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METHODS OF BEAM OPTICS

by

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Abstract

In the following report we give a survey of linear machine theory. Our investigations are restricted to coasting beam betatron motion but coupling is taken into account in a general way. The equations of motion for on- and off-momentum particles are derived und written in canonical form. From the canonicity it follows, that all transfer matrices are automatically symplectic. Eigenvector methods are introduced to study the stability behaviour.To investigate the influence of coupling, generalised lattice functions are defined and canonical perturbation techniques are applied.

Contents

1 Introduction

The description of the motion of charged particles in a circular accelerator is a complex physical problem. When approaching this kind of complicated system, physicists try first to find a certain regime of the parameters for which the system behaviour is described by a simple model. Once such a model has been established it may serve as a basis which allows one to take into account the full complexity of the system by means of a perturbation treatment.

This is the way one proceeds in describing the beam dynamics in accelerators. The first step in a series of simplifications is to assume that the particle density in a beam is low enough so that the interaction of a single particle with all the other particles in the beam can be neglected. The motion of a particle is then determined by "external" forces only. In fact there are many operation modes of real accelerators for which this single particle model is a rather satisfactory description of the beam behaviour. Moreover one can say that in high energy accelerators the external forces acting on a particle are always much stronger than the inner forces arising from the interaction among the particles. Thus collective effects are approached later by a perturbation treatment.

The single particle model describes the motion of a charged particle in an accelerator as oscillations in three degrees of freedom around a particle trajectory which closes on itself, the so called closed orbit. One distinguishes between the transverse motion and the variation of the time with respect to a reference time at which the particle passes a certain point in the accelerator. The independent variable of the motion is the path length along the closed orbit. The transverse motion is called betatron oscillation. The variation of the particle time associated with a change in the particle energy imposed by rf cavity resonators is called synchrotron oscillation.

Though the single particle model is already a considerable simplification, the remaining forces and interactions are still very complicated. That is because the forces acting on a particle are in general nonlinear. The motion of a particle under the influence of nonlinear forces is a problem which has never been completely solved. Therefore accelerators are designed so that the forces acting on the particles are as linear as possible. Furthermore, most of the particles are in the center of the beam performing small amplitude oscillations and therefore are only slightly influenced by the weak nonlinearities. Therefore it is reasonable to develop a model in which all the forces have been linearized, the linear accelerator theory.

In the linear model, one considers quasi harmonic oscillations in three degrees of freedom under the influence of linear restoring forces. We call these synchro-betatron oscillations. The coupling between the two transverse modes and the longitudinal mode is not very strong in most real accelerators. This allows, as a further simplification, the neglect of these couplings and the treatment of horizontal, vertical and longitudinal motion as three independent oscillation modes. The result is the uncoupled linear machine theory which was subject of the first two introductory lectures about transverse and longitudinal motion in this course. Despite all the simplifications, the uncoupled linear machine theory provides a powerful tool for describing, analysing and interpreting the beam behaviour in an accelerator. It is the basis of machine design. Also the operation of the machine, measurement of beam properties, adjustments and optimization of parameters follow the concepts of the linear uncoupled theory.

For certain operating conditions however, the description of beam behaviour in terms of linear uncoupled theory is insufficient. However it is still an excellent starting point for the treatment of more sophisticated effects like the impact of machine nonlinearities, interaction among the particles in the beam and coupling between the oscillation modes. All of this is the reason why the uncoupled linear machine theory is considered as the backbone of accelerator physics.

Following this concept, it will be demonstrated how the theory of uncoupled betatron motion can be extended to the case of coupled transverse betatron oscillations. For this purpose we start with the equation of motion for a single particle in the accelerator. We then specialize to a coasting beam with no accelerating fields. In this case the particles have constant energy (we do not consider at this point the energy loss by the emission of synchrotron radiation in the arcs) and there is no coupling between the "frozen" longitudinal motion and the transverse motion. But the transverse betatron oscillations are influenced by a constant deviation from the ideal particle energy.

After linearization of the appropriate equations of motion, we arrive at the linear model of transverse coupled machine theory. The coupling between the two transverse modes comes from longitudinal magnetic fields or from fields which don't have a midplane symmetry with respect to the beam. These equations (for solenoids and skew quadrupoles) will be solved exactly. Furthermore, in order to study the influence of coupling, we introduce generalized lattice functions. Besides the exact treatment of coupling, a very useful procedure is to consider coupling effects as a perturbation of the uncoupled theory. In this form the results are especially well suited for analysing machine measurements and for calculating adjustments and corrections for machine operations.

In the second part of this course, the impact on the transverse oscillations of a deviation of the particle energy from the ideal energy,-called chromaticity - will be examined. Taking into account chromatic effects in the transverse motion is the lowest order step of acknowledging that transverse and longitudinal motion are coupled. The concept of chromaticity has a great practical significance for the operation of accelerators.

2 Derivation of the Equations of Motion

2.1 The Lagrangian for a Charged Particle

As a starting point we consider the relativistic Lagrangian of a charged particle of charge e and mass m_0 in an electromagnetic field (see e.g. [1]) :

$$L = -m_0 c^2 \cdot \sqrt{1 - \frac{v^2}{c^2}} + e(\dot{\vec{r}} \cdot \vec{A}) - e \cdot \varphi \; ; \tag{2.1}$$

$$(v = |\dot{\vec{r}}|)$$

where $\vec{r}$ is the position vector and $\vec{A}$ and φ are the vector and scalar potentials from which the electric field $\vec{\epsilon}$ and the magnetic field $\vec{B}$ are derived as

$$\vec{\epsilon} = -\text{grad}\;\varphi - \frac{\partial \vec{A}}{\partial t} \; ; \tag{2.2a}$$

$$\vec{B} = \text{curl}\;\vec{A} \; . \tag{2.2b}$$

As usual, the equations of motion are derived from the Euler-Lagrange equations and in Cartesian coordinates we have

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\vec{r}}} - \frac{\partial L}{\partial \vec{r}} = 0 \; . \tag{2.3}$$

2.2 Introduction of the Natural Coordinates x, y, s

The position vector $\vec{r}$ in eqn. (2.1) refers to a fixed coordinate system. However, in accelerator physics, it is useful to introduce the natural coordinates x, y, s in a suitable curvilinear coordinate system. With this in mind we assume that an ideal closed design orbit exists which describes the path of a particle of constant energy E_0, i.e. we neglect energy variations due to cavities and to radiation loss. In addition we assume that there are no field errors or correction magnets. We also assume that the design orbit comprises piecewise flat curves which lie either in the horizontal or vertical plane so that it has no torsion. The design orbit which will be used as the reference system will in the following be described by the vector $\vec{r}_0(s)$ where s is the length along the design orbit. An arbitrary particle orbit $\vec{r}(s)$ is then described by the deviation $\delta\vec{r}(s)$ of the particle orbit $\vec{r}(s)$ from the design orbit $\vec{r}_0(s)$:

$$\vec{r}(s) = \vec{r}_0(s) + \delta\vec{r}(s) \ . \tag{2.4}$$

The vector $\delta\vec{r}$ can as usual [2] be described using an orthogonal coordinate system ("dreibein") accompanying the particles which travel along the design orbit and comprising

a unit normal vector	$\vec{\nu}(s)$,
a unit tangent vector	$\vec{\tau}(s)$
and a unit binormal vector	$\vec{\beta}(s) = \vec{\tau}(s) \times \vec{\nu}(s)$.

We require that the vector $\vec{\nu}(s)$ is directed outwards if the motion takes place in the horizontal plane and upwards if the motion takes place in the vertical plane.

Choosing the direction of $\vec{\nu}(s)$ in this way, implies that the curvature K(s) appearing in the Fresnet formulae:

$$\vec{\tau}(s) = \frac{d}{ds}\vec{r}_0(s) \equiv \vec{r}_0{}'(s) \ ; \tag{2.5}$$

$$\frac{d}{ds}\vec{\tau} = -K(s) \cdot \vec{\nu}(s) \ ; \tag{2.6a}$$

$$\frac{d}{ds}\vec{\nu} = +K(s) \cdot \vec{\tau}(s) \ ; \tag{2.6b}$$

$$\frac{d}{ds}\vec{\beta} = 0 \tag{2.6c}$$

is always positive in the horizontal plane and negative in the vertical plane if and only if the centre of curvature lies above the reference trajectory.

In this natural coordinate system we can represent $\delta\vec{r}(s)$ as:

$$\delta\vec{r}(s) = (\delta\vec{r} \cdot \vec{\nu}) \cdot \vec{\nu} + (\delta\vec{r} \cdot \vec{\beta}) \cdot \vec{\beta}$$

(since the "dreibein" accompanies the particle the $\vec{\tau}$- component of $\delta\vec{r}$ is always zero by definition).

However this representation has the disadvantage that the direction of the normal vector $\vec{\nu}(s)$ changes discontinuously if the design orbit is going over from the vertical plane to the horizontal plane and vice versa. Therefore, it is advantageous to introduce new unit vectors $\vec{\tau}$, $\vec{\epsilon_x}$ and $\vec{\epsilon_z}$ which change their directions continuously. This is achieved by putting

$$\vec{\epsilon_x}(s) = \begin{cases} +\vec{\nu}(s) & \text{if the orbit lies in the horizontal plane ;} \\ -\vec{\beta}(s) & \text{if the orbit lies in the vertical plane ;} \end{cases}$$

$$\vec{\epsilon_y}(s) = \begin{cases} +\vec{\beta}(s) & \text{if the orbit lies in the horizontal plane ;} \\ +\vec{\nu}(s) & \text{if the orbit lies in the vertical plane .} \end{cases}$$

Thus, the orbit-vector $\vec{r}(s)$ can be written in the form

$$\vec{r}(x,y,s) = \vec{r}_0(s) + x(s) \cdot \vec{\epsilon}_x(s) + y(s) \cdot \vec{\epsilon}_y(s) \tag{2.7}$$

and the Fresnet formulae (2.6) now read as:

$$\frac{d}{ds}\vec{e}_x(s) = +K_x(s)\cdot\vec{\tau}(s) ; \tag{2.8a}$$

$$\frac{d}{ds}\vec{e}_y(s) = +K_y(s)\cdot\vec{\tau}(s) ; \tag{2.8b}$$

$$\frac{d}{ds}\vec{\tau}(s) = -K_x(s)\cdot\vec{e}_x(s) - K_y(s)\cdot\vec{e}_y(s) \tag{2.8c}$$

where we assume that

$$K_x(s)\cdot K_y(s) = 0 \tag{2.9}$$

and where $K_x(s), K_y(s)$ designate the curvatures in the x-direction and in the y-direction respectively.

From eqns. (2.4), (2.6) and (2.7) one then has

$$\begin{aligned}\dot{\vec{r}} &= \dot{s}\cdot\left[\frac{d\vec{r}_0}{ds} + x\cdot\frac{d\vec{e}_x}{ds} + y\cdot\frac{d\vec{e}_y}{ds}\right] + \dot{x}\cdot\vec{e}_x + \dot{z}\cdot\vec{e}_z \\ &= \vec{\tau}\cdot\dot{s}\cdot(1 + x\cdot K_x + y\cdot K_y) + \dot{x}\cdot\vec{e}_x + \dot{y}\cdot\vec{e}_y\end{aligned}$$

so that for the expressions

$$\sqrt{1-\frac{v^2}{c^2}} \quad \text{and} \quad (\dot{\vec{r}}\cdot\vec{A})$$

in eqn. (2.1) we have

$$\begin{aligned}\sqrt{1-\frac{v^2}{c^2}} &= \left\{1 - \frac{1}{c^2}\cdot\left[\dot{x}^2 + \dot{y}^2 + (1 + K_x\cdot x + K_y\cdot y)^2\cdot\dot{s}^2\right]\right\}^{1/2} ; \\ (\dot{\vec{r}}\cdot\vec{A}) &= \dot{x}\cdot A_x + \dot{y}\cdot A_y + \dot{s}\,(1 + K_x\cdot x + K_y\cdot y)\cdot A_s\end{aligned}$$

with

$$\vec{A} = A_x\cdot\vec{e}_x + A_y\cdot\vec{e}_y + A_s\cdot\vec{\tau} .$$

In the new coordinate system x, y, s, the Lagrangian in eqn. (2.1) then becomes

$$\begin{aligned}L(x,y,s,\dot{x},\dot{y},\dot{s},t) &= -m_0c^2\left\{1 - \frac{1}{c^2}\left[\dot{x}^2 + \dot{y}^2 + (1 + K_x\cdot x + K_y\cdot y)^2\cdot\dot{s}^2\right]\right\}^{1/2} \\ &\quad + e\cdot\{\dot{x}\cdot A_x + \dot{y}\cdot A_y + \dot{s}\,(1 + K_x\cdot x + K_y\cdot y)\cdot A_s\} - e\varphi\end{aligned} \tag{2.10}$$

and eqn. (2.2) leads to

$$\epsilon_x = -\frac{\partial\varphi}{\partial x} - \cdot\frac{\partial A_x}{\partial t} ; \tag{2.11a}$$

$$\epsilon_y = -\frac{\partial\varphi}{\partial y} - \cdot\frac{\partial A_y}{\partial t} ; \tag{2.11b}$$

$$\epsilon_s = -\frac{\partial\varphi}{\partial s} - \cdot\frac{\partial A_s}{\partial t} \tag{2.11c}$$

and

$$B_x = \frac{1}{h} \cdot \left\{ \frac{\partial}{\partial y}(h \cdot A_s) - \frac{\partial}{\partial s} A_y \right\} ; \qquad (2.12a)$$

$$B_y = \frac{1}{h} \cdot \left\{ \frac{\partial}{\partial s} A_x - \frac{\partial}{\partial x}(h \cdot A_s) \right\} ; \qquad (2.12b)$$

$$B_s = \frac{\partial}{\partial x} A_y - \frac{\partial}{\partial y} A_x \qquad (2.12c)$$

with

$$h = 1 + K_x \cdot x + K_y \cdot y . \qquad (2.13)$$

Finally the equations of motion

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0 ; \qquad (2.14a)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} = 0 ; \qquad (2.14b)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{s}} - \frac{\partial L}{\partial s} = 0 \qquad (2.14c)$$

take the form

$$\frac{d}{dt}(m_0\gamma \cdot \dot{x}) = m_0\gamma \cdot h \cdot \dot{s}^2 \cdot K_x + e \cdot \{\dot{y} \cdot B_s - \dot{s} \cdot h \cdot B_y\} + e \cdot E_x ; \qquad (2.15a)$$

$$\frac{d}{dt}(m_0\gamma \cdot \dot{y}) = m_0\gamma \cdot h \cdot \dot{s}^2 \cdot K_y + e \cdot \{-\dot{x} \cdot B_s + \dot{s} \cdot h \cdot B_x\} + e \cdot E_y ; \qquad (2.15b)$$

$$\frac{d}{dt}(m_0\gamma \cdot h \cdot \dot{s}) = -m_0\gamma\,(K_x \cdot \dot{x} + K_y \cdot \dot{y}) \cdot \dot{s} + e \cdot \{\dot{x} \cdot B_y - \dot{y} \cdot B_x\} + e \cdot E_s . \qquad (2.15c)$$

2.3 Specialization to Coasting Beam Betatron Motion

In eqn. (2.15) the first two equations describe the transverse motion (betatron oscillations) and the last equation describes the longitudinal motion.

