DEUTSCHES ELEKTRONEN-SYNCHROTRON DESY

DESY 87-159
December 1987

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ISSN 0418-9833

NOTKESTRASSE 85 · 2 HAMBURG 52

DESY 87-159
December 1987

ISSN 0418-9833

SOLUTION OF THE LATTICE ϕ^4 THEORY IN 4 DIMENSIONS [1)]

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ABSTRACT

Some recent analytical and numerical studies of the one-component ϕ^4 theory on a 4-dimensional hypercubic lattice are reviewed. Taken together, the results obtained provide a complete solution of the model in the sense that most low energy amplitudes can be calculated with reasonable accuracy in those parts of the phase diagram, where the ultra-violet cutoff Λ satisfies $\Lambda \geqslant 2m$ ($\Lambda = 1/a$, a: lattice spacing, m: physical particle mass). Further topics discussed include the issue of "triviality" and a possible upper bound on the Higgs meson mass.

1. INTRODUCTION

Although the one-component ϕ^4 theory has so far not found any direct application in elementary particle physics, it has been used for many years as a guinea-pig to test and develop new ideas in quantum field theory. Among today's motivations to study the lattice regularized ϕ^4 theory are the following.

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1) Lectures given at the Nato Advanced Study Institute on "Non-Perturbative Quantum Field Theory", Cargèse (1987)

(a) There is overwhelming evidence /1-17/ that this model is "trivial" in 4 dimensions, i.e. that its continuum limit is a free field theory. As I will explain later (sect. 2), "trivial" field theories can nevertheless serve as accurate mathematical models for interacting elementary particles. However, "triviality" implies an upper bound on the interaction strength and one of the questions one would like to answer is, where exactly this bound lies and whether a non-perturbative (strong interaction) sector is excluded, in particular.

(b) In the limit of vanishing gauge coupling, the SU(2) Higgs model (which is an important part of the standard electro-weak theory) reduces to three copies of Maxwell fields and the 4-component ϕ^4 theory. By studying the latter, one thus hopes to get some insight into how the Higgs model behaves, especially when the scalar self-coupling is large and perturbation theory is not reliable. In particular, it is possible, at least for small gauge coupling, that the "triviality" of the scalar sector implies the "triviality" of the full Higgs model /18,19/, and this would then give rise to an upper bound on the Higgs meson mass /20-31/.

(c) Because of its simplicity, the ϕ^4 theory is an ideal laboratory to test improved numerical simulation algorithms /8,32,50-52/, to learn how the systematical errors in these calculations can be controlled and to develop new methods to extract the more elusive quantities of physical interest (such as scattering amplitudes) from the numerical data /33/. To a large extent, the present excitement in this field is due to the fact that accurate numerical simulations are feasible with the available computer power and, as we shall see, that detailed analytical "predictions" exist, which can be immediately compared with the "experimental" results.

In these lectures, I would first like to expand a little on points (a) and (b) above and I will then proceed to explain in outline how the one-component ϕ^4 theory in the symmetric phase can be solved analytically /34/. Of course, by a solution I do not mean that an exact and explicit formula for (say) the scattering matrix can be given, but that most low energy quantities can be calculated with respectable accuracy by combining renormalized perturbation theory with data obtained from the "high temperature" expansion. It is important that these expansions are only used in regions of the parameter space where they really apply, i.e. no analytic extrapolations are performed and an effort is made to estimate the systematic errors which arise when truncating the expansions at a finite order.

The analytic solution of the ϕ^4 theory can be extended to the broken symmetry phase of the model /35/, but before explaining how this goes (sect. 6), I shall review the numerical work of Montvay and Weisz /33/ on the 4-dimensional Ising model, which is a limiting case of the ϕ^4 theory. Their results agree very well with the analytic solution. In addition, they have made a detailed finite size analysis, which enabled them, for the first time in a numerical simulation, to determine a scattering matrix element (the S-wave scattering length). The conclusion from this beautiful "experiment" is that within errors the analytic solution of the ϕ^4 theory in the symmetric phase is correct and that a complete quantitative understanding of the model has hence been achieved. Simulations in the broken symmetry phase are already on the way and hopefully result in a similar confirmation of the analytic solution.

2. THE MEANING OF "TRIVIALITY"

The action of the lattice ϕ^4 theory may be written in the form

$$(2.1) \quad S = a^4 \sum_x \left\{ \frac{1}{2}(\partial_\mu \phi_o)^2 + \frac{1}{2} m_o^2 \phi_o^2 + \frac{g_o}{4!}\phi_o^4 \right\},$$

where "a" denotes the lattice spacing, $\phi_o(x)$ $(x/a \in \mathbb{Z}^4)$ is a real scalar field and $\partial_\mu \phi_o$ the nearest neighbor lattice derivative of ϕ_o. For stability we require $g_o \geq 0$ and we also assume that the bare mass parameter m_o^2 is in the range where the reflection symmetry $\phi_o \to -\phi_o$ is not spontaneously broken (the discussion below is however equally valid in the broken symmetry phase).

Let us now define a wave function renormalization constant Z_R, a renormalized mass m_R and a renormalized coupling g_R through

$$(2.2) \quad \Gamma^{(2)}(p,-p) = -Z_R^{-1}\{m_R^2 + p^2 + O(p^4)\} \qquad (p \to 0),$$

$$(2.3) \quad \Gamma^{(4)}(0,0,0,0) = -Z_R^{-2}\, g_R,$$

where $\Gamma^{(n)}(p_1,\dots,p_n)$ denotes the n-point vertex function of ϕ_o. The renormalized parameters m_R, g_R are well-defined functions of a, m_o^2 and g_o, which (by dimensional analysis) are of the form

$$(2.4) \quad m_R = \frac{1}{a}\, r(a^2 m_o^2, g_o),$$

$$(2.5) \quad g_R = s(a^2 m_o^2, g_o).$$

Using Lebowitz' inequality, one may show that $g_R \geq 0$ and, by definition, we also have $m_R > 0$ throughout the symmetric phase region.

