12), (2.13). Their explicit expressions are given in the Appendix (see formulae (A.1)-(A.8)). After a lengthy calculation one finds that: i) the terms of highest order in s in eq. (2.12), i.e. the terms of order 1, cancel if and only if

$$\alpha = 1; \tag{2.15}$$

ii) independently of the values of $\beta, \gamma, \ldots$ in (2.11), the terms of order s^{-1} and of order s^{-2} cancel throughout; and iii) the contributions to the term of order s^{-3} only depend on β, and are given by

$$\begin{aligned} \Sigma_s = \sum_{j=1}^{p-1-\lambda} \Big[& \frac{4\beta(\beta+1)(\beta+2)}{(s-j-1)^3}\, j c_j c_{s-j-1} \\ & - \frac{\beta(\beta+1)(\beta+2)(\beta+3)}{(s-j)^3}\, j c_j c_{s-j}\Big] + \frac{2\beta^2(\beta-1)}{p} c_p^2 + O(p^{-2}), \end{aligned} \tag{2.16}$$

where

$$\lambda = \begin{cases} 1, & s = 2p; \\ 0, & s = 2p+1, \end{cases} \tag{2.17}$$

while the full expression (2.12) turns out to be ($\alpha = 1$)

$$c_s = \frac{1}{s(s^2+4\bar{\omega}^2)}\Big\{4\bar{\omega}\Sigma_s + s(s^2+4\bar{\omega}^2)c_s + 2 i s \Sigma_s\Big\} + O(s^{-3}). \tag{2.18}$$

Thus, we have proved that under the condition that c_s be analytical as a function of s^{-1} as $s \to \infty$, (2.11), the equation of recurrence (2.6) implies that $\alpha = 1$ and then it is fulfilled up to terms of order s^{-3}. Let us analyze these terms and see if they can be also made disappear for a convenient value of p. Substituting (2.16) into (2.18) one gets

$$\begin{aligned} 0 = \frac{2}{s(2\bar{\omega} - is)} \Big[& \frac{\beta(\beta-1)}{s} + \beta(\beta+1)(\beta+2) \\ & \times \sum_{j=1}^{p-1-\lambda} \frac{j c_j}{c_s}\Big(\frac{4c_{s-j-1}}{(s-j-1)^3} - \frac{(\beta+3)c_{s-j}}{(s-j)^2}\Big)\Big] + O(s^{-4}). \end{aligned} \tag{2.19}$$

This can be simplified

$$0 = \frac{2\beta(\beta-1)}{s(2\bar{\omega} - is)}\Big[\frac{1}{s} + (\beta+1)(\beta+2)\sum_{j=1}^{p-1-\lambda} \frac{j c_j c_{s-j}}{c_s (s-j)^3}\Big] + O(s^{-4}). \tag{2.20}$$

Actually, we have not been completely precise concerning the order of the term affected by the summation sign in (2.16), (2.19) and (2.20). In fact, we could not be so, because its order depends on the asymptotic behaviour of the c_j's. However, it is clear now that it is at most of order $O(s^{-1})$ (not taking into account the coefficients in front of it) as a consequence of the fact that the coefficient β in the expansion (2.11) must necessarily be

$$\beta = 0 \qquad \text{or} \qquad \beta = 1 \tag{2.21}$$

in order to satisfy the recurrence relation, which is now cast under the form (2.20). In fact, we have proved that the fulfillment of the recursion equation for the c_s's implies that

$$\frac{c_{s-1}}{c_s} \sim 1 + \frac{\beta}{s} + \cdots, \qquad s \to \infty, \tag{2.22}$$

with $\beta = 0 or 1$. But this on its turn provides a limitation for the possible expressions of c_s as an expansion in terms of s^{-1}. It is immediate to see that if

$$c_s = x_0 + \frac{x_k}{s^k} + \frac{x_{k+1}}{s^{k+1}} + \cdots; \qquad x_0 \neq 0, \tag{2.23}$$

then one obtains

$$x_s = \frac{c_{s-1}}{c_s} = 1 + \frac{k x_k}{x_0}\frac{1}{s^{k+1}} + \cdots, \tag{2.24}$$

while for

$$c_s = \frac{\alpha_k}{s^k} + \frac{\alpha_{k+1}}{s^{k+1}} + \dots, \qquad \alpha_k \neq 0, \tag{2.25}$$

the result is

$$X_s = \frac{c_{s-1}}{c_s} = 1 + \frac{k}{s} + \left[\frac{k(k+1)}{2} + \frac{\alpha_{k+1}}{\alpha_k}\right]\frac{1}{s^2} + \dots \tag{2.26}$$

Therefore, it turns out that in the case $\beta = 0$ any expansion of the form (2.23) fulfills the recurrence equation up to terms of order $O(s^{-4})$, while in the case $\beta = 1$, c_s must be given by (2.25) with k=1. These are all the possible solutions and we see *a posteriori* that, in all cases, the terms affected by the summation sign in eqs. (2.16), (2.19) and (2.20) are, in fact, at most of order $O(s^{-1})$ (not taking into account the coefficients in front). Further, from eq. (2.20) we also deduce that for every $\epsilon > 0$ there exists an $s_\epsilon \in \mathcal{N}$ such that for $s \geq s_\epsilon$ one has

$$|X_s - 1| < \varepsilon, \qquad \forall\, s \geq s_\varepsilon . \tag{2.27}$$

Summing up, it has been proven in this Section that if c_s is asymptotically an analytic function of s^{-1} (a sensible condition in view of the form of the recurrence relation (2.6)) then all possible solutions of the recursion formula for the c_s 's are given by (2.23) and (2.25), the last for k=1. In all cases we obtain a series $g(\rho)$ (2.3) with radius of convergence $\rho_g = 1$. The verification of the analyticity assumption for the c_s depends on the initial values c_1, c_2, c_3, c_4 which do not obey the general recursion formula (2.6), valid for $s \geq 5$, and which are completely fixed by the parameters $\bar{\omega}, \bar{l}$ and $\bar{m}$ (2.5). In the following Section we shall analyze this dependence on the initial values and some exact and aproximate solutions will be explicitly worked out.