Now we assume that the electric field $\vec{\epsilon}$ vanishes (cavities turned off) and that the magnetic field $\vec{B}$ is time independent. Then γ and v become constants and the field $\vec{B}$ depends only on s and x, y.

In this case it is useful to introduce the arc length s of the design orbit as independent variable. Then, using the relation

$$\frac{d}{dt} = v \cdot \frac{d}{dl} = v \cdot \frac{ds}{dl} \cdot \frac{d}{ds} = v \cdot \frac{1}{l'} \cdot \frac{d}{ds} \qquad (2.16)$$

with

$$dl = |d\vec{r}| \qquad (2.17)$$

we get from eqn. (2.15):

$$x'' - \frac{l''}{l'} \cdot x' = K_x \cdot h - (1 - \delta) \cdot \frac{e}{p_0} \cdot l' \cdot \{h \cdot B_y - y' \cdot B_s\} ; \qquad (2.18a)$$

$$y'' - \frac{l''}{l'} \cdot y' = K_y \cdot h + (1 - \delta) \cdot \frac{e}{p_0} \cdot l' \cdot \{h \cdot B_x - x' \cdot B_s\} ; \qquad (2.18b)$$

$$\begin{aligned}\frac{l''}{l'} &= \frac{1}{h}\left\{K'_x \cdot x + K'_y \cdot y + 2\left(K_x \cdot x' + K_y \cdot y'\right)\right\} - \\ &\quad \frac{1}{h} \cdot (1-\delta) \cdot \frac{\epsilon}{p_0} \cdot l' \cdot \left[x' \cdot B_y - y' \cdot B_x\right] \ , \end{aligned} \tag{2.18c}$$

where

$$p = m_0 \gamma v$$

is the momentum and where we have introduced the relative momentum deviation

$$\delta = \frac{p - p_0}{p} \equiv \frac{\Delta p}{p}$$

(p_0 is the momentum of the particle corresponding to the energy E_0).

Here the third equation (2.18c) which represents the longitudinal motion can in fact also be obtained from eqns. (2.18a,b) by multiplying eqn. (2.18a) with x' and eqn. (2.18b) with y', by adding these two equations and by taking into account the relations

$$\begin{aligned}(l')^2 &= (x')^2 + (y')^2 + (1 + K_x \cdot x + K_y \cdot y)^2 \ ; \\ l' \cdot l'' &= x'x'' + y'y'' + h \cdot \left(K_x \cdot x' + K_y \cdot y' + K'_x \cdot x + K'_y \cdot y\right)\end{aligned}$$

or

$$\begin{aligned}(x')^2 + (y')^2 &= (l')^2 - h^2 \ ; \\ x'x'' + y'y'' &= l' \cdot l'' - h \cdot \left(K_x \cdot x' + K_y \cdot y' + K'_x \cdot x + K'_y \cdot y\right) \ .\end{aligned}$$

Thus eqn. (2.18c) for the s motion is redundant and we can restrict our consideration to the betatron motion (according to eqns. (2.18a,b)) alone.

2.4 The Canonical Form of the Equations of Motion

It is interesting that eqn. (2.18c) can be used to eliminate the term (l'/l'') appearing in eqns. (2.18a,b). The equations of motion then obtained can be used in the theory of transport systems.

In order to avoid the accidental introduction of dissipative terms (see Ref. [3]) when making (inevitable) approximations it is useful to write the equations of motion in canonical form. This will ensure that the phase space density is conserved during tracking calculations.

For that purpose, we first remark that we can write the equations (2.18a,b) of betatron oscillations in Lagrangian form

$$\frac{d}{dt}\frac{\partial \hat{L}}{\partial x'} - \frac{\partial \hat{L}}{\partial x} = 0 \ ; \tag{2.19a}$$

$$\frac{d}{dt}\frac{\partial \hat{L}}{\partial y'} - \frac{\partial \hat{L}}{\partial y} = 0 \tag{2.19b}$$

with the Lagrangian

$$\hat{L}(x, y, x', y', s) = l' + (1-\delta) \cdot \frac{\epsilon}{p_0} \cdot \{x' \cdot A_x + y' \cdot A_y + h \cdot A_s\} \tag{2.20}$$

where l' is given by

$$l' = +\sqrt{(x')^2 + (y')^2 + (1 + K_x \cdot x + K_y \cdot y)^2}\ . \tag{2.21}$$

Defining now the momentum variables

$$p_x = \frac{\partial \hat{L}}{\partial x'} = \frac{x'}{l'} + (1-\delta) \cdot \frac{\epsilon}{p_0} \cdot A_x\ ; \tag{2.22}$$

$$p_y = \frac{\partial \hat{L}}{\partial y'} = \frac{y'}{l'} + (1-\delta) \cdot \frac{\epsilon}{p_0} \cdot A_y\ ; \tag{2.23}$$

we construct the Hamiltonian in the usual way :

$$\begin{aligned} \hat{H}(x, p_x, y, p_y; s) &= p_x \cdot x' + p_y \cdot y' - \hat{L} \qquad (2.24) \\ &= -\left\{1 - \left[p_x - (1-\delta) \cdot \frac{\epsilon}{p_0} A_x\right]^2 - \left[p_y - (1-\delta) \cdot \frac{\epsilon}{p_0} A_y\right]^2\right\}^{1/2} \\ &\qquad \times (1 + K_x \cdot x + K_y \cdot y) \\ &\quad - (1-\delta) \cdot \frac{\epsilon}{p_0} (1 + K_x \cdot x + K_y \cdot y) \cdot A_s\ . \end{aligned}$$

The equations of motion now take the form:

$$\begin{aligned} \frac{d}{ds}\, x &= +\frac{\partial \hat{H}}{\partial p_x}\ ; \quad \frac{d}{ds}\, p_x = -\frac{\partial \hat{H}}{\partial x}\ ; \\ \frac{d}{ds}\, y &= +\frac{\partial \hat{H}}{\partial p_y}\ ; \quad \frac{d}{ds}\, p_y = -\frac{\partial \hat{H}}{\partial y} \end{aligned}$$

or, in matrix form :

$$\frac{d}{ds}\, \vec{z} = \underline{S} \cdot \frac{\partial \hat{H}}{\partial \vec{z}} \tag{2.25a}$$

with

$$\underline{S} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{2.25b}$$

and

$$\vec{z} = \begin{pmatrix} x \\ p_x \\ z \\ p_z \end{pmatrix}\ . \tag{2.25c}$$

2.5 Description of the Magnetic Field

In order to utilize the Hamiltonian of (2.24), the magnetic field $\vec{B}$ and the corresponding vector potential,

$$\vec{A} = \vec{A}(x, y, s), \tag{2.26}$$

(eqn. (2.12)) for commonly occurring types of accelerator magnet must be given.

The (time-independent) field $\vec{B}$ obeys the Maxwell equations

$$\begin{aligned} \text{div } \vec{B} &= 0 ; & (2.27a) \\ \text{curl } \vec{B} &= 0 . & (2.27b) \end{aligned}$$

In our natural coordinate system (x,y,s) these equations read as

$$\begin{aligned} \frac{\partial B_x}{\partial y} &= \frac{\partial B_y}{\partial x} ; & (2.28a) \\ \frac{\partial B_x}{\partial s} &= \frac{\partial}{\partial x}(h \cdot B_s) ; & (2.28b) \\ \frac{\partial B_y}{\partial s} &= \frac{\partial}{\partial y}(h \cdot B_s) , & (2.28c) \end{aligned}$$

where h is given by eqn. (2.13).

Using the freedom to select a gauge, we can choose any vector potential which leads to the correct form of the fields. Suitable vector potentials are as follows and have been chosen for their simplicity.

2.5.1 Bending Magnet

If the curvatures K_x and K_y of the design orbit are given, the magnetic bending field on the design orbit, $B_x^{(0)}(s)$ and $B_y^{(0)}(s)$:

$$\begin{aligned} B_x^{(0)}(s) &= B_x(0, 0, s) ; & (2.29a) \\ B_y^{(0)}(s) &= B_y(0, 0, s) & (2.29b) \end{aligned}$$

can be easily calculated from eqn. (2.18) if we notice that the design orbit

$$x(s) = y(s) \equiv 0 \tag{2.30}$$

is a solution of the equations of motion for

$$p = p_0 \tag{2.31}$$

by definition. Thus we get:

$$\begin{aligned} \frac{\epsilon}{p_0} \cdot B_x^{(0)} &= -K_y ; & (2.32a) \\ \frac{\epsilon}{p_0} \cdot B_y^{(0)} &= +K_x . & (2.32b) \end{aligned}$$

The corresponding vector potential can be written as

$$\begin{aligned} \frac{\epsilon}{p_0} \cdot A_s &= -\frac{1}{2}(1 + K_x \cdot x + K_y \cdot y) ; & (2.33a) \\ A_x &= A_y = 0 . & (2.33b) \end{aligned}$$

2.5.2 Quadrupole

The quadrupole fields are

$$B_x = y \cdot \left(\frac{\partial B_y}{\partial x}\right)_{x=y=0} ; \quad (2.34a)$$

$$B_y = x \cdot \left(\frac{\partial B_y}{\partial x}\right)_{x=y=0} , \quad (2.34b)$$

so that we may use the vector potential

$$A_s = \left(\frac{\partial B_y}{\partial x}\right)_{x=y=0} \cdot \frac{1}{2}\left(y^2 - x^2\right) ; \quad (2.35a)$$

$$A_x = A_y = 0 . \quad (2.35b)$$

In the following we rewrite the term $(e/p_0) \cdot A_s$ in (2.24) as

$$\frac{e}{p_0} A_s = \frac{1}{2}k \cdot (y^2 - x^2) ; \quad (2.36a)$$

$$k = \frac{e}{p_0} \cdot \left(\frac{\partial B_y}{\partial x}\right)_{x=y=0} . \quad (2.36b)$$

2.5.3 Skew Quadrupole

The fields are

$$B_x = -\frac{1}{2} \cdot \left(\frac{\partial B_y}{\partial y} - \frac{\partial B_x}{\partial x}\right)_{x=y=0} \cdot x ; \quad (2.37a)$$

$$B_y = +\frac{1}{2} \cdot \left(\frac{\partial B_y}{\partial y} - \frac{\partial B_x}{\partial x}\right)_{x=y=0} \cdot y . \quad (2.37b)$$

Thus we may use

$$A_s = -\frac{1}{2}\left(\frac{\partial B_y}{\partial y} - \frac{\partial B_x}{\partial x}\right)_{x=y=0} \cdot xy ; \quad (2.38a)$$

$$A_x = A_y = 0 , \quad (2.38b)$$

and we write

$$\frac{e}{p_0} A_s = -N \cdot xy ; \quad (2.39a)$$

$$N = \frac{1}{2} \cdot \frac{e}{p_0} \left(\frac{\partial B_y}{\partial y} - \frac{\partial B_x}{\partial x}\right)_{x=y=0} \cdot xy . \quad (2.39b)$$

2.5.4 Sextupole

$$B_x = xy \cdot \left(\frac{\partial^2 B_y}{\partial x^2}\right)_{x=y=0} ; \quad (2.40a)$$

$$B_y = \frac{1}{2}\left(x^2 - y^2\right) \cdot \left(\frac{\partial^2 B_y}{\partial x^2}\right)_{x=y=0} . \quad (2.40b)$$

so that

$$\frac{e}{p_0} A_s = -\lambda \cdot \frac{1}{6} \left(x^3 - 3xy^2 \right) \; ; \; A_x = A_y = 0 \tag{2.41a}$$

with

$$\lambda = \frac{e}{p_0} \left(\frac{\partial^2 B_y}{\partial x^2} \right)_{x=y=0} . \tag{2.41b}$$

2.5.5 Solenoid Fields

The field components in the current free region are given by[11]:

$$B_x(x, y, s) = x \cdot \sum_{\nu=0}^{\infty} b_{2\nu+1} \cdot (x^2 + y^2)^{\nu} \; ; \tag{2.42a}$$

$$B_y(x, y, s) = z \cdot \sum_{\nu=0}^{\infty} b_{2\nu+1} \cdot (x^2 + y^2)^{\nu} \; ; \tag{2.42b}$$

$$B_s(x, y, s) = \sum_{\nu=0}^{\infty} b_{2\nu} \cdot (x^2 + y^2)^{\nu} \tag{2.42c}$$

where for consistency with Maxwell's equations the coefficients b_{μ} obey the recursion equations:

$$b_{2\nu+1}(s) = -\frac{1}{(2\nu+2)} \cdot b'_{2\nu}(s) \; ; \tag{2.43a}$$

$$b_{2\nu+2}(s) = +\frac{1}{(2\nu+2)} \cdot b'_{2\nu+1}(s) \; ; \tag{2.43b}$$

$$(\nu = 0, 1, 2, \ldots)$$

and where

$$b_0(s) \equiv B_s(0, 0, s) \; . \tag{2.44}$$

The vector potential leading to the solenoid field of eqn. (2.25) is then:

$$A_x(x, y, s) = -y \cdot \sum_{(\nu=0}^{\infty} \frac{1}{2\nu+2)} \cdot b_{(2\nu)}(s) \cdot r^{2\nu} \; ; \tag{2.45a}$$

$$A_y(x, y, s) = +x \cdot \sum_{\nu=0}^{\infty} \frac{1}{(2\nu+2)} \cdot b_{(2\nu)}(s) \cdot r^{2\nu} \; ; \tag{2.45b}$$

$$A_s(x, y, s) = 0 \; . \tag{2.45c}$$

Thus we can write :

$$\frac{e}{p_0} A_x = -\frac{1}{2} R_0(s) \cdot y - \frac{1}{16} R_0''(s) \cdot (x^2 + y^2) \cdot y + \ldots \; ; \tag{2.46a}$$

$$\frac{e}{p_0} A_y = -\frac{1}{2} R_0(s) \cdot x - \frac{1}{16} R_0''(s) \cdot (x^2 + y^2) \cdot x - \ldots \tag{2.46b}$$

with

$$R(s) = \frac{e}{p_0} \cdot b_0(s) \tag{2.47}$$

$$\equiv \frac{e}{p_0} \cdot B_s(0, 0, s) \; .$$

2.6 Series Expansion of the Hamiltonian

The eqns. (2.33), (2.36), (2.39), (2.41) and (2.46) can now be combined as

$$\frac{\epsilon}{p_0}A_s = -\frac{1}{2}(1 + K_x \cdot x + K_y \cdot y) + \frac{1}{2}k \cdot (y^2 - x^2) - N \cdot xy - \frac{1}{6}\lambda \cdot (x^3 - 3xy^2) . \quad (2.48)$$

Together with eqns.(2.46a,b) all the components of the vector potential $\vec{A}$ appearing in the Hamiltonian (2.24) are now known.