If it exists at all, the continuum limit of the lattice theory is obtained by fixing m_R, g_R and sending the cutoff mass $\Lambda = 1/a$ to infinity. This assumes, in particular, that for given m_R, g_R and arbitrarily large Λ, bare parameters $m_o^2(\Lambda)$, $g_o(\Lambda)$ exist such that eqs. (2.4), (2.5) hold. In a "trivial" theory, this precondition is only fulfilled if $g_R = 0$. In other words, for all $g_R > 0$, eqs. (2.4), (2.5) imply an upper bound on the cutoff Λ of the form

$$(2.6) \quad \ln(\Lambda/m_R) \leq f(g_R),$$

where $f(g_R)$ is continuous and

$$(2.7) \quad \lim_{g_R \to 0} f(g_R) = \infty.$$

Thus, if one insists on taking the cutoff to infinity, one also has to scale g_R to zero so that in the end one is left with a free field theory.

The lattice ϕ^4 theory is most likely trivial /1-17/, but a completely rigorous proof of triviality is still missing. The solution of the one-component model, which I shall discuss later, also implies triviality and moreover yields an estimate for the function $f(g_R)$, which enters the triviality bound (2.6).

An obvious question is, whether a trivial theory is necessarily useless for the description of interacting elementary particles. The answer is definitely no here, because the bound (2.6) is often not very restrictive from a practical point of view. For example, in case of the ϕ^4 theory, we shall see that

$$f(g_R) \underset{g_R \to 0}{\sim} 16\pi^2/3g_R, \tag{2.8}$$

and for $g_R = 1$ (which is sufficiently small for (2.8) to apply), the triviality bound hence becomes

$$\Lambda/m_R \leqslant 7\cdot 10^{22}. \tag{2.9}$$

Thus, even for reasonably large couplings, the cutoff Λ can be pushed to very high values which may be orders of magnitude beyond the experimentally accessible energy region. In such an instance, the presence of the cutoff has no practical relevance, i.e. at low energies E, the theory behaves effectively like a continuum theory. Of course, cutoff effects are not totally absent, but since they are of order E^2/Λ^2 /36/, they are usually completely negligible.

Still, a trivial theory can only be a valid description of elementary particles and their interactions up to some finite energy scale and thus cannot by itself be a fundamental theory. It is however conceivable that trivial theories arise by integrating out the high energy degrees of freedom of an underlying ultra-violet stable theory. In that case, the triviality bound (2.6) provides an upper bound on the energy scale where "new physics" has to set in.

3. THE ϕ^4 THEORY AS A LIMIT OF THE SU(2) HIGGS MODEL

The Higgs sector of the standard model of electro-weak interactions is described by the (euclidean) action

$$S = S_G + S_H, \tag{3.1}$$

$$S_G = \int d^4x \; \frac{1}{4} W^a_{\mu\nu} W^a_{\mu\nu}, \tag{3.2}$$

$$S_H = \int d^4x \, \{ D_\mu\phi^\dagger \cdot D_\mu\phi + \frac{\lambda}{2}(\phi^\dagger\cdot\phi - \frac{v^2}{2})^2 \}, \tag{3.3}$$

where ϕ is an SU(2) doublet and

$$W^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + g\,\epsilon^{abc} W^b_\mu W^c_\nu, \tag{3.4}$$

$$D_\mu\phi = (\partial_\mu + g\, W^a_\mu \frac{\sigma^a}{2i})\,\phi \tag{3.5}$$

(σ^a are the Pauli matrices and the indices a,b,c,... run from 1 to 3). For convenience, I here use a continuum notation, but everything what follows, with obvious modifications, also applies to the standard lattice version of the model (e.g. ref. /28/).

In the Higgs phase, i.e. for large positive v^2, the model describes a triplet of heavy vector bosons ("W bosons") and a neutral scalar particle (the "Higgs boson"). At tree level of perturbation theory, the masses of these particles are

$$m_W = \frac{1}{2} g v, \tag{3.6}$$

$$m_H = \sqrt{\lambda}\, v. \tag{3.7}$$

The physical values of g and v are approximately given by

$$g \simeq 0.65, \tag{3.8}$$

(3.9) $v \simeq 250$ GeV.

The Higgs self-coupling λ, on the other hand, proved to be a very elusive parameter so that today its value is essentially unknown (experimental bounds on the Higgs meson mass are given in ref. /37/, for example).

It is conceivable that λ is in fact quite large. In this case, the Higgs particle would be heavy, perhaps $m_H \simeq 1$ TeV, and the perturbation expansion in powers of λ would become unreliable. Thus, non-perturbative methods are required to determine the properties of the Higgs model in this situation and one obvious possibility then is to apply the numerical simulation technique to the latticised model (see /38,39/ for reviews and /28-30/ for recent papers in this field). These simulations are done with the complete model including all fields and interactions as listed at the beginning of this section. They are therefore rather complicated and it is not easy to obtain solid results in a short time.

At this point, it is useful to note that the gauge coupling g is actually rather small (the relevant expansion parameter is $g^2/4\pi \simeq 1/30$). Thus, as has been proposed by Dashen and Neuberger some time ago /25/, the solution of the Higgs model at large λ may be attempted by first expanding in powers of g at fixed λ, v and then evaluating the coefficients in this expansion by numerical simulation or any other non-perturbative method.

