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H.J. Timme
II. Institut für Theoretische Physik, Universität Hamburg

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ON THE ITERATION OF RENORMALIZATION GROUP TRANSFORMATIONS IN A FOUR DIMENSIONAL HIERARCHICAL SU(2) LATTICE GAUGE THEORY MODEL

Hans-Jörg Timme *

II. Institut für Theoretische Physik der Universität Hamburg

Abstract. For the example of a hierarchical $SU(2)$ lattice gauge theory model in 4 dimensions we show that renormalization group transformations can be controlled by simple bounds on polymer activities, which are valid for all field configurations. Thus it is not necessary to distinguish large and small field regions, and to use different bounds for them (stability bounds for large fields or bounds on effective actions in case of small fields).

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1. Introduction

A treatment of exact renormalization group transformations [1] in Euclidean quantum field theory with the help of expansion methods of statistical mechanics was performed by Kupiainen and Gawędzki. They have recently been able to treat some renormalizable models, in particular massless lattice ϕ^4-theory in 4 dimensions, which is asymptotically free at large distances [2]. Other models were treated by other authors, and results for pure nonabelian gauge theories in 4 dimensions are also available [3].

The most serious problem is the "large field problem", that is the control of the nonperturbative contributions to the effective action. This problem was solved by a technique originally proposed by Gallavotti and collaborators [4]. It starts by splitting the space into large field domains ("islands"), and their complement, called small field region. In the small field region the lattice fields stay bounded in a suitable way and standard perturbation theory converges. It seems fair to say that the resulting formalism is complicated. We would like to propose a different strategy which is simpler. We were motivated by our desire to iterate exact renormalization group transformations in order to derive a single computable expansion (of the phase cell type) for the full theory, as outlined in [5]. To make this feasible, essential simplifications over the approach of [2] appeared mandatory.

In a single renormalization group step one starts from a theory which lives on a lattice Λ_j of lattice spacing a_j, and one wishes to compute a new effective action[1] H_{j-1} (or Boltzmannian $Z_{j-1} = e^{-H_{j-1}}$) for a theory on a lattice Λ_{j-1} of lattice spacing $a_{j-1} = La_j$ ($L > 1$, integer). To control convergence of the expansions mentioned above one must state bounds on the pieces of the action H_j, or on related quantities, and show that they iterate through the renormalization group transformations. That is, H_{j-1} should satisfy the same bounds.

In the small field region, effective actions will be holomorphic in the fields in a complex strip. But in the large field domains, standard expansions for the action H_{j-1} will not converge, moreover H_{j-1} need not necessarily have good locality properties there[2]. Therefore one works with bounds on the activities $A_{j-1}(P|\phi)$ in the polymer representation of the Boltzmannian $Z_{j-1}(\Lambda_{j-1}|\phi)$. Our idea is to work with uniform bounds for such activities throughout. They are valid for small as well as for large fields. More precisely we write the activities A_j as a sum of an explicit term A_j^0, which is determined by the running coupling constants and is what one would have if one started from a local theory on Λ_j (ϕ^4 or whatever), plus a correction term R_j. Bounds are stated for $R_j(P|\phi)$. By a judicious choice of normalization factors, the leading terms in the expansion of the action for small fields $\phi \approx 0$ are easy to extract from the activities. In this way the running coupling constants are obtained. See Mack's Cargèse lectures [7] for a discussion of these points.

In Section 2 of this paper we will illustrate how all this works in a simple model, the hierarchical $SU(2)$ lattice gauge theory model (in 4 dimensions). The existence of a continuum limit for this model was shown by V.F. Müller and J. Schiemann [8]. We redo the main part of the analysis - i.e. the proof that suitable bounds which control the recursion relations for the running coupling constants iterate under renormalization group transformations - without splitting the gauge field configurations into large and small field regions. This analytical result

[1] In statistical mechanics it is called Hamiltonian instead of action.
[2] For some models it has been shown that the Hamiltonian is not well behaved [6].

applies in all lattices Λ_j when the running coupling constant is still small enough (Sections 4 and 5). A numerical study of the renormalization group flow will be presented in Section 3.

2. Hierarchical SU(2) Lattice Gauge Theory Model - Summary of Results

In a hierarchical model, there are no polymers other than monomers. In a pure gauge theory, the monomer is a plaquette p of the lattice Λ_j, and the corresponding activity $A_j(p|U)$ depends on the lattice gauge field U through the parallel transporter u around the plaquette p

$$A_j(p|U) = g_j(u) \qquad \text{with} \qquad u = \prod_{b\in\partial p} U(b) \tag{2.1}$$

g_j is a positive and normalized class function on the gauge group G :

$$g_j(u) \geq 0 \tag{2.2a}$$
$$g_j(vuv^{-1}) = g_j(u) \tag{2.2b}$$
$$g_j(e) = 1 \tag{2.2c}$$

The hierarchical model is constructed in such a way that renormalization group transformations lead to the simple Migdal recursion relation [9]

$$g_{j-1}(u) = \left[\mathcal{N}_j^{-1}\int_G \prod_{i=1}^{L^2-1} dv_i \; g_j(uv_1^{-1}) \prod_{k=1}^{L^2-1} g_j(v_k v_{k+1}^{-1})\right]^{L^{D-2}} = \left[\frac{g_j^{*L^2}(u)}{g_j^{*L^2}(e)}\right]^{L^{D-2}} \tag{2.3}$$

dv_i are normalized Haar measures on the group and $*$ denotes the corresponding convolution product. The constant $\mathcal{N}_j$ is chosen to maintain the normalization condition. L is the scale factor which determines the lattice spacing of the block lattice Λ_{j-1} in units of the lattice spacing of Λ_j. In $D = 4$ dimensions we get for the simplest choice $L = 2$

$$g_{j-1}(u)|_{L=2} = \left[\frac{g_j^{*\,4}(u)}{g_j^{*\,4}(e)}\right]^4 \tag{2.4}$$

One can realize (2.4) as an exact block spin transformation on a special 4 dimensional lattice which is hierarchically organized in the following way [10,11]. Consider the 4 dimensional hypercubic lattice as a stack of two dimensional lattice planes, connected by bonds b on which the temporal gauge $U(b) = 1$ is chosen. On the plaquettes perpendicular to these planes one sets the gauge coupling equal to zero or to infinity, depending on whether the plaquettes lie on the boundary or in the interior of a block of the superimposed block lattice. As a consequence, the bonds on the block surfaces outside the two dimensional layers have to be identified, whereas the bonds inside the blocks but between the layers are independent. Hence (2.3) factorizes in $L^{D-2} = 4$ identical integrations over the gauge fields on the internal bonds of the two dimensional layers. A graphical representation of this blocking procedure is given in [11].

For simplicity we consider in this paper the following Migdal recursion[1]

$$g_{j-1}(u)|_{L=\sqrt{2}} = \left(\mathcal{N}_j^{-1}\,[g_j * g_j](u)\right)^2 = [\mathcal{N}_j^{-1}\int_G dv\; g_j(v)g_j(uv^{-1})]^2 \tag{2.5}$$

which is to an extent the square root of (2.4), because the scaling factor L corresponds to $\sqrt{2}$. Note, however, that (2.4) can be obtained by the composition [8]

$$g_{j-1}(u)|_{L=2} = \left[\frac{[\mathcal{N}_j^{-1}\; g_j * g_j] * [\mathcal{N}_j^{-1}\; g_j * g_j](u)}{[\mathcal{N}_j^{-1}\; g_j * g_j] * [\mathcal{N}_j^{-1}\; g_j * g_j](e)}\right]^4 \tag{2.6}$$

Thus it is sufficient to study (2.5).

We consider the gauge group $G = SU(2)$. We start with a Wilson action for the theory on the original (finest) lattice Λ_N. Thus we have

$$g_N(u) = g_W(\beta_N, u) \quad , \tag{2.7}$$

where the Wilson activity g_W has the explicit form

$$g_W(\beta, u) = \exp\{-\frac{1}{2}\beta\; tr(1-u)\} = \exp\{-2\beta \sin^2\frac{\theta}{2}\} \tag{2.8}$$

The classes of $SU(2)$ elements u are parametrized by the rotation angle θ

$$u = ve^{i\theta\sigma_3}v^{-1} \quad , \quad \sigma_3 = \text{Pauli matrix} \quad , \quad v \in SU(2) \tag{2.9}$$

Because of the property

$$\frac{1}{2} tr\; u = \cos\theta \quad , \tag{2.10}$$

the activities g_j can be written as functions of θ. In order to apply complex analysis, we analytically continue in this angle, such that the g_j become holomorphic in strips $|Im\;\theta| < \beta_j^{-1/4}$. Due to (2.9), this corresponds to continuation in the group. Bearing in mind that for complex θ the argument u is no longer an element of $SU(2)$ but of $SL(2,C)$, we will not distinguish between $g_j(u)$ and $g_j(\theta)$ in the following.

We write g_j as a sum of what we would have if we wanted to start on the lattice Λ_j with a Wilson activity, and a correction term r_j

$$g_j(u) = g_W(\beta_j, u) + r_j(u) \tag{2.11}$$

We impose the renormalization conditions ($'$ = derivative with respect to θ)

$$r_j(0) = 0 \quad , \quad r_j''(0) = 0 \tag{2.12}$$

It follows that β_j is determined by g_j.

$$\beta_j = -\left.\frac{\partial^2}{\partial\theta^2}\ln g_j(\theta)\right|_{\theta=0} \tag{2.13}$$

β_j is called the running coupling constant.

[1] The same recursion relation holds for the $O(4)$-symmetric hierarchical Heisenberg ferromagnet in two dimensions [12]. Note, however, that in statistical mechanical models the $g_j(v)$ are no longer class functions.

The activities g_j and r_j respectively are even and 2π-periodic functions in θ. Moreover, they will be holomorphic in the strips $|Im\ \theta| < \beta_j^{-1/4}$, and real valued for real arguments θ. If we start from the Wilson action, these properties are obviously true for $j = N$, and will follow for $j < N$ by the recursion relation (2.5). As we will see later on, the marginal coupling β_j will be decreased under renormalization group transformations. This implies that the analyticity domains $|Im\ \theta| < \beta_j^{-1/4}$ of the activities g_j will need to expand. But this is indeed the case. Consider the transformation (2.5) in the form

$$g_{j-1}(u) = \left\{ N_j^{-1} \int_G dv\ g_j(u^{1/2}v) g_j(u^{1/2}v^{-1}) \right\}^2 \quad , \qquad (2.14)$$

and define the central angles θ_+, θ_- of the arguments $u^{1/2}v, u^{1/2}v^{-1}$ by

$$\frac{1}{2} tr(u^{1/2}v) = \cos\theta_+ \quad , \quad \frac{1}{2} tr(u^{1/2}v^{-1}) = \cos\theta_-$$

Using an explicit parametrization (see App. A), one can easily show that $|Im\ \theta_\pm(\theta)| < \beta_j^{-1/4}$ for $|Im\ \theta| < 2\beta_j^{-1/4}$. Thus the new analyticity strip could be chosen at least twice as wide than the old one.

Now we want to state the main result of this paper: a single and simple bound, which governs the small as well as the large field behaviour of the activities.