3. Solutions valid for $r > r_S$.

An exact solution of the radial equation (2.2) is given by (2.3) with

$$c_s = c, \qquad s = 1, 2, 3, \dots \tag{3.1}$$

In fact, substituting for

$$c_1 = c_2 = c_3 = c_4 \equiv c \tag{3.2}$$

in the recursion formula (2.6) for s=5 one gets $c_5 = c_4$ and, in general, $c_s = c_{s-1}$, provided that $c_1 = c_2 = \dots = c_{s-1}$. The sum of the series $g(\rho)$ is given by

$$g(\rho) = \frac{c\rho}{1-\rho} = c\left(\frac{r}{2GM} - 1\right). \tag{3.3}$$

Actually, it is not possible to provide an exact realization of the condition (3.1) in terms of the parameters $\bar{\omega}, \bar{l}$ and $\bar{m}$, and only approximate solutions can be obtained in this way. In fact, imposing (3.2) one gets the relations

$$c_1 = -\frac{\bar{b}_1}{2\bar{\omega} - i}, \qquad c_2 = c_1 - \frac{c_1^2 + 2ic_1 - \bar{b}_2}{4(\bar{\omega} - i)},$$
$$c_3 = c_2 + \frac{ic_2 - \bar{b}_2}{3(2\bar{\omega} - 3i)}, \qquad c_4 = c_3 + \frac{1}{8(\bar{\omega} - 2i)}. \tag{3.4}$$

It is the last condition the one which cannot be exactly satisfied, although it can be approximated to any desired order. The remaining three equations (3.4) imply

$$\bar{b}_1 = (1 + 2i\bar{\omega})\bar{b}_3, \qquad \bar{b}_3^{\,2} - 2\bar{b}_3 + \bar{b}_2 = 0, \tag{3.5}$$

whose solutions are

$$\bar{l} = -3 + \sqrt{3}, \qquad \bar{m}^2 = 2 - \sqrt{3} + (1 + 2i\bar{\omega})[1 + (4 - 2\sqrt{3})^{1/2}],$$
$$c_1 = c_2 = c_3 = -i\,[1 + (4 - 2\sqrt{3})^{1/2}], \tag{3.6a}$$

and

$$\bar{l} = -3 - \sqrt{3}, \qquad \bar{m}^2 = 2 + \sqrt{3} + (1 + 2i\bar{\omega})[1 - (4 + 2\sqrt{3})^{1/2}],$$
$$c_1 = c_2 = c_3 = -i\,[1 - (4 - 2\sqrt{3})^{1/2}]. \tag{3.6b}$$

A numerical analysis carried out both for small and for large values of $\bar{\omega}$ provides the following explicit, approximate solutions of the radial equation (2.2).

i) In the range of $\bar{\omega}$ between $\bar{\omega} = 0$ and $\omega - 0.3$ one gets

$$f_{\omega l}(r) \simeq \alpha \exp\left\{ i\bar{\omega}\left[\frac{r}{2GM} + \ln\left(\frac{r}{2GM}-1\right) + 0.2\left(\frac{r}{2GM}-1 - 0.1\ln\frac{r}{2GM}\right)\right] + 1.4\left(\frac{r}{2GM} - 1 - 0.1\ln\frac{r}{2GM}\right)\right\}. \tag{3.7}$$

This solution corresponds to (3.6a).

ii) In the other extreme, for $\bar{\omega} \geqslant 10$, one obtains a couple of solutions

$$f_{\omega l}(r) \simeq \alpha \exp\left\{ i\left[\bar{\omega}\left(\frac{r}{2GM} + \ln\left(\frac{r}{2GM}-1\right)\right) + 0.02\left(\frac{r}{2GM}-1\right)\right] \pm 1.73\left(\frac{r}{2GM}-1\right)\right\}, \tag{3.8}$$

where the plus sign corresponds to (3.6a) and the minus sign to (3.6b), respectively. Notice the important fact that the second of these solutions is convergent at $r \to \infty$ due to the presence of the factor exp(-1.73(r/2GM-1)).

Another solution of the radial equation (2.2) is given by (2.3) with

$$c_s = \frac{c}{s}, \quad s = 1, 2, 3, \ldots \tag{3.9}$$

This can be seen by substituting

$$c_1 = c, \quad c_2 = \frac{c}{2}, \quad c_3 = \frac{c}{3}, \quad c_4 = \frac{c}{4}, \tag{3.10}$$

in the equation of recurrence (2.6) for s=5. One gets $c_5 = c/5$ and, proceeding further, for the general c_s one obtains $c_s = c/s$, provided that $c_k = c/k$, k=1,2,....,s-1. As before, one must now express the initial conditions (3.10) in terms of the parameters $\bar{\omega}, \bar{l}$ and $\bar{m}$. Starting from (2.4), one has

$$c = -\frac{b_1}{2\bar{\omega}-i}, \qquad c^2 - 2(\bar{\omega}-2i)c - b_2 = 0,$$
$$2c^2 + 5ic - b_3 = 0, \qquad c^2 + 2ic - 1 = 0, \tag{3.11}$$

which can be solved and yield

$$\bar{\omega} = 0, \quad \bar{l} = 0, \quad \bar{m}^2 = 0, \quad c = -i. \tag{3.12}$$

Substituting this result into (2.3) we obtain

$$g(\rho) = c\sum_{s=1}^{\infty}\frac{\rho^s}{s} = c\ln\frac{r}{2GM} \tag{3.13}$$

and

$$f_{c0}(r) = \alpha\frac{r}{2GM} \tag{3.14}$$

recovering the already known result for $\omega = 0, l = 0$.