To lowest order in g, the gauge field W^a_μ and the Higgs field ϕ decouple. Furthermore, the gauge action (3.2) reduces to the action for a triplet of non-interacting Maxwell fields and the Higgs action (3.3) becomes the action of an O(4) symmetric φ^4 theory:

(3.10) $$S_H = \int d^4x \left\{ \frac{1}{2} \partial_\mu \varphi \cdot \partial_\mu \varphi + \frac{\lambda}{8} (\varphi \cdot \varphi - v^2)^2 \right\},$$

(3.11) $$\phi = \frac{1}{\sqrt{2}} \begin{pmatrix} \varphi_2 + i\varphi_1 \\ \varphi_4 - i\varphi_3 \end{pmatrix}, \qquad \varphi_\alpha \text{ real.}$$

Since the limit $g \to 0$ is taken in the Higgs phase, the parameters in (3.10) are such that the O(4) symmetry is spontaneously broken. The associated Goldstone bosons are the former W bosons with a longitudinal spin polarization (the transversely polarized W bosons become the "photons", which are described by the gauge action).

The Higgs particle corresponds to a radial excitation of the scalar field and remains massive for g = 0. However, since it can decay into any even number of Goldstone bosons, it is actually a resonance with a decay width given by

(3.12) $$\Gamma_H/m_H = 3\lambda/32\pi + O(\lambda^2).$$

Thus, for large λ the Higgs particle is presumably a broad resonance.

Besides the Higgs mass m_H, there is another physical scale F in the φ^4 theory, which is associated with the dynamics of the Goldstone bosons. Suppose the vacuum expectation value of φ_α is in the 4-direction and let

(3.13) $$A^a_\mu = \varphi_4 \partial_\mu \varphi_a - \varphi_a \partial_\mu \varphi_4, \qquad a = 1,2,3,$$

be the conserved currents, which generate the spontaneously broken symmetries. Then, F is defined by the matrix element

(3.14) $$\langle 0 | A^a_\mu(0) | p,b \rangle = i p_\mu \delta^{ab} F,$$

where $|p,b\rangle$ denotes the state of a single Goldstone boson with momentum p and symmetry label b. The normalizations are such that

(3.15) $$\langle q,a | p,b \rangle = 2|\vec{q}|(2\pi)^3 \delta^{ab} \delta(\vec{q} - \vec{p})$$

and $p_\mu = (i|\vec{p}|, \vec{p})$ (the time derivative in eq. (3.13) is with respect to euclidean time). Eq. (3.14) defines F non-perturbatively and there is also no normalization ambiguity, because the normalization of the currents A_μ^a is fixed by the associated Ward identities. Incidentally, by a simple application of these identities, it is possible to show that /40/

$$(3.16) \quad F = \langle 0|\varphi_4|0\rangle,$$

provided φ_α is renormalized in such a way that the Goldstone pole in the two-point function of φ_α has unit residue. In particular, $F = v + O(\lambda)$.

So far I have discussed what happens at $g = 0$. If the gauge coupling is now switched on again, the most important effect is that the gauge bosons and the Goldstone bosons become massive and combine to form the W vector bosons as indicated above. To first order in g, the vector boson mass is proportional to g and one may actually show that /25/

$$(3.17) \quad m_W^2 = \frac{1}{4} g^2F^2 + O(g^4 \ln g^2).$$

The proof of this nice formula is based solely on the O(4) Ward identities at $g = 0$ and it is therefore an exact result valid for all values of v^2 and λ. It also holds literally on the lattice (the lattice artefacts only show up at order g^4). Essentially, eq. (3.17) should be considered a form of the Goldstone theorem.

Closed expressions to first order in g^2 could perhaps also be derived for other physical quantities such as the WW scattering amplitude, but I would now like to proceed to discuss another issue, which is how triviality gives rise to an upper bound on the Higgs meson mass.

In view of eq. (3.7), a possible definition of a renormalized Higgs self-coupling λ_R at $g = 0$ is

$$(3.18) \quad \lambda_R = m_H^2/F^2.$$

The triviality bound (2.6) for the (lattice regularized) φ^4 theory with action (3.10) then reads

$$(3.19) \quad \ln(\Lambda/m_H) \leq f(m_H^2/F^2).$$

At least for small λ_R and presumably in the whole range of λ_R, the function $f(\lambda_R)$ is monotonically increasing when λ_R is made smaller so that (3.19) may be rewritten in the form

$$(3.20) \quad m_H^2/F^2 \leq f^{-1}(\ln(\Lambda/m_H)).$$

Finally, using eq. (3.17) to eliminate the scale F, one obtains

$$(3.21) \quad m_H^2/m_W^2 \leq \frac{4}{g^2} f^{-1}(\ln(\Lambda/m_H)).$$

Since g and m_W are measured, eq. (3.21) provides an upper bound on the Higgs mass if we require that Λ is greater than (say) $2m_H$ (for lower values of Λ, the low energy properties of the Higgs model would be strongly influenced by non-universal cutoff effects). Of course, it may also be sensible to require that Λ is beyond the Planck scale or some other huge mass, in which case the bound (3.21) would be more stringent.