Furthermore, since

$$\begin{aligned} |p_x - (1-\delta) \cdot \frac{e}{p_0}A_x| &= |\frac{x'}{l'}| \ll 1 ; \\ |p_y - (1-\delta) \cdot \frac{e}{p_0}A_y| &= |\frac{y'}{l'}| \ll 1 \end{aligned}$$

the square root

$$\left\{1 - \left[p_x - (1-\delta) \cdot \frac{\epsilon}{p_0}A_x\right]^2 - \left[p_y - (1-\delta) \cdot \frac{e}{p_0}A_y\right]^2\right\}^{1/2}$$

in (2.24) can be expanded in a series :

$$\begin{aligned} &\left\{1 - \left[p_x - (1-\delta) \cdot \frac{\epsilon}{p_0}A_x\right]^2 - \left[p_y - (1-\delta) \cdot \frac{\epsilon}{p_0}A_y\right]^2\right\}^{1/2} = \\ &1 - \frac{1}{2} \cdot \left[p_x - (1-\delta) \cdot \frac{\epsilon}{p_0}A_x\right]^2 - \frac{1}{2} \cdot \left[p_y - (1-\delta) \cdot \frac{\epsilon}{p_0}A_y\right]^2 + ... \end{aligned}$$

so that in practice the particle motion can be conveniently calculated to varies orders of approximation.

In the following we shall use a series expansion of the Hamiltonian up to third order in the variables x, p_x, y, p_y and δ . Then we obtain, using eqns. (2.46) and (2.48) :

$$\hat{H} = -\delta \cdot (K_x \cdot x + K_y \cdot y) + H_0 + H_{11} + H_{12} \quad (2.49)$$

with

$$\begin{aligned} H_0 &= \frac{1}{2}\left[p_x + \frac{1}{2}R \cdot y\right]^2 + \frac{1}{2}\left[p_y - \frac{1}{2}R \cdot x\right]^2 & (2.50a) \\ &+ \frac{1}{2}\left(K_x^2 + k\right) \cdot x^2 + \frac{1}{2}\left(K_y^2 - k\right) \cdot y^2 + N \cdot xy ; \\ \frac{1}{\delta} \cdot H_{11} &= -\frac{1}{2}\left(K_x^2 + k\right) \cdot x^2 - \frac{1}{2}\left(K_y^2 - k\right) \cdot y^2 - N \cdot xy & (2.50b) \\ &- \frac{1}{2}R \cdot y\left[p_x + \frac{1}{2}R \cdot y\right] + \frac{1}{2}R \cdot x\left[p_y - \frac{1}{2}R \cdot x\right] ; \\ H_{12} &= \frac{1}{2}(K_x \cdot x + K_y \cdot y) \cdot \left[p_x^2 + p_y^2\right] + \frac{1}{6}\lambda \cdot \left(x^3 - 3xy^2\right) & (2.50c) \end{aligned}$$

(a constant term, $-(1-\delta)$, in the Hamiltonian, which has no influence on the motion has been dropped).

2.7 Introduction of Dispersion

The presence of the linear term

$$-\delta \cdot (K_x \cdot x + K_y \cdot y)$$

means that the design orbit

$$x = y \equiv 0$$

is no longer a solution of the equations of motion (2.25).

In order to eliminate this term we now write

$$\vec{z} = \vec{z_0} + \tilde{\vec{z}} \tag{2.51}$$

by introducing a new reference orbit $\vec{z_0}$ which is a periodic solution of the equations of motion (2.25) :

$$\frac{d}{ds}\,\vec{z_0} = \underline{S} \cdot \frac{\partial}{\partial \vec{z_0}}\,\hat{H}(\vec{z_0}, s)\ ; \tag{2.52a}$$

$$\vec{z_0}(s + L) = \vec{z_0}(s)\ . \tag{2.52b}$$

If we take into account only the first two terms on the right hand side of (2.49) (neglecting third order terms in x, p_x, y, p_y and δ) we can write

$$\vec{z_0} = \delta \cdot \vec{D} \tag{2.53a}$$

with

$$\vec{D} = \begin{pmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \end{pmatrix} , \tag{2.53b}$$

where the "dispersion-vector" $\vec{D}$ approximately satisfies the differential equations:

$$D_1' = D_2 + \frac{1}{2}R \cdot D_3\ ; \tag{2.54a}$$

$$D_2' = +\frac{1}{2}R \cdot \left[D_4 - \frac{1}{2}R \cdot D_1\right] - \left(K_x^2 + k\right) \cdot D_1 - N \cdot D_3 + K_x\ ; \tag{2.54b}$$

$$D_3' = D_4 - \frac{1}{2}R \cdot D_1\ ; \tag{2.54c}$$

$$D_4' = -\frac{1}{2}R \cdot \left[D_2 + \frac{1}{2}R \cdot D_3\right] - \left(K_y^2 - k\right) \cdot D_3 - N \cdot D_1 + K_y\ . \tag{2.54d}$$

2.8 The Free Betatron Oscillations Around the Dispersion Orbit

In eqn. (2.51) we have decomposed the whole orbit vector $\vec{z}$ into two components, where by construction, the first component $\vec{z_0}$ designates the "dispersion-orbit" so that the second component $\tilde{\vec{z}}$ describes the free betatron oscillations around this new reference orbit.

The differential equations for $\tilde{\vec{z}}$:

$$\begin{aligned}
\frac{d}{ds}\,\tilde{\vec{z}} &= \frac{d}{ds}\,\vec{z} - \frac{d}{ds}\,\vec{z}_0 \\
&= \underline{S}\cdot\frac{\partial}{\partial\vec{z}}\,\hat{H}(\vec{z},s) - \underline{S}\cdot\frac{\partial}{\partial\vec{z}_0}\hat{H}(\vec{z}_0,s) \\
&= \underline{S}\cdot\frac{\partial}{\partial\tilde{\vec{z}}}\,\hat{H}(\vec{z}_0+\tilde{\vec{z}},s) - \underline{S}\cdot\frac{\partial}{\partial\vec{z}_0}\,\hat{H}(\vec{z}^0,s)
\end{aligned}$$

can again be written in canonical form :

$$\frac{d}{ds}\,\tilde{\vec{z}} = \underline{S}\cdot\frac{\partial}{\partial\tilde{\vec{z}}}\,H(\tilde{\vec{z}},s) \tag{2.55a}$$

with the Hamiltonian H

$$H(\tilde{\vec{z}},s) = \hat{H}(\vec{z}_0+\tilde{\vec{z}},s) - \hat{H}(\vec{z}_0,s) - \tilde{\vec{z}}\cdot\frac{\partial}{\partial\vec{z}_0}\,\hat{H}(\vec{z}_0,s)\ . \tag{2.55b}$$

Taking into account only second order terms in x, p_x, y, p_y so that the theory is linear, the Hamiltonian H becomes :

$$H(\tilde{\vec{z}},s) = H_0(\tilde{\vec{z}},s) + H_1(\tilde{\vec{z}},s) \tag{2.56}$$

where H_0 is given by eqn. (2.50a) and where for H_1 one gets:

$$\begin{aligned}
\frac{1}{\delta}\cdot H_1(\tilde{\vec{z}},s) &= -\frac{1}{2}\left(K_x^2+k\right)\cdot\tilde{x}^2 - \frac{1}{2}\left(K_y^2-k\right)\cdot\tilde{y}^2 - N\cdot\tilde{x}\tilde{y} \\
&\quad -\frac{1}{2}R\cdot\tilde{y}\left[\tilde{p}_x+\frac{1}{2}R\cdot\tilde{y}\right] + \frac{1}{2}R\cdot\tilde{x}\left[\tilde{p}_y-\frac{1}{2}R\cdot\tilde{x}\right] \\
&\quad +\frac{1}{2}\left(K_x\cdot D_1 + K_y\cdot D_3\right)\cdot\left[\tilde{p}_x^2+\tilde{p}_y^2\right] \\
&\quad +\left(K_x\cdot\tilde{x}+K_y\cdot\tilde{y}\right)\cdot\left[D_2\cdot\tilde{p}_x + D_4\cdot\tilde{p}_y\right] \\
&\quad +\frac{\lambda}{2}\cdot\left[D_1\cdot\tilde{x}^2 - D_1\cdot\tilde{y}^2 - 2\tilde{x}\tilde{y}\cdot D_3\right]\ .
\end{aligned} \tag{2.57}$$

Putting (2.56) into (2.55a) and using (2.50a) and (2.57) one has:

$$\frac{d}{ds}\,\tilde{\vec{z}} = \underline{A}(s)\cdot\tilde{\vec{z}} \tag{2.58a}$$

$$\underline{A}(s) = \underline{A}_0(s) + \delta\cdot\underline{B}(s) \tag{2.58b}$$

with

$$\underline{A}_0 = ((A_{ik}^{(0)}))\ :$$

$$\begin{aligned}
A_{12}^{(0)} &= 1\ ; \\
A_{13}^{(0)} &= +\frac{1}{2}\cdot R\ ; \\
A_{21}^{(0)} &= -\frac{1}{4}\cdot R^2 - \left(K_x^2+k\right)\ ; \\
A_{23}^{(0)} &= -N\ ;
\end{aligned}$$

$$
\begin{aligned}
A_{24}^{(0)} &= +\frac{1}{2}R \ ; \\
A_{31}^{(0)} &= -\frac{1}{2}\cdot R \ ; \\
A_{34}^{(0)} &= 1 \ ; \\
A_{41}^{(0)} &= -N \ ; \\
A_{42}^{(0)} &= -\frac{1}{2}\cdot R \ ; \\
A_{43}^{(0)} &= -\frac{1}{4}\cdot R^2 - \left(K_y^2 - k\right) \ ; \\
A_{ik}^{(0)} &= 0 \quad \text{otherwise}
\end{aligned}
\tag{2.59}
$$

and

$$
\underline{B} = ((\delta B_{ik})) \ ;
$$

$$
\begin{aligned}
B_{11} &= K_x \cdot D_2 \ ; \\
B_{12} &= K_x \cdot D_1 + K_y \cdot D_3 \ ; \\
B_{13} &= -\frac{1}{2}\cdot R + K_y \cdot D_2 \ ; \\
B_{21} &= \left(K_x^2 + k\right) + \frac{1}{2}\cdot R^2 - \lambda \cdot D_1 \ ; \\
B_{22} &= -K_x \cdot D_2 \ ; \\
B_{23} &= +N + \lambda \cdot D_3 \ ; \\
B_{24} &= -\frac{1}{2}\cdot R - K_x \cdot D_4 \ ; \\
B_{31} &= +\frac{1}{2}\cdot R + K_x \cdot D_4 \ ; \\
B_{33} &= +K_y \cdot D_4 \ ; \\
B_{34} &= K_x \cdot D_1 + K_y \cdot D_3 \ ; \\
B_{41} &= +N + \lambda \cdot D_3 \ ; \\
B_{42} &= +\frac{1}{2}\cdot R - K_y \cdot D_2 \ ; \\
B_{43} &= \left(K_y^2 - k\right) + \frac{1}{2}\cdot R^2 + \lambda \cdot D_1 \ ; \\
B_{44} &= -K_y \cdot D_4 \ ; \\
B_{ik} &= 0 \quad \text{otherwise} \ .
\end{aligned}
\tag{2.60}
$$

Remark:
From eqn. (2.58) we get:

$$
\begin{aligned}
\tilde{x}' &= \tilde{p}_x + \frac{1}{2}R\cdot\tilde{y} + \\
&\quad \delta\cdot\left[K_x D_2 \cdot \tilde{x} - (K_x D_1 + K_y D_3)\cdot\tilde{p}_x - \left(-\frac{1}{2}R + K_y D_2\right)\cdot\tilde{y}\right] \ ;
\end{aligned}
\tag{2.61a}
$$

$$\tilde{y}' = \tilde{p}_y - \frac{1}{2}R\cdot\tilde{x} + \delta\cdot\left[K_yD_4\cdot\tilde{y} + (K_xD_1 + K_yD_3)\cdot\tilde{p}_y + \left(+\frac{1}{2}R + K_xD_4\right)\cdot\tilde{x}\right] . \tag{2.61b}$$

Therefore, if $\tilde{p}_x, \tilde{p}_y$ are known, x' and z' can be calculated and vice versa..

The variation of the magnetic fields with s is in good approximation represented by a "box"-like shape, i.e. we assume that the fields have sharp edges.

Since $\tilde{p}_x(s)$ and $\tilde{p}_y(s)$ are continuous functions of s, it follows then from eqns. (2.61a,b) that x' and z' make a jump at the ends of a solenoid and a bending magnet (the step due to bends vanishes for on-energy particles with $\delta = 0$).

Note, that the terms $K_x'\cdot x$ and $K_y'\cdot y$ appearing in the Lagrange equations (2.18) and which are discontinous at the ends of a bending magnet do not appear in (2.58) and have been absorbed in p_x and p_y. The same happens with the terms $R_0'\cdot x$ and $R_0'\cdot y$ which arise from eqn. (2.42) and which are discontinous at the ends of "sharp edged" solenoid fields.

2.9 Definition of the Transfer Matrix

Because the equations of motion (2.58) are linear, the solution can be written in the form:

$$\vec{\tilde{z}}(s) = \underline{M}(s, s_0)\ \vec{\tilde{z}}(s_0) \tag{2.62}$$

which defines the transfer matrix $\underline{M}(s, s_0)$.

With respect to (2.58), $\underline{M}(s, s_0)$ is determined by the differential equations

$$\frac{d}{ds}\underline{M}(s, s_0) = \underline{A}(s)\cdot\underline{M}(s, s_0) ; \tag{2.63a}$$

$$\underline{M}(s_0, s_0) = \underline{1} . \tag{2.63b}$$

Since the variables $\tilde{x}, \tilde{p}_x, \tilde{y}, \tilde{p}_y$ are canonical, the transfer matrix is symplectic:

$$\underline{M}^T(s, s_0)\cdot\underline{S}\cdot\underline{M}(s, s_0) = \underline{S} \tag{2.64}$$

as is shown in Appendix A.

An approximate way to solve eqns. (2.63) is described in Appendix B (thin lens approximation) .

The symplecticity condition (2.64) ensures that the transfer matrix , $\underline{M}(s, s_0)$, contains complete information about the stability of the betatron motion. We shall discuss this in chapter 3.3 .

Finally, we mention for later considerations that the coefficient matrix $\underline{A}(s)$ of eqn. (2.58) satisfies the condition

$$\underline{A}^T(s)\cdot\underline{S} + \underline{S}\cdot\underline{A}(s) = \underline{0} . \tag{2.65}$$

This can be obtained by differentiating eqn (2.64) with respect to s and using eqn. (2.63b).

Eqns. (2.58) and (2.63) together with (2.59) and (2.60) are now the defining equations of coupled betatron oscillations around the dispersion orbit and they will serve as the starting point for the developments to follow.