Theorem. *Assume that β_j is large enough and that the activity g_j is of the form* (2.11) *with a correction term r_j that obeys the following bounds*

$$|r_j(\theta)| \le 2\beta_j\ |\sin^4 \frac{\theta}{2}|\ |gw(\beta_j, \theta)| \quad \textit{for} \quad |Im\ \theta| \le \beta_j^{-1/4} \qquad (2.15)$$

and

$$|\rho_j| \le \beta_j \quad , \quad |\sigma_j| \le \mathcal{O}(\beta_j) \quad , \quad |\lambda_j - \frac{1}{2}\rho_j^2| \le \mathcal{O}(\beta_j) \qquad (2.16)$$

for the parametrization

$$r_j(\theta) = \left[\rho_j \sin^4 \frac{\theta}{2} + \sigma_j \sin^6 \frac{\theta}{2} + \lambda_j \sin^8 \frac{\theta}{2} + \mathcal{O}(\beta_j^{5/2} \sin^{10} \frac{\theta}{2}) \right] gw(\beta_j, \theta)$$

Then

$$0 < \beta_j - \beta_{j-1} = \mathcal{O}(1) \qquad (2.17)$$

and r_{j-1} will obey the same bounds (2.15), (2.16) *with $j-1$ substituted for j. The new coupling constants $\beta' = \beta_{j-1}$, $\rho' = \rho_{j-1}$, $\sigma' = \sigma_{j-1}$ are given in terms of $\beta = \beta_j$, $\rho = \rho_j$, $\sigma = \sigma_j$ by the following formulae*

$$\beta' = \beta - \frac{3}{4} - \frac{5}{8}\rho\beta^{-1} + \frac{3}{32}\beta^{-1} + \frac{15}{64}\rho\beta^{-2} - \frac{105}{128}\sigma\beta^{-2} - \frac{25}{64}\rho^2\beta^{-3} + \mathcal{O}(\beta^{-2}) \qquad (2.18a)$$

$$\rho' = \frac{1}{4}\rho - \frac{1}{2}\beta + \frac{3}{4} + \frac{7}{16}\rho\beta^{-1} + \frac{21}{32}\sigma\beta^{-1} + \frac{11}{32}\rho^2\beta^{-2} + \mathcal{O}(\beta^{-1}) \qquad (2.18b)$$

$$\sigma' = \frac{1}{16}\sigma - \frac{1}{4}\beta + \frac{1}{8}\rho + \mathcal{O}(\beta^0) \qquad (2.18c)$$

The error terms $\mathcal{O}(\beta_j^{-n})$ in equations (2.18) mean that the whole remainder (including perturbative and nonperturbative contributions) is bounded by *const* β_j^{-n} with some constant independent of j.

Remarks. (i) If we start with a Wilson action $g_N = gw(\beta_N)$ then $r_N = 0$. Therefore the assumptions of the theorem are satisfied for $j = N$ if β_N is large enough, and the bound (2.15) will iterate as long as β_j stays large enough. The flow of coupling constants $\beta_j, \rho_j, \sigma_j$ can be obtained by solving the relations (2.18) with initial conditions (see [8]).

(ii) The theorem can be stated for the $L = 2$ recursion (2.4) as well. Splitting the recursion into two subsequent $L = \sqrt{2}$ recursions, see (2.6), makes it possible to get the relations for the running couplings almost immediately. For the first $L = \sqrt{2}$ transformation the relations (2.18) have to be multiplied by 1/2, and for the second by a factor 2. Inserting the first relations into the second, one gets

$$\beta' = \beta - \frac{13}{8} - \frac{15}{16}\rho\beta^{-1} - \frac{5}{128}\beta^{-1} + \frac{15}{256}\rho\beta^{-2} - \frac{945}{512}\sigma\beta^{-2} - \frac{285}{256}\rho^2\beta^{-3} + \mathcal{O}(\beta^{-2})$$

$$\rho' = \frac{1}{16}\rho - \frac{5}{8}\beta + \frac{47}{32} + \frac{81}{128}\rho\beta^{-1} + \frac{63}{256}\sigma\beta^{-1} + \frac{33}{256}\rho^2\beta^{-2} + \mathcal{O}(\beta^{-1})$$

$$\sigma' = \frac{1}{256}\sigma - \frac{21}{64}\beta + \frac{5}{128}\rho + \mathcal{O}(\beta^0)$$

In addition, it is possible to show that the bound (2.15) can be replaced by

$$|r_j(\theta)| \le \frac{4}{3}\beta_j\ |\sin^4 \frac{\theta}{2}|\ |gw(\beta_j, \theta)| \quad \text{for} \quad |Im\ \theta| \le \beta_j^{-1/4}$$

3. Numerical Study of the Renormalization Group Flow

It is known that the effective activities g_j are driven to the high temperature fixed point $\lim_{j\to-\infty} g_j = 1$ by the Migdal-Kadanoff recursion formulae [11]. This holds for $g_N = gw(\beta_N)$ with arbitrary $\beta_N > 0$. In order to give an example of the renormalization group flow of the hierarchical $SU(2)$ lattice gauge theory model and to illustrate the form of the correction terms, we will present some results from a computer investigation. The numerical iteration of the Migdal recursion (2.5) is an easy task because every recursion step involves only two integrations (see Appendix A for the parametrization of the Haar measure).

If one starts from a Wilson activity, one is driven towards a universal (continuum) trajectory within about 3 renormalization group steps, see Figure 1.

On this trajectory, the activities are considerably smaller in the large field region (θ not small) than the corresponding Wilson activities if β_j is large. This is in agreement with our bounds (2.15). Consider for example the first iteration step $N \to N-1$, Figure 2.

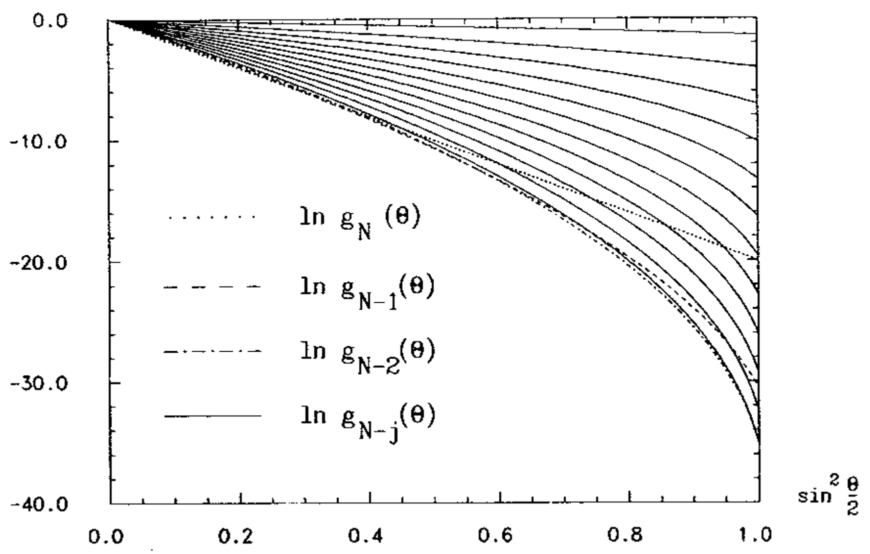

Fig. 1. Flow of effective actions $\ln g_{N-j}$ ($j = 3, 5, ..., 23, 25$) under the Migdal recursion (2.15) towards the high temperature fixed point $\lim_{j \to -\infty} g_{N-j} = 1$ for $g_N = g_W(\beta_N = 10)$.

We have chosen $g_N = g_W(\beta_N)$ with $\beta_N = 10$. The effective activity g_{N-1} is given by

$$g_{N-1}(\theta) = \left[\frac{I_1(2\beta_N \cos\frac{\theta}{2})}{\cos\frac{\theta}{2}\, I_1(2\beta_N)}\right]^2 \qquad \text{with} \qquad \beta_{N-1} = \beta_N \frac{I_2(2\beta_N)}{I_1(2\beta_N)} < \beta_N$$

Thus we will get a negative correction term r_{N-1} if we subtract $g_W(\beta_{N-1})$ from g_{N-1}. It is bounded from below by $-\frac{1}{2}\beta_{N-1}\sin^4\frac{\theta}{2}\, g_W(\beta_{N-1}, \theta)$.

The evolution of the correction terms is shown in Figure 3. r_{N-27} is already suppressed by two orders of magnitude compared with r_{N-26}. It turns out that the bound (2.15) is valid not only for large β_j, but more generally. However, it is difficult to prove the iteration of this bound analytically for all values of β_j because one would need to control the flow of the running coupling constant apart from weak coupling or high temperature expansions.

Figure 4 shows the flow of the marginal β-coupling and compares it with the prediction of the recursion relations (2.18) which are valid only for sufficiently large β_j. One sees that the exact value of the running coupling constant starts deviating from third order perturbation theory quite early (around $\beta_j = 5$). After 30 iteration steps β_{N-j} becomes too small and the perturbation expansion breaks down (yielding negative values for β_{N-j} and increasing irrelevant couplings ρ_{N-j}, σ_{N-j}).

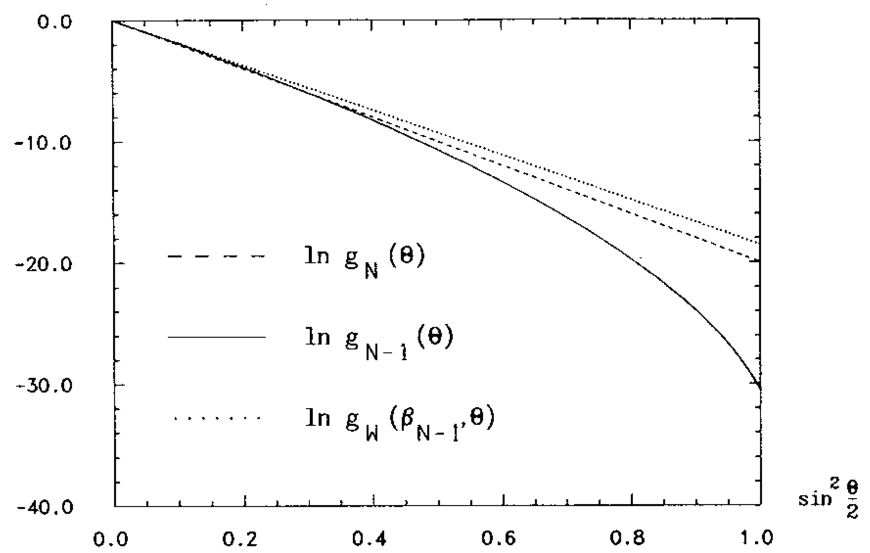

Fig. 2. The effective action and their Wilson part for the first renormalization group step $N \to N-1$ with $g_N = g_W(\beta_N)$ for $\beta_N = 10$.