To extract actual numbers from eq. (3.21), one needs the function $f(\lambda_R)$, which is defined in the (pure) φ^4 theory with action (3.10). Unfortunately, only the asymptotic form of $f(\lambda_R)$ for $\lambda_R \to 0$ is known presently, but there is little doubt that $f(\lambda_R)$ will soon be determined in the full range by the analytic method, which I shall explain later for the one-component model, and by numerical simulations (see /31,41/ for first attempts in this direction). Finally, I would like to remark that in the derivation of the bound (3.21), we have neglected the correction term in eq. (3.17) and, of course, we have also discarded the influence of the fermions and the other fields in the standard model, which are not included in the Higgs action (3.1)-(3.3).

4. SOLUTION OF THE ONE-COMPONENT MODEL IN THE SYMMETRIC PHASE

I now sketch how the lattice ϕ^4 theory defined in sect. 2 can be solved analytically in the symmetric phase region. A more detailed discussion is given in ref. /34/.

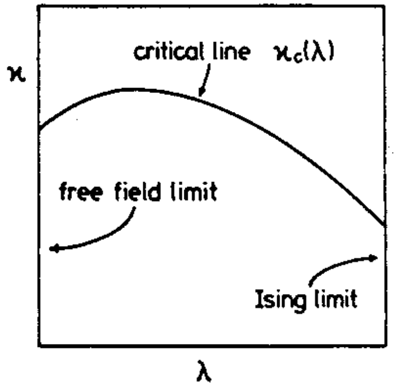

Fig. 1. Qualitative plot of the phase diagram of the lattice model with action (4.1). For $\lambda \to \infty$, the theory reduces to the Ising model.

For what follows, it is convenient to rewrite the action (2.1) in the form

$$S = \sum_x \Big\{ -\varkappa \sum_{\mu=0}^{3} \big(\phi(x)\phi(x+\hat\mu) + \phi(x)\phi(x-\hat\mu) \big) + \phi(x)^2 + \lambda\,(\phi(x)^2 - 1)^2 \Big\}, \tag{4.1}$$

where $\varkappa \geq 0$, $0 \leq \lambda \leq \infty$ and the lattice spacing "a" has been set equal to one for convenience, i.e. I shall use lattice units from now on. The relation between the old and the new notation is

$$\phi_o = \sqrt{2\varkappa}\;\phi, \tag{4.2}$$

$$m_o^2 = (1-2\lambda)/\varkappa - 8, \tag{4.3}$$

$$g_o = 6\lambda/\varkappa^2. \tag{4.4}$$

The phase diagram of the model (4.1) is displayed in Fig. 1. There are two phases separated by a second order critical line $\varkappa = \varkappa_c(\lambda)$. Here we are interested in the region $\varkappa < \varkappa_c(\lambda)$,

which corresponds to an unbroken reflection symmetry $\phi \to -\phi$.

From the action (4.1) one derives in the usual way the correlation functions $\langle \phi(x_1) \ldots \phi(x_n) \rangle$ and the n-point vertex functions $\Gamma^{(n)}(p_1,\ldots,p_n)$ in momentum space. Suppose now we define Z_R, m_R, g_R as before through eqs. (2.2), (2.3). The immediate goal in what follows then is, to calculate these quantities as a function of the bare parameters $\varkappa$ and λ. As we shall see later, the solution of this problem also leads to a reasonably accurate determination of the low energy properties of the model, at least in the region $\Lambda \geq 2m_R$ ($\Lambda = 1$ in lattice units).

For $\varkappa = 0$, the field variables at different points of the lattice decouple and the model becomes soluble. Relying on this fact, it is easy to derive an expansion of Z_R, m_R, g_R in powers of $\varkappa$, e.g. for m_R we have

$$m_R = \frac{1}{\sqrt{\varkappa}} \sum_{\nu=0}^{\infty} m_R^{(\nu)}(\lambda)\,\varkappa^\nu. \tag{4.5}$$

This expansion has been known for a long time in statistical mechanics, where it is called the "high temperature expansion". It is convergent for $\varkappa < \varkappa_c$ and the expansion coefficients can be worked out in a mechanical way to a high order. In particular, the series for Z_R, m_R, g_R have been tabulated by Baker and Kincaid /3/ up to 10th order.

As one can see from eq. (4.5), m_R becomes large for $\varkappa \to 0$ so that one expects the expansion to be practically useful when m_R is not too small. Still, since the first 10 terms in the high temperature series are known, it is possible to perform a careful convergence analysis, and one then finds that the truncation error stays reasonably small up to $\varkappa = 0.95\,\varkappa_c$ which corresponds to $m_R \simeq 0.5$ (estimates for $\varkappa_c$ are given in ref. /34/). Some results at $\varkappa = 0.95\,\varkappa_c$ obtained in this way are listed in Table 1. The data show that Z_R is surprisingly close to $1/2\varkappa$, which is the lowest order term in the weak coupling perturbation expansion of this quantity. The renormalized coupling g_R is monotonically rising with λ and reaches a maximal value of about 41 at $\lambda = \infty$. This is

Table 1. Values of Z_R, m_R, g_R as a function of λ at $\varkappa = 0.95\,\varkappa_c$ as calculated from the high temperature expansion

λ	$\varkappa$	$2\varkappa Z_R$	m_R	g_R
0.00	0.1188	1.0	0.649	0.0
0.01	0.1206	1.0000(2)	0.639(1)	3.57(5)
0.10	0.1298	0.9990(5)	0.599(6)	16(1)
1.00	0.1267	0.990(2)	0.54(1)	34(4)
∞	0.0710	0.973(4)	0.49(1)	41(6)

actually not such a big value, at least, it is only about 2/3 of the tree level unitarity bound and renormalized perturbation theory should in general still be applicable at these values of the coupling (the "natural" expansion parameter in perturbation theory is $\alpha_R = g_R/16\pi^2$).