It can also be demonstrated that an arbitrary combination of the two exact solutions which we have found above, namely

$$c_s = a + \frac{b}{s}, \quad s = 1, 2, 3, \ldots \tag{3.15}$$

yields also an exact solution of eq. (2.2). Observe that this is not at all immediate, because the equation of recurrence (2.6) is not linear, owing to the presence of the Σ_s terms. This third type of exact solutions can be also expressed in terms of the $\bar{\omega}, \bar{l}$ and $\bar{m}$. From (2.4) we get

$$a + b = -\frac{b_1}{2\bar{\omega}-i}, \qquad b = \frac{c_1^2 + 2ic_1 - b_2}{2(\bar{\omega}-i)},$$
$$2c_1 b + i(c_1 + 4b) - b_3 = 0, \qquad b^2 + 2ib - 1 = 0. \tag{3.16}$$

The solutions to these equations are

$$a = i\bar{l}, \qquad b = -i,$$
$$\bar{l} = -1 \pm \sqrt{1 + 2i\bar{\omega}},$$
$$\bar{m}^2 = -2 + 2(1+i\bar{\omega})(2 \mp \sqrt{1+2i\bar{\omega}}), \qquad \bar{\omega} \text{ arbitrary}. \tag{3.17}$$

This yields for the general coefficient of the expansion (2.3)

$$c_s = i\left(\bar{l} - \frac{1}{s}\right). \tag{3.18}$$

Again the series is convergent and its sum is

$$g(r) = i\left[\bar{l}\left(\frac{r}{2GM}-1\right) - \ln\frac{r}{2GM}\right]. \tag{3.19}$$

Thus we obtain the following exact solution of the radial equation (2.2)

$$f_{\omega l}(r) = \alpha \exp\left\{ i\bar{\omega}\left[\frac{r}{2GM} + \ln\left|\frac{r}{2GM}-1\right|\right] - \left(\pm\sqrt{1+2i\bar{\omega}} - 1\right)\left(\frac{r}{2GM}-1\right) + \ln\frac{r}{2GM}\right\}. \tag{3.20}$$

It is very interesting to notice that for any $\tilde{\omega} \in R$, $\tilde{\omega} \neq 0$ when we take the plus sign in the square root we get a convergent solution at $r \to \infty$. In fact, for every $\tilde{\omega} \neq 0$ the real part of $(1+2i\tilde{\omega})^{1/2} - 1$ is positive and this makes $f_{\tilde{\omega}}(r)$ tend to zero as r tends to infinity. For $\tilde{\omega} = \alpha + i\beta, 0 < \beta < 1$, (3.20) provides us with a whole family of solutions of (2.2) with real $m > 0$ and converging both at $r = r_S$ (Kruskal coordinates) and at $r = +\infty$. This is a very remarkable family of solutions. It had even been speculated that such solutions would not exist.

As before, we have carried out a numerical check on the validity of the exact solution (3.20) for different values of $\tilde{\omega}$. The accuracy and the stability of the coefficients are remarkable and the recursion formula (2.6) can actually be used with negligible errors to obtain the values of the c_s up to very high s. A standard Fortran program with double precision complex variables does the job pretty well. In particular, the check that $c_{100} - c_{10} = 0.09i$ has an accuracy of 10^{-9}.

All what has been done in this and in the preceding Section by taking the plus sign in front of the i in (2.3) can be repeated for -i. This corresponds to taking the alternative sign for i in (1.6), (1.7). There is no problem in doing this for $r > r_S$ and the number of solutions given above is duplicated in this region.

We now turn to study what happens in the interior region $r_S/2 < r < r_S$, i.e. $-1 < \rho < 0$. As has been proven in Section 2, the series given above are also convergent here, and the corresponding function $g(\rho)$ they define is valid, in principle, in this region inside the event horizon. However, as everybody knows, the coordinates (t,r) are singular at $r = r_S$ and the differential equation (1.4) makes no sense at this point. It actually makes sense again for $0 < r < r_S$ and (1.4) has an exact solution given, for example, by (3.20) there, but the question is: how can we make sure that the solution (3.20) for $r > r_S$ and the solution (3.20) for $r_S/2 < r < r_S$ are *the same* solution, namely that the second is the one into which the first converts (i.e. it remains intact in this case) after having crossed the event horizon ?. This question is very relevant in the study of gravitational collapse.

In order to answer it, one has to abandon the singular coordinates (t,r) and use regular ones, such as the Kruskal coordinates

$$t' = \frac{U+V}{2}, \qquad r' = \frac{V-U}{2} \tag{3.21}$$

or, equivalently, the coordinates U, V defined through the pair of Eddington-Finkelstein coordinates (u,r) and (v,r) by

$$U = -\exp\left(-\frac{u}{4GM}\right), \qquad V = \exp\left(\frac{v}{4GM}\right), \tag{3.22}$$

where

$$u = t - r_*, \qquad v = t + r_* . \tag{3.23}$$

In the coordinates (3.22) the Schwarzschild metric (1.2) becomes

$$ds^2 = \frac{4(2GM)^3}{r} \exp\left(-\frac{r}{2GM}\right) dU dV - r^2 (d\theta^2 + \sin^2\theta d\varphi^2). \tag{3.24}$$

Its only singularity is now the origin $r = 0$. This is a true physical singularity (it reflects, for instance, in the curvature scalar) and cannot be eliminated by further coordinate transformations.