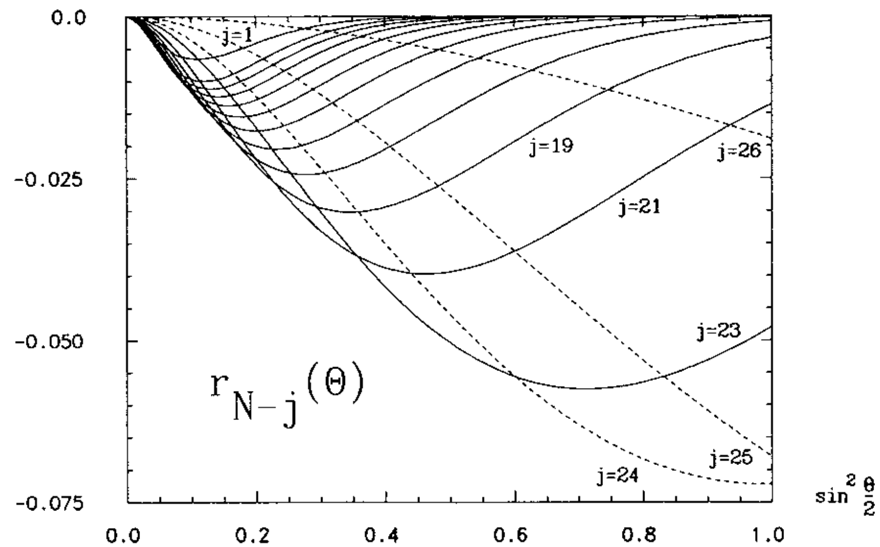

Fig. 3. Evolution of the correction terms r_{N-j} corresponding to the effective actions of Fig. 1. The curves are for $j = 1, 3, ..., 21, 23$ and $j = 24, 25, 26$.

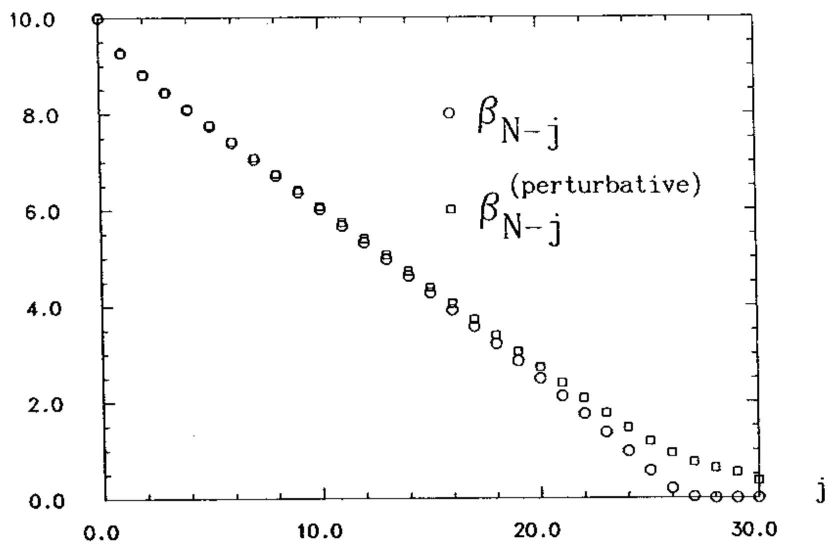

Fig. 4. Flow of the running coupling constant β_{N-j} for $\beta_N = 10$ in comparison with the third order perturbation expansion for the weak coupling regime (β_{N-j} large).

4. Proof of the Theorem - Reproduction of the Bound (2.15)

We assume that the bound (2.15) is valid for a certain index j.

The new correction term r_{j-1} is given by the recursion relation (2.14)

$$r_{j-1} = [\mathcal{N}_j^{-1} g_j * g_j]^2 - gw(\beta_{j-1}) \tag{4.1}$$

where the normalization factor $\mathcal{N}_j$ and the new coupling β_{j-1} have to be chosen in such a way that the renormalization conditions $r_{j-1}(0) = 0$, $r''_{j-1}(0) = 0$ are fulfilled. The perturbation expansion (Sect. 5) yields

$$\mathcal{N}_j = \mathcal{N}_{j0}[1 + \mathcal{O}(\beta_j^{-1})] \quad , \quad \mathcal{N}_{j0} = [gw(\beta_j) * gw(\beta_j)](0) \tag{4.2}$$

and

$$\beta_{j-1} < \beta_j \quad \text{with} \quad \beta_j - \beta_{j-1} = \mathcal{O}(1)\ . \tag{4.3}$$

Let us first note that a bound of the form (2.15) is quite natural for r_{j-1}. To see this, consider the function $r_{j-1}(\theta) \sin^{-4}\frac{\theta}{2}\, gw(\beta_{j-1},\theta)^{-1}$. Due to the renormalization conditions, it is analytic in the whole strip $|Im\ \theta| < \beta_{j-1}^{-1/4}$ (remember that r_{j-1} is even and 2π-periodic in θ). Thus one can apply the maximum principle to get

$$|r_{j-1}(\theta)| \le \left\{ \max_{\substack{\theta\in[0,\pi] \\ |Im\ \theta| = \beta_{j-1}^{-1/4}}} |r_{j-1}(\theta) \sin^{-4}\frac{\theta}{2}\, gw(\beta_{j-1},\theta)^{-1}| \right\} |\sin^4\frac{\theta}{2}|\, |gw(\beta_{j-1},\theta)| \tag{4.4}$$

The restriction $|Im\ \theta| = \beta_{j-1}^{-1/4}$ is essential, because it will enable us to estimate $|r_{j-1}|$ by splitting the correction term (4.1) as follows. Inserting the split (2.11) we see that

$$\begin{aligned} r_{j-1} = {} & [\mathcal{N}_j^{-1} gw(\beta_j) * gw(\beta_j)]^2 - gw(\beta_{j-1}) + \\ & + \text{products of convolution integrals containing } r_j \end{aligned} \tag{4.5}$$

This will be estimated by use of the triangle inequality. No explicit knowledge of $\mathcal{N}_j, \beta_{j-1}$ which goes beyond (4.2), (4.3) will be required.

There are contributions of two kinds to r_{j-1}. First, we have contributions which are generated by the Wilson activities $gw(\beta_j)$ and $gw(\beta_{j-1})$ alone (apart from the normalization factor $\mathcal{N}_j$). These contributions can be bounded in the strip $|Im\ \theta| \le \beta_{j-1}^{-1/4}$ by

$$\frac{1}{2} C_W \beta_{j-1}\, |\sin^4\frac{\theta}{2}|\, |gw(\beta_{j-1},\theta)|$$

with some constant $C_W > 1$ (see the estimate (4.12) at the end of this section). Contributions of a second type owe their existence to a nonvanishing old correction term r_j. Note that the terms r_j are *irrelevant* pieces of the activities g_j. To see this, remember that u is the parallel transporter around a plaquette, and consider the leading part of r_j

$$\sin^4\frac{\theta}{2} = [\frac{1}{4} tr(1-u)]^2 \sim (F^{\mu\nu} F^*_{\mu\nu})^2$$

It has canonical dimension $d = 8$. Hence we can expect that it will be scaled down by a factor $L^{4-d} = L^{-4} = \frac{1}{4}$ in every recursion step. Since we are dealing with the "square root" of a renormalization group transformation with scale factor $L = 2$, we have set $L = \sqrt{2}$ here. Thus the irrelevant correction term gives a contribution that is bounded by

$$(\frac{1}{4} C_r)\ 2\beta_{j-1}\, |\sin^4\frac{\theta}{2}|\, |gw(\beta_{j-1},\theta)|$$

which is essentially the old bound (2.15) but scaled down by a factor $\frac{1}{4}C_r$ less than 1. It turns out that $1 - \frac{1}{4}C_r > \frac{1}{4}C_W$ for our choice of (2.15). Hence the bound will iterate under renormalization group steps.

Let us begin with estimates on the convolution integrals that contain the irrelevant r_j terms. The basic bound is the following inequality on convoluted Wilson activities:

Lemma. *For large enough β_j, β_{j-1} with $|\beta_j - \beta_{j-1}| = \mathcal{O}(1)$ the following bound is valid in the strip $|Im\ \theta| \le \beta_{j-1}^{-1/4}$:*

$$\mathcal{N}_{j0}^{-1}(|gw(\beta_j)| * |gw(\beta_j)|)(\theta) \le |gw(\beta_{j-1},\theta)|^{1/2}[1 + \mathcal{O}(\beta_j^{-1/2})] \exp\left(-\frac{1}{4}\beta_j Re \sin^4\frac{\theta}{2}\right) \tag{4.6}$$

The proof of this lemma will be given in Appendix C.

We introduce the following quantities, which are expectation values of the expressions in { }:

$$\mathcal{E}_1(u) = \frac{1}{(|gw(\beta_j)| * |gw(\beta_j)|)(u)} \int_G dv\ |gw(\beta_j, u^{1/2}v) gw(\beta_j, u^{1/2}v^{-1})| \cdot \left\{ \frac{1}{2}\left|\frac{1}{4}tr(1-u^{1/2}v)\right|^2 + \frac{1}{2}\left|\frac{1}{4}tr(1-u^{1/2}v^{-1})\right|^2 \right\} \tag{4.7}$$

$$\mathcal{E}_2(u) = \frac{1}{(|gw(\beta_j)| * |gw(\beta_j)|)(u)} \int_G dv\ |gw(\beta_j, u^{1/2}v) gw(\beta_j, u^{1/2}v^{-1})| \cdot \left\{ \left|\frac{1}{4}tr(1-u^{1/2}v)\right|^2 \left|\frac{1}{4}tr(1-u^{1/2}v^{-1})\right|^2 \right\} \tag{4.8}$$

We use the bound (2.15), i.e.

$$|r_j(u^{1/2}v^{\pm1})| \le 2\beta_j \left|\frac{1}{4}tr(1-u^{1/2}v^{\pm1})\right|^2 |gw(\beta_j, u^{1/2}v^{\pm1})| \quad ,$$

to get

$$(|gw(\beta_j)| * |r_j| + |r_j| * |gw(\beta_j)|)(u) \le 4\beta_j\ \mathcal{E}_1(u)\ |gw(\beta_j)| * |gw(\beta_j)|(u)$$
$$(|r_j| * |r_j|)(u) \le 4\beta_j^2\ \mathcal{E}_2(u)\ |gw(\beta_j)| * |gw(\beta_j)|(u)$$

To proceed further, one calculates $\mathcal{E}_1, \mathcal{E}_2$ (see App. C) and inserts the inequality of the lemma. This yields the following generalizations of the bound (4.6):

$$\mathcal{N}_j^{-1}(|gw(\beta_j)| * |r_j| + |r_j| * |gw(\beta_j)|)(\theta) \le \le \frac{1}{4}\beta_j\ |\sin^4\frac{\theta}{2}|\ |gw(\beta_{j-1},\theta)|^{1/2}\ [1+\mathcal{O}(\beta_j^{-1/2})] \exp\left(-\frac{1}{4}\beta_j Re \sin^4\frac{\theta}{2}\right) + + \mathcal{O}(\beta_j^{-1/2})\ |gw(\beta_{j-1},\theta)|^{1/2} \tag{4.9}$$

$$\mathcal{N}_j^{-1}(|r_j| * |r_j|)(\theta) \le \le \left(\frac{1}{8}\beta_j |\sin^4\frac{\theta}{2}|\right)^2 |gw(\beta_{j-1},\theta)|^{1/2}\ [1+\mathcal{O}(\beta_j^{-1/2})] \exp\left(-\frac{1}{4}\beta_j Re \sin^4\frac{\theta}{2}\right) + + \mathcal{O}(\beta_j^{-1/2})\ |gw(\beta_{j-1},\theta)|^{1/2} \tag{4.10}$$