With the help of the high temperature expansion we have thus been able to solve the theory in the region $\varkappa \leq 0.95\,\varkappa_c$, which corresponds approximately to $\Lambda/m_R \leq 2$ (see Fig. 2). To get closer to the critical line, i.e. closer to the continuum limit, we shall use the renormalization group equations. One of these equations is usually written as

$$(4.6) \qquad -\Lambda\left(\frac{\partial g_R}{\partial \Lambda}\right)_\lambda = \beta,$$

where β is the Callan-Symanzik β-function. In lattice units, $\Lambda = 1$ by definition and the proper form of eq. (4.6) then is

$$(4.7) \qquad m_R\left(\frac{\partial g_R}{\partial m_R}\right)_\lambda = \beta.$$

This equation describes the evolution of g_R as one moves towards the critical line at fixed λ. Similar equations exist for Z_R and $\varkappa$. Thus, if we knew the β function (and the other Callan-Symanzik coefficients), we could easily calculate Z_R, m_R, g_R at (say) point B of Fig. 2 by integrating the re-

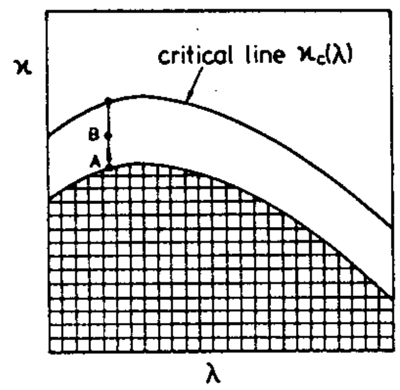

Fig. 2. Same as Fig. 1, but showing the region where the high temperature expansion applies (cross-hatched area)

normalization group equations using the known values of these quantities at point A as initial data.

The crucial observation now is, that as we have noted above, the coupling g_R is already in the perturbative domain along the line $\varkappa = 0.95\,\varkappa_c$ where the integration of the renormalization group equations is started. Thus, we may employ renormalized perturbation theory to calculate $\beta(g_R)$, at least during the initial steps of the integration. In fact, since β is positive, eq. (4.7) drives g_R to smaller values as m_R decreases and perturbation theory hence becomes an ever better approximation the closer one is to the critical line. Thus, in this way it is possible to compute Z_R, m_R, g_R as a function of $\varkappa, \lambda$ everywhere in the white area below the critical line in Fig. 2.

For illustration, some results obtained by integrating the renormalization group equations are listed in Table 2. The errors quoted derive from the errors in the initial data at $\varkappa = 0.95\,\varkappa_c$. As can be seen from Table 1, the errors are maximal for $\lambda = \infty$, in particular, the estimated accuracy in g_R is never worse than 15 %. The data in Table 2 smoothly join the high temperature curves at the matching point $\varkappa = 0.95\,\varkappa_c$

Table 2. Values of $\varkappa$, Z_R, g_R as a function of m_R at $\lambda = \infty$ (Ising model)

m_R	$\varkappa$	$2\varkappa Z_R$	g_R
0.40	0.0722(2)	0.973(7)	35(5)
0.20	0.0741(4)	0.975(9)	24(2)
0.10	0.0746(4)	0.976(9)	18(1)
0.05	0.0747(4)	0.974(9)	15.0(8)
0.01	0.0748(4)	0.972(9)	10.6(4)

and they also agree well with the Monte Carlo data of ref. /33/ (cf. sect. 5).

In the limit $\Lambda/m_R \to \infty$, λ fixed, the coupling g_R eventually goes to zero according to the implicit asymptotic formula

$$m_R/\Lambda = C_1 (\beta_1 g_R)^{17/27} e^{-1/\beta_1 g_R} \{1 + O(g_R)\}, \quad (4.8)$$

where C_1 is a constant (depending on λ) and $\beta_1 = 3/16\pi^2$ is the one-loop coefficient of the β function. Eq. (4.8) is just the asymptotic form of the general solution of the renormalization group equation (4.7), which one obtains when the initial value of g_R is sufficiently close to the origin (which we have argued to be the case). The triviality of the ϕ^4 theory is essentially a consequence of the scaling law (4.8). This can be seen more clearly from Fig. 3, where I have plotted the curves of constant g_R in the $\varkappa,\lambda$-plane.[2)] Along these curves, only the cutoff Λ (in units of m_R) changes while the low energy physics is fixed. Now it turns out that the maximal value of Λ/m_R is attained in the Ising limit ($\lambda = \infty$) where the curves end, and the triviality bound (2.6) is thus given by

2) I shall later explain how to obtain the curves in the broken symmetry phase $\varkappa > \varkappa_c$.

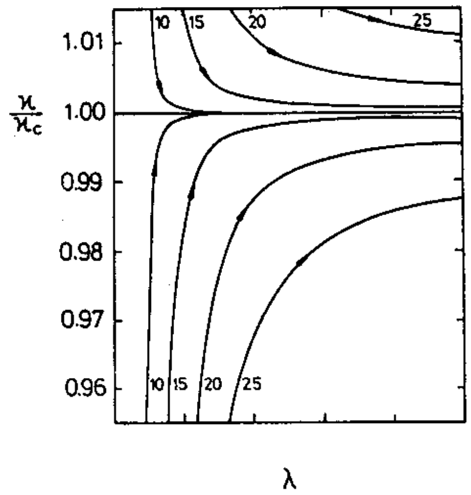

Fig. 3. Curves of constant coupling g_R = 10,15,20,25 in the plane of bare parameters. The arrows are in the direction of increasing cutoff. All curves end at the Ising line (for g_R = 10, the distance to the critical line is so small that it cannot be resolved in this drawing)

$$\ln(\Lambda/m_R) \leqslant \frac{1}{\beta_1 g_R} - \frac{17}{27}\ln(\beta_1 g_R) + C + O(g_R), \quad (4.9)$$

where

$$C = -\ln C_1(\infty) = -1.5(2). \quad (4.10)$$

The correction terms in eq. (4.9) are negligible for $g_R \leqslant 10$ and for the larger values of g_R, the triviality bound can be read off from Table 2.