The coordinate change from (t,r) to (U,V) is singular at $r = r_S$ (that is, the singularity $r = r_S$ of the differential equation (1.2) has been eliminated by performing a coordinate transformation which is singular at this same point). In principle, the definition of (U,V) from (t,r) given by (3.22), (3.23) is valid only for $r > r_S$. In the other regions of space-time the coordinates U, V are the ones to be used. However, since we have taken the logarithm of the absolute value in the definition of r_* (1.5), one would think that the same relations (3.22), (3.23) would also hold for $0 < r < r_S$. An easy calculation shows that, starting from (3.24), one gets the Schwarzschild metric (1.2) whith a *global* minus sign. Such a metric is mathematically (and physically) indistinguishable from the Schwarzschild one (1.2). In particular, the radial Klein-Gordon equation one gets from it is exactly the same (1.4). Notwithstanding that, one has to be, again, very careful on crossing the coordinate singularity $r = r_S$.

The correct way to proceed is to make consistent use of the nonsingular coordinates (U,V) (or (t',r')). The solution of the radial Klein-Gordon equation corresponding to the metric (3.24) -let us call it g(U,V) will be valid in the whole enlarged domain of space-time covered by the Kruskal coordinates (regions I, II, III and IV of the Kruskal diagram). In region I, that is $U < 0, V > 0$, or $r > r_S$, the solution we are considering here let us call it $f_I(t,r)$- expressed in terms of (t,r) is given by (3.20). At the event horizon $r = r_S$ the solution can only be expressed in terms of (U,V) but it has no translation in terms of (t,r). In the black-hole region II, namely $U > 0, V > 0$, or $0 < r < r_S$, the solution $g(U,V)$ will be expressed in terms of (t,r) by $f_{II}(t,r)$. Summing up,

$$\text{I}: \quad g(U,V) = \left(1 - \frac{2GM}{r}\right) f_I(t,r), \qquad U = -\exp\left(-\frac{t-r_*}{4GM}\right), \quad V = \exp\left(\frac{t+r_*}{4GM}\right). \tag{3.25a}$$

$$\text{II}: \quad g(U,V) = \overline{\left(1 - \frac{2GM}{r}\right)} f_{II}(t,r), \qquad U = \exp\left(-\frac{t-r_*}{4GM}\right), \quad V = \exp\left(\frac{t+r_*}{4GM}\right), \tag{3.25b}$$

where r_* is given by (1.5). The change of sign in U in (3.25b) is the natural implementation of the global change of sign in the Schwarzschild metric ds^2 found above. Eqs. (3.25) lead to

$$f_{II}(t, r(r_*)) = \frac{r(r_* - \sigma)}{r} \exp\left[\frac{r(r_* - \sigma) - r}{2GM}\right] \times f_I(t+\sigma, r(r_* - \sigma)), \qquad \sigma \equiv 2GM \ln(-1). \tag{3.26}$$

For f_I given by (3.20) this can be written explicitly as

$$f_{II}(t,r) = \alpha \frac{r(r_*-\sigma)}{r} \exp\left[-\frac{r(r_*-\sigma)-r}{2GM}\right] \exp\Big\{-i\omega[(t-r_*)+\sigma] \tag{3.27}$$
$$-\left(\pm\sqrt{1+2i\bar{\omega}}-1\right)\left[\frac{\phi(r_*-\sigma)}{2GM}-1\right]+\ln\left[\frac{\phi(r_*-\sigma)}{2GM}\right]\Big\},$$

where $r = \phi(r_*)$ is the inversion of (1.5). Put in another form, in terms of the function $r = \psi(r_*)$ inverse of $r_* = r + 2GM\ln(r/2GM - 1), r < r_S$, we can write

$$f_{II}(t,r) = \alpha\, e^{2\pi\bar{\omega}} \frac{\psi(\bar{r}_*)}{r} \exp\left[\frac{\psi(\bar{r}_*)-r}{2GM}\right]$$
$$\times \exp\left\{-i\omega u - \left(\pm\sqrt{1+2i\bar{\omega}}-1\right)\left[\frac{\psi(\bar{r}_*)}{2GM}-1\right]+\ln\left[\frac{\psi(\bar{r}_*)}{2GM}\right]\right\}. \tag{3.28}$$

This is the solution corresponding to (3.20) after crossing the event horizon $r = r_S$, having chosen a determination of the logarithm, and in terms of the Eddington-Finkelstein coordinates (u, r). The coordinate u is the one relevant when going from region I to region II through $r = r_S$ along a geodesic given by v=const. A similar analysis could be made for the transit from region III (the white-hole region $U < 0, V > 0$) to region I. In fact, putting -i instead of i in (3.20) (or in any other of the ansätze which we shall encounter in this paper) one gets a new, independent solution in region I. In it, the other Eddington-Finkelstein coordinate v (3.23) appears. This is the solution relevant for the cross-over from region III to region I, along a geodesic u=const. [6]

The ansatz (2.3)-(2.7) appears to be well suited in order to study the different aspects of quantum scalar fields in Schwarzschild space-time for: i) as has been proven, it provides us with analytic solutions in the entire region extending from r_S to ∞ plus the interior region $r_S/2 < r < r_S$; ii) considered as a function of $\bar{\omega}$ it does not develop singularities. Notice that when $r \to r_S$ it behaves as

$$f_{\omega l}(r) = \alpha \exp\left\{i\omega r_* + \frac{\bar{m}^2+\bar{l}+1}{1+2i\bar{\omega}}\left(1-\frac{2GM}{r}\right)+O\left(\left(1-\frac{2GM}{r}\right)^2\right)\right\}, \tag{3.29}$$

while when $r \to \infty$, it behaves as

$$f_{\omega l}(r) \sim \beta_r(\bar{m},\bar{\omega},\bar{l})\exp(-i\omega r_*), \tag{3.30}$$

where β_r is a well defined function of $\bar{\omega}, \bar{l}, m$ for any value $r \to \infty$. As has been demonstrated, β_∞ exists for entire families of values of the parameters.