It follows that the correction term r_{j-1} can be estimated by

$$|r_{j-1}(\theta)| \le \left|\left\{\mathcal{N}_j^{-1}[gw(\beta_j) * gw(\beta_j)](\theta)\right\}^2 - gw(\beta_{j-1},\theta)\right| + + \frac{1}{2}\beta_j\ |\sin^4\frac{\theta}{2}|\ |gw(\beta_{j-1},\theta)|\ [1+\mathcal{O}(\beta_j^{-1/2})] \cdot \cdot \exp\left(-\frac{1}{2}\beta_j Re\sin^4\frac{\theta}{2}\right) \int_0^1 ds\ [1 + s\frac{1}{8}\beta_j|\sin^4\frac{\theta}{2}|]^3 + + \mathcal{O}(\beta_j^{-1/2})\ |gw(\beta_{j-1},\theta)| \tag{4.11}$$

In order to get the simple form for the third term, we have used the large field damping factor $\exp(-\frac{1}{4}\beta_j Re\sin^4\frac{\theta}{2})$ of the lemma to dominate unwanted powers of $2\beta_j|\sin^4\frac{\theta}{2}|$:

$$\left(\frac{1}{8}\beta_j|\sin^4\frac{\theta}{2}|\right)^k \exp\left(-\frac{1}{4}\beta_j Re\sin^4\frac{\theta}{2}\right) \le \mathcal{O}(1) \qquad \text{for} \qquad |Im\ \theta| \le \beta_{j-1}^{-1/4}\ ,\ k=0,1,2$$

In the second term of (4.11) such a domination has to be done more carefully, see later on.

Now we insert (4.11) into (4.4) and make use of an estimate for the contributions of the Wilson activities (see App. C)

$$\max_{\theta:|Im\ \theta|=\beta_{j-1}^{-1/4}} \left|\sin^{-4}\frac{\theta}{2}\ gw(\beta_{j-1},\theta)^{-1}\left(\left\{\mathcal{N}_j^{-1}[gw(\beta_j)*gw(\beta_j)](\theta)\right\}^2 - gw(\beta_{j-1},\theta)\right)\right| \le \le \frac{1}{2}\beta_j\ \frac{e^{\frac{1}{4}}-1}{\frac{1}{4}} + \mathcal{O}(\beta_j^{1/2}) \tag{4.12}$$

We get

$$|r_{j-1}(\theta)| \le \left[\frac{1}{2}\beta_j\ \frac{e^{\frac{1}{4}}-1}{\frac{1}{4}} + \mathcal{O}(\beta_j^{1/2})\right] |\sin^4\frac{\theta}{2}|\ |gw(\beta_{j-1},\theta)| + + \frac{1}{2}\beta_j\ [1+\mathcal{O}(\beta_j^{-1/2})]\ |\sin^4\frac{\theta}{2}|\ |gw(\beta_{j-1},\theta)| \cdot \cdot\left\{\max_{\theta:|Im\ \theta|=\beta_{j-1}^{-1/4}} \exp\left(-\frac{1}{2}\beta_j Re\sin^4\frac{\theta}{2}\right)\int_0^1 ds\ [1+s\frac{1}{8}\beta_j|\sin^4\frac{\theta}{2}|]^3\right\} + + \mathcal{O}(\beta_j^{-1/2})\left\{\max_{\theta:|Im\ \theta|=\beta_{j-1}^{-1/4}}|\sin^{-4}\frac{\theta}{2}|\right\}\ |\sin^4\frac{\theta}{2}|\ |gw(\beta_{j-1},\theta)| \tag{4.13}$$

We note that $|\sin^{-4}\frac{\theta}{2}|$ is bounded by $\sinh^{-4}(\beta_{j-1}^{-1/4}/2) = \mathcal{O}(\beta_j)$ for $|Im\ \theta| = \beta_{j-1}^{-1/4}$. Finally one arrives at

$$|r_{j-1}(\theta)| \le 2\beta_j\ [\frac{5}{6} + \mathcal{O}(\beta_j^{-1/2})]\ |\sin^4\frac{\theta}{2}|\ |gw(\beta_{j-1},\theta)|$$
$$\le 2\beta_{j-1}|\sin^4\frac{\theta}{2}|\ |gw(\beta_{j-1},\theta)|$$

This is (2.15) for $j-1$, provided that the running coupling constant $\beta_j^{-1/2}$ is sufficiently small.

Remark. For the case $L=2$ the same method (with nearly the same bounds) is applicable if one uses the split (2.6) and considers separately the recursions

$$g_{j-1/2} = \mathcal{N}_j^{-1}\ g_j * g_j \qquad , \qquad g_{j-1} = [\mathcal{N}_{j-1/2}^{-1}\ g_{j-1/2} * g_{j-1/2}]^4$$

5. Proof of the Theorem - Perturbation Expansion in the Running Coupling Constant $\beta_j^{-1/2}$

In view of the form of the bounds (2.15) it is natural to rewrite equation (2.11) as

$$g_j(u) = gw(\beta_j, u)[1 + \tilde{r}_j(\theta)] \tag{5.1}$$

The bound (2.15) becomes

$$|\tilde{r}_j(\theta)| \leq 2\beta_j |\sin^4 \frac{\theta}{2}| \qquad \text{for} \qquad |Im\ \theta| \leq \beta_j^{-1/4} \tag{5.2}$$

Because the $\tilde{r}_j(\theta)$ are 2π-periodic, even, and analytic in the strips $|Im\ \theta| < \beta_j^{-1/4}$, there exist unique functions $\check{R}_j$ with

$$\check{R}_j(\sin^2 \frac{\theta}{2}) = \tilde{r}_j(\theta)\ . \tag{5.3}$$

These functions $\check{R}_j(z)$ are analytic in regions

$$\Omega(\eta_j) = \left\{ z \left| \left(\frac{Re\ z - 1}{\cosh \eta_j} \right)^2 + \left(\frac{Im\ z}{\sinh \eta_j} \right)^2 < 1 \right. \right\} \quad ; \quad \eta_j = \beta_j^{-1/4} \tag{5.4}$$

It follows from definition (5.3), renormalization condition (2.12), and bound (5.2) that the renormalization conditions

$$\check{R}_j(0) = 0 \quad , \quad \check{R}'_j(0) = 0 \tag{5.5}$$

and the bounds

$$|\check{R}_j(z)| \leq 2\beta_j |z|^2 \qquad \text{for} \qquad z \in \overline{\Omega(\eta_j)} \tag{5.6}$$

hold. A Taylor expansion with remainder can be written for $\check{R}_j$

$$\check{R}_j(z) = \sum_{n=2}^{4} a_n z^n + \check{\Re}_j^5(z) \tag{5.7a}$$

$$\check{\Re}_j^5(z) = z^5 \int_0^1 dt\ \frac{(1-t)^4}{4!} \check{R}_j^{(5)}(zt) \tag{5.7b}$$

The coefficients a_n are determined by Cauchy's integral formula

$$a_n = \frac{1}{2\pi i} \oint_{|\xi|=\kappa} \frac{\check{R}_j(\xi)}{\xi^{n+1}} d\xi$$

with some $\kappa < \cosh \eta_j - 1 = \mathcal{O}(\beta_j^{-1/2})$, and can be estimated with help of (5.6):

$$|a_n| \leq \frac{1}{\kappa^n} \max_{|\xi|=\kappa} |\check{R}_j(\xi)| \leq 2\beta_j \kappa^{2-n} = \mathcal{O}(\beta_j^{n/2}) \tag{5.8}$$

The function $z^{-5}\check{\Re}_j^5(z)$ is analytic in the region $\Omega(\eta_j)$. Thus it can be bounded in $\overline{\Omega(\eta_j)}$ by means of the maximum principle:

$$|z^{-5}\check{\Re}_j^5(z)| = \left| z^{-5}\left\{ \check{R}_j(z) - \sum_{n=2}^{4} a_n z^n \right\} \right| \leq \max_{\xi \in \partial\Omega(\eta_j)} \left| \xi^{-5}\left\{ \check{R}_j(\xi) - \sum_{n=2}^{4} a_n \xi^n \right\} \right|$$
$$\leq \max_{\xi \in \partial\Omega(\eta_j)} \left\{ 2\beta_j |\xi|^{-3} + \sum_{n=2}^{4} |a_n|\ |\xi|^{n-5} \right\} = \mathcal{O}(\beta_j^{5/2}) \tag{5.9}$$

Then, using equations (5.3) and (5.7), we get for the correction term

$$\tilde{r}_j(\theta) = \rho_j \sin^4 \frac{\theta}{2} + \sigma_j \sin^6 \frac{\theta}{2} + \lambda_j \sin^8 \frac{\theta}{2} + \mathcal{O}(\beta_j^{5/2} \sin^{10} \frac{\theta}{2}) \tag{5.10}$$

with

$$\rho_j = \mathcal{O}(\beta_j) \quad , \quad \sigma_j = \mathcal{O}(\beta_j^{3/2}) \quad , \quad \lambda_j = \mathcal{O}(\beta_j^2) \tag{5.11}$$

One can insert this Taylor expansion into the recursion relation which gives g_{j-1}. The result involves integrals ($n, m \geq 0$ but $n, m \neq 1$)

$$F_j^{nm}(u) = \int dv\ gw(\beta_j, u^{1/2}v)[\frac{1}{4}tr(1 - u^{1/2}v)]^n gw(\beta_j, u^{1/2}v^{-1})[\frac{1}{4}tr(1 - u^{1/2}v^{-1})]^m \tag{5.12}$$

These integrations can be done in closed form (see App. A). They yield modified Bessel functions. By Taylor expansion in $z = \sin^2 \frac{\theta}{2} = \frac{1}{4}tr(1-u)$ one obtains their (unnormalized) contributions to the $\sin^{2k} \frac{\theta}{2}$ terms of the new activities g_{j-1} :

$$\frac{\partial^k}{\partial z^k} F^{nm}(z) \Big|_{z=0} = (-1)^{n+m+k} \left(\frac{1}{4}\right)^{n+m} \sum_{\nu=0}^{\min\{[\frac{n+m}{2}],k\}} a_{2\nu}^{nm} (2\nu)! \binom{k}{\nu} (-1)^\nu \frac{\beta^{k-(n+m)-1}}{\sqrt{4\pi\beta}} \cdot$$
$$\cdot \left\{ \sum_{\mu=0}^{l} \frac{(-1)^\mu}{\mu!} \frac{\Gamma(k+\mu+3/2)}{\Gamma(k-\mu+3/2)} \frac{\Gamma(k-2\nu-\mu-1/2)}{\Gamma(k-n-m-\mu-1/2)} \left(\frac{1}{4\beta}\right)^\mu + \mathcal{O}(\beta^{-l-1}) \right\} \tag{5.13}$$

with

$$F^{nm}(z)|_{z=\frac{1}{4}tr(1-u)} = F^{nm}(u) \quad ; \quad a_{2\nu}^{nm} = \sum_{\substack{i=0\\ i+j=2\nu}}^{n} \sum_{j=0}^{m} \binom{n}{i}\binom{m}{j}(-1)^j$$