An important result of the discussion so far is that the coupling g_R is always less than about 2/3 of the tree level unitarity bound when $\Lambda \geqslant 2m_R$. In other words, whenever the cutoff is sufficiently high for the theory to have essentially cutoff independent low energy properties, the coupling is necessarily small and renormalized perturbation theory should be applicable. In particular, the scattering matrix for low energy processes can be computed in this way to a respectable accuracy and a complete solution of the ϕ^4 theory in the symmetric phase has thus been achieved.

5. NUMERICAL SIMULATION OF THE ISING MODEL

For the analytical solution of the ϕ^4 theory, the Ising limit is the most difficult case, because g_R assumes its largest values there. On the other hand, numerical simulations of the Ising model are relatively easy and a significant comparison between "theory" and "experiment" is hence feasible.

The numerical work, which I am now going to describe, has been done by Montvay and Weisz /33/. They chose two values of $\varkappa$ in the symmetric phase, approximately corresponding to m_R=0.5 and m_R=0.2. The lattices considered were of the form $L^3 \times T$ with (ordinary) periodic boundary conditions in all directions and

$$T = 12, \quad L = 4,6,\ldots,12 \quad \text{for } m_R \simeq 0.5,$$
(5.1)
$$T = 24, \quad L = 8,10,\ldots,20 \quad \text{for } m_R \simeq 0.2$$

(T is time, L is space). On each lattice, several million field configurations were generated using a standard Metropolis algorithm. The reason for having L vary over a range of values is that in this way a detailed finite size analysis is possible, as I will explain later. In particular, the results quoted below are, within errors, infinite volume values.

It is not easy to determine m_R in a Monte Carlo simulation, because the finite extent of the lattice implies that the momentum p in eq. (2.2) is quantized with a lowest non-zero value, which may not be sufficiently small to suppress the $O(p^4)$ terms (for the lattices (6.1), for example, one has $p^2 > m_R^2$ if $p \neq 0$). A more readily accessible quantity

Table 3. Results from a numerical simulation of the Ising model /33/ and comparison with the analytic solution /34/. In the Monte Carlo calculation, $\varkappa$ is given, while for the analytic calculation m_R is taken as the independent parameter

	Monte Carlo	analytic solution
$\varkappa$	0.07102	0.0710(2)
m_R	0.4923(5)	0.4923
$2\varkappa Z_R$	0.970(3)	0.973(7)
g_R	44(4)	42(7)
$\varkappa$	0.07400	0.0740(3)
m_R	0.2148(5)	0.2148
$2\varkappa Z_R$	0.962(7)	0.975(9)
g_R	25(2) [3)]	24(2)

is the physical mass m, which, for all L, is defined through the exponential decay of the two-point function of $\phi(x)$ in the time direction. As I will discuss shortly, the L-dependence of m is weak and well understood. Furthermore, for $L = \infty$ we have

$$m_R = 2 \sinh m/2 \, (1 + O(g_R^2)), \qquad (5.2)$$

where the $O(g_R^2)$ correction has been calculated and was found to be negligible ($< 10^{-4}$ for $g_R < 44$).

Some results obtained by Montvay and Weisz, for the two values of $\varkappa$ considered, are listed in Table 3, where I have used eq. (5.2) to eliminate m in favour of m_R. Within the quoted errors, the agreement with the analytic solution is

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3) This number includes an analytically calculated finite size correction of Δg_R = 1.7 at L = 18. The error quoted is statistical only.

perfect. Thus, the qualitative assumptions on which the analytic solution is based (for example, that renormalized perturbation theory may be applied when $g_R \lesssim 41$) appear to be justified and little doubt remains that the solution is in fact correct.

I would now like to digress a little and discuss the lattice size dependence of the particle mass m(L) and the lowest two-particle energy W(L), which I shall define later. First note that m(L) is an eigenvalue of the transfer matrix and is hence independent of T by definition (in a Monte Carlo simulation, T must however be large so that the exponential decay of the two-point function of ϕ can be followed over a significant distance). As a function of L, the finite size mass shift

$$\delta_1 = [m(L) - m(\infty)]/m(\infty) \tag{5.3}$$

decays exponentially according to

$$\delta_1 = -\frac{3}{2m^2}\int_{-\pi}^{\pi}\frac{d^3q}{(2\pi)^3\,2\omega(q)}\,e^{-\omega(q)L}\,F(q) + O(e^{-\bar{m}L}), \tag{5.4}$$

where $\omega(q)$ denotes the energy of a single particle with momentum q, F(q) an elastic forward scattering amplitude and $\bar{m} > m$. All quantities m, $\omega(q)$ and F(q) on the right hand side of eq. (5.4) are defined and evaluated at $L = \infty$. I have first presented this formula at Cargèse 1983 /42/ and since then provided a detailed proof /43/ (the lattice corrections have been discussed by Münster /48/).