3 On-Momentum Betatron Oscillations

The equations of motion (2.58) or (2.63) are valid for arbitrary momentum deviations. In the following we investigate in more detail only the case of on-momentum betatron oscillations ($\delta \equiv 0$). The influence of momentum deviations shall then be discussed in chapt. 4.

3.1 The Equations of Motion for On-Momentum Particles

The equations of motion for on-momentum particles read as (eqn.(2.58) with $\delta \equiv 0$):

$$\frac{d}{ds}\,\tilde{\vec{z}} = \underline{A}_0(s) \cdot \tilde{\vec{z}} \qquad (3.1)$$

(from now on we write $\vec{z}$ and x, p_x, z, p_z instead of $\tilde{\vec{z}}$ and $\tilde{x}, \tilde{p_x}, \tilde{z}, \tilde{p_z}$) or , written in components :

$$\begin{aligned}
x' &= p_x + \frac{1}{2} R \cdot y \,; && (3.2a)\\
p_x' &= -\left[\left(K_x^2 + k\right) + \frac{1}{4}R^2\right] \cdot x - N \cdot y + \frac{1}{2} R \cdot p_y \,; && (3.2b)\\
y' &= p_y - \frac{1}{2} R \cdot x \,; && (3.2c)\\
p_y' &= -N \cdot x - \frac{1}{2} R \cdot p_x - \left[\left(K_y^2 - k\right) + \frac{1}{4}R^2\right] \cdot y \,. && (3.2d)
\end{aligned}$$

In detail, one has:

a) $k \neq 0$; $N = R = K_x = K_y = 0$: quadrupole;
b) $N \neq 0$; $k = R = K_x = K_y = 0$: skew quadrupole;
c) $K_x^2 + K_y^2 \neq 0$; $k = N = R = 0$: bending magnet;
d) $R \neq 0$; $k = N = K_x = K_y = 0$: solenoid.

Eliminating p_x and p_y we obtain from eqn.(3.2):

$$\begin{aligned}
x'' + \left[K_x^2 + k\right] \cdot x + \left(N - \frac{1}{2} R'\right) \cdot y - R \cdot y' &= 0 \,; && (3.3a)\\
y'' + \left[K_y^2 - k\right] \cdot y + \left(N + \frac{1}{2} R'\right) \cdot x + R \cdot x' &= 0 \,. && (3.3b)
\end{aligned}$$

The connection between p_x, p_y and x', y' is given by eqn. (3.2a,c).

3.2 Coupled Betatron Motion

3.2.1 The Sources of Coupling in Circular Accelerators

There are two kinds of magnetic fields which produce a coupling between the two transverse oscillation modes: the field of a solenoid magnet and the field of a rotated quadrupole magnet.

Usually, coupling is an undesired effect in high energy accelerators. Firstly, it tends to make the behaviour of the beam more difficult to understand and the operation of the machine becomes more complicated. In electron accelerators where, because of radiation

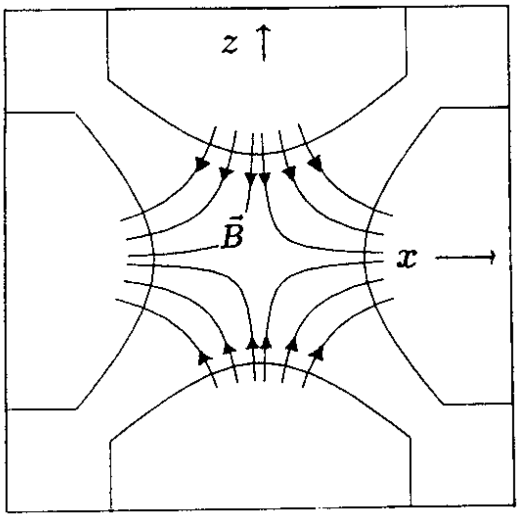

Figure 1: Cross Section of a Skew Quadrupole

damping, the beams are flat, coupling causes the beam to become rounder. For these reasons coupling generating elements are avoided in the design of accelerators.

Unfortunately they are not completely unavoidable. There are sources of coupling which are installed deliberately in an accelerator: The most important example is the detector field of a colliding beam detector which is usually solenoidal. Sometimes a round electron beam is more desirable than the naturally flat one. Then skew quadrupole - quadrupole magnets which are rotated by 45 degrees about the beam axis - may be installed. In nonplanar machines the occurrance of rotated dipole magnets can introduce a coupling too. Coupling is also introduced by the weak focussing magnets which bend at the same time in horizontal and vertical direction. There is also the possibility that the periodic solution of the beam coordinate system in a nonplanar machine differs from the plane to which the quadrupole magnets are aligned.

Besides such systematic sources of coupling, coupling occurs in every real machine as the result of distortions and imperfections. The dominating effect comes from small tilts of the quadrupole magnets due to alignment tolerances, longtime drifts of the supports or thermal effects. A vertical deviation of the closed orbit in a sextupole field also gives rise to a skew quadrupole component and causes coupling. There are numerous smaller effects, for example a vertical angle between the longitudinal axis of a dipole magnet and the beam axis.

Systematically and accidently generated coupling require different treatment by the model describing the beam behaviour. Strong systematic coupling fields can only be treated by the exact formalism developed in the following sections. Nonsystematic sources of coupling can be much better taken into account by perturbation theory which will be demonstrated in chapter 6.

3.2.2 Skew Quadrupole

The first beam transport element which causes coupling which we will consider is the skew quadrupole. This is a normal quadrupole rotated about its longitudinal axis by 45 degrees. Fig. 1 shows the cross section of a skew quadrupole. The magnetic field lines are hyperbolas with the x and y-axis as asymptotes. The magnetic field vector $\vec{B}$ is given by eqn. (2.37).

With respect to (3.3) the equations of motion for a skew quadrupole read as

$$x'' + N \cdot y = 0 ;$$

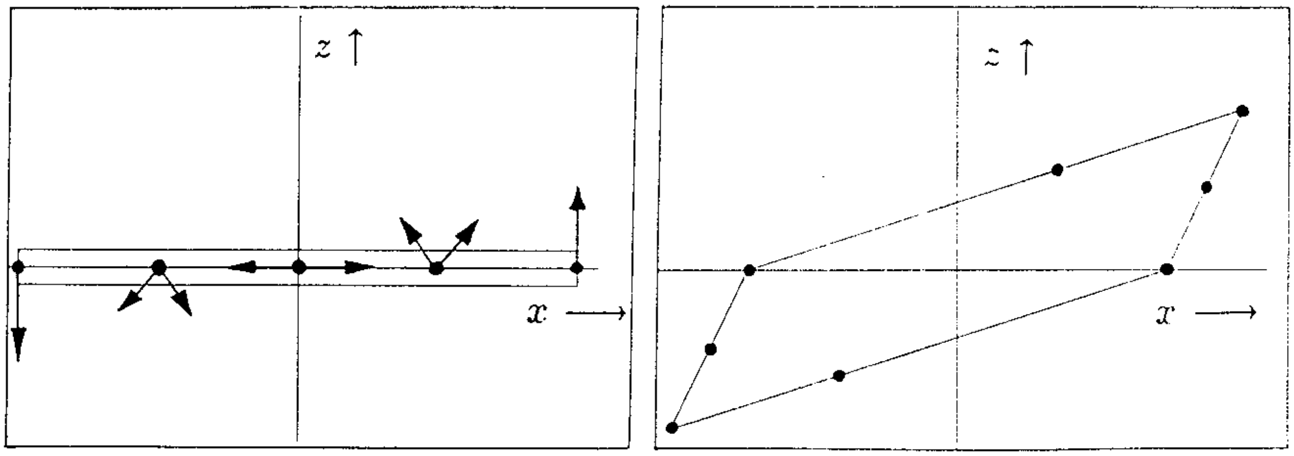

Figure 2: Effect of a skew quadrupole on a flat beam. The arrows indicate the slope of the particles behind the skew quadrupole (left). The beam is observed after some downstream drift behind the skew quadrupole. It is transformed into a parallelogram (right).

$$y'' + N \cdot x \;=\; 0 \;.$$

They are easily integrated if one writes :

$$\begin{aligned}(x+y)'' + N \cdot (x+y) &= 0 \;;\\ (x-y)'' - N \cdot (x-y) &= 0 \;.\end{aligned}$$

The solution expressed by a transport matrix M is given by :

$$\begin{pmatrix} \frac{\cos\sqrt{N}s+\cosh\sqrt{N}s}{2} & \frac{\sin\sqrt{N}s-\sinh\sqrt{N}s}{2\sqrt{N}} & \frac{\cos\sqrt{N}s-\cosh\sqrt{N}s}{2} & \frac{\sin\sqrt{N}s-\sinh\sqrt{N}s}{2\sqrt{N}} \\ -\frac{\sqrt{N}(\sin\sqrt{N}s-\sinh\sqrt{N}s)}{2} & \frac{\cos\sqrt{N}s+\cosh\sqrt{N}s}{2} & -\frac{\sqrt{N}(\sin\sqrt{N}s+\sinh\sqrt{N}s)}{2} & \frac{\cos\sqrt{N}s-\cosh\sqrt{N}s}{2} \\ \frac{\cos\sqrt{N}s-\cosh\sqrt{N}s}{2} & \frac{\sin\sqrt{N}s-\sinh\sqrt{N}s}{2\sqrt{N}} & \frac{\cos\sqrt{N}s+\cosh\sqrt{N}s}{2} & \frac{\sin\sqrt{N}s-\sinh\sqrt{N}s}{2\sqrt{N}} \\ -\frac{\sqrt{N}(\sin\sqrt{N}s+\sinh\sqrt{N}s)}{2} & \frac{\cos\sqrt{N}s-\cosh\sqrt{N}s}{2} & -\frac{\sqrt{N}(\sin\sqrt{N}s-\sinh\sqrt{N}s)}{2} & \frac{\cos\sqrt{N}s-\cosh\sqrt{N}s}{2} \end{pmatrix} \quad (3.4)$$

Fig. 3 shows schematically the effect of a skew quadrupole on a flat beam. The particles at the right and left side of the beam get kicked upwards and downwards respectively. The particles in the center of the beam don't get kicked but they have a horizontal slope and are moving away from the centre. Particles at intermediate positions in the beam leave the skew quadrupole with a horizontal and vertical slope. After some drift space the flat beam is transformed into a parallelogram in the $x-y$-plane.

3.2.3 Solenoid Fields

The other coupling element is the solenoid. This is a somewhat special beam transport device. That is because the endfields have a strong transverse component perpendicular to the major beam direction while the central field is parallel to the beam. Therefore the endfields in a solenoid are an essential contribution to the total effect of this element.

The linearized equation of motion read as (see eqn. (3.3)) :

$$x'' - R \cdot y' - \frac{1}{2}R' \cdot y = 0 \; ; \tag{3.5a}$$

$$y'' + R \cdot x' + \frac{1}{2}R' \cdot x = 0 \; , \tag{3.5b}$$

where R is given by eqn. (2.47) .

Solution is conveniently obtained by introducing complex variables ξ and η [4] :

$$\xi \;=\; x + i \cdot y \; ; \tag{3.6a}$$

$$\eta \;=\; p_x + i \cdot p_y \tag{3.6b}$$

in which we get from (3.5) and (3.2a,c) :

$$\xi'' + iR \cdot \xi' + \frac{i}{2}R' \cdot \xi = 0 \; ; \tag{3.7a}$$

$$\eta = \xi' + \frac{i}{2} \cdot \xi \; . \tag{3.7b}$$

Transforming into a rotating system

$$\bar{\xi} \equiv \bar{x} + i \cdot \bar{y} \;=\; \xi \cdot e^{i\theta(s)} \tag{3.8a}$$

$$\bar{\eta} \equiv \bar{p}_x + i \cdot \bar{p}_y \;=\; \eta \cdot e^{i\theta(s)} \tag{3.8b}$$

we obtain:

$$\bar{\xi}'' - \theta'^2 \cdot \bar{\xi} + R \cdot \theta' \cdot \bar{\xi} + i \cdot (R - 2\theta') \cdot \bar{\xi}' + i \cdot (\frac{1}{2}R' - \theta'') \cdot \bar{\xi} = 0 \; ; \tag{3.9a}$$

$$\bar{\eta} = \bar{\xi}' - i \cdot [\theta'(s) - \frac{1}{2}R] \cdot \bar{\xi} \; . \tag{3.9b}$$

We see that that system is decoupled in x and y by choosing

$$\theta(s) = \frac{1}{2} \cdot \int_0^s ds' R(s') \tag{3.10}$$

which yields

$$\bar{\xi}'' + \frac{1}{4}R^2 \cdot \bar{\xi} \;=\; 0 \; ; \tag{3.11a}$$

$$\bar{\eta} \;=\; \bar{\xi}' \tag{3.11b}$$

or

$$\bar{x}'' + \frac{1}{4}R^2 \cdot \bar{x} \;=\; 0 \; ; \quad \bar{p}_x = \bar{x}' \; ; \tag{3.12}$$

$$\bar{y}'' + \frac{1}{4}R^2 \cdot \bar{y} \;=\; 0 \; ; \quad \bar{p}_y = \bar{y}' \; . \tag{3.13}$$

So in this coordinate system a solenoid looks just like a focussing quadrupole for each plane. For an hard edge solenoid with $\theta = \frac{1}{2}R \cdot l$ (l being the solenoid length) the solution can be expressed using the following transport matrix $\underline{M}$:

$$\begin{pmatrix} \bar{x} \\ \bar{p}_x \\ \bar{y} \\ \bar{p}_y \end{pmatrix} = \begin{pmatrix} \cos\theta & \frac{2}{R}\sin\theta & 0 & 0 \\ \frac{-R}{2}\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \frac{2}{R}\sin\theta \\ 0 & 0 & \frac{-R}{2}\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \bar{x}_0 \\ \bar{p}_{x0} \\ \bar{y}_0 \\ \bar{p}_{y0} \end{pmatrix} \; . \tag{3.14}$$

Now from (3.8) we have

$$\begin{pmatrix} \bar{x}_0 \\ \bar{p}_{x0} \\ \bar{y}_0 \\ \bar{p}_{y0} \end{pmatrix} = \begin{pmatrix} x_0 \\ p_{x0} \\ y_0 \\ p_{y0} \end{pmatrix} \tag{3.15}$$

and

$$\begin{pmatrix} \bar{x} \\ \bar{p}_x \\ \bar{y} \\ \bar{p}_y \end{pmatrix} = \underline{T} \cdot \begin{pmatrix} x \\ p_x \\ y \\ p_y \end{pmatrix} \tag{3.16}$$

with the rotation matrix

$$\underline{T} = \begin{pmatrix} \cos\theta & 0 & -\sin\theta & 0 \\ 0 & \cos\theta & 0 & -\sin\theta \\ +\sin\theta & 0 & \cos\theta & 0 \\ 0 & +\sin\theta & 0 & \cos\theta \end{pmatrix} . \tag{3.17}$$

Therefore the transport matrix for the coordinates in the machine coordinate system is obtained by multiplying eqn. (3.14) with the matrix $\underline{T}^{-1}$ from the the left side :

$$\underline{M} = \underline{T}^{-1} \cdot \bar{\underline{M}} . \tag{3.18}$$

Here the first matrix $\underline{T}^{-1}$ describes a rotation in the $x - y$-plane and the second one, $\bar{\underline{M}}$ describes the focussing in the rotating coordinate system.