Note that

$$\frac{\partial^k}{\partial z^k} F_j^{nm}(z) \Big|_{z=0} = \mathcal{O}(\beta_j^{k-(n+m)-3/2})$$

Because of $F_j^{00}(0) = [gw(\beta_j) * gw(\beta_j)](0)$ it follows immediately that the normalization factor $\mathcal{N}_j$ obeys the bound (4.2). Let us also consider the coupling constants $\rho_j, \sigma_j, \lambda_j$ multiplying the moments F^{nm}. With $\mathcal{N}_j^{-1} = \mathcal{O}(\beta_j^{3/2})$ and (5.8) we get

$$\mathcal{N}_j^{-1}\ a_n a_m\ \frac{\partial^k}{\partial z^k} F_j^{nm}(z) \Big|_{z=0} = \mathcal{O}(\beta_j^{k-\frac{1}{2}(n+m)})$$

Therefore, to obtain the running coupling β_{j-1} up to terms of order $\mathcal{O}(\beta_j^{-1})$, only moments F^{nm} with $n+m \leq 4$ are needed. It can be shown (App. B) that the contributions of the integrals which contain the remainder term $\mathfrak{R}_j^5(\sin^2\frac{\theta}{2})$ of (5.10) are at most of order $\mathcal{O}(\beta_j^{k-5/2})$. Collecting all the contributions, one arrives at recursion relations for the marginal β_j coupling and for the irrelevant couplings $\rho_j, \sigma_j, \lambda_j$ etc.

$$\beta' = \beta - \frac{3}{4} - \frac{5}{8}\rho\beta^{-1} - \frac{105}{128}\sigma\beta^{-2} + \frac{3}{32}\beta^{-1} + \frac{15}{64}\rho\beta^{-2} + \frac{115}{512}\rho^2\beta^{-3} - \frac{315}{256}\lambda\beta^{-3} + \mathcal{O}(\beta^{-3/2}) \tag{5.14a}$$

$$\rho' = \frac{1}{4}\rho - \frac{1}{2}\beta + \frac{21}{32}\sigma\beta^{-1} + \frac{3}{4} + \frac{7}{16}\rho\beta^{-1} - \frac{101}{256}\rho^2\beta^{-2} + \frac{189}{128}\lambda\beta^{-2} + \mathcal{O}(\beta^{-1/2}) \tag{5.14b}$$

$$\sigma' = \frac{1}{16}\sigma - \frac{1}{4}\beta + \frac{1}{8}\rho - \frac{9}{64}\rho^2\beta^{-1} + \frac{9}{32}\lambda\beta^{-1} + \mathcal{O}(\beta^{1/2}) \tag{5.14c}$$

$$\lambda' = \frac{1}{64}\lambda + \frac{1}{8}\beta^2 - \frac{1}{8}\rho\beta + \frac{3}{128}\rho^2 + \mathcal{O}(\beta^{3/2}) \tag{5.14d}$$

Inserting the conditions (2.16b) and (2.16c), the system of equations (5.14) simplifies to (2.18). Because of (2.16a), β_{j-1} is smaller than β_j by an amount of order $\mathcal{O}(1)$, provided that β_j is large enough. This is relation (2.17) and (4.3) respectively. Finally, it is easy to see that the conditions (2.16) iterate under the recursion relations (5.14), cp. [8]. Thus the theorem is proven.

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Appendix A. Perturbation Expansion for Convolution Integrals

Consider the moments

$$F^{nm}(z)|_{z=\frac{1}{4}tr(1-u)} = \int_G dv\; gw(\beta, u^{1/2}v)gw(\beta, u^{1/2}v^{-1}) \cdot [\frac{1}{4}tr(1-u^{1/2}v)]^n\, [\frac{1}{4}tr(1-u^{1/2}v^{-1})]^m \tag{A.1}$$

which have been introduced in (5.12). In order to state the recursion relations for the couplings $\beta_j, \rho_j, \sigma_j, \lambda_j$, etc., one has to calculate the coefficients $\frac{\partial^k F^{nm}}{\partial z^k}(0)$. This can be done rigorously by perturbation theory up to a certain order in the small running coupling constant $\beta_j^{-1/2}$, if the remainder (including perturbative and nonperturbative parts) is bounded and small enough. Especially, we want to show that equation (5.13) holds.

Define the generating function

$$\begin{aligned} F(\beta,\beta';z)|_{z=\frac{1}{4}tr(1-u)} &= \int_G dv\; gw(\beta, u^{1/2}v)gw(\beta', u^{1/2}v^{-1}) \\ &= \int_G dv\; \exp\left(-\frac{1}{2}\beta\; tr(1-u^{1/2}v) - \frac{1}{2}\beta' tr(1-u^{1/2}v^{-1})\right) \end{aligned} \tag{A.2}$$

Then obviously

$$F^{nm}(z) = \left(-\frac{1}{2}\frac{\partial}{\partial\beta}\right)^n \left(-\frac{1}{2}\frac{\partial}{\partial\beta'}\right)^m F(\beta,\beta';z)\Big|_{\beta=\beta'} \tag{A.3}$$

We can parametrize the $SU(2)$ elements u and v as points on the surface of the four dimensional unit sphere

$$SU(2) = \left\{ a_0\mathbf{1} + ia_j\sigma_j \mid a_0^2 + a_ja_j = 1 \right\} \quad ; \quad \sigma_j = \text{Pauli matrices}$$

Without loss of generality we associate with $u^{1/2}, v$ the following unit 4-vectors

$$(w_0, w_j) = (\cos\frac{\theta}{2}, 0, 0, \sin\frac{\theta}{2})$$
$$(v_0, v_j) = (\cos\phi, 0, \sin\phi\sin\chi, \sin\phi\cos\chi)$$

Therefore

$$\frac{1}{2}tr(u^{1/2}v) = w_0v_0 - w_jv_j = \cos\frac{\theta}{2}\cos\phi - \sin\frac{\theta}{2}\sin\phi\cos\chi \tag{A.4a}$$

$$\frac{1}{2}tr(u^{1/2}v^{-1}) = w_0v_0 + w_jv_j = \cos\frac{\theta}{2}\cos\phi + \sin\frac{\theta}{2}\sin\phi\cos\chi \tag{A.4b}$$

Note that

$$\frac{1}{2}tr\; u = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} = 1 - 2\sin^2\frac{\theta}{2} = \cos\theta \tag{A.5}$$

The (normalized) Haar measure is in terms of polar coordinates

$$dv = \frac{1}{\pi}d\phi\sin^2\phi\; d\chi\sin\chi$$

Now the computation of F is straightforward (we set $\beta_\pm = \beta \pm \beta'$):

$$F(\beta,\beta';\sin^2\frac{\theta}{2}) = \frac{1}{\pi}\int_0^\pi d\phi \sin^2\phi \int_0^\pi d\chi \sin\chi \cdot$$
$$\cdot \exp\{-\beta_+[1-\cos\frac{\theta}{2}\cos\phi] - \beta_-[\sin\frac{\theta}{2}\sin\phi\cos\chi]\}$$
$$= \frac{2}{\pi\beta_- \sin\frac{\theta}{2}}\int_0^\pi d\phi \sin\phi \sinh(\beta_- \sin\frac{\theta}{2}\sin\phi)\exp\{-\beta_+[1-\cos\frac{\theta}{2}\cos\phi]\}$$

Expanding the sinh in a power series, we obtain integral representations of modified Bessel functions:

$$F(\beta,\beta';\sin^2\frac{\theta}{2}) = \frac{2e^{-\beta_+}}{\pi\beta_-\sin\frac{\theta}{2}}\sum_{\nu=0}^{\infty}\frac{(\beta_-\sin\frac{\theta}{2})^{2\nu+1}}{(2\nu+1)!}\int_0^\pi d\phi \sin^{2(\nu+1)}\phi\; e^{+\beta_+\cos\frac{\theta}{2}\cos\phi}$$
$$= e^{-\beta_+}\sum_{\nu=0}^{\infty}\frac{\Gamma(\nu+3/2)}{\Gamma(3/2)}\frac{2^{\nu+1}}{(2\nu+1)!}\beta_-^{2\nu}\left(\sin^2\frac{\theta}{2}\right)^\nu \frac{I_{\nu+1}(\beta_+\cos\frac{\theta}{2})}{(\beta_+\cos\frac{\theta}{2})^{\nu+1}} \quad (A.6)$$

Let us look at derivatives $\frac{\partial^k}{\partial z^k}$ with $z = \sin^2\frac{\theta}{2}$. Using

$$\frac{\partial^l}{\partial z^l}\frac{I_{\nu+1}(\beta_+\sqrt{1-z})}{(\beta_+\sqrt{1-z})^{\nu+1}} = (-\frac{1}{2}\beta_+^2)^l\,\frac{I_{\nu+l+1}(\beta_+\sqrt{1-z})}{(\beta_+\sqrt{1-z})^{\nu+l+1}}$$

and

$$\frac{\partial^k}{\partial z^k}\left\{z^\nu\,\frac{I_{\nu+1}(\beta_+\sqrt{1-z})}{(\beta_+\sqrt{1-z})^{\nu+1}}\right\}\Bigg|_{z=0} = \sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}\, i!\, 2^{i-k}\,\beta_+^{k-2i-1}\, I_{k+1}(\beta_+)\,\delta_{i,\nu}$$

we get

$$\frac{\partial^k}{\partial z^k}F(\beta,\beta';z)\Big|_{z=0} = 2^{1-k}\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}\beta_-^{2i}\beta_+^{k-2i-1}e^{-\beta_+}I_{k+1}(\beta_+) \quad (A.7)$$

It can be shown that the following expansion

$$e^{-\beta_+}I_{k+1}(\beta_+) = \frac{1}{\sqrt{2\pi\beta_+}}\sum_{\mu=1}^{l}\frac{(-1)^\mu}{\mu!}\frac{\gamma(k+\mu+3/2;2\beta_+)}{\Gamma(k-\mu+3/2)}\left(\frac{1}{2\beta_+}\right)^\mu + R_l(k+1,\beta_+) \quad (A.8)$$

with remainder

$$R_l(k+1,\beta_+) = \frac{2(2\beta_+)^{k+1}(-1)^{l+1}}{\Gamma(1/2)\Gamma(k-l+1/2)}\int_0^1 dx\; e^{-2\beta_+x^2}(x^2)^{k+l+2}\int_0^1 dt\frac{(1-t)^l}{l!}(1-x^2t)^{k-l-\frac{1}{2}} \quad (A.9)$$

holds (for complex β_+ and $Re\,(k+1) > -\frac{1}{2}$).