4. Solutions developed around $r = \infty$.

A solution to eq. (2.2) as a power series of 2GM/r is given by

$$f_{\omega l}(r) = \alpha \exp\left\{i\left[kr_* + b_0 \ln\left(\frac{r}{2GM}\right)+\sum_{s=1}^{\infty} b_s\left(\frac{2GM}{r}\right)^s\right]\right\}, r > r_S, \tag{4.1}$$

By direct substitution into (2.2), we find the values for the first coefficients

$$b_0 = \mu\bar{k}, \qquad b_1 = \frac{\mu}{2}\left[\bar{k}(\mu-2)+\frac{\bar{l}}{\mu\bar{k}}+i\right],$$
$$b_2 = \frac{\mu}{4}\left[-\bar{k}(\mu^2-\mu+2)-\frac{\bar{l}+1}{\bar{k}}+\frac{1}{\mu\bar{k}}\right]$$
$$-2i\left(\mu+\frac{\bar{l}}{\bar{m}^2}\right), \tag{4.2}$$
$$b_3 = \frac{1}{3}\left[2\left(1-\frac{b_0}{\bar{k}}\right)b_2+\frac{b_1}{2\bar{k}}(b_1+2b_0)-\frac{1}{2\bar{k}}\right.$$
$$\left.+\frac{i}{\bar{k}}(-3b_2+5b_1+b_0)\right],$$

where

$$\bar{k} = 2GMk, \qquad \mu = \frac{m^2}{2k^2}. \tag{4.3}$$

For $s \geq 4$ the general b_s is given by the recursion formula

$$b_s = \frac{1}{s}\Big\{(1-\mu)(s-1)b_{s-1}+2\mu(s-2)b_{s-2}-\mu(s-3)b_{s-3}$$
$$+\frac{1}{2\bar{k}}\Big[\sum_{j=1}^{s-2} j(s-j-1)b_j b_{s-j-1}-2\sum_{j=1}^{s-3} j(s-j-2)b_j b_{s-j-2}$$
$$+\sum_{j=1}^{s-4} j(s-j-3)b_j b_{s-j-3}\Big]-\frac{i}{2\bar{k}}\Big[s(s-1)b_{s-1} \tag{4.4}$$
$$-(2s-1)(s-2)b_{s-2}+(s-1)(s-3)b_{s-3}\Big]\Big\}, \qquad s \geq 4.$$

The procedure will be exactly the same as before. The starting hypothesis was that b_s is analytic as a function of 1/s, for s big enough. We write

$$x_s \equiv \frac{b_{s-1}}{b_s} = \alpha_0 + \frac{\alpha_1}{s} + \frac{\alpha_2}{s^2} + \cdots \tag{4.5}$$

Substituting this expression into (4.3), we get

$$\begin{aligned} b_s = \frac{1}{s}\Big\{ & b_s\big[(1-\mu)(s-1)x_s + 2\mu(s-2)x_s x_{s-1} - \mu(s-3)x_s x_{s-1} x_{s-2}\big] \\ & + \frac{1}{2k}\Big[2\sum_{j=1}^{p-1-\lambda} j(s-j-1)b_j b_{s-j-1}(1-2x_{s-j-1} + x_{s-j-1}x_{s-j-2}) \\ & + 4\sum_{j=1}^{p-1-\lambda} j b_j b_{s-j-2}(1-x_{s-j-2}) + S_s\Big] \\ & - i\frac{b_s}{2k} x_s\big[s(s-1) - (2s-1)(s-2)x_{s-1} + (s-1)(s-3)x_{s-1}x_{s-2}\big]\Big\}, \end{aligned} \tag{4.6}$$

where p and λ are given by (2.17), and

$$\begin{aligned} S_s &= 2p(p-1)b_p b_{p-1} - 2(p-1)^2 b_{p-1}^2, \qquad & s = 2p, \\ S_s &= p^2 b_p^2 - (p-1)^2 b_{p-1}^2, & s = 2p+1. \end{aligned} \tag{4.7}$$

Substituting for the different x_s's the expansions given in the Appendix, after a straightforward but tedious calculation we find that, order by order in $1/s$, the terms on the r.h.s. of (4.5) vanish, provided that the corresponding coefficient α in (4.4) is equal to 1. Namely, for

$$\alpha_0 = \alpha_1 = \cdots = \alpha_i = 1, \tag{4.8}$$

the recurrence equation (4.3) is satisfied up to terms of order $s^{-(i+1)}$, i.e., we have

$$b_s = b_s + O(s^{-(i+1)}). \tag{4.9}$$

Notice that

$$\frac{b_s}{b_{s-1}} = 1 - \frac{1}{s} + O(s^{-2}), \tag{4.10}$$

i.e., for any $\epsilon > 0$ there exists an $s_\epsilon \in N$ such that for every $s > s_\epsilon$ one has

$$\left|\frac{b_s}{b_{s-1}} - 1\right| < \varepsilon. \tag{4.11}$$

The radius of convergence of the series solution is again $\rho_g = 1$.