Thus the transport matrix of a solenoid is given by

$$\underline{M} = \begin{pmatrix} \cos^2\theta & \frac{2}{R}\sin\theta\cos\theta & \sin\theta\cos\theta & \frac{2}{R}\sin^2\theta \\ -\frac{R}{2}\sin\theta\cos\theta & \cos^2\theta & -\frac{R}{2}\sin^2\theta & \sin\theta\cos\theta \\ -\sin\theta\cos\theta & -\frac{2}{R}\sin^2\theta & \cos^2\theta & \frac{2}{R}\sin\theta\cos\theta \\ \frac{R}{2}\sin^2\theta & -\sin\theta\cos\theta & -\frac{R}{2}\sin\theta\cos\theta & \cos^2\theta \end{pmatrix} . \tag{3.19}$$

3.2.4 Description of the Motion in Terms of the Transfer Matrix

As in the uncoupled case, we can combine beam transport matrices by multiplying them and we obtain the transport matrix for the whole lattice or the revolution matrix $\underline{M}$ in this way:

$$\underline{M} = \prod_{i=1}^{n} \underline{M}_i .$$

At this point we might say that we have solved our problem. We are able to calculate particle trajectories in a lattice with coupling elements. But single trajectories don't tell us in an obvious way about the optical properties of a lattice because they depend both on initial conditions and the lattice properties. The transport matrices which are free from initial conditions don't give us a clear picture either. But in the uncoupled case you have seen how a beta or envelope function gives us an immediate overview of the beam optics properties of the whole lattice. It is therefore desirable to have an extension of such lattice functions to the coupled case.

3.3 Eigenvectors for the Particle Motion; Floquet-Theorem.

In order to obtain more information about the focussing properties of a lens system with coupling elements we first investigate the eigenmotion of the particles.

To begin, we note that from the symplecticity condition of eqn. (2.64) and with the aid of arbitrary solution vectors $\vec{z_1}$ and $\vec{z_2}$ of eqn. (3.1) one can construct a constant of motion for the betatron oscillation, the so called Lagrange invariant

$$W[\vec{z_1}(s), \vec{z_2}(s)] = \vec{z_2}^T(s) \cdot \underline{S} \cdot \vec{z_1}(s) \ .$$

Indeed :

$$\begin{aligned} W[\vec{z_1}(s),\ \vec{z_2}(s)] &= [\underline{M}(s,s_0)\ \vec{z_2}(s_0)]^T \cdot \underline{S} \cdot [\underline{M}(s,s_0)\ \vec{z_1}(s_0)] \\ &= \vec{z_2}^T(s_0) \cdot \underline{M}^T(s,s_0)\ \underline{S}\ \underline{M}(s,s_0) \cdot \vec{z_1}(s_0) \\ &= \vec{z_2}^T(s_0) \cdot \underline{S}\ \vec{z_1}(s_0) \\ &= W[\vec{z_1}(s_0),\ \vec{z_2}(s_0)] \\ &= \text{const.} \end{aligned} \tag{3.20}$$

With the help of this invariant we are in a position to study the eigenvalue spectrum of the one turn transfer matrix $\underline{M}(s_0+L,s_0)$:

$$\begin{aligned} \underline{M}(s_0+L,s_0)\ \vec{v}_\mu(s_0) &= \lambda_\mu \cdot \vec{v}_\mu(s_0) \ ; \\ (\mu &= 1,2,3,4) \ . \end{aligned} \tag{3.21}$$

The spectrum of eigenvalues λ_μ $(\mu = 1,2,3,4)$ will then allow us to study the stability of the betatron motion.

We carry out this investigation in several steps[5]:

1)We construct the Lagrange invariant with arbitrary eigenvectors $\vec{v}_\mu(s_0)$ and $\vec{v}_\nu(s_0)$ of $\underline{M} \equiv \underline{M}(s_0+L,s_0)$. This gives :

$$\begin{aligned} W[\vec{v_\mu}(s_0),\ \vec{v_\nu}(s_0)] &= W[\underline{M}\vec{v_\mu}(s_0),\ \underline{M}\vec{v_\nu}(s_0)] \\ &= W[\lambda_\mu \cdot \vec{v_\mu}(s_0),\ \lambda_\nu \cdot \vec{v_\nu}(s_0)] \\ &= \lambda_\mu\lambda_\nu \cdot W[\vec{v_\mu}(s_0),\ \vec{v_\nu}(s_0)] \ , \end{aligned} \tag{3.22}$$

from which it follows that

$$\lambda_\mu \cdot \lambda_\nu \neq 1 \implies W[\vec{v_\mu},\ \vec{v_\nu}] \equiv \vec{v_\nu}^T \cdot \underline{S} \cdot \vec{v_\mu} = 0 \ ; \tag{3.23a}$$

$$\vec{v_\nu}^T \cdot \underline{S} \cdot \vec{v_\mu} \neq 0 \implies \lambda_\mu \cdot \lambda_\nu = 1 \ , \tag{3.23b}$$

so that the eigenvectors of $\underline{M}$ can be separated into two groups

$$(\vec{v}_k,\ \vec{v}_{-k}) \ ; \ k = I,\ II$$

with the properties

$$\underline{M}\ \vec{v}_k = \lambda_k \cdot \vec{v}_k \ ; \ \ \underline{M}\ \vec{v}_{-k} = \lambda_{-k} \cdot \vec{v}_{-k} \ ; \ \ \lambda_k \cdot \lambda_{-k} = 1 \ ; \tag{3.24a}$$

$$\left\{ \begin{array}{l} (\vec{v}_{-k})^T \cdot \underline{S} \cdot \vec{v}_k = -(\vec{v}_k)^T \cdot \underline{S} \cdot \vec{v}_{-k} \neq 0 \ ; \\ (\vec{v}_\mu)^T \cdot \underline{S} \cdot \vec{v}_\nu = 0 \ \ \text{otherwise} \ ; \end{array} \right. \tag{3.24b}$$

$$(k = I, II) \ .$$

In the following we put :

$$\begin{cases} \lambda_k = \epsilon^{i \cdot 2\pi Q_k} ; \\ \lambda_{-k} = \epsilon^{i \cdot 2\pi Q_{-k}} ; \end{cases} \qquad (3.25)$$
$$(k = I, II) .$$

Then according to eqn. (3.24a) :

$$Q_{-k} = -Q_k . \qquad (3.26)$$

where Q_k can be either real or complex.

2) Eqn.(3.24a) shows that the eigenvalues of $\underline{M}(s_0 + L, s_0)$ always appear in reciprocal pairs

$$(\lambda_k,\ \lambda_{-k} = 1/\lambda_k) ; \qquad (3.27)$$
$$(k = I, II) .$$

Since $\underline{M}(s_0 + L, s_0)$ is real, then λ^* as well as λ is an eigenvalue.

For the eigenvalue spectrum of $\underline{M}(s_0 + L, s_0)$ there are then the following possibilities :

a) All 4 eigenvalues are complex with unit absolute value and therefore lie on a unit circle in the complex plane :

$$|\lambda_k| = |\lambda_{-k}| = 1 ;$$
$$(k = I, II) .$$

Then :

$$Q_k \ \ \text{real} : \qquad (3.28)$$
$$\lambda_k = \lambda^*_{-k} ; \quad \vec{v}_{-k} = (\vec{v}_k)^* .$$

b) One reciprocal pair is real and the others lie on a unit circle :

$$\lambda_I = \lambda_I^* ; \quad \lambda_{-I} = \lambda^*_{-I} ; \quad \lambda_{-I} = 1/\lambda_I ;$$
$$\lambda_{-II} = \lambda^*_{II} ; \quad |\lambda_{II}| = |\lambda_{-II}| = 1 .$$

c) Both reciprocal pairs are real :

$$\lambda_I = \lambda_I^* ; \quad \lambda_{-I} = \lambda^*_{-I} ; \quad \lambda_{-I} = 1/\lambda_I ;$$

$$\lambda_{II} = \lambda^*_{II} ; \quad \lambda_{-II} = \lambda^*_{-II} ; \quad \lambda_{-II} = 1/\lambda_{II} .$$

d) One eigenvalue e.g. λ_I is complex and does not lie on the unit circle :

$$|\lambda_I| \neq 1 ; \quad \lambda_I \neq \lambda_I^* .$$

Then we must have :

$$\lambda_{-I} = 1/\lambda_I$$

and

$$\lambda_{II} = \lambda_I^* \; ;$$
$$\lambda_{-II} = 1/\lambda_I^*$$

or

$$\lambda_{II} = 1/\lambda_I^* \; ;$$
$$\lambda_{-II} = \lambda_I^* \; .$$

In the following it will become clear that only case a) leads to stable particle motion.

3) We define :

$$\vec{v}_\mu(s) = \underline{M}(s, s_0) \; \vec{v}_\mu(s_0) \; . \tag{3.29}$$

Then the vector $\vec{v}_\mu(s)$ is an eigenvector of the matrix $\underline{M}(s + L, s)$ with the eigenvalue λ_μ:

$$\underline{M}(s + L, s) \; \vec{v}_\mu(s) \;\; = \;\; \lambda_\mu \cdot \vec{v}_\mu(s) \; . \tag{3.30}$$

<u>Proof:</u>

$$\begin{aligned}
\underline{M}(s + L, s) \; \vec{v}_\mu(s) &= \underline{M}(s + L, s) \cdot \underline{M}(s, s_0) \vec{v}_\mu(s_0) \\
&= \underline{M}(s + L, s_0 + L) \cdot \underline{M}(s_0 + L, s_0) \; \vec{v}_\mu(s_0) \\
&= \underline{M}(s, s_0) \cdot \underline{M}(s_0 + L, s_0) \; \vec{v}_\mu(s_0) \\
&= \lambda_\mu \cdot \underline{M}(s, s_0) \; \vec{v}_\mu(s_0) \\
&= \lambda_\mu \cdot \; \vec{v}_\mu(s) \; ; \quad q.e.d.
\end{aligned}$$

The eigenvector $\vec{v}_\mu(s)$ thus has the same eigenvalue as $\vec{v}_\mu(s_0)$: The eigenvalue is therefore independent of s.

4) We put

$$\vec{v}_\mu(s) = \vec{u}_\mu(s) \cdot e^{i \cdot 2\pi Q_\mu \cdot (s/L)} \; . \tag{3.31a}$$

Then :

$$\vec{u}_\mu(s + L) = \vec{u}_\mu(s) \; . \tag{3.31b}$$

<u>Proof:</u>

We put eqn. (3.31a) into (3.30). Using eqn. (3.25) we obtain :

$$\vec{u}_\mu(s + L) \cdot e^{i \cdot 2\pi Q_\mu \cdot (s + L)/L} = e^{i \cdot 2\pi Q_\mu} \cdot \vec{u}_\mu(s) \cdot e^{i \cdot 2\pi Q_\mu \cdot s/L} \; .$$

One now gets eqn. (3.31b) when one cancels the factor

$$\epsilon^{i \cdot 2\pi Q_\mu \cdot (s+L)/L} = \epsilon^{i \cdot 2\pi Q_\mu} \cdot \epsilon^{i \cdot 2\pi Q_\mu \cdot s/L}$$

on each side.

Eqn.(3.31) is a statement of the Floquet theorem : Vectors $\vec{v}_\mu(s)$ are special solutions of the equations of motion (2.58) which can be expressed as the product of a periodic function $\vec{u}_\mu(s)$ and a harmonic function

$$\epsilon^{i \cdot 2\pi Q_\mu \cdot (s/L)} .$$

5) The general solution of the equation of motion (2.58) is a linear combination of the special solutions (3.31a) and can be therefore written in the form

$$\vec{z}(s) = \sum_{k=I,II} \left\{ A_k \cdot \vec{u}_k(s) \cdot \epsilon^{i \cdot 2\pi Q_k \cdot (s/L)} + A_{-k} \cdot \vec{u}_{-k}(s) \cdot \epsilon^{i \cdot 2\pi Q_{-k} \cdot (s/L)} \right\} . \quad (3.32)$$

We now see that the amplitude of the betatron oscillations only remain limited and the particle motion under control if the Q_k are real, i.e. if all eigenvalues, as already predicted, lie on the unit circle :

$$|\lambda_k| = |\lambda_{-k}| = 1 \; ; \quad (k = I,\ II) \; ; \quad (3.33)$$

(Stability criterion) .

On the contrary, if at least one of the exponents Q_k is complex, according to (3.26) either Q_k or Q_{-k} has a positive imaginary part. In this case the components of $\vec{z}(s)$ grow exponentially and the motion is unstable.

6) In the following, we always assume that the stability condition (3.33) is satisfied. Then from eqn. (3.28):

$$\vec{v}_{-k} = (\vec{v}_k)^* \; ; \quad (k = I,\ II) , \quad (3.34)$$

and (3.5b) simplifies to $\left(\vec{v}^+ = (\vec{v}^{\,T})^*\right)$:

$$\left\{ \begin{array}{l} \vec{v}_k^+(s_0) \cdot \underline{S} \cdot \vec{v}_k(s_0) = -\vec{v}_{-k}^+(s_0) \cdot \underline{S} \cdot \vec{v}_{-k}(s_0) \neq 0 \; ; \\ \vec{v}_\mu^+(s_0) \cdot \underline{S} \cdot \vec{v}_\nu(s_0) = 0 \;\; \text{otherwise} . \end{array} \right. \quad (3.35)$$

$$(k = I, II) .$$

Thus the terms $\vec{v}_\mu^+(s_0) \cdot \underline{S} \cdot \vec{v}_\mu(s_0)$ in eqn. (3.34) are pure imaginary :

$$\begin{aligned} \left[\vec{v}_\mu^+(s_0) \cdot \underline{S} \cdot \vec{v}_\mu(s_0)\right]^+ &= \vec{v}_\mu^+(s_0) \cdot \underline{S}^+ \cdot \vec{v}_\mu(s_0) \\ &= -\left[\vec{v}_\mu^+(s_0) \cdot \underline{S} \cdot \vec{v}_\mu(s_0)\right] \end{aligned}$$

(since $\underline{S}^+ = -\underline{S}$), so that in future the vectors $\vec{v}_k(s_0)$ and $\vec{v}_{-k}(s_0)$ $(k = I, II)$ can be normalised as:

$$\vec{v}_k^+(s_0) \cdot \underline{S} \cdot \vec{v}_k(s_0) = -\vec{v}_{-k}^+(s_0) \cdot \underline{S} \cdot \vec{v}_{-k}(s_0) = i \; ; \qquad (k = I, II) \; . \tag{3.36}$$

From the validity of the symplecticity condition (2.63) it then follows that the vectors $\vec{v}_k(s)$, und $\vec{v}_{-k}(s)$ (k=I, II) satisfy the conditions (3.35), (3.36) also at position s :

$$\begin{cases} \vec{v}_k^+(s) \cdot \underline{S} \cdot \vec{v}_k(s) = -\vec{v}_{-k}^+(s) \cdot \underline{S} \cdot \vec{v}_{-k}(s) = i \; ; \\ \vec{v}_\mu^+(s) \cdot \underline{S} \cdot \vec{v}_\nu(s) = 0 \quad \text{otherwise} \; . \end{cases} \tag{3.37}$$

<u>Remark:</u>

In order to prove the statement that if λ is an an eigenvalue of the symplectic matrix $\underline{M}$ then λ^{-1} is also an eigenvalue we have used the Lagrange invariant W.

We now show that this statement can also be obtained from the characteristic polynomial

$$P(\lambda) = \det[\underline{M} - \lambda \cdot \underline{1}]$$

in a direct manner.

For that purpose we first recognise that from the symplectic condition (2.64) it follows that:

$$\underline{M}^{-1} = -\underline{S}\underline{M}^T\underline{S} \; .$$

Then we have (for $\lambda \neq 0$) :

$$\begin{aligned} [\underline{M} - \lambda \cdot \underline{1}] &= \underline{M} \cdot [\underline{1} - \lambda \cdot \underline{M}^{-1}] \\ &= \underline{M} \cdot [-\underline{S}^2 + \lambda \cdot \underline{S}\underline{M}^T\underline{S}] \\ &= \underline{M}\underline{S} \cdot \lambda \cdot [\underline{M}^T - \lambda^{-1} \cdot \underline{1}] \cdot \underline{S} \\ &= \lambda \cdot \underline{M}\underline{S} \cdot [\underline{M} - \lambda^{-1} \cdot \underline{1}]^T \cdot \underline{S} \; . \end{aligned}$$

But from the last equation we can see that:

$$P(\lambda) = 0 \quad \Leftrightarrow \quad P(\lambda^{-1}) = 0$$

so that

$$\lambda = \text{eigenvalue of } \underline{M} \quad \Leftrightarrow \quad \lambda^{-1} = \text{eigenvalue of } \underline{M}$$

(note that $\lambda \neq 0$ for all eigenvalues of $\underline{M}$, since $\underline{M}$ is nonsingular as can be seen from eqn. (2.64)).

We also recognise that λ and λ^{-1} have the same multiplicity.

3.4 Optics Calculation in the Presence of Coupling

3.4.1 Optics Calculation in the Uncoupled Case

Now we are ready to introduce lattice functions in the coupled case. But let us first remember how we introduced lattice functions in the uncoupled case. Assume that we have a number of particles which at the same point $s = 0$ in the lattice occupy an area in two dimensional phase space. We then can draw an ellipse around them (fig 5a). This ellipse may be generated by two vectors in phase space $\vec{z}_1, \vec{z}_2$. Any point on this curve can be represented by a vector $\vec{z}(0)$:

$$\vec{z}(0) = \sqrt{\epsilon} \cdot [\vec{z}_1(0) \cos\phi - \vec{z}_2(0) \sin\phi] \tag{3.38}$$

with

$$\vec{z} \equiv \begin{pmatrix} z \\ p_z \end{pmatrix} = \begin{pmatrix} z \\ z' \end{pmatrix} ,$$

$(z \equiv x, y)$ and

$$\begin{aligned} \vec{z}_1 &= \begin{pmatrix} z_1 \\ z_1' \end{pmatrix} ; \\ \vec{z}_2 &= \begin{pmatrix} z_2 \\ z_2' \end{pmatrix} \end{aligned}$$

as shown in Fig. 5b. The factor $\sqrt{\epsilon}$ is introduced to characterize the size of the ellipse. We normalize the generating vectors $\vec{z}_{1,2}$ as:

$$(\vec{z}_1)^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \vec{z}_2 = 1$$

or

$$z_1 \cdot z_2' - z_1' \cdot z_2 = 1 \, . \tag{3.39}$$

The area of the ellipse is then $\pi\epsilon$. This expression however is just the Lagrange invariant W. Thus while the ellipse moves through the lattice still enclosing all the particles, and is in general parametrised as,

$$\vec{z}(s) = \sqrt{\epsilon} \cdot [\vec{z}_1(s) \cos\phi - \vec{z}_2(s) \sin\phi] \ , \tag{3.40}$$

its area is preserved and therefore the particle density is also preserved. We see that the existence of the invariant W is a manifestation of Liouville's theorem.

We now introduce lattice functions by writing the first components z_1, z_2 of the generating phase space vectors $\vec{z}_{1,2}$ as the product of an envelope function $\sqrt{\beta}$ and phase functions $\cos\Phi$ and $\sin\Phi$:

$$z_1(s) = \sqrt{\beta(s)} \cos\Phi(s) \ ; \tag{3.41a}$$

$$z_2(s) = \sqrt{\beta(s)} \sin\Phi(s) \ . \tag{3.41b}$$

One does the same for the second components z_1' and z_2' introducing an angle envelope γ and an angle phase function $\tilde{\Phi}$:

$$z_1'(s) = \sqrt{\gamma(s)} \cos \tilde{\Phi}(s) \; ; \qquad (3.42a)$$

$$z_2'(s) = \sqrt{\gamma(s)} \sin \tilde{\Phi}(s) \; . \qquad (3.42b)$$

Our point, Ξ, representing the phase ellipse can be written as

$$\Xi = \begin{pmatrix} \sqrt{\epsilon}(\sqrt{\beta} \cos(\Phi + \phi) \\ \sqrt{\epsilon}(\sqrt{\gamma} \cos(\tilde{\Phi} + \phi) \end{pmatrix} . \qquad (3.43)$$

The normalisation condition expressed in lattice functions gives

$$\sqrt{\beta} \cos \Phi \cdot (\sqrt{\beta} \sin \Phi)' - \sqrt{\beta} \sin \Phi \cdot (\sqrt{\beta} \cos \Phi)' = \beta \Phi' = 1 \; . \qquad (3.44)$$

That means that the envelope and the phase function are related by

$$\Phi(s) = \Phi(s_0) + \int_{s_0}^{s} \frac{ds}{\beta} \; . \qquad (3.45)$$

Using the abreviation

$$\alpha = -\frac{1}{2} \cdot \beta' \qquad (3.46)$$

we find that γ is related to β by

$$\gamma = \frac{\beta^2 \cdot \Phi'^2 + \alpha^2}{\beta} = \frac{1 + \alpha^2}{\beta} \qquad (3.47)$$

and the two different phase functions are related by

$$\tilde{\Phi} = \Phi - \arctan \alpha^{-1} \; . \qquad (3.48)$$

Fig. 5c shows how the lattice functions characterize the shape of the phase ellipse and thus the focussing properties of the lattice. The ellipse is defined by the three functions $E = \sqrt{\epsilon\beta}$, $A = \sqrt{\epsilon\gamma}$, $G = -\sqrt{\epsilon}\frac{\alpha}{\sqrt{\beta}}$. The area Γ enclosed by the ellipse is

$$\Gamma = \pi E \sqrt{A^2 - G^2} = \pi\epsilon\sqrt{\beta}\sqrt{\frac{\gamma^2 - \alpha^2}{\beta}} = \pi\epsilon \; . \qquad (3.49)$$

(An alternative way of introducing the lattice functions $\alpha, \beta, \gamma, \Phi$ (also called "Twiss-parameters") can be found in Ref. [2].)

3.4.2 The Coupled Case

This procedure can be generalized to the case of coupled motion (see also [5]).

Let us consider the motion of a single particle in a lattice with coupling elements. We expect that there are still two distinct oscillation modes denoted by I, II. But they are no

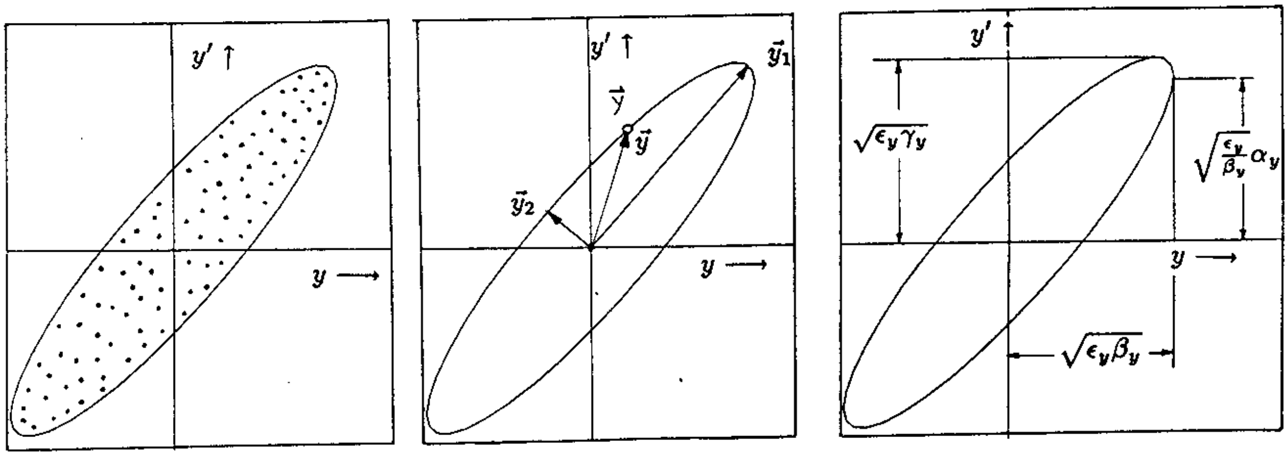

Figure 3:

a) Particles occupying some area in phase space observed at some point in the lattice may be enclosed by an ellipse b) This ellipse is generated by two phase space vectors $\vec{z}_1$ and $\vec{z}_2$. The motions of these generating vectors characterize the motion of all of these particles through the lattice. c) The resulting generating trajectories in turn can be expressed in terms of lattice functions which describe the shape of the phase ellipse everywhere in the lattice.

longer expected to correspond to pure horizontal and vertical motion. The motion starts at some point $\hat{\vec{z}}(0)$ in the four dimensional $x - x' - z - z'$ space at position s=0 :

$$\hat{\vec{z}} \equiv \begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix} = \underline{U}^{-1} \cdot \begin{pmatrix} x \\ p_x \\ y \\ p_y \end{pmatrix} \tag{3.50}$$

with the matrix

$$\underline{U} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{2}R & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{1}{2}R & 0 & 0 & 1 \end{pmatrix} \tag{3.51}$$

which connects the vector $\hat{\vec{z}}$ with the phase space vector $\vec{z}$. It will be confined to a toroidal surface in this space. Every point on this torus is generated by four independent vectors $\hat{\vec{z}}_i$:

$$\hat{\vec{z}}_i = \begin{pmatrix} x_i \\ x_i' \\ y_i \\ y_i' \end{pmatrix} .$$

In analogy to the uncoupled case we consider them as two pairs $\hat{\vec{z}}_{1,2}$ and $\hat{\vec{z}}_{3,4}$. The first pair we assign to mode $'I'$ (instead of $'x'$) and the second one to mode $'II'$ (instead of $'y'$). Our starting point $\hat{\vec{z}}(0)$ is then expressed in terms of these vectors by two amplitudes and two phases:

$$\hat{\vec{z}}(0) = \sqrt{\epsilon_I} \cdot \left[\hat{\vec{z}}_1(0)\cos\phi_I - \hat{\vec{z}}_2(0)\sin\phi_I\right] + \sqrt{\epsilon_{II}} \cdot \left[\hat{\vec{z}}_3(0)\cos\phi_{II} - \hat{\vec{z}}_4(0)\sin\phi_{II}\right] . \tag{3.52}$$

With the help of the transfer matrix

$$\underline{\hat{M}}(s,0) = \underline{U}^{-1}(s)\underline{M}(s,0)\underline{U}(0)$$

the vector $\hat{\vec{z}}$ at position s is:

$$\hat{\vec{z}}(s) = \sqrt{\epsilon_I}\cdot\left[\hat{\vec{z}}_1(s)\cos\phi_I - \hat{\vec{z}}_2(s)\sin\phi_I\right] + \sqrt{\epsilon_{II}}\cdot\left[\hat{\vec{z}}_3(s)\cos\phi_{II} - \hat{\vec{z}}_4(s)\sin\phi_{II}\right] \tag{3.53}$$

with

$$\hat{\vec{z}}_k(s) = \underline{\hat{M}}(s,0)\ \hat{\vec{z}}_k(0)\quad ;\quad (k=1,2,3,4)\ . \tag{3.54}$$

In contrast to the uncoupled case, the vectors $\hat{\vec{z}}_{1,2}$ have y, y'-components and the vectors $\hat{\vec{z}}_{3,4}$ have x, x'-components. In close analogy to the uncoupled case we define lattice functions by

$$x_1 = \sqrt{\beta_{xI}}\cos\Phi_{xI}\ ;\quad x_2 = \sqrt{\beta_{xI}}\sin\Phi_{xI}\ ; \tag{3.55a}$$
$$x_1' = \sqrt{\gamma_{xI}}\cos\tilde{\Phi}_{xI}\ ;\quad x_2' = \sqrt{\gamma_{xI}}\sin\tilde{\Phi}_{xI} \tag{3.55b}$$

belonging to $\hat{\vec{z}}_{1,2}$ and there is a second set of horizontal lattice functions

$$x_3 = \sqrt{\beta_{xII}}\cos\Phi_{xII}\ ;\quad x_4 = \sqrt{\beta_{xII}}\sin\Phi_{xII}\ ; \tag{3.56a}$$
$$x_3' = \sqrt{\gamma_{xII}}\cos\tilde{\Phi}_{xII}\ ;\quad x_4' = \sqrt{\gamma_{xII}}\sin\tilde{\Phi}_{xII} \tag{3.56b}$$

belonging to $\hat{\vec{z}}_{3,4}$. We do the same for the vertical plane:

$$y_1 = \sqrt{\beta_{yI}}\cos\Phi_{yI}\ ;\quad y_2 = \sqrt{\beta_{yI}}\sin\Phi_{yI}\ ; \tag{3.57a}$$
$$y_1' = \sqrt{\gamma_{yI}}\cos\tilde{\Phi}_{yI}\ ;\quad y_2' = \sqrt{\gamma_{yI}}\sin\tilde{\Phi}_{yI} \tag{3.57b}$$

and

$$y_3 = \sqrt{\beta_{yII}}\cos\Phi_{yII}\ ;\quad y_4 = \sqrt{\beta_{yII}}\sin\Phi_{yII}\ ; \tag{3.58a}$$
$$y_3' = \sqrt{\gamma_{yII}}\cos\tilde{\Phi}_{yII}\ ;\quad y_4' = \sqrt{\gamma_{yII}}\sin\tilde{\Phi}_{yII}\ . \tag{3.58b}$$