Proof. Consider

$$e^{-z}I_\nu(z) = e^{-z}\frac{(\frac{z}{2})^\nu}{\Gamma(1/2)\Gamma(\nu+1/2)}\int_0^\pi d\omega\; e^{z\cos\omega}\sin^{2\nu}\omega$$

This representation of the modified Bessel functions is valid for $Re\;\nu > -\frac{1}{2}$. We substitute $x = \sin\frac{\omega}{2}$

$$e^{-z}I_\nu(z) = \frac{2(2z)^\nu}{\Gamma(1/2)\Gamma(\nu+1/2)}\int_0^1 dx\; e^{-2zx^2}x^{2\nu}(1-x^2)^{\nu-\frac{1}{2}}$$

A Taylor expansion with remainder yields

$$(1-x^2)^{\nu-\frac{1}{2}} = \sum_{\mu=0}^{l}\frac{1}{\mu!}\left[\frac{\partial^\mu}{\partial(x^2)^\mu}(1-x^2)^{\nu-\frac{1}{2}}\Big|_{x^2=0}\right]x^{2\mu} +$$
$$+ x^{2(l+1)}\int_0^1 dt\,\frac{(1-t)^l}{l!}\left[\frac{\partial^{l+1}}{\partial(x^2)^{l+1}}(1-x^2)^{\nu-\frac{1}{2}}\Big|_{x^2=x^2t}\right]$$

As a result we get

$$e^{-z}I_\nu(z) = \frac{2(2z)^\nu}{\Gamma(1/2)\Gamma(\nu+1/2)}\sum_{\mu=0}^{l}\binom{\nu-\frac{1}{2}}{\mu}(-1)^\mu\int_0^1 dx\; e^{-2zx^2}x^{2(\nu+\mu)} + R_l(\nu,z) \quad ,$$

and after substitution of $w = 2zx^2$

$$e^{-z}I_\nu(z) = \frac{(2z)^{\nu-1}}{\Gamma(1/2)\Gamma(\nu+1/2)}\sum_{\mu=0}^{l}\binom{\nu-\frac{1}{2}}{\mu}(-1)^\mu\int_0^{2z}dw\; e^{-w}\left(\frac{w}{2z}\right)^{\nu+\mu-\frac{1}{2}} + R_l(\nu,z)$$
$$= \frac{1}{\sqrt{2\pi z}}\sum_{\mu=0}^{l}(-1)^\mu\binom{\nu-\frac{1}{2}}{\mu}\frac{\gamma(\nu+\mu+1/2;2z)}{\Gamma(\nu+1/2)}\left(\frac{1}{2z}\right)^\mu + R_l(\nu,z)$$
$$= \frac{1}{\sqrt{2\pi z}}\sum_{\mu=0}^{l}\frac{(-1)^\mu}{\mu!}\frac{\gamma(\nu+\mu+1/2;2z)}{\Gamma(\nu-\mu+1/2)}\left(\frac{1}{2z}\right)^\mu + R_l(\nu,z) \quad .$$

We have used the incomplete Gamma function of Legendre

$$\gamma(\alpha;x) = \int_0^x e^{-t}t^{\alpha-1}dt$$

Thus (A.8) follows if we set $z = \beta_+$, $\nu = k+1$.

Remark. The representation of $e^{-\beta_+}I_{k+1}(\beta_+)$ by the convergent series

$$e^{-\beta_+}I_{k+1}(\beta_+) = \sum_{\mu=0}^{\infty}\frac{(-1)^\mu}{\mu!}\frac{\gamma(k+\mu+3/2;2\beta_+)}{\Gamma(1/2)\Gamma(k-\mu+3/2)}\left(\frac{1}{2\beta_+}\right)^{\mu+\frac{1}{2}} \quad (A.10)$$

goes beyond perturbation theory in the physical running coupling constant $\beta_j^{-1/2}$ (remember $\beta_+ = 2\beta_j$). Expanding (A.10), we get the decomposition

$$e^{-\beta_+}I_{k+1}(\beta_+) = \sum_{\mu=0}^{\infty}\frac{(-1)^\mu}{\mu!}\frac{\Gamma(k+\mu+3/2)}{\Gamma(1/2)\Gamma(k-\mu+3/2)}\left(\frac{1}{\sqrt{2}}\beta_+^{-1/2}\right)^{2\mu+1} +$$
$$+ \quad \text{nonperturbative contributions} \quad (A.11)$$

Note that the perturbation series in (A.11) also converges.

Substitution of the incomplete Gamma function of Legendre $\gamma(k+\mu+3/2;2\beta_+)$ in (A.8) by the Gamma functions $\Gamma(k+\mu+3/2)$ gives an additional remainder term which is exponentially small in β_+, provided that $\beta_+ = 2\beta_j$ is large enough. Its derivatives are exponentially small too. Thus the β_+,β_- differentiations of (A.7) can be performed easily (n_- even, $2k \geq n_-$):

$$\frac{\partial^{n_+}}{\partial\beta_+^{n_+}}\frac{\partial^{n_-}}{\partial\beta_-^{n_-}}\frac{\partial^k}{\partial z^k}F(\beta,\beta';z)\Big|_{\substack{z=0=\beta_-\\ \beta_+=2\beta}} = 2^{1-k}\binom{k}{n_-/2}(-1)^{k-\frac{n_-}{2}}\,(n_-)!\cdot$$
$$\cdot\left\{\sum_{\mu=0}^{l}\frac{(-1)^\mu}{\mu!}\frac{2^{-\mu}}{\sqrt{2\pi}}\frac{\Gamma(k+\mu+3/2)}{\Gamma(k-\mu+3/2)}\frac{\Gamma(k-n_--\mu-1/2)}{\Gamma(k-n_+-n_--\mu-1/2)}(2\beta)^{k-n_+-n_--\mu-\frac{5}{2}}+\right.$$
$$\left.+\,\mathcal{O}(\beta^{k-n_+-n_--l-\frac{5}{2}})\right\} \qquad (A.12)$$

Here we have used equation (A.9) and

$$\left|\frac{\partial^i}{\partial\beta_+^i}\left\{\beta_+^{-(k+1)}R_l(k+1,\beta_+)\right\}\right| \leq$$
$$\leq \frac{2^{k+i+1}}{\Gamma(1/2)}\frac{\Gamma(k+l+i+5/2)}{\Gamma(k-l+1/2)}\frac{1}{(l+1)!}\max\left\{1,\frac{l+1}{k+\frac{1}{2}}\right\}\left(\frac{1}{2\beta_+}\right)^{k+l+i+\frac{5}{2}} \qquad (A.13)$$

to bound the derivatives of the rest term.
As a last step, one inserts formula (A.12) into

$$\frac{\partial^k}{\partial z^k}F^{nm}(z)\Big|_{z=0} = \left(-\frac{1}{2}\right)^{n+m}\left[\frac{\partial}{\partial\beta_+}+\frac{\partial}{\partial\beta_-}\right]^n\left[\frac{\partial}{\partial\beta_+}-\frac{\partial}{\partial\beta_-}\right]^m\frac{\partial^k F}{\partial z^k}(\beta,\beta';0)\Big|_{\beta=\beta'}$$
$$= \left(-\frac{1}{2}\right)^{n+m}\sum_{\nu=0}^{[\frac{n+m}{2}]}\left\{\sum_{\substack{i=0\\ i+j=2\nu}}^{n}\sum_{j=0}^{m}\binom{n}{i}\binom{m}{j}(-1)^j\right\}$$
$$\left(\frac{\partial}{\partial\beta_+}\right)^{n+m-2\nu}\left(\frac{\partial}{\partial\beta_-}\right)^{2\nu}\frac{\partial^k F}{\partial z^k}(\beta,\beta';0)\Big|_{\beta_-=0,\beta_+=2\beta} \qquad (A.14)$$

to get equation (5.13).

Appendix B. Bounds for Convolution Integrals

Consider now the convolution integrals which contain the remainder term of expansion (5.10). For technical reasons and in order to simplify the notation, we use

$$\tilde{r}_j(\theta) = \sum_{n=2}^{5} a_n\sin^{2n}\frac{\theta}{2} + \tilde{\Re}_j^6(\sin^2\frac{\theta}{2}) \quad ; \quad a_n = \mathcal{O}(\beta_j^{\frac{n}{2}}) \qquad (B.1)$$

instead of (5.10). The remainder $\tilde{\Re}_j^6(\sin^2\frac{\theta}{2})$ is analytic in the strip $|Im\,\theta| < \eta_j = \beta_j^{-1/4}$. It obeys (cp. (5.9))

$$|\tilde{\Re}_j^6(\sin^2\frac{\theta}{2})| \leq \mathcal{O}(\beta_j^3)\left|\sin^2\frac{\theta}{2}\right|^6 \quad \text{for} \quad |Im\,\theta| \leq \eta_j \qquad (B.2)$$

Introduce the functions

$$\sin^2\frac{\theta_+}{2} = \frac{1}{4}tr(1-u^{1/2}v) \quad , \quad \sin^2\frac{\theta_-}{2} = \frac{1}{4}tr(1-u^{1/2}v^{-1}) \qquad (B.3)$$

of the central angles θ_+,θ_-, and define the integrals

$$M_j^n(\sin^2\frac{\theta}{2}) = \int_G dv\;\exp\left\{-2\beta_j\sin^2\frac{\theta_+}{2} - 2\beta_-\sin^2\frac{\theta_-}{2}\right\}\cdot$$
$$\cdot\, a_n\left[\sin^{2n}\frac{\theta_+}{2}\,\tilde{\Re}_j^6(\sin^2\frac{\theta_-}{2}) + \tilde{\Re}_j^6(\sin^2\frac{\theta_+}{2})\sin^{2n}\frac{\theta_-}{2}\right] \qquad (B.5a)$$

$$M_j(\sin^2\frac{\theta}{2}) = \int_G dv\;\exp\left\{-2\beta_j\sin^2\frac{\theta_+}{2} - 2\beta_j\sin^2\frac{\theta_-}{2}\right\}\tilde{\Re}_j^6(\sin^2\frac{\theta_+}{2})\tilde{\Re}_j^6(\sin^2\frac{\theta_-}{2}) \quad (B.5b)$$

Clearly, M_j^n and M_j are analytic in the region $\Omega(2\eta_j)$. We want to show the following bounds

$$\mathcal{N}_j^{-1}\left|\frac{\partial^k M_j^n}{\partial z^k}(0)\right| \leq \mathcal{O}(\beta_j^{k-3}) \quad \text{for} \quad n = 0,2,3,4,5 \qquad (B.6a)$$

$$\mathcal{N}_j^{-1}\left|\frac{\partial^k M_j}{\partial z^k}(0)\right| \leq \mathcal{O}(\beta_j^{k-6}) \qquad (B.6b)$$

Together with (5.13), these bounds lead to the recursion relations for the running coupling constants (5.14). Due to Cauchy's estimate and (B.2) we get with $p = \mathcal{O}(\beta_j^{-1})$

$$\left|\frac{\partial^k M_j^n}{\partial z^k}(0)\right| \leq \frac{k!}{p^k}\max_{|z|=p}|M_j^n(z)| \leq \frac{k!}{p^k}\,\mathcal{O}(\beta_j^{3+\frac{n}{2}})\max_{|\sin^2\frac{\theta}{2}|=p}[\tilde{F}^{n6}(\theta)+\tilde{F}^{6n}(\theta)] \qquad (B.7a)$$

$$\left|\frac{\partial^k M_j}{\partial z^k}(0)\right| \leq \frac{k!}{p^k}\max_{|z|=p}|M_j(z)| \leq \frac{k!}{p^k}\,\mathcal{O}(\beta_j^6)\max_{|\sin^2\frac{\theta}{2}|=p}\tilde{F}^{66}(\theta) \qquad (B.7b)$$

where we have used moments of the type

$$\tilde{F}^{nm}(\theta) = \int_G dv\;\exp\{-2\beta Re[\sin^2\frac{\theta_+}{2}+\sin^2\frac{\theta_-}{2}]\}\left|\sin^2\frac{\theta_+}{2}\right|^n\left|\sin^2\frac{\theta_-}{2}\right|^m \qquad (B.8)$$

Suppose that n, m are even. Then we can calculate these moments explicitly by introducing the generating function

$$\widetilde{F}(\beta,\beta',\gamma,\gamma';\theta) = \int_G dv\ \exp\{-2\beta Re \sin^2\frac{\theta_+}{2} - 2\beta' Re\sin^2\frac{\theta_-}{2} - $$
$$- 2\gamma Im\sin^2\frac{\theta_+}{2} - 2\gamma' Im \sin^2\frac{\theta_-}{2}\} \quad (B.9)$$

We set $\beta_\pm = \beta \pm \beta'$, $\gamma_\pm = \gamma \pm \gamma'$, and obtain (cp. (A.6))

$$\widetilde{F}(\beta,\beta',\gamma,\gamma';\theta) = e^{-\beta_+}\sum_{\nu=0}^{\infty}\frac{\Gamma(\nu+3/2)}{\Gamma(3/2)}\frac{2^{\nu+1}}{(2\nu+1)!}\alpha_-^{2\nu}\frac{I_{\nu+1}(\alpha_+)}{\alpha_+^{\nu+1}} \quad (B.10)$$

with the abbreviations

$$\alpha_+ = \beta_+ Re\cos\frac{\theta}{2} + \gamma_+ Im\cos\frac{\theta}{2}$$
$$\alpha_- = \beta_- Re\sin\frac{\theta}{2} + \gamma_- Im\sin\frac{\theta}{2} \quad (B.11)$$

From the simple identities

$$\left|\sin^2\frac{\theta_\pm}{2}\right|^n = \sum_{i=0}^{\frac{n}{2}}\binom{n/2}{i}\left(Re\sin^2\frac{\theta_\pm}{2}\right)^{2i}\left(Im\sin^2\frac{\theta_\pm}{2}\right)^{n-2i}$$

and

$$\frac{\partial^l}{\partial\beta^l} = \left(\frac{\partial}{\partial\beta_+}+\frac{\partial}{\partial\beta_-}\right)^l = \sum_{i=0}^{l}\binom{l}{i}\frac{\partial^i}{\partial\beta_+^i}\frac{\partial^{l-i}}{\partial\beta_-^{l-i}} \qquad \text{etc.}$$

one gets immediately

$$\widetilde{F}^{nm}(\theta) = \sum_{\substack{i_+,i_-,j_+,j_-\ge 0\\ i_++i_-+j_++j_-=n+m}} \{\text{combinatorial factors}\}\cdot$$
$$\cdot\ \partial^{i_+}_{\beta_+}\partial^{i_-}_{\beta_-}\partial^{j_+}_{\gamma_+}\partial^{j_-}_{\gamma_-}\widetilde{F}(\beta,\beta',\gamma,\gamma';\theta)\Big|_{\substack{\beta_-=\gamma_+=\gamma_-=0\\ \beta_+=2\beta}} \quad (B.12)$$

It remains to show that every derivation of the generating function $\widetilde{F}$ will give a small factor β_j^{-1}. Therefore we carry out the differentiations with respect to β_- and γ_- explicitly and use, on the other hand, analyticity of $\widetilde{F}$ in β_+, γ_+ to apply Cauchy's estimate:

$$\partial^{i_-}_{\beta_-}\partial^{j_-}_{\gamma_-}\widetilde{F}(\beta,\beta',\gamma,\gamma';\theta)\Big|_{\beta_-=\gamma_-=0} = \left(Re\sin\frac{\theta}{2}\right)^{i_-}\left(Im\sin\frac{\theta}{2}\right)^{j_-}\cdot$$
$$\cdot\ e^{-\beta_+}\frac{\Gamma(l+3/2)}{\Gamma(3/2)}\frac{2^{l+1}}{(2l+1)!}(2l)!\,\frac{I_{l+1}(\alpha_+)}{\alpha_+^{l+1}}\Big|_{i_-+j_-=2l} \quad (B.13)$$

Because of (B.10) and (B.11b) the sum $i_- + j_-$ has to be even. Note that

$$\max_{|\sin^2\frac{\theta}{2}|=p=\mathcal{O}(\beta_j^{-1})}\left|\left(Re\sin\frac{\theta}{2}\right)^{i_-}\left(Im\sin\frac{\theta}{2}\right)^{j_-}\right| = \mathcal{O}(\beta_j^{-\frac{1}{2}(i_-+j_-)})$$

Furthermore

$$\left|\partial^{i_+}_{\beta_+}\partial^{j_+}_{\gamma_+}\left\{e^{-\beta_+}\frac{I_{l+1}(\alpha_+)}{\alpha_+^{l+1}}\right\}\Big|_{\gamma_+=0,\beta_+=2\beta}\right| \le$$
$$\le \frac{i_+!}{q^{i_+}}\frac{j_+!}{s^{j_+}}\max_{|\xi-2\beta|=q}\max_{|\zeta|=s}\left|e^{-\xi}\frac{I_{l+1}(\xi\ Re\cos\frac{\theta}{2}+\zeta\ Im\cos\frac{\theta}{2})}{(\xi\ Re\cos\frac{\theta}{2}+\zeta\ Im\cos\frac{\theta}{2})^{l+1}}\right|$$
$$\le \frac{i_+!}{q^{i_+}}\frac{j_+!}{s^{j_+}}\max_{|\xi-2\beta|=q}\max_{|\zeta|=s}\left\{e^{-Re\ \xi}\frac{I_{l+1}(Re\ \xi\ Re\cos\frac{\theta}{2}+Re\ \zeta\ Im\cos\frac{\theta}{2})}{(Re\ \xi\ Re\cos\frac{\theta}{2}+Re\ \zeta\ Im\cos\frac{\theta}{2})^{l+1}}\right\}$$
$$\le \frac{i_+!}{q^{i_+}}\frac{j_+!}{s^{j_+}}\mathcal{O}(\beta_j^{-l-\frac{3}{2}}) \quad \text{for} \quad q,s=\mathcal{O}(\beta_j),\ |\theta|=\mathcal{O}(\beta_j^{-\frac{1}{2}}) \quad (B.14)$$

Thus we get $(n, m = 0,2,4,6)$

$$\max_{|\sin^2\frac{\theta}{2}|=p}\widetilde{F}^{nm}(\theta) \le \mathcal{O}(\beta_j^{-\frac{1}{2}(i_-+j_-)})\mathcal{O}(\beta_j^{-(i_++j_+)})\mathcal{O}(\beta_j^{-\frac{1}{2}(i_-+j_-)-\frac{3}{2}}) = \mathcal{O}(\beta_j^{-(n+m)-\frac{3}{2}})$$

Finally, the estimates (B.7) give

$$\left|\frac{\partial^k M_j^n}{\partial z^k}(0)\right| \le \mathcal{O}(\beta_j^{k-3-\frac{n}{2}-\frac{3}{2}}) \qquad \text{for} \qquad n=0,2,4$$
$$\left|\frac{\partial^k M_j}{\partial z^k}(0)\right| \le \mathcal{O}(\beta_j^{k-6-\frac{3}{2}})$$

In order to establish (B.6), we use $\mathcal{N}_j^{-1} = \mathcal{O}(\beta_j^{3/2})$ and the rough estimates $\widetilde{F}^{3,6}(\theta) \le \mathcal{O}(1)\ \widetilde{F}^{2,6}(\theta)$, $\widetilde{F}^{5,6}(\theta) \le \mathcal{O}(1)\ \widetilde{F}^{4,6}(\theta)$ yielding

$$\left|\frac{\partial^k M_j^3}{\partial z^k}(0)\right| \le \mathcal{O}(\beta_j^{k-5}) \quad , \quad \left|\frac{\partial^k M_j^5}{\partial z^k}(0)\right| \le \mathcal{O}(\beta_j^{k-6})\,.$$

Appendix C. Proof of the Lemma and its Generalizations

Proof of the lemma. We consider the auxiliary function

$$h(\theta) = \mathcal{N}_{j0}^{-1}\, (|gw(\beta_j)| * |gw(\beta_j)|)(\theta)\; |gw(\beta_{j-1},\theta)|^{-1/2} \exp\left(+\frac{1}{4}\beta_j Re \sin^4\frac{\theta}{2}\right)$$

in the strip $|Im\ \theta| \le \beta_{j-1}^{-1/4}$. Inserting the expressions

$$[gw(\beta_j) * gw(\beta_j)](0) = \mathcal{N}_{j0} = F(\beta_j,\beta_j;0) = 2e^{-2\beta_j}\frac{I_1(2\beta_j)}{2\beta_j}$$

$$(|gw(\beta_j)| * |gw(\beta_j)|)(\theta) = \widetilde{F}^{00}(\theta) = \widetilde{F}(\beta_j,\beta_j,0,0;\theta) = 2e^{-2\beta_j}\frac{I_1(2\beta_j Re\cos\frac{\theta}{2})}{2\beta_j Re\cos\frac{\theta}{2}}$$

$$|gw(\beta_{j-1},\theta)|^{-1/2} = \exp\left(+\beta_{j-1} Re\sin^2\frac{\theta}{2}\right)$$

we get

$$h(\theta) = \frac{I_1(2\beta_j Re\cos\frac{\theta}{2})}{Re\cos\frac{\theta}{2}\; I_1(2\beta_j)} \exp\left(+\beta_{j-1} Re\sin^2\frac{\theta}{2} + \frac{1}{4}\beta_j Re\sin^4\frac{\theta}{2}\right)$$

Provided that θ is not too large, say $Re\cos\frac{\theta}{2} \ge \mathcal{O}(\beta_j^{-1/2})$, one can apply the asymptotic expansion

$$I_1(z) = \frac{1}{\sqrt{2\pi z}}\, e^z\, [1 + \mathcal{O}(z^{-1})]$$

for the modified Bessel function I_1. This leads to

$$\begin{aligned} h(\theta) &= [1+\mathcal{O}(\beta_j^{-1/2})]\left(Re\cos\frac{\theta}{2}\right)^{-3/2} \exp\left\{-\beta_j Re\left[2(1-\cos\frac{\theta}{2}) - \sin^2\frac{\theta}{2} - \frac{1}{4}\sin^4\frac{\theta}{2}\right]\right\}\cdot \\ &\quad\cdot \exp\left\{-(\beta_j-\beta_{j-1}) Re\sin^2\frac{\theta}{2}\right\} \\ &\le [1+\mathcal{O}(\beta_j^{-1/2})]\left(Re\cos\frac{\theta}{2}\right)^{-3/2} \exp\left\{-\beta_j Re\left[8\sin^6\frac{\theta}{4} - 4\sin^8\frac{\theta}{4} + \mathcal{O}(\beta_j^{-1})\sin^2\frac{\theta}{2}\right]\right\} \end{aligned}$$

The maximum of the function on the right hand side lies in the small field region and is of order $1 + \mathcal{O}(\beta_j^{-1/2})$. Outside this region, the function decreases monotonically. Even in the domain of validity of the used asymptotic formula, it becomes exponentially small for sufficiently large fields θ. (For $Re\cos\frac{\theta}{2} = \mathcal{O}(\beta_j^{-1/2})$ it is at least of order $\mathcal{O}(\beta_j^{3/2}e^{-\frac{3}{4}\beta_j})$.) Thus $h(\theta) \le 1 + \mathcal{O}(\beta_j^{-1/2})$, which yields the lemma.

***Proof of the bounds* (4.9) *and* (4.10).** Now we want to show that the generalizations (4.9) and (4.10) of (4.6) hold for $|Im\ \theta| \le \beta_{j-1}^{-1/4}$. Therefore we consider the expectation values $\mathcal{E}_1, \mathcal{E}_2$ introduced in Sect. 4. Obviously (cf. (B.3))

$$\mathcal{E}_i(\theta) \le \cosh^{4i}\frac{\beta_{j-1}^{-1/4}}{4} \qquad \text{for} \qquad i = 1,2 \tag{C.1}$$

In order to calculate these functions exactly, we simply use the formula (B.8) and (B.12), (B.13) respectively. However, in contrast to the discussion of Appendix B, the β_+, γ_+ differentiations of $\widetilde{F}$ have to be performed explicitly now. The combinatorial factor in (B.12) follows by the binomial theorem.
Applying the well known formula $I'_\nu(x) = I_{\nu\pm1}(x) \pm \frac{\nu}{x}I_\nu(x)$, one gets the following decompositions

$$\frac{\partial}{\partial x}\frac{I_\nu(x)}{x^\nu} = \frac{I_{\nu+1}(x)}{x^\nu} \qquad\qquad \frac{\partial^2}{\partial x^2}\frac{I_\nu(x)}{x^\nu} = \frac{I_\nu(x)}{x^\nu} - (2\nu+1)\frac{I_{\nu+1}(x)}{x^{\nu+1}}$$

where each term on the right hand sides is nonsingular for $x \to 0$. Thus the expectation values become ($\beta = \beta_j$)

$$\mathcal{E}_1 = \frac{1}{4}\left[1 + a^2 + b^2\right] + \frac{I_2(2\beta a)}{2\beta a\, I_1(2\beta a)}\left\{\frac{1}{4}\left[-4\beta a^2 - 3(a^2+b^2) + c^2 + d^2\right]\right\} \tag{C.2}$$

$$\begin{aligned} \mathcal{E}_2 = &\frac{1}{16}\left[1 + 6a^2 + 2b^2 + (a^2+b^2)^2\right] + \frac{1}{8}\beta^{-1}\left[3(a^2+b^2) + c^2 - d^2\right] + \\ &+ \frac{5}{64}\beta^{-2}\left[3a^2 + 6b^2 + 2(c^2-d^2)\right] + \\ &+ \frac{I_2(2\beta a)}{2\beta a\, I_1(2\beta a)}\left\{-\frac{1}{2}\beta a^2(1+a^2+b^2) - \frac{1}{8}\left[9a^2 + 3b^2 + 3(a^2+b^2)^2\right] - \right. \\ &\qquad - \frac{1}{8}(1+a^2-b^2)(c^2-d^2) - \frac{1}{2}\beta^{-1}\left[3(a^2+b^2) + c^2 - d^2\right] - \\ &\qquad \left. - \frac{5}{16}\beta^{-2}\left[3a^2 + 6b^2 + 2(c^2-d^2)\right]\right\} + \\ &+ \frac{I_3(2\beta a)}{(2\beta a)^2\, I_1(2\beta a)}\left\{\frac{15}{16}b^4 - \frac{5}{8}b^2(c^2-d^2) + \frac{3}{16}(c^2+d^2)^2\right\} \end{aligned} \tag{C.3}$$

with the abbreviations

$$\begin{aligned} a &= \frac{\partial\alpha_+}{\partial\beta_+} = Re\cos\frac{\theta}{2} \quad , \qquad b = \frac{\partial\alpha_+}{\partial\gamma_+} = Im\cos\frac{\theta}{2} \\ c &= \frac{\partial\alpha_-}{\partial\beta_-} = Re\sin\frac{\theta}{2} \quad , \qquad d = \frac{\partial\alpha_-}{\partial\gamma_-} = Im\sin\frac{\theta}{2} \quad . \end{aligned} \tag{C.4}$$

In the small field region $\theta = \mathcal{O}(\beta^{-1/4})$ we have $1 - a = \mathcal{O}(\beta^{-1/2})$, $a^2 = \mathcal{O}(1)$, $b^2 = \mathcal{O}(\beta^{-1})$, $c^2 = \mathcal{O}(\beta^{-1/2})$, and $d^2 = \mathcal{O}(\beta^{-1/2})$. According to the expansions

$$\begin{aligned} \frac{I_2(2\beta a)}{2\beta a\, I_1(2\beta a)} &= \frac{1}{2\beta a}\left[1 - \frac{3}{4}\frac{1}{a}\beta^{-1} + \frac{3}{32}\frac{1}{a^2}\beta^{-2} + \mathcal{O}(a^{-3}\beta^{-3})\right] \\ \frac{I_3(2\beta a)}{(2\beta a)^2\, I_1(2\beta a)} &= \left(\frac{1}{2\beta a}\right)^2\left[1 + \mathcal{O}(a^{-1}\beta^{-1})\right] \quad , \end{aligned}$$

$\mathcal{E}_1, \mathcal{E}_2$ can be expressed as

$$\mathcal{E}_i(\theta) = \left(\frac{1}{4}\left[(1-a)^2 + b^2\right]\right)^i + \mathcal{O}(\beta^{-1/2-i}) = \left(\frac{1}{16}\left|\sin^4\frac{\theta}{2}\right|\right)^i + \mathcal{O}(\beta^{-1/2-i})$$

Hence we write

$$\mathcal{E}_i = \left(\frac{1}{16}\left|\sin^4\frac{\theta}{2}\right|\right)^i + \chi_i(\theta)$$

In a last step we combine these expectation values with the bounds (4.6). The functions χ_i are at most of order $\mathcal{O}(1)$. Because of their small field estimates one has therefore

$$|\chi_i(\theta)|\exp\left(-\frac{1}{4}\beta Re\sin^4\frac{\theta}{2}\right) \le \mathcal{O}(\beta^{-1/2-i}) \quad ,$$

which proves (4.9) and (4.10).

Proof of the estimate (4.12). The left hand side of inequality (4.12) is known explicitly if we use (4.2) for the normalization factor. Using

$$[gw(\beta_j) * gw(\beta_j)](\theta) = F(\beta_j,\beta_j;\theta) = 2e^{-2\beta_j}\frac{I_1(2\beta_j\cos\frac{\theta}{2})}{2\beta_j\cos\frac{\theta}{2}}$$

we get

$$\mathcal{N}_j^{-1}[gw(\beta_j) * gw(\beta_j)](\theta)\; gw(\beta_{j-1},\theta)^{-1/2} = [1+\mathcal{O}(\beta_j^{-1})]\frac{I_1(2\beta_j\cos\frac{\theta}{2})}{\cos\frac{\theta}{2}\, I_1(2\beta_j)}\exp\left(+\beta_{j-1}\sin^2\frac{\theta}{2}\right)$$

For $Re\cos\frac{\theta}{2} \ge \mathcal{O}(\beta_j^{-1/2})$ this becomes

$$[1+\mathcal{O}(\beta_j^{-1/2})]\left(\cos\frac{\theta}{2}\right)^{-3/2}\exp\left(-2\beta_j\left[1-\cos\frac{\theta}{2}\right] + \beta_{j-1}\sin^2\frac{\theta}{2}\right) =$$
$$= [1+\mathcal{O}(\beta_j^{-1/2})]\left(\cos\frac{\theta}{2}\right)^{-3/2}\exp\left(-4\beta_j\sin^4\frac{\theta}{4} - (\beta_j-\beta_{j-1})\sin^2\frac{\theta}{2}\right)$$

Thus

$$\left|\left\{\mathcal{N}_j^{-1}[gw(\beta_j) * gw(\beta_j)](\theta)\; gw(\beta_{j-1},\theta)^{-1/2}\right\}^2 - 1\right|$$

is bounded by a constant of order $\mathcal{O}(1)$ for all θ with $|Im\ \theta| = \eta_{j-1} = \beta_{j-1}^{-1/4}$, which implies that the left hand side of (4.12) is of order $\mathcal{O}(\beta_j)$. However, this requires $|\sin^4\frac{\theta}{2}| = (\sin^2\frac{w}{2} + \sinh^2\frac{\eta_{j-1}}{2})^2 = \mathcal{O}(\beta_j^{-1})$. Hence we may restrict the further analysis to the small field region $Re\ \theta = \mathcal{O}(\beta_j^{-1/2})$:

$$\text{l.h.s. of (4.12)} = \max_{|Im\ \theta|=\eta_{j-1}}\left|\sin^{-4}\frac{\theta}{2}\left([1+\mathcal{O}(\beta_j^{-1/2})]\, e^{-8\beta_j\sin^4\frac{\theta}{4}} - 1\right)\right|$$
$$\le \mathcal{O}(\beta_j^{1/2}) + \max_{|Im\ \theta|=\eta_{j-1}}\frac{1}{2}\left|\frac{e^{-8\beta_j\sin^4\frac{\theta}{4}} - 1}{8\sin^4\frac{\theta}{4}}\right|$$

where we have used $\sin^4\frac{\theta}{2} = 16\sin^4\frac{\theta}{4}\cos^4\frac{\theta}{4}$. Finally the fundamental theorem of calculus gives

$$\left|\frac{e^{-8\beta_j\sin^4\frac{\theta}{4}} - 1}{8\sin^4\frac{\theta}{4}}\right| = \beta_j\left|\int_0^1 dt\; e^{-8\beta_j\sin^4\frac{\theta}{4}\,t}\right| \le \beta_j\int_0^1 dt\; e^{-8\beta_j Re\sin^4\frac{\theta}{4}\,t}$$
$$\le \beta_j\int_0^1 dt\; e^{-8\beta_j\left[-\frac{1}{32}\eta_{j-1}^4 + \mathcal{O}(\beta_j^{-3/2})\right]t} = \beta_j[1+\mathcal{O}(\beta_j^{-1/2})]\int_0^1 dt\; e^{\frac{1}{4}t}$$
$$= \beta_j\,\frac{e^{\frac{1}{4}} - 1}{\frac{1}{4}}\,[1+\mathcal{O}(\beta_j^{-1/2})]$$

We end up with

$$\text{l.h.s. of (4.12)} \le \frac{1}{2}\beta_j\,\frac{e^{\frac{1}{4}} - 1}{\frac{1}{4}} + \mathcal{O}(\beta_j^{1/2}).$$

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