Summing up, there is only one exact solution to the radial equation (2.2): the one given by (4.1), namely

$$x_s = \frac{b_{s-1}}{b_s} = \sum_{i=0}^{\infty} s^{-i} = \frac{s}{s-1}. \tag{4.12}$$

In fact, it is easy to prove that,

$$b_0, b_1 \text{ arbitrary}, \qquad b_s = \frac{b_1}{s}, \quad s \geq 1, \tag{4.13}$$

is an exact solution of the recurrence equation (4.5), (4.6). The corresponding solution $f_{\omega l}(r)$ turns out to be

$$f_{\omega\ell}(r) = x\exp\Big\{i\Big[kr_* + b_0\ln\frac{r}{2GM} - b_1\ln\Big(1-\frac{2GM}{r}\Big)\Big]\Big\}. \tag{4.14}$$

As before, $b_2 = b_1/2$ and $b_3 = b_1/3$ impose restrictions on the values of the parameters $\bar{\omega}, \bar{l}$ and $\bar{m}$ wich actually lead to a solution of the type (4.12).

Changing the sign in front of i in (4.1) we get a second exact solution of (2.2) which is independent of the above one. Both are valid in the whole region outside the event horizon $r_S < r < +\infty$. For $Imb_1 < 0$ the solution (4.14) is also convergent as $r \to r_S$:

$$f_{\omega\ell}(r) \sim x\exp\Big[ikr_* - ib_1\ln\Big(1-\frac{2GM}{r}\Big)\Big]. \tag{4.15}$$

For $Imb_0 > 0$ the solution is convergent as $r \to +\infty$

$$f_{\omega\ell}(r) \sim x\exp\Big[ikr_* - ib_0\ln\frac{2GM}{r}\Big]. \tag{4.16}$$

For Imb_0 and Imb_1 positive we obtain an exact solution valid in the whole compact region $r_S \leq r \leq +\infty$.

5. Solutions valid for $r < r_S$.

Once more, the same procedure can be employed to investigate the convergence of the series solutions of (2.2) given by

$$f_{\omega\ell}(r) = \alpha \exp\left\{ i \left[k r_* + a_0 \ln \frac{r}{2GM} + \sum_{s=1}^{\infty} a_s \left(\frac{r}{2GM}\right)^s \right] \right\}, \tag{5.1}$$

with

$$a_0 = -i, \quad a_1 = i\bar{\ell}, \quad a_2 = \frac{k}{2} + \frac{i}{4}\bar{\ell}(\bar{\ell}+2),$$
$$a_3 = \frac{k}{3} + \frac{i}{9}\left[\bar{m}^2 + \bar{\ell}\left(3 + \frac{5}{2}\bar{\ell} + \bar{\ell}^2\right)\right], \tag{5.2}$$

and with the general a_s being given by

$$a_s = \frac{1}{s^2}\Big\{ (s-1)(2s-1)a_{s-1} - (s-1)(s-2)a_{s-2}$$
$$+ i\Big[2k\big((s-2)a_{s-2} - (s-3)a_{s-3}\big) + \sum_{j=1}^{s-1} j(s-j) a_j a_{s-j}$$
$$- 2\sum_{j=1}^{s-2} j(s-j-1) a_j a_{s-j-1} + \sum_{j=1}^{s-3} j(s-j-2) a_j a_{s-j-2} \Big] \Big\}, \quad s \geq 4. \tag{5.3}$$

Exactly the same considerations as before apply here. By postulating that a_s is analytic in s as $s \to \infty$, we may write

$$x_s = \frac{a_{s-1}}{a_s} = \alpha_0 + \frac{\alpha_1}{s} + \frac{\alpha_2}{s^2} + \ldots \tag{5.4}$$

Eq. (5.3) can be written in terms of the x_s :

$$a_s = \frac{1}{s^2}\Big\{ a_s \big[(s-1)(2s-1)x_s - (s-1)(s-2)x_s x_{s-1} \big]$$
$$+ i \Big[2k\, a_s x_s x_{s-1} \big((s-2) - (s-3)x_{s-2} \big)$$
$$+ 2\sum_{j=1}^{p-\lambda} j(s-j) a_j a_{s-j} \left(1 - 2x_{s-j} + x_{s-j} x_{s-j-1}\right)$$
$$+ 4\sum_{j=1}^{p-\lambda} j\, a_j a_{s-j-1} \left(1 - x_{s-j-1}\right) + S_s \Big] \Big\}, \tag{5.5}$$

where p and λ are given by (2.17), and

$$S_s = p^2 a_p^2 - (p-1)^2 a_{p-1}^2, \qquad s = 2p,$$
$$S_s = 2p^2 a_p^2 - 2p(p-1) a_p a_{p-1}, \qquad s = 2p+1. \tag{5.6}$$

Substituting (5.4) and the expressions given in the Appendix into (5.5), after another somehow tedious calculation we find a similar result as in the preceding Section, namely setting

$$\alpha_0 = \alpha_1 = \cdots = \alpha_i = 1, \tag{5.7}$$

the recursion relation (5.3) is satisfied up to terms of order $s^{-(i+1)}$, i.e., we have

$$a_s = a_s + O\left(s^{-(i+1)}\right). \tag{5.8}$$

Also, the convergence radius for the series in (5.1) is exactly equal to 1, because

$$\frac{a_s}{a_{s-1}} = 1 - \frac{1}{s} + O(s^{-2}). \tag{5.9}$$

It is not difficult to prove by direct substitution that

$$a_0, a_1 \ \text{arbitrary}, \qquad a_s = \frac{a_1}{s}, \quad s \geq 1, \tag{5.10}$$

is an exact solution of the recurrence equations (5.1), (5.3). The corresponding solution of (2.2) is

$$f_{\omega\ell}(r) = \alpha \exp\left[i k r_* + \ln\frac{r}{2GM} + \bar{\ell} \ln\left(1 - \frac{r}{2GM}\right)\right]. \tag{5.11}$$

The conditions $a_2 = a_1/2$ and $a_3 = a_1/3$ can be expressed in terms of the parameters $\bar{\omega}$, $\bar{l}$ and $\bar{m}$

$$\frac{\bar{k}}{2} + i\frac{\bar{l}^2}{4} = 0,$$
$$\frac{\bar{k}}{3} - \frac{i}{9}\left(\bar{m}^2 + \frac{5}{2}\bar{l}^2 + \bar{l}^3\right) = 0. \tag{5.12}$$

We obtain

$$\bar{k} = -\frac{i}{2}\bar{l}^2,$$
$$\bar{m}^2 = -\bar{l}^2(\bar{l}+1). \tag{5.13}$$

Notice that for $\bar{l} = 0$ we have $\bar{k} = \bar{m} = 0$ and the very simple solution

$$f_{\omega l}(r) = \alpha \frac{r}{2GM}. \tag{5.14}$$

Another example, for $\bar{l}^2 = 2i$ we have $\bar{k} = 1, \bar{m}^2 = 2^{3/4} - i(2 + 2^{3/4})$, and the solution

$$f_{\omega l}(r) = \alpha \frac{r}{2GM}\left(1 - \frac{r}{2GM}\right)^{2^{-1/4}} \exp\left(i\left(\frac{r}{2GM} + (1 + 2^{-1/4})\ln\left(1 - \frac{r}{2GM}\right)\right)\right) \quad (5.15)$$

All of them are convergent solutions as $r \to 0$. Moreover, the second solution, (5.15), is also convergent at $r = r_S$. In general, we observe that the solution (5.11) is convergent as $r \to 0$ due to the presence of the factor r/2GM. If $Re\bar{l} < 0$, it is convergent in the compact region $0 \leq r \leq r_S$.

In the present case, the connection between the part of the global solution $g(U, V)$ (3.25) valid in region II (the one above) and the part valid in the subregion $r_S < r < 2r_S$ of I is done in a way which is completely analogous to the procedure described in Section 3, eqs. (3.25), (3.26). The only modification to be taken into account is that now we start from f_{II} (given by (5.11)) and we have to obtain the corresponding f_I by making use of (3.25). Thus, the expression (3.26) has to be inverted.

Alternatively, we could also have started from the independent solution given by (5.11) with -i instead of +i, and could have envisaged this solution as being defined in region III; going ahead, we would then obtain the corresponding solution in region I, through the relations similar to (3.26), for the transit $III \to I$ along a geodesic u=const., as has already been described to the end of Section 3.

6. Discussions and conclusions.

Let us recapitulate what we have done. The aim was to find solutions of the radial Klein-Gordon equation corresponding to a spinless particle of mass m in Schwarzschild space-time, as created by a black hole of mass M. We tried to find any such solution which can be expressed as a power series expansion around i) $r = r_S \equiv 2GM$ (Sections 2 and 3), ii) $r = +\infty$ (Section 4), or iii) r=0 (Section 5). The first step was just to obtain the recursion formulae for the coefficients of the series expansions in each case [5].

The second step has been to investigate which of these formal series expansions actually yield convergent, exact solutions of the radial Klein-Gordon-Schwarzschild equation, to determine the convergence radius of each power series expansion and to sum them, thus providing closed, algebraic expressions for the solutions. Of course, all the coefficients of the series expansions and the algebraic expressions for the solutions themselves are given in terms of the parameters m, M, ω and l. Actually, they always appear in the combinations $\bar{m} = 2GMm$, $\bar{\omega} = 2GM\omega$ and $\bar{l} = l(l + 1)$.

In case i) we have obtained in this way two essentially different exact solutions (2.3), (3.3) and (2.3), (3.13), and also a very interesting combination of them (2.3), (3.19). In cases ii) and iii) the solution was completely specified by the above conditions of being given by a convergent series expansion . It is given by (4.14) and by (5.11), respectively.

The third step in the whole procedure has been to match the parameters $\bar{m}, \bar{\omega}$ and $\bar{l}$ with the first coefficients of the power series expansions in each case. In fact, this imposes restrictions on the values of these parameters which lead to exact, convergent solutions. The

situation is different in each case. Sometimes the parameters are completely determined in this way, and we obtain just one solution corresponding to these precise values of the parameters: this is the case in (3.14). Sometimes, they have remained undetermined and a one-parameter family of solutions has been obtained, as in (3.20) and (5.11), (5.13). And it has also once ocurred that the matching yields incompatible equations, so that only approximate solutions are obtained. This was the case in (3.4) which yielded the approximate solutions (3.7) and (3.8).

Summing up, equations (3.7), (3.8), (3.14), (3.20), (3.28), (4.14) and (5.11) contain the main results of this work. All of them are solutions of the Klein-Gordon-Schwarzschild radial equation. The first two are only approximate solutions, valid for the region $r_S < r < +\infty$; the second of them is also convergent at $r = +\infty$. (3.14) is an exact solution valid in the same region. For complex $\bar{\omega} = \alpha + i\beta$, $0 < \beta < 1$, (3.20),(3.28) provide us with a remarkable one-parameter family of exact solutions of (2.2) for *real mass $m > 0$ and converging in the whole of space-time $0 < r \leq +\infty$*. Such a kind of solution had been long searched for in the literature [7]. Of course, in every case the event horizon $r = r_S$ remains singular in the sense that the variable r_* blows up there and we are faced up with the known problem of having to choose good coordinates (such as the Kruskal ones) in order to come across this event (see the long explanation towards the end of Section 3). Notice, however, that our additional function $g(\rho)$ is free from these difficulty, it never worsens the situation. For particular values of the parameters, (4.14) is a solution valid in the compact region $r_S \leq r \leq +\infty$. (5.11), valid for $0 \leq r < r_S$, together with its continuation to $r_S < r \leq 2r_S$, provides a family of solutions valid in the whole interior region, actually for $0 \leq r \leq 2r_S$. As a final remarck, these solutions are the relevant ones in order to study gravitational collapse, which involves crossing the event horizon by going from region I to the black-hole region II along a geodesic v=const. All the previous solutions are duplicate by substituting -i for i in the initial ansatz, thus providing also the way to study the transition from the white-hole region III to the region I along a geodesic u=const.

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Appendix.

Starting from the expansion of x_s for s big, given by (2.11), after a boresome but straightforward calculation we obtain

$$x_s x_{s-1} = \alpha^2 + \frac{2\alpha\beta}{s} + (\alpha\beta + 2\alpha\gamma + \beta^2)\frac{1}{s^2} + (\alpha\beta + 2\alpha\gamma + \beta^2 + 2\alpha\delta + 2\beta\gamma)\frac{1}{s^3}$$

$$+(\alpha\beta + 3\alpha\gamma + \beta^2 + 3\alpha\delta + 3\beta\gamma + 2\alpha\epsilon + 2\beta\delta + \gamma^2)\frac{1}{s^4} + O(s^{-5}). \qquad (A.1)$$

and

$$\begin{aligned} x_s x_{s-1} x_{s-2} = \alpha^3 + \frac{3\alpha^2\beta}{s} + (\alpha\beta + \alpha\gamma + \beta^2)\frac{3\alpha}{s^2} \\ +(5\alpha^2\beta + 6\alpha^2\gamma + 6\alpha\beta^2 + 3\alpha^2\delta + 6\alpha\beta\gamma + \beta^3)\frac{1}{s^3} \\ +(9\alpha^2\beta + 15\alpha^2\gamma + 12\alpha\beta^2 + 9\alpha^2\delta + 3\alpha^2\epsilon \\ +18\alpha\beta\gamma + 6\alpha\beta\delta + 3\alpha\gamma^2 + 3\beta^3 + 3\beta^2\gamma)\frac{1}{s^4} + O(s^{-5}), \end{aligned} \qquad (A.2)$$

and

$$\begin{aligned} x_s x_{s-1} x_{s-2} x_{s-3} = \alpha^4 + \frac{4\alpha^3\beta}{s} + (3\alpha\beta + 2\alpha\gamma + 3\beta^2)\frac{2\alpha^2}{s^2} \\ +(7\alpha^2\beta + 6\alpha^2\gamma + 2\alpha^2\delta + 9\alpha\beta^2 + 6\alpha\beta\gamma + 2\beta^3)\frac{2\alpha}{s^3} + (36\alpha^3\beta + 42\alpha^3\gamma + 53\alpha^2\beta^2 \\ +18\alpha^3\delta + 4\alpha^3\epsilon + 54\alpha^2\beta\gamma + 12\alpha^2\beta\delta + 6\alpha^2\gamma^2 + 18\alpha\beta^3 + 12\alpha\beta^2\gamma + \beta^4)\frac{1}{s^4} + O(s^{-5}). \end{aligned} \qquad (A.3)$$

On the other hand, it is clear that as $s \to \infty$ then also $p \to \infty$, so that

$$x_p = \frac{c_{p-1}}{c_p} = 1 + \frac{\beta}{p} + \frac{\gamma}{p^2} + \frac{\delta}{p^3} + \frac{\epsilon}{p^4} + O(p^{-5}). \qquad (A.4)$$

The series expansions needed in order to be substituted into eq. (2.13) are the following

$$x_p^2 = \alpha^2 + \frac{2\alpha\beta}{p} + \frac{2\alpha\gamma + \beta^2}{p^2} + \frac{2\alpha\delta + 2\beta\gamma}{p^3} + O(p^{-4}), \qquad (A.5)$$

and

$$\begin{aligned} x_p^2 x_{p-1}^2 = \alpha^4 + \frac{4\alpha^3\beta}{p} + (\alpha\beta + 2\alpha\gamma + 3\beta^2)\frac{2\alpha^2}{p^2} \\ +(\alpha^2\beta + 2\alpha^2\gamma + 3\alpha\beta^2 + 2\alpha^2\delta + 6\alpha\beta\gamma + 2\beta^3)\frac{2\alpha}{p^3} + O(p^{-4}), \end{aligned} \qquad (A.6)$$

and

$$\begin{aligned} \frac{1}{x_{p+1}} = \frac{1}{\alpha} - \frac{\beta}{\alpha^2 p} + \frac{\alpha\beta - \alpha\gamma + \beta^2}{\alpha^3 p^2} \\ +(-\alpha^2\beta + 2\alpha^2\gamma - \alpha^2\delta - 2\alpha\beta^2 + 2\alpha\beta\gamma - \beta^3)\frac{1}{\alpha^4 p^3} + O(p^{-4}), \end{aligned} \qquad (A.7)$$

and

$$\frac{x_p}{x_{p+1}} = 1 + \frac{\beta}{\alpha p^2} + \frac{2\alpha\gamma - \alpha\beta - \beta^2}{\alpha^2 p^3} + O(p^{-4}). \qquad (A.8)$$

The term $x_p x_{p-1}$ can be read off from (A.1) for s=p.

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