The trajectory $\hat{\vec{z}}(s)$ is expressed in terms of these lattice functions as:

$$\hat{\vec{z}} = \begin{pmatrix} \sqrt{\epsilon_I}\sqrt{\beta_{xI}}\cos(\Phi_{xI}+\phi_I) + \sqrt{\epsilon_{II}}\sqrt{\beta_{xII}}\cos(\Phi_{xII}+\phi_{II}) \\ \sqrt{\epsilon_I}\sqrt{\gamma_{xI}}\cos(\tilde{\Phi}_{xI}+\phi_I) + \sqrt{\epsilon_{II}}\sqrt{\gamma_{xII}}\cos(\tilde{\Phi}_{xII}+\phi_{II}) \\ \sqrt{\epsilon_I}\sqrt{\beta_{yI}}\cos(\Phi_{yI}+\phi_I) + \sqrt{\epsilon_{II}}\sqrt{\beta_{yII}}\cos(\Phi_{yII}+\phi_{II}) \\ \sqrt{\epsilon_I}\sqrt{\gamma_{yI}}\cos(\tilde{\Phi}_{yI}+\phi_I) + \sqrt{\epsilon_{II}}\sqrt{\gamma_{yII}}\cos(\tilde{\Phi}_{yII}+\phi_{II}) \end{pmatrix}\ . \tag{3.59}$$

We wish to normalize the generating vectors so that, as for the uncoupled case, the factors ϵ_I and ϵ_{II} again characterize the size of the torus. But now we have to distinguish between the trajectory slopes x', y' contained in the vector $\hat{\vec{z}}$ and the canonical momenta p_x, p_y used in the phase space vector $\vec{z}$. Thus the normalization condition in the coupled case is more complicated:

$$(\vec{z}_1)^T\underline{S}\vec{z}_2 = (\hat{\vec{z}}_1)^T\underline{U}^T\underline{S}\underline{U}\hat{\vec{z}}_2 = 1\ ; \tag{3.60a}$$
$$(\vec{z}_3)^T\underline{S}\vec{z}_4 = (\hat{\vec{z}}_3)^T\underline{U}^T\underline{S}\underline{U}\hat{\vec{z}}_4 = 1 \tag{3.60b}$$

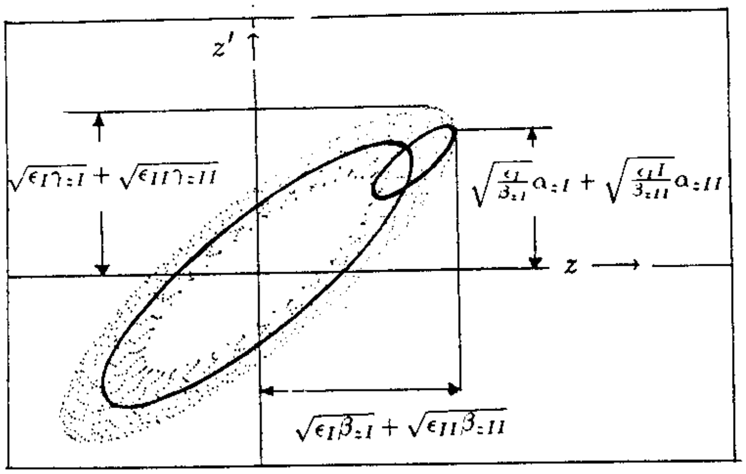

Figure 4: Two sets of lattice functions for each plane describe the projection of a four dimensional torus on the $x - x'$ and $y - y'$-planes.

or

$$\beta_x \Phi_x' + \beta_y \Phi_y' - R \cdot \sqrt{\beta_x \beta_y} \sin(\Phi_x - \Phi_y) = 1 \; . \tag{3.61}$$

Outside of a solenoid field the normalization condition is simpler with

$$\beta_x \Phi_x' + \beta_y \Phi_y' = 1 \tag{3.62}$$

for mode I and II respectively. Again γ is related to β by

$$\gamma = \frac{\beta^2 \cdot \Phi'^2 + \alpha^2}{\beta} \tag{3.63}$$

for x, y and for modes I, II respectively.

What is the meaning of these four sets of lattice functions? If the four dimensional torus on which our particle, represented by the vector $\hat{z}$, is moving is projected onto the $x - x'$-plane or on the $y - y'$-plane we get an area filled with possible coordinates x, x' or y, y'. This area is obtained by the superposition of the two ellipses defined by the two sets of lattice functions:

$$E_{xI} = \sqrt{\epsilon_I \beta_{xI}} \; ; \quad A_{xI} = \sqrt{\epsilon_I \gamma_{xI}} \; ; \quad G_{xI} = -\sqrt{\epsilon_I} \frac{\alpha_{xI}}{\sqrt{\beta_{xI}}} \; ; \tag{3.64a}$$

$$E_{xII} = \sqrt{\epsilon_I \beta_{xII}} \; ; \quad A_{xII} = \sqrt{\epsilon_{II} \gamma_{xII}} \; ; \quad G_{xII} = -\sqrt{\epsilon_{II}} \frac{\alpha_{xII}}{\sqrt{\beta_{xII}}} \; . \tag{3.64b}$$

The same thing happens for the projection onto the $y - y'$-plane. This is shown in Fig. 6. We see from eqn. (3.59) that as in the uncoupled case, the lattice functions describe the focussing properties of the lattice: Mode I and mode II lattice functions tell us where the the particle oscillation amplitudes or the trajectory slopes are large or small just in the same way as for the uncoupled case.

The areas of the two ellipses

$$\Gamma_{xI} = \pi \cdot E_{xI} \sqrt{A_{xI}^2 - G_{xI}^2} = \pi \epsilon_I \beta_{xI} \cdot \mid \Phi_{xI}' \mid \; ; \tag{3.65a}$$

$$\Gamma_{xII} = \pi \cdot E_{xII} \sqrt{A_{xII}^2 - G_{xII}^2} = \pi \epsilon_{II} \beta_{xII} \cdot \mid \Phi_{xII}' \mid \tag{3.65b}$$

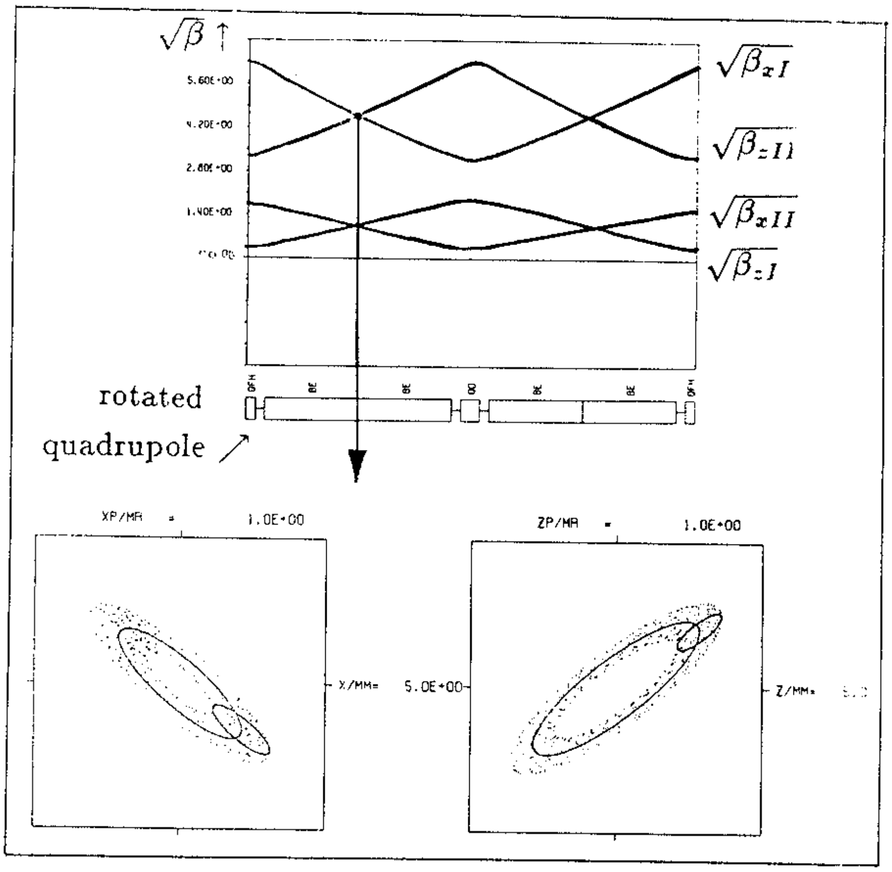

Figure 5: Coupled Lattice functions for a FODO-cell with a rotated quadrupole.

are not preserved during the motion through the lattice but the sum or difference (due to the sign of Φ') of the horizontal and vertical ellipse areas of mode I and II respectively :

$$\pi\epsilon_I \cdot (\beta_{xI}\Phi'_{xI} + \beta_{yI}\Phi'_{yI}) = \pi\epsilon_I \; ; \tag{3.66a}$$
$$\pi\epsilon_{II} \cdot (\beta_{xII}\Phi'_{xII} + \beta_{yII}\Phi'_{yII}) = \pi\epsilon_{II} \tag{3.66b}$$

is a constant of motion because of the normalization of the generating vectors. (This expression is only valid outside a solenoid.) Thus in the coupled case the invariants $\epsilon_I, \epsilon_{II}$ replace the horizontal and vertical invariants which we obtain in the uncoupled case.

3.4.3 Periodic Lattice Functions

In a circular machine the lattice functions are only useful if they are periodic in the machine circumference. This is the case if the generating vectors $\hat{\vec{z}}_{1,..4}$ differ from their initial values by only a phase factor after we have transformed them once around the machine. In order to find the correct initial values of these generating vectors we first write the phase space vector $\hat{\vec{z}}$ from eq (3.52) in complex form defining

$$\hat{\vec{v}}_I = \frac{1}{\sqrt{2}} \cdot (\hat{\vec{z}}_1 + i \cdot \hat{\vec{z}}_2) \; ; \hat{\vec{v}}_{II} = \frac{1}{\sqrt{2}} \cdot (\hat{\vec{z}}_3 + i \cdot \hat{\vec{z}}_4) \; . \tag{3.67}$$

We obtain

$$\hat{\vec{z}} = \sqrt{\frac{\epsilon_I}{2}} \cdot \hat{\vec{v}}_I e^{i\phi_I} + \sqrt{\frac{\epsilon_{II}}{2}} \cdot \hat{\vec{v}}_{II} e^{i\phi_{II}} + \text{conj.compl.} \tag{3.68}$$

If the vectors $\hat{\vec{v}}_{I,II}$ are eigenvectors of the revolution matrix

$$\underline{\hat{M}}(s+L,s) = \underline{U}(s) \cdot \underline{M}(s+L,s) \cdot \underline{U}^{-1}(s) \tag{3.69}$$

with eigenvalues $\lambda_{I,II} = e^{2\pi i Q_{I,II}}$, (note, that $\underline{\hat{M}}(s+L,s)$ and $\underline{M}(s+L,s)$ have the same eigenvalues because they are connected by a similarity transformation; according to the normalization conditions (3.56) one has $\hat{\vec{v}}_k = \underline{U} \cdot \vec{v}_k, k = I, II$) our initial point $\hat{\vec{z}}$ transforms after one revolution into

$$\hat{\vec{z}} = \sqrt{\frac{\epsilon_I}{2}} \cdot \hat{\vec{v}}_I e^{i(\phi_I + 2\pi Q_I)} + \sqrt{\frac{\epsilon_{II}}{2}} \cdot \hat{\vec{v}}_{II} e^{i(\phi_{II} + 2\pi Q_{II})} + \text{c.c.} \tag{3.70}$$

or

$$\begin{aligned} \hat{\vec{z}} = & \sqrt{\epsilon_I} \left\{ \hat{\vec{z}}_1 \cos(\phi_I + 2\pi Q_I) + \hat{\vec{z}}_2 \sin(\phi_I + 2\pi Q_I) \right\} \\ + & \sqrt{\epsilon_{II}} \left\{ \hat{\vec{z}}_3 \cos(\phi_{II} + 2\pi Q_{II}) + \hat{\vec{z}}_4 \sin(\phi_{II} + 2\pi Q_{II}) \right\} \; . \end{aligned} \tag{3.71}$$

We see that the generating vectors satisfy the periodicity condition if they are obtained from the eigenvectors of the revolution matrix. From the periodic generating vectors we then obtain the periodic lattice functions.

For the calculation of coupled beam optics in a circular machine we now have the following scheme:

- Find the eigenvectors and the eigenvalues of the revolution matrix at s_0. The eigenvalues must be complex and must have the absolute value 1.

- Form the generating vectors by $\hat{\vec{z}}_{1,3} = \frac{1}{\sqrt{2}}\left(\hat{\vec{v}}_{I,II} + \hat{\vec{v}}^*_{I,II}\right)$; $\hat{\vec{z}}_{2,4} = -\frac{i}{\sqrt{2}}\left(\hat{\vec{v}}_{I,II} - \hat{\vec{v}}^*_{I,II}\right)$.
- Normalize those vectors according to $(\vec{z}_{1,3})^T \underline{S}\vec{z}_{2,4} = 1$ with $\vec{z}_i = \underline{U}\hat{\vec{z}}_i$.
- Calculate the initial lattice functions at s_0:

$$\left(\begin{array}{llll} \beta_{xI} = x_1^2 + x_2^2 & \gamma_{xI} = x_1'^2 + x_2'^2 & \alpha_{xI} = -(x_1x_1' + x_2x_2') & \Phi_{xI} = \arctan(x_2/x_1) \\ \beta_{xII} = x_1^3 + x_4^2 & \gamma_{xII} = x_3'^2 + x_4'^2 & \alpha_{xII} = -(z_3z_3' + z_4z_4') & \Phi_{xII} = \arctan(x_4/x_3) \\ \beta_{yI} = x_1^2 + z_2^2 & \gamma_{yI} = z_1'^2 + z_2'^2 & \alpha_{yI} = -(z_1z_1' + z_2z_2') & \Phi_{yI} = \arctan(z_2/z_1) \\ \beta_{yII} = z_1^3 + z_4^2 & \gamma_{yII} = z_3'^2 + z_4'^2 & \alpha_{yII} = -(z_3z_3' + z_4z_4') & \Phi_{yII} = \arctan(z_4/z_3) \end{array}\right) \tag{3.72}$$

The lattice functions elsewhere in the ring are obtained the same way after transporting the generating vectors through the lattice.

We also identify the exponent $Q_{I,II}$ of the eigenvalues of $\underline{\hat{M}}$ (revolution matrix for the vector $\hat{\vec{z}}$) as the total machine tune because after one revolution, the phases $\phi_{I,II}$ of a particle are changed by $2\pi Q_{I,II}$ and we interpret this as the total phase advance of the four oscillation modes:

$$\Phi_{xI}(s+L) + \Phi_{xI}(s) = \Phi_{yI}(s+L) + \Phi_{yI}(s) = 2\pi Q_I \tag{3.73a}$$

$$\Phi_{xII}(s+L) + \Phi_{xII}(s) = \Phi_{yII}(s+L) + \Phi_{yII}(s) = 2\pi Q_{II} \tag{3.73b}$$

3.4.4 The Four Dimensional Phase Ellipsoid

So far we have considered the motion of just one particle characterized by the values of the two invariants ϵ_I and ϵ_{II} and two initial values of the phases ϕ_I, ϕ_{II}. Let us now consider a bunch of particles which are distributed on the ϵ_I-ϵ_{II}-plane inside an elliptical boundary as shown in Fig. 8. An arbitrary particle on the surface of the bunch is described by the maximum values of the two invariants ε_I, ε_{II}, an angle χ and the initial values of the phases $\phi_{I,II}$ [5,6]:

$$\hat{\vec{z}} = \cos\chi\sqrt{\varepsilon_I}\cdot\left(\hat{\vec{z}}_1\cos\phi_I - \hat{\vec{z}}_2\sin\phi_I\right) + \sin\chi\sqrt{\varepsilon_{II}}\cdot\left(\hat{\vec{z}}_3\cos\phi_{II} - \hat{\vec{z}}_4\sin\phi_{II}\right) \tag{3.74}$$

or, using the lattice functions:

$$\hat{\vec{z}} = \cos\chi\sqrt{\varepsilon_I}\begin{pmatrix} \sqrt{\beta_{xI}}\cos(\Phi_{xI}+\phi_I) \\ \sqrt{\gamma_{xI}}\cos(\tilde{\Phi}_{xI}+\phi_I) \\ \sqrt{\beta_{yI}}\cos(\Phi_{yI}+\phi_I) \\ \sqrt{\gamma_{yI}}\cos(\tilde{\Phi}_{yI}+\phi_I) \end{pmatrix} + \sin\chi\sqrt{\varepsilon_{II}}\begin{pmatrix} \sqrt{\beta_{xII}}\cos(\Phi_{xII}+\phi_{II}) \\ \sqrt{\gamma_{xII}}\cos(\tilde{\Phi}_{xII}+\phi_{II}) \\ \sqrt{\beta_{yII}}\cos(\Phi_{yII}+\phi_{II}) \\ \sqrt{\gamma_{yII}}\cos(\tilde{\Phi}_{yII}+\phi_{II}) \end{pmatrix} . \tag{3.75}$$

If we project this ellipsoid onto the $x - x'$ (or $z - z'$) plane we obtain an ellipse defined by:

a) the maximum value of x:

$$x_{max} \equiv E_x = \sqrt{\varepsilon_I \cdot \beta_{xI} - \varepsilon_{II}\cdot\beta_{xII}} \tag{3.76a}$$

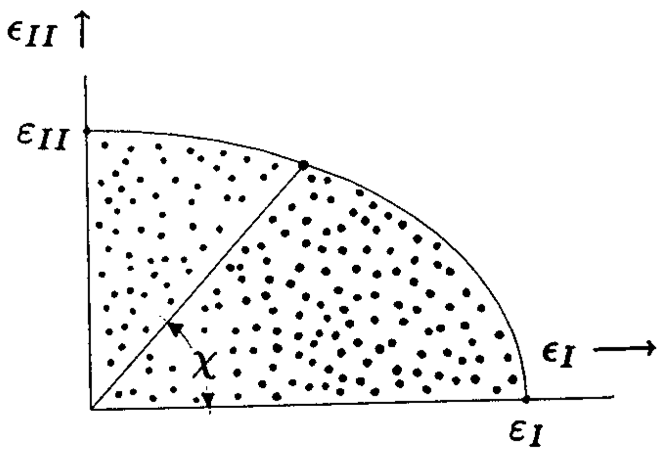

Figure 6: Distribution of particles characterized by the two invariants $\epsilon_{I,II}$

for

$$\cos\chi = \frac{\sqrt{\varepsilon_I \cdot \beta_{xI}}}{\sqrt{\varepsilon_I \cdot \beta_{xI} + \varepsilon_I I \cdot \beta_{xII}}} ;$$
$$\sin\chi = \frac{\sqrt{\varepsilon_{II} \cdot \beta_{xII}}}{\sqrt{\varepsilon_I \cdot \beta_{xI} + \varepsilon_I I \cdot \beta_{xII}}}$$

and

$$\phi_{I,II} = -\Phi_{xI,II} ;$$

b) the maximum value of x':

$$x'_{max} \equiv A_x = \sqrt{\varepsilon_I \cdot \gamma_{xI} + \varepsilon_{II} \cdot \gamma_{xII}} \tag{3.76b}$$

for

$$\cos\chi = \frac{\sqrt{\varepsilon_I \cdot \gamma_{xI}}}{\sqrt{\varepsilon_I \cdot \gamma_{xI} + \varepsilon_I I \cdot \gamma_{xII}}} ;$$
$$\sin\chi = \frac{\sqrt{\varepsilon_{II} \cdot \gamma_{xII}}}{\sqrt{\varepsilon_I \cdot \gamma_{xI} + \varepsilon_I I \cdot \gamma_{xII}}}$$

and

$$\phi_{I,II} = -\tilde{\Phi}_{xI,II}$$

and

c) the value of the slope x' at $(x = x_{max})$:

$$x'(x = x_{max}) \equiv G_x = -\frac{\varepsilon_I \cdot \alpha_{xI} + \varepsilon_{II} \cdot \alpha_{xII}}{\sqrt{\varepsilon_I \cdot \beta_{xI} + \varepsilon_{II} \cdot \beta_{xII}}} . \tag{3.77}$$

This ellipse is shown in Fig. 9. Its area may be considered as the horizontal emittance (as it would be obtained by a measurement).

$$\varepsilon_x = E_x \cdot \sqrt{A_x^2 - G_x^2} = \varepsilon_I \cdot \beta_{xI}\Phi'_{xI} + \varepsilon_{II} \cdot \beta_{xII}\Phi'_{xII} \tag{3.78}$$

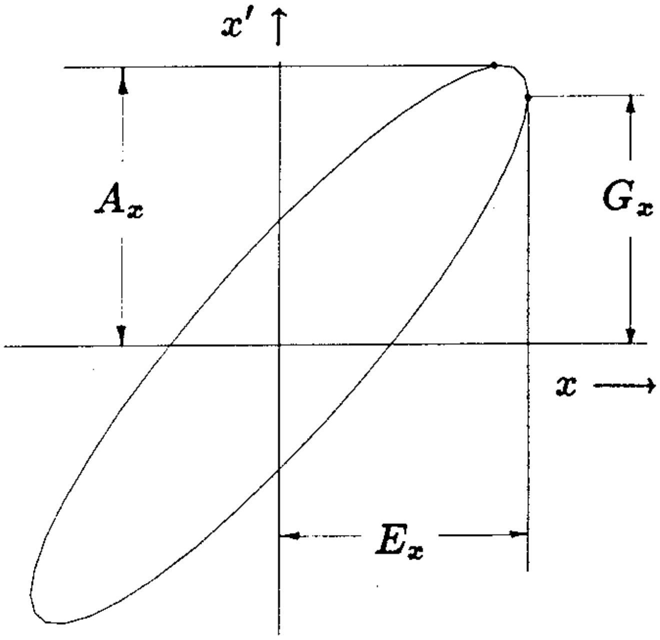

Figure 7: Projection of the four dimensional phase ellipsoid onto the $x - x'$-plane

which is not a constant of motion. In the same way we can define a vertical emittance.

Another interesting projection is the projection onto the $x - z$-plane (see Fig. 10). This projection yields also an ellipse which is , besides E_x, E_y, determined by the value of z for $x = E_x$:

$$G_{xz} = \frac{\varepsilon_I \sqrt{\beta_{xI}\beta_{yI}} \cos(\Phi_{xI} - \Phi_{yI}) + \varepsilon_{II}\sqrt{\beta_{xII}\beta_{yII}} \cos(\Phi_{xII} - \Phi_{yII})}{\sqrt{\varepsilon_I \beta_{xI} + \varepsilon_{II}\beta_{xII}}} . \tag{3.79}$$

We can express the angle that the major axis of the ellipse makes with the x-axis in terms of these three parameters:

$$\begin{aligned} \tan 2\psi &= \frac{2E_x G_{xz}}{E_x^2 - E_y^2} \\ &= 2 \cdot \frac{\varepsilon_I \sqrt{\beta_{xI}\beta_{yI}} \cos(\Phi_{xI} - \Phi_{yI}) + \varepsilon_{II}\sqrt{\beta_{xII}\beta_{yII}} \cos(\Phi_{xII} - \Phi_{yII})}{\varepsilon_I(\beta_{xI} - \beta_{yI}) + \varepsilon_{II}(\beta_{xII} - \beta_{yII})} . \end{aligned} \tag{3.80}$$

3.5 Perturbation Treatment of Coupling

3.5.1 Introductory Remark

Again, at this point we might say that we have solved our problem. We know how to transform single trajectories through a lattice with coupling elements, furthermore we have lattice functions which describe the optical properties of this lattice and we know how to calculate these lattice functions. Suppose however, we have designed a linear machine with no coupling at all. In reality the machine will have slightly misaligned quadrupole magnets, vertical closed orbit deviations in the sextupole magnets and field errors. All produce coupling. These sources of coupling are distributed randomly over the lattice. In most cases these coupling effects will distort the beam behaviour only very little because the random

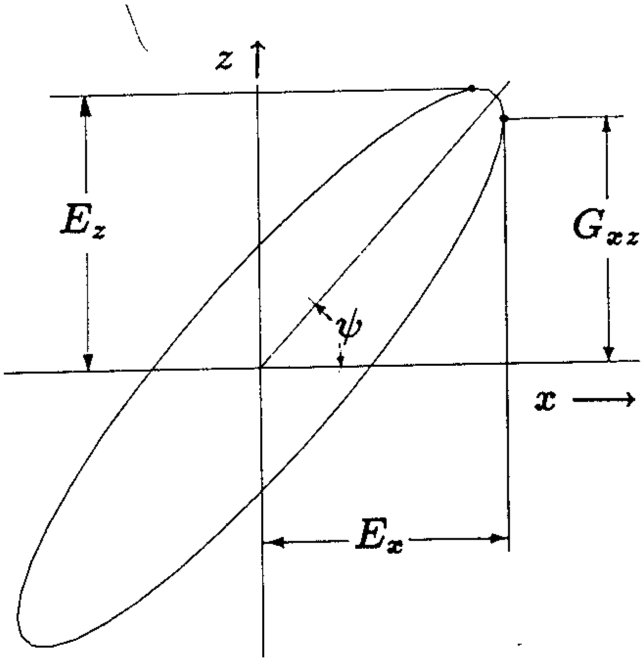

Figure 8: Projection of the four dimensional phase ellipsoid onto the $x-z$-plane

errors are small. But if the errors oscillate around the machine with the same frequency as the beam does when it performs betatron oscillations, these small distortions can act coherently on the beam and we get a resonant enhancement of the coupling. Because the random distortions are ring periodic, such resonances occur only if the the sum or the difference of the undistorted machine tunes is an integer multiple of the revolution frequency. So far we have not developed a formalism which can handle this situation. But perturbation theory offers an adequate procedure to handle small random distortions. That is what we will consider next.

3.5.2 Variation of Constants

Consider a linear lattice with small skew quadrupole distortions distributed in some way over the lattice. The equation of motion is derived from the Hamilton function we already know:

$$H = \frac{1}{2}[p_x^2 + p_y^2] + \frac{1}{2}k \cdot [x^2 - y^2] + N \cdot xy \ . \tag{3.81}$$

(for simplicity we only investigate the case where coupling is produced by skew quadrupoles). We separate this into an undistorted Hamiltonian H_0:

$$H_0 = \frac{1}{2}[p_x^2 + p_y^2] + \frac{1}{2}k \cdot [x^2 - y^2] \tag{3.82}$$

and a perturbation H_1 containing the skew quadrupole factor N:

$$H_1 = N \cdot xy \ . \tag{3.83}$$

Suppose we have already solved the uncoupled problem described by H_0 with the solution expressed in terms of the uncoupled lattice functions ($z \equiv x, y$)

$$z^{(0)}(s) = \sqrt{2I}\sqrt{\beta(s)}\cos[\Phi(s) + \phi] \ ; \tag{3.84a}$$

$$p_z^{(0)}(s) = -\sqrt{\frac{2I}{\beta(s)}} \cdot \{\alpha\cos[\Phi(s) + \phi] + \sin[\Phi(s) + \phi]\} \ . \tag{3.84b}$$

I, ϕ are constants of motion. We now want to express the solution for the full problem $z(s)$ in terms of the same lattice functions β, Φ. Therefore we must allow the constants I, ϕ to become dependent of s:

$$z(s) = \sqrt{2I(s)}\sqrt{\beta(s)}\cos[\Phi(s) + \phi(s)] \; ; \tag{3.85a}$$

$$p_z(s) = -\sqrt{\frac{2I(s)}{\beta(s)}} \cdot \{\alpha \cos[\Phi(s) + \phi(s)] + \sin[\Phi(s) + \phi(s)]\} \; . \tag{3.85b}$$

This is the concept of variation of constants as introduced into accelerator physics by [7,8].

Inserting this ansatz into the Hamilton equations of motion we obtain differential equations for the varying constants:

$$\frac{dz}{ds} = \frac{\partial z}{\partial s} + \frac{\partial z}{\partial I} \cdot \frac{dI}{ds} + \frac{\partial z}{\partial \phi} \cdot \frac{d\phi}{ds} = \frac{\partial H}{\partial p_z} = \frac{\partial H_0}{\partial p_z} + \frac{\partial H_1}{\partial p_z} \; ; \tag{3.86a}$$

$$\frac{dp_z}{ds} = \frac{\partial p_z}{\partial s} + \frac{\partial p_z}{\partial I} \cdot \frac{dI}{ds} + \frac{\partial p_z}{\partial \phi} \cdot \frac{d\phi}{ds} = -\frac{\partial H}{\partial z} = -\frac{\partial H_0}{\partial z} - \frac{\partial H_1}{\partial z} \; . \tag{3.86b}$$

Then, taking into account the relations

$$\left(\frac{\partial}{\partial s} z\right)_{I,\phi} = +\frac{\partial H_0}{\partial p_z} \; ; \tag{3.87a}$$

$$\left(\frac{\partial}{\partial s} p_z\right)_{I,\phi} = -\frac{\partial H_0}{\partial z} \tag{3.87b}$$

which result from the unperturbed equations of motion with the Hamiltonian H_0 we get

$$\begin{pmatrix} +\partial H_1/\partial p_z \\ -\partial H_1/\partial z \end{pmatrix} = \begin{pmatrix} \partial z/\partial I & \partial z/\partial \phi \\ \partial p_z/\partial I & \partial p_z/\partial \phi \end{pmatrix} \begin{pmatrix} I' \\ \phi' \end{pmatrix} \tag{3.88}$$

With

$$\frac{\partial z}{\partial \phi}\frac{\partial p_z}{\partial I} - \frac{\partial p_z}{\partial \phi}\frac{\partial z}{\partial I} = 1 \tag{3.89}$$

we obtain

$$\begin{pmatrix} I' \\ \phi' \end{pmatrix} = \begin{pmatrix} -\partial p_z/\partial \phi & +\partial z/\partial \phi \\ +\partial p_z/\partial I & -\partial z/\partial I \end{pmatrix} \begