DEUTSCHES ELEKTRONEN-SYNCHROTRON DESY

DESY 88-040
April 1988

RENORMALIZATION THEORY FOR USE IN CONVERGENT EXPANSIONS OF EUCLIDEAN QUANTUM FIELD THEORY

by

A. Pordt

II. Institut für Theoretische Physik, Universität Hamburg

ISSN 0418-9833

NOTKESTRASSE 85 · 2 HAMBURG 52

DESY 88-040
April 1988

ISSN 0418-9833

RENORMALIZATION THEORY FOR USE IN CONVERGENT EXPANSIONS OF EUCLIDEAN QUANTUM FIELD THEORY[1]

Andreas Pordt

II. Institut für Theoretische Physik der Universität Hamburg

1. Introduction

Renormalization is necessary for perturbation and cluster expansion methods of Euclidean quantum field theory. Bounds for Feynman graphs show that divergencies for renormalizable theories come from subgraphs with a small number of low frequency external lines [1,2]. Renormalization group suggests a way to avoid such divergencies by introducing running coupling constants. The tree expansion introduced by G. Gallavotti and F. Nicolò [3-6] represents the expansion in powers of running coupling constants instead of renormalized coupling constants. Running coupling constants are related by renormalization group equations and the bare coupling constants have to be chosen such that the renormalized coupling constants are finite if an ultraviolet cutoff is removed.

Nonperturbative effects coming from large field contributions cause the divergence of perturbation expansions. Cluster expansion methods take into account such nonperturbative effects and provide convergent series expansions for Euclidean Greens functions. A polymer system on the multigrid is a natural example of a phase cell cluster expansion (see G. Mack's lecture and [7]).

In section 2 of this paper the use of renormalization with running coupling constants for perturbation theory is discussed. The resummation of Feynman graph expansion in terms of running coupling constants is presented . Section 3 discusses the renormalization for a polymer system on the multigrid.

2. Renormalization with running coupling constants

Consider the following partition function for a d-dimensional scalar field theory

$$Z(\Psi) = \frac{1}{\mathcal{N}} \int \prod_{z \in \mathbf{R}^d} d\Phi(z) \exp\{-\frac{1}{2}(\Phi, v^{-1}\Phi) - V(\Phi + \Psi)\} \tag{1}$$

[1] Cargèse lectures July 1987, to appear in: G.'t Hooft et al., Nonperturbative Quantum Field Theory, Plenum Press, N.Y. 1988.

where the normalization constant $\mathcal{N}$ is determined by $Z(0) = 1$. The free propagator v is the Yukawa potential with mass m

$$v = (-\triangle + m^2)^{-1}.$$

Then the free part of the interaction is

$$\frac{1}{2}(\Phi, v^{-1}\Phi) = \frac{1}{2}\int \Phi(z)(-\triangle + m^2)\Phi(z).$$

With the Gaussian measure

$$d\mu_v(\Phi) = (\det 2\pi v)^{-\frac{1}{2}} \exp\{-\frac{1}{2}(\Phi, v^{-1}\Phi)\} \prod_{z \in \mathbf{R}^d} d\Phi(z)$$

we can write

$$Z(\Psi) = \int d\mu_v(\Phi) \exp\{-V(\Phi + \Psi) - c\} \tag{2}$$

where

$$c = -\ln \int d\mu_v(\Phi) \exp\{-V(\Phi)\}$$

$Z(\Psi)$ is the generating functional for free propagator amputated Greens functions. The (connected) n-point free propagator amputated Greens functions $F(z_1, ..., z_n)$ ($F_c(z_1, ..., z_n)$) are defined by

$$\begin{aligned} F(z_1, ..., z_n) &= \frac{\delta^n}{\delta\Psi(z_1)\dots\delta\Psi(z_n)} Z(\Psi)|_{\Psi=0} \\ F_c(z_1, ..., z_n) &= \frac{\delta^n}{\delta\Psi(z_1)\dots\delta\Psi(z_n)} \ln Z(\Psi)|_{\Psi=0}. \end{aligned} \tag{3}$$

$Z(\Psi)$ and $\ln Z(\Psi)$ can be represented as a formal Taylor expansion in Ψ

$$\begin{aligned} Z(\Psi) &= 1 + \sum_{n\geq 1} \frac{1}{n!} \int_{z_1,...,z_n \in \mathbf{R}^d} dz_1 \dots dz_n \; F(z_1, ..., z_n)\Psi(z_1)\dots\Psi(z_n) \\ \ln Z(\Psi) &= \sum_{n\geq 1} \frac{1}{n!} \int_{z_1,...,z_n \in \mathbf{R}^d} dz_1 \dots dz_n \; F_c(z_1, ..., z_n)\Psi(z_1)\dots\Psi(z_n). \end{aligned} \tag{4}$$

For renormalization group calculations split the free propagator v into propagators v^j which obey the bound

$$|v^j(z_1, z_2)| \leq c_1 L^{j(d-2)} \exp\{-c_2 L^j |z_1 - z_2|\} \tag{5}$$

where $L > 1$ is a fixed scale factor and c_1, c_2 are positive constants. For example

$$v^j(z_1, z_2) = \int_{-\infty}^{\infty} \frac{dp}{(2\pi)^d} \frac{e^{ip(z_1 - z_2)}}{p^2} [e^{-L^{-2j}p^2} - e^{-L^{-2(j-1)}p^2}]. \tag{6}$$

Then we obtain for the massless free propagator $v = (-\triangle)^{-1}$

$$v = \sum_{j=-\infty}^{\infty} v^j.$$

For the massive free propagator $v = (-\Delta + m^2)^{-1}$, $m^2 > 0$, we can find similarly

$$v = \sum_{j=0}^{\infty} v^j$$

such that the bound (5) for v^j is fulfilled. Let us consider a partition function with ultraviolet and infrared cutoff

$$Z_k^{(N)}(\Psi) = \int d\mu_{\sum_{j=k}^{N} v^j}(\Phi) \exp\{-V^{(N)}(\Phi + \Psi) + e_k^{(N)}\}. \tag{7}$$

$V^{(N)}$ is called bare interaction and $e_k^{(N)}$ is fixed by imposing $Z_k^{(N)}(0) = 1$. The effective interaction $V_k^{(N)}$ is defined by

$$Z_k^{(N)}(\Psi) = \exp\{V_k^{(N)}(\Psi)\}. \tag{8}$$

Suppose that the bare interaction $V^{(N)}$ depends on a finite number of parameters $\{\lambda_\alpha^{(N)}\}$. A theory is called renormalizable if one can choose the finite bare parameters $\{\lambda_\alpha^{(N)}\}$ such that the effective theory is well defined when the ultraviolet cutoff is removed ($N \to \infty$), i.e. Greens functions exist for the partition functions

$$Z_k(\Psi) = \lim_{N\to\infty} Z_k^{(N)}(\Psi)$$

for all k .

$Z_k^{(N)}$ and $Z_{k-1}^{(N)}$ are related by the renormalization group equations

$$Z_{k-1}^{(N)}(\Psi) = \int d\mu_{v^k}(\Phi) Z_k^{(N)}(\Phi + \Psi) e^{\delta e_k^{(N)}} \tag{9}$$

where

$$\delta e_k^{(N)} = -\ln \int d\mu_{v^k}(\Phi) Z_k^{(N)}(\Phi).$$

The effective interaction and connected Greens functions are related by

$$V_k^{(N)}(\Psi) = -\sum_{n\geq 1} \frac{1}{n!} \int_{z_1,\dots,z_n\in\mathbf{R}^d} F_{k,c}^{(N)}(z_1,\dots,z_n)\Psi(z_1)\dots\Psi(z_n). \tag{10}$$

If we start with a local bare interaction $V^{(N)}$ we obtain a nonlocal effective interaction $V_k^{(N)}$. But the effective interaction is almost local since we have integrated fields with covariances v^j , $j = k+1, k+2, \dots, N$, and v^j has decay length of order L^{-j} . Therefore the coefficients $F_{k,c}^{(N)}$ of the effective interaction show exponentially decay with decay length of order $L^{-(k+1)}$. Using Taylor expansion for external fields we can represent the effective interaction in a local form

$$V_k^{(N)}(\Psi) = -\sum_{n\geq 1} \frac{1}{n!} \sum_{m_2,\dots,m_n\in\mathbf{N}^d} \int_{z\in\mathbf{R}^d} F_{k,c,m_2,\dots,m_n}^{(N)}(z)\Psi(z) \prod_{a=2}^{n} \partial_z^{m_a}\Psi(z) \tag{11a}$$

where

$$F_{k,c,m_2,\dots,m_n}^{(N)}(z) = \int_{z_1,\dots,z_n\in\mathbf{R}^d} \frac{(z_2-z)^{m_2}\dots(z_n-z)^{m_n}}{m_2!\dots m_n!} F_{k,c}^{(N)}(z_1,\dots,z_n). \tag{11b}$$