It is of course possible to compute $\omega(q)$ and F(q) in renormalized perturbation theory. Taking the first order expressions and inserting the values of m_R, g_R at $\varkappa = 0.07102$ as given by Table 3, one obtains curve "a" in Fig. 4. The agreement with the Monte Carlo data at L = 6,8,10,12 (the points with the small error bars in Fig. 4) is very good although perhaps a bit fortuitous given that only the first order perturbative formulae were used and that the error term

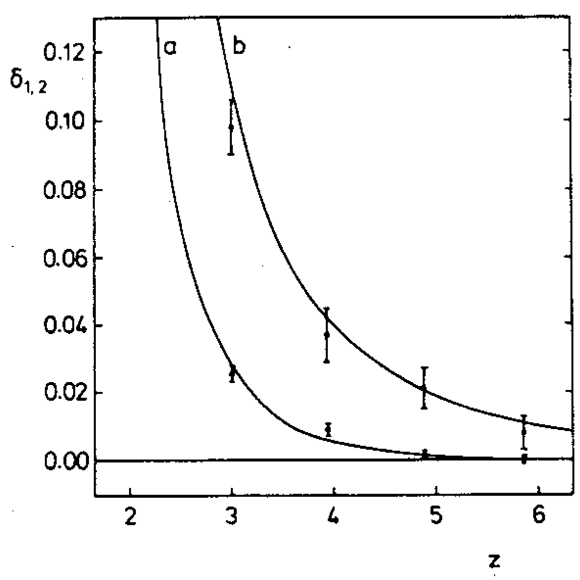

Fig. 4. Finite size energy shifts δ_1 and δ_2 as a function of $z = m(L)L$. Curves a and b correspond to eqs. (5.4) and (5.5) evaluated at $\varkappa = 0.07102$ ($m(\infty) \simeq 0.49$ and $a_0 \simeq -0.68$ at this point)

in eq. (5.4) was also neglected. Anyway, eq. (5.4) certainly gives the right order of magnitude for δ_1 and there is no doubt that the finite size effects are negligible compared to the statistical errors beyond say z = 6.

Another quantity considered by Montvay and Weisz is the two-particle energy W(L), which is the lowest energy above the vacuum in the sector of even states under $\phi \to -\phi$. The corresponding energy eigenstate describes two particles, which, being confined to the finite lattice, are in a stationary scattering state. Thus, W(L) is essentially equal to 2m with a finite size correction δ_2 given by

$$\delta_2 = [W(L) - 2m(L)]/2m(\infty)$$

$$= -\frac{2\pi a_0}{m^2 L^3}\left\{1 + c_1\frac{a_0}{L} + c_2\frac{a_0^2}{L^2}\right\} + O(L^{-6}), \tag{5.5}$$

(5.6) $c_1 = -2.837297,$

(5.7) $c_2 = 6.375183,$

where a_o denotes the S-wave scattering length.[4] A proof of eq. (5.5) in the framework of quantum field theory has been given recently /43/, but for the case of non-relativistic hard spheres in a periodic box, the formula has actually been derived much earlier by Huang and Yang /44/ (see also refs. /45,46/).

If we use renormalized perturbation theory to two loops to compute a_o at $\varkappa = 0.07102$, eq. (5.5) yields curve b in Fig. 4. Again the Monte Carlo data of Montvay and Weisz agree very well with the theoretical prediction. In fact, the scattering length a_o could have been extracted from the data by fitting them with eq. (5.5). It has thus been demonstrated that a calculation of S-matrix elements through numerical simulation is possible in certain cases, the main difficulties being that very accurate data are required and that several lattices of variable size must be considered. Of course one hopes to apply the method to other models such as the Higgs model or even QCD.

A plot similar to Fig. 4 could also be produced at $\varkappa = 0.07400$, which corresponds to $m(\infty) = 0.21$ approximately. The picture would look less impressive in this case, because the errors are larger and because the maximal value of z would only be around 4 for the lattices considered. Within these limitations, the agreement between theory and experiment is however equally good.

6. SOLUTION OF THE ONE-COMPONENT MODEL IN THE BROKEN SYMMETRY PHASE

For $\varkappa > \varkappa_c(\lambda)$, the reflection symmetry $\phi \to -\phi$ of the action (4.1) is spontaneously broken and the field ϕ acquires

4) It is possible to develop a full-fledged scattering theory for euclidean lattice field theories /49/ and a_o is hence a completely well-defined quantity for all $\varkappa, \lambda$.

a non-zero vacuum expectation value

(6.1) $v = \langle\phi\rangle > 0.$

If we define Z_R and m_R as in the symmetric phase (eq. (2.2)), the renormalized vacuum expectation value is given by

(6.2) $v_R = v\, Z_R^{-1/2}$

and a renormalized coupling may be introduced through

(6.3) $g_R = 3m_R^2/v_R^2$

(to first order in g_R, this definition is equivalent to eq. (2.3)).

As in the symmetric phase, the first goal now is to compute Z_R, m_R and g_R as a function of $\varkappa$ and λ. However, since there is no known practical expansion for $\varkappa \to \infty$, which could play the rôle the high temperature expansion did in our analysis of the symmetric phase, a different strategy is needed.