We have used the multiindex notation $m = (m^1, \dots, m^d) \in \mathbf{N}^d$, $m! = \prod_{\mu=1}^d m^\mu!$ and $|m| = \sum_{\mu=1}^d m^\mu$. Define a localization operator $\mathcal{L}$ by

$$\mathcal{L}V_k^{(N)}(\Psi) = -\sum_{n\geq 1} \frac{1}{n!} \sum_{\substack{m_2,\dots,m_n\in\mathbf{N}^d :\\ \sum_{a=2}^n |m_a|\leq K_n}} \int_{z\in\mathbf{R}^d} F_{k,c,m_2,\dots,m_n}^{(N)}(z)\Psi(z) \prod_{a=2}^{n} \partial_z^{m_a}\Psi(z) \tag{12}$$

with $K_n \in \{-1\} \cup \mathbf{N}$. For example for a theory with reflection symmetry $\Psi \to -\Psi$

$$\mathcal{L}V_k^{(N)}(\Psi) = -\frac{1}{2}\int_z m_k^{(N)}(z)^2\Psi(z)^2 - \frac{1}{4!}\int_z \lambda_k^{(N)}(z)^4\Psi(z)^4 - \frac{1}{2}\sum_{\substack{\mu,\nu=1\\ \mu\leq\nu}}^{d} \int_z \beta_{k,\mu\nu}^{(N)}(z)\Psi(z)\partial_{z^\mu}\partial_{z^\nu}\Psi(z) \tag{13}$$

where

$$m_k^{(N)}(z)^2 = \int_{z_2} F_{k,c}^{(N)}(z,z_2)$$

$$\lambda_k^{(N)}(z) = \int_{z_2,z_3,z_4} F_{k,c}^{(N)}(z,z_2,z_3,z_4)$$

$$\beta_{k,\mu\nu}^{(N)}(z) = \int_{z_2} (z_2^\mu - z^\mu)(z_2^\nu - z^\nu) F_{k,c}^{(N)}(z,z_2).$$

For Euclidean invariant effective actions we obtain $m_k^{(N)}(z)^2 = m_k^{(N)2}$, $\lambda_k^{(N)}(z) = \lambda_k^{(N)}$, $\beta_{k,\mu\nu}^{(N)}(z) = \beta_k^{(N)}\delta_{\mu\nu}$. Let us consider the nonlocal remainder term

$$(1-\mathcal{L})V_k^{(N)}(\Psi).$$

The 2nd order term in Ψ is

$$\frac{1}{2}\sum_{m\in\mathbf{N}^d : |m|=4} \int_{z_1,z_2} \frac{(z_2-z_1)^m}{m!} F_{k,c}^{(N)}(z_1,z_2)\Psi(z_1)\partial_z^m\Psi(z)|_{z=z_1+\Theta(z_2-z_1)}$$

for some $\Theta \in [0,1]$. Using that $F_{k,c}^{(N)}(z_1,z_2)$ is smaller than a factor $\exp\{-const\cdot L^{k+1}|z_1-z_2|\}$ we gain a factor $L^{-4(k+1)}$ for the remainder term by

$$|z_2 - z_1|^m \exp\{-const\cdot L^{k+1}|z_1-z_2|\} \leq O(1)\cdot L^{-4(k+1)}.$$

In next renormalization group steps fields with covariances v^j for $j \leq k$ are integrated. Then external fields Ψ are replaced by $v^j(z,\cdot)$ and the factor $\partial_z^m\Psi(z)$ in (12) is replaced by $\partial_z^m v^j(z,\cdot)$. This gives a factor L^{4j}. Therefore we get a convergence producing factor $L^{4(j-k-1)}$. In summary the effective interactions are split into a local and a nonlocal term. The local term is parametrized by running coupling constants and the nonlocal term gives small contributions because of the convergence producing factors. The renormalization group procedure starts with a local interaction $V^{(N)}$ such that $\mathcal{L}V^{(N)} = V^{(N)}$. For example for the localization operator given by (13) we take

$$V^{(N)}(\Psi) = -\frac{1}{2}m^{(N)2}\int_z \Psi(z)^2 - \frac{1}{4!}\lambda^{(N)}\int_z \Psi(z)^4 - \frac{1}{2}\sum_{\mu=1}^{d}\beta^{(N)}\int_z \Psi(z)\partial_\mu^2\Psi(z). \tag{14}$$

In the rest of this section I will show the relation of renormalization by running coupling constants and perturbative renormalization. Let

$$v = \sum_{j: j \le N} v^j$$

be the free propagator with UV-cutoff. For notational simplicity we consider only the case of renormalization for the quartic interaction term. Generalizations are easily obtained. Consider the partition function

$$Z(\Psi) = \int d\mu_v(\Phi) \exp\{-\frac{\lambda_N}{4!} \int_z (\Phi + \Psi)(z)^4 + e_N\}. \qquad (15)$$

The partition function for an effective theory is

$$Z_k(\Psi) = \int d\mu_{v \le k+1}(\Phi) \exp\{-\frac{\lambda_N}{4!} \int_z (\Phi + \Psi)(z)^4 + e_k\}. \qquad (16)$$

The running coupling constant is defined by

$$\lambda_k = - \int_{z_2, z_3, z_4} \frac{\delta^4}{\delta\Psi(z)\delta\Psi(z_2)\delta\Psi(z_3)\delta\Psi(z_4)} \ln Z_k(\Psi)|_{\Psi=0}. \qquad (17)$$

Define a new partition function by

$$\bar{Z}_k(\Psi) = Z_k(\Psi) \exp\{\frac{\lambda_{k+1}}{4!} \int_z \Psi(z)^4\} \qquad (18)$$

for all $k \le N$. We set $\lambda_{N+1} = 0$. The renormalization group flow for the running coupling constant is given by

$$\delta\lambda_j = \lambda_{j+1} - \lambda_j = \int_{z_2, z_3, z_4} \frac{\delta^4}{\delta\Psi(z)\delta\Psi(z_2)\delta\Psi(z_3)\delta\Psi(z_4)} \ln \bar{Z}_j(\Psi)|_{\Psi=0}. \qquad (19)$$

The recursion relation for the partition function $\bar{Z}_k$ is

$$\bar{Z}_{k-1}(\Psi) = \int d\mu_{v^k}(\Phi^k) \bar{Z}_k(\Phi^k + \Psi) \exp\{-\frac{\lambda_k}{4!}[\int_z (\Phi^k + \Psi)^4 - \int \Psi^4] - \frac{\delta\lambda_k}{4!} \int (\Phi^k + \Psi)^4 - \delta e_{k-1}\} \qquad (20)$$

with $\delta e_k = e_{k+1} - e_k$.

The Feynman graph expansion for the partition function is

$$Z(\Psi) = \exp\{\sum_{G \in \mathcal{F}_c} I(G)\}. \qquad (21)$$

$\mathcal{F}_c$ is the set of all connected Feynman graphs and $I(G)$ is the Feynman integral for the Feynman graph G. Each line in the Feynman graph corresponds to a free propagator v with UV-cutoff and each vertex corresponds to the negative bare coupling constant $-\lambda_N$. Removing the cutoff ($N \to \infty$) the Feynman integral is no longer convergent for all Feynman graphs G. In standard perturbation theory the bare interaction is splitted in a renormalized interaction and a counterterm. The vertices

in the Feynman graphs corresponds to the renormalized coupling constant or to a counterterm insertion. The counterterm is chosen such that the divergencies in each Feynman graph are cancelled. We want to show how to renormalize by using running coupling constants. Assign to each propagator in a Feynman graph an integer $j \le N$. Interprete this line in a Feynman integral as propagator v^j. The sum over all assignments reproduce the Feynman integral. Denote by $G^{\ge i}$ the subgraph of G when we omit all propagator lines v^j with $j < i$. $G^{\ge i}$ consists of connected components. Denote by C^i_γ a connected component of $G^{\ge i}$ which contains at least one propagator v^i. The set of all connected Feynman graphs with propagartors $v^N, v^{N-1}, ..., v^{k+1}$ containing at least one propagator is denoted by $\bar{\mathcal{F}}^k_c$. The partition functions are represented by

$$\bar{Z}_k(\Psi) = \exp\{\sum_{G \in \bar{\mathcal{F}}^k_c} \bar{I}(G)\} \qquad (22)$$

for all $k < N$. The rules for computation of a renormalized Feynman integral $\bar{I}(G)$ are

(i) Vertex rule : From each vertex in G emerges at least one propagator line v^j, $k-1 \le j \le N$. If m is the maximal index of all propagator lines emerging from a vertex in G then this vertex corresponds to the negative running coupling constant $-\lambda_m$.

(ii) Subtraction rule : Suppose that the vertices of the Feynman graph G are labelled by 1,...,n. A vertex of a subgraph G' of G is called maximal if all other vertices of G' have larger labels. For a connected component C^j_γ of G with four external lines let $\mathcal{R}(C^j_\gamma)$ be the graph obtained from C^j_γ by linking external lines which emerge from not minimal vertices to the minimal vertex of C^j_γ. Then replace for each connected component C^j_γ by $C^j_\gamma - \mathcal{R}(C^j_\gamma)$.

3. Renormalization theory on the multigrid

For an introduction of multigrid methods I refer the reader to G.Mack's lecture. Euclidean quantum field theory can be represented as a polymer system on a multigrid Λ. The field Φ and free propagator v are transformed to the multigrid by the following split

$$\Phi(z) = \int_{x \in \Lambda} \mathcal{A}(z, x)\varphi(x)$$
$$v(z_1, z_2) = \sum_j \int_{x_1, x_2 \in \Lambda^j} \mathcal{A}(z_1, x_1) v(x_1, x_2) \mathcal{A}(z_2, x_2) \equiv \sum_j v^j(z_1, z_2) \qquad (23)$$

for $z, z_1, z_2 \in \mathbf{R}^d$. Free propagators v_j obey the bound (5). Suppose that for all finite subsets X of Λ partition functions $Z(X)$ are defined. Then activities $A(P)$ are defined by

$$Z(X) = \sum_{X = \sum P} \prod_P A(P) \qquad (24)$$

for all finite subsets X of Λ. The sum runs over all disjoint partitions of X. For convergence properties we need that $A(P)$ is small if P contains blocks $x \in \Lambda^j, y \in \Lambda^k$ with $|j - k|$ large. In the last section we have seen how to gain convergence factors $L^{-\epsilon|j-k|}, \epsilon > 0$, by introducing running coupling constants. In this section

I will show how convergence factors for activities $A(P)$ are produced by imposing renormalization conditions.

Feynman graphs are naturally defined on a multigrid. Introduce $\mathcal{A}$-lines which connect points $z \in \mathbf{R}^d$ with blocks $x \in \Lambda$ on the multigrid and v-lines which connect blocks x, y on layers Λ^j of the multigrid. By split (23) each propagator line in a Feynman graph connecting vertices $z_1, z_2 \in \mathbf{R}^d$ can be represented by two $\mathcal{A}$-lines which connect z_1, z_2 by vertices $x_1, x_2 \in \Lambda^j$ respectively and a v-line connecting x_1 and x_2. A Feynman graph on a multigrid is called point connected if the graph is connected when vertices on the multigrid which represent same blocks are identified. A Feynman graph expansion for activities consists of point connected Feynman graphs. For renormalization the following definition is useful. A Feynman graph on a multigrid is called k-vertically irreducible if it is point connected and it is not possible by cutting less than k+1 $\mathcal{A}$-lines to separate vertices $x \in \Lambda^j, y \in \Lambda^{j'}, j \neq j'$, in the Feynman graph. Vertices x, y are separated if there is no path in the Feynman graph which connects x and y. Feynman graphs on a multigrid can be decomposed into k-vertically irreducible components. The point is that a k-vertically irreducible component needs no renormalization if k is large enough (for a renormalizable theory). Correspondingly, activities can be expressed by k-vertically irreducible activities. In the following the definition for k-vertically irreducible activities is given.

Let the partition function for a finite subset X of Λ be of the following form

$$Z(X|\Psi) = \int d\mu_{v_X}(\Phi) \exp\{-V(X|\Phi + \Psi) + \int_{x \in X} j(x)(\varphi + \psi)(x)\} \qquad (25)$$

where

$$v_X(z_1, z_2) = \int_{x_1, x_2 \in X} \mathcal{A}(z_1, x_1) v(x_1, x_2) \mathcal{A}(z_2, x_2).$$

$j(x)$ is an external source and $V(X|\Phi + \Psi)$ an X-dependent interaction. We will use the notations

$$X^j = X \cap \Lambda^j, \quad X^{\geq j} = X^j + X^{j+1} + \ldots, \quad X^{<j} = X^{j-1} + X^{j-2} + \ldots .$$

Define new partition functions

$$Z_j(X|\Psi) = \int d\mu_{v_{X^{\geq j+1}}}(\Phi) \exp\{-[V(X|\Phi + \Psi) - V(X^{<j+1}|\Phi + \Psi)] + \int_{x \in X^{\geq j+1}} j(x)(\varphi + \psi)(x)\}. \qquad (26)$$

We have

$$Z_k(X|\Psi) = \begin{cases} 1, & \text{if } k \geq h(x) = \max\{j | X^j \neq \emptyset\} \\ Z(X|\Psi), & \text{if } k < l(X) = \min\{j | X^j \neq \emptyset\}. \end{cases}$$

Let $\tau : \Lambda \to \mathbf{N}$ ($\mathbf{N}$ = set of all natural numbers) be a function with finite support supp $\tau = \tau^{-1}(\mathbf{N} - \{0\})$. τ can be considered as a finite collection of blocks x with multiplicity $\tau(x)$. We will use the notation

$$\tau! = \prod_{x \in \Lambda} \tau(x)!, \quad |\tau| = \sum_{x \in \Lambda} \tau(x), \quad \frac{\delta^\tau}{\delta\psi^\tau} = \prod_{x \in \Lambda} \frac{\delta^{\tau(x)}}{\delta\psi(x)^{\tau(x)}}.$$

For a polymer P and a collection τ with supp $\tau \subset \Lambda^{<l(P)}$ we call $(P|\tau)$ an extended polymer and define a partition function and activity by

$$\begin{aligned} Z(P|\Psi) &= \frac{1}{\tau!} \frac{\delta^\tau}{\delta\psi^\tau} Z_{l(P)-1}(P + \text{supp } \tau|\Psi)|_{\Psi^{<l(P)}=0} \\ A(P|\tau) &= \frac{1}{\tau!} \frac{\delta^\tau}{\delta\psi^\tau} A_{l(P)-1}(P + \text{supp } \tau|\Psi)|_{\Psi^{<l(P)}=0} \end{aligned} \qquad (27)$$

where A_j is the activity for the partition function Z_j and $\Psi^{<j}$ is defined by

$$\Psi^{<j}(z) = \int_{x \in \Lambda^{<j}} \mathcal{A}(z, x)\psi(x).$$

Partition functions and activities for extended polymers are related by

$$Z(X|\tau) = \sum_{n \geq 1} \sum_{X = \sum_{i=1}^n P_i} \sum_{X = \sum_{i=1}^n \tau_i} \prod_{i=1}^n A(P_i|\tau_i). \qquad (28)$$

Let $(P_1|\tau_1), \ldots, (P_n|\tau_n)$ be extended polymers. We call $[(P_1|\tau_1), \ldots, (P_n|\tau_n)]$ a k-partition ($k \in \mathbf{N}$) of $(X|\tau)$ into extended polymers if

$$\bigcup_{i=1}^n \text{supp } \tau_i \subset X + \Lambda^{<l(X)}, \quad X = \sum_{i=1}^n P_i, \quad \tau = \sum_{a=1}^n \tau_a^{<l(X)}, \quad |\tau_a|_X| \leq k$$

for all $a \in \{1, \ldots, n\}$. $B^{[k]}(X|\tau)$ is the set of all k-partitions of $(X|\tau)$ into extended polymers. For extended polymers $(P|\tau)$ k-vertically irreducible activities $A^{[k]}(P|\tau)$ are defined by

$$Z(X|\tau) = \sum_{n \geq 1} \sum_{[(P_1|\tau_1), \ldots, (P_1|\tau_n)] \in B^{[k]}(X|\tau)} N\{\prod_{i=1}^n A^{[k]}(P_i|\tau_i) \frac{\delta^{\tau_i|x}}{\delta j^{\tau_i|x}}\} \qquad (29)$$

for all finite subsets X of Λ and collections τ with supp $\tau \subset \Lambda^{<l(X)}$. The symbol $N\{\ldots\}$ means that all derivatives operate inside the bracket. In Feynman graph expansion $A^{[k]}(P|\tau)$ consists of k-vertically irreducible Feynman graphs.

For sets $Q_1, \ldots, Q_n$ a graph $\gamma(Q_1, \ldots, Q_n)$ consists of vertices $1, \ldots, n$ and lines (ij) if $i \neq j$ and $Q_i \cap Q_j \neq \emptyset$. A k-partition $[(P_1|\tau_1), \ldots, (P_n|\tau_n)]$ is called connected if $\gamma(P_1, \ldots, P_n, \text{supp } \tau_1|_X, \ldots, \text{supp } \tau_n|_X)$ is connected. The set of all k-partitions of $(X|\tau)$ into extended polymers is denoted by $B_c^{[k]}(X|\tau)$. Then we obtain the following representation for activities

$$A(X|\tau) = \sum_{n \geq 1} \sum_{[(P_1|\tau_1), \ldots, (P_1|\tau_n)] \in B_c^{[k]}(X|\tau)} N\{\prod_{i=1}^n A^{[k]}(P_i|\tau_i) \frac{\delta^{\tau_i|x}}{\delta j^{\tau_i|x}}\} \qquad (30)$$

For extended Polymers $(P_1|\tau_1), \ldots, (P_m|\tau_m)$ and collections $\sigma_1, \ldots, \sigma_n$ we call $[(P_1|\tau_1), \ldots, (P_m|\tau_m)]$ a k-vertically irreducible connected partition of $(R|\tau)$ with respect to $\sigma_1, \ldots, \sigma_n$ if

(i) $R = \sum_{a=1}^m P_a$, $\tau = \sum_{a=1}^n \sigma_a^{<l(R)} + \sum_{b=1}^m \tau_a^{<l(R)}$ and

$$\bigcup_{a=1}^n \text{supp } \sigma_a \cup \bigcup_{b=1}^m \text{supp } \tau_b \subseteq R + \text{supp } \tau$$

(ii) $\gamma(P_1, ..., P_m, \text{supp } \sigma_1|_R, ..., \text{supp } \sigma_n|_R)$ is connected

(iii) There exists no $I \subseteq \{1, ..., m\}$ such that $(\text{supp } \sigma_b) \cap R' = \emptyset$ for $R' = \sum_{a \in I} P_a$ and all $b \in \{1, ..., m\} - I$ and

$$|\sum_{a \in I} \tau_a^{<l(R')} + \sum_{b \in I:\, (\text{supp } \sigma_b) \cap R' \neq \emptyset} \sigma_b^{<l(R')}| \leq k.$$

The set of all k-vertically irreducible connected partitions of $(R|\tau)$ with respect to $\sigma_1, ..., \sigma_n$ is denoted by $C^{[k]}(R\|\sigma_1, ..., \sigma_n|\tau)$. Define

$$A^{[k]}(R\|\sigma_1, ..., \sigma_n|\tau) = \sum_{m \geq 1} \sum_{[(P_1|\tau_1), ..., (P_m|\tau_m)] \in C^{[k]}(R\|\sigma_1, ..., \sigma_n|\tau)} N\{\prod_{i=1}^{m} A^{[k]}(P_i|\tau_i) \frac{\delta^{\tau_i|_R}}{\delta j^{\tau_i|_R}}\}. \quad (31)$$

Then the activities $A(Q|\tau)$ obey the following recursion relations

$$A(X|\tau) = \sum_{\tau', \tau'':\, \tau = \tau' + \tau''} \sum_{R:\, \emptyset \neq R \subseteq X} \sum_{n \geq 0} \sum_{\substack{\sigma_1, ..., \sigma_n: \\ \sum_{i=1}^{n} \sigma_a^{<l(X)} = \tau'}} \sum_{X - R = \sum_{i=1}^{n} Q_a} N\{A^{[k]}(R\|\sigma_1, ..., \sigma_n|\tau'') \prod_{i=1}^{n} A(Q_a|\sigma_a) \frac{\delta^{\sigma_a|_R}}{\delta j^{\sigma_a|_R}}\}. \quad (32)$$

For the first factor in the bracket of Eq. (32) we have enough supression factors if k is chosen sufficiently large. The product in the bracket is small if we impose renormalization conditions. For example impose the renormalization condition

$$\int_{z_2} \frac{\delta^2}{\delta\Psi(z)\delta\Psi(z_2)} A(P|\Psi)|_{\substack{\Psi=0 \\ j=0}} = 0 \quad (33)$$

for all polymers P and $z \in \mathbf{R}^d$. Then for a collection σ consisting of blocks x_1 and x_2 we obtain

$$A(P|\sigma)|_{\substack{\Psi=0 \\ j=0}} = \frac{1}{\sigma!} \int_{z_1, z_2 \in \mathbf{R}^d} \mathcal{A}(z_1, x_1)\mathcal{A}(z_2, x_2) \frac{\delta^2}{\delta\Psi(z_1)\delta\Psi(z_2)} A(P|\Psi)|_{\substack{\Psi=0 \\ j=0}} =$$
$$= \frac{1}{2} \frac{1}{\sigma!} \int_{z_1, z_2 \in \mathbf{R}^d} [\mathcal{A}(z_1, x_1) - \mathcal{A}(z_2, x_1)][\mathcal{A}(z_2, x_2) - \mathcal{A}(z_1, x_2)] \frac{\delta^2}{\delta\Psi(z_1)\delta\Psi(z_2)} A(P|\Psi)|_{\substack{\Psi=0 \\ j=0}}. \quad (34)$$

From the differences of the $\mathcal{A}$-kernels we can extract a factor $|z_1 - z_2|^2 L^{j_1 + j_2}$ for $x_1 \in \Lambda^{j_1}$, $x_2 \in \Lambda^{j_2}$. Using the exponential decay for funtional derivative of $A(P|\Psi)$ we obtain a factor $L^{-2l(P)}$. Thus a factor $L^{j_1 + j_2 - 2l(P)}$ is gained by using the renormalization condition (33).

Acknowledgement

The author wishes to thank the Deutsche Forschungsgemeinschaft for financial support of the work reported here, and the Hamburgische Wissenschaftliche Stiftung for a grant which made the visit to the Cargèse Advanced Study Institute possible.

References

1. J.Feldman, J.Magnen, V.Rivasseau and R. Sénéor, Bounds on Completely Convergent Euclidean Feynman Graphs, Comm. Math. Phys. 98 , 273-288 (1985).
2. J.Feldman, J.Magnen, V.Rivasseau and R. Sénéor, Bounds on Renormalized Feynman Graphs, Comm. Math. Phys. 100 , 23-55 (1985).
3. G.Gallavotti , The Structure of Renormalization Theory: Renormalization, Form Factors and Resummation in Scalar Field Theory , in : Critical Phenomena, Random Systems, Gauge Theories , K.Osterwalder and R.Stora (Eds.) , Les Houches 1984.
4. G.Gallavotti, Renormalization Theory and Ultraviolet Stability for Scalar Fields via Renormalization Group Methods , Rev. Mod. Phys. 57 ,471-561 (1985).
5. G.Gallavotti and F.Nicolò , Renormalization Theory in Four Dimensional Scalar Fields (I) , Comm. Math. Phys. 100 , 545-590 (1985) .
5. G.Gallavotti and F.Nicolò , Renormalization Theory in Four Dimensional Scalar Fields (II) , Comm. Math. Phys. 101 , 247-282 (1985) .
7. G.Mack and A.Pordt , Convergent Perturbation Expansions for Euclidean Quantum Field Theory , Comm. Math. Phys. 97 , 267-298 (1985).