The basic idea is as follows /35/. As we have discussed in sect. 4, the renormalized coupling g_R in the symmetric phase scales to zero as one approaches the critical line in such a way that the limit

(6.4) $$C_1(\lambda) = \lim_{\varkappa \to \varkappa_c} m_R\,(\beta_1 g_R)^{-17/27}\, e^{1/\beta_1 g_R}$$

exists (cf. eq. (4.8)). Similarly, a constant $C_1'(\lambda)$ may be defined by approaching $\varkappa_c(\lambda)$ from the broken symmetry phase. Both constants are defined at the critical line and it is therefore not surprising that they can be given an interpretation in terms of the critical (massless) theory. It then turns out that $C_1'(\lambda)$ is actually proportional to $C_1(\lambda)$ with a proportionality constant, which is exactly given by

(6.5) $C_1'(\lambda) = e^{1/6}\, C_1(\lambda)$

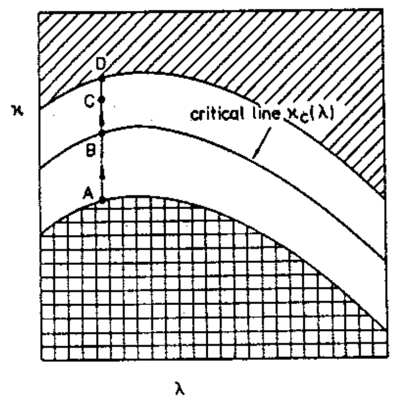

Fig. 5. Qualitative plot of the phase diagram of the lattice ϕ^4 theory. The integration of the renormalization group equations is started at e.g. point A at the boundary of the high temperature region (cross-hatched area) and follows the line λ = constant towards point B, where C_1 and C_1' are determined. The integration can then be carried on to (say) point C in the broken symmetry phase.

for our choice of renormalization conditions.

$C_1(\lambda)$ can be calculated for all λ to a reasonable estimated accuracy from the solution of the model in the symmetric phase. Thus, $C_1'(\lambda)$ is also known and may be used as initial datum for the integration of the renormalization group equations along the lines λ = constant in the broken symmetry phase starting at $\varkappa = \varkappa_c$ (see Fig. 5). Since the β-function (and the other Callan-Symanzik coefficients) are only known in perturbation theory, the integration must be stopped when g_R becomes large (point D in Fig. 5). Thus, in this way the theory can only be solved in a narrow band above the critical line, but as it turns out, this band includes the whole region $\Lambda \geqslant 2m_R$. The shaded area in Fig. 5, where the theory remains unsolved, is therefore not a very interesting region since there, similarly to the high temperature region in the symmetric phase, the physics at scales of m_R is strongly influenced by non-universal cutoff effects.

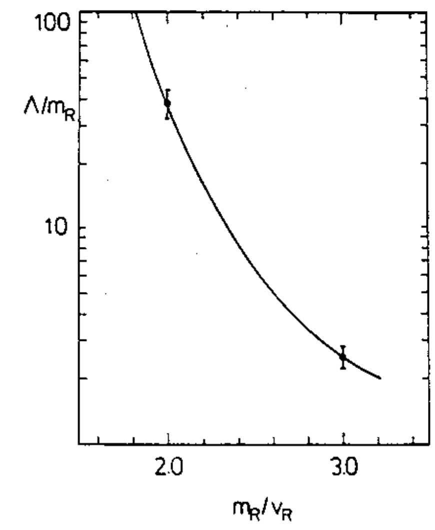

Fig. 6. Maximal value of the ultra-violet cutoff Λ in units of m_R for given m_R/v_R. The size of the estimated errors in the calculation is indicated at two representative points.

The most conspicuous feature of the solution of the model in the broken symmetry phase obtained along these lines is that, concerning the scaling behaviour, there is practically no difference to what happens in the symmetric phase. In particular, for $\Lambda \geqslant 2m_R$, the renormalized coupling g_R does not exceed a maximal value of about 2/3 of the tree level unitarity bound and renormalized perturbation theory should hence give an essentially correct description of the particle interactions at low energies in this region. Furthermore, as shown by Fig. 3, the flow of the curves of constant coupling g_R in the plane of bare parameters also looks similar on both sides of the critical line, a marked difference being that the interval of $\varkappa$ corresponding to $\Lambda \geqslant 2m_R$ is about a factor of 3 smaller in the broken symmetry phase.

In Fig. 6, I have plotted the triviality bound (2.6), where, instead of the coupling g_R, I have taken the ratio

m_R/v_R as the independent variable (cf. eq. (6.3)). One expects that a similar result will be obtained for the O(4) symmetric ϕ^4 theory and, as discussed in sect. 3, this will then lead to an upper bound on the Higgs meson mass. Fig. 6 applies to the one-component model and is therefore not amenable to such an interpretation. However, it is interesting to note that if we insert $m_R = m_H$, $v_R = 250$ GeV and assume $\Lambda \geqslant 2m_H$ for the purpose of illustration, the bound $m_H \leqslant 800$ GeV is obtained, which is actually not far from what other people have found earlier /20-31/.

7. CONCLUDING REMARKS

The analytic solution of the one-component ϕ^4 theory in the broken symmetry phase has not yet been checked by a large scale numerical simulation, but such calculations are on the way, one computing the effective action /50/ and another one /51/ employing the highly efficient up-dating algorithm of Swendson and Wang /52/. Work is also in progress on the physically more interesting O(4) symmetric model, which I expect to be soluble in the same way as the one-component model. In particular, the Goldstone modes in the broken symmetry phase should not give rise to any great difficulties for the analytic approach. Still, a substantial amount of labour remains to be done, especially so since the Baker-Kincaid tables /3/ are only for the one-component model and the high temperature series for z_R, m_R and g_R must hence be newly derived /53/.

ACKNOWLEDGEMENTS

The bulk of these lectures is based on joint work with Peter Weisz, whom I would like to thank for the enjoyable collaboration. I am also indebted to Istvan Montvay for discussions on various aspects of the Monte Carlo simulation technique and I finally thank the organizers of the Cargèse Summer School for their kind invitation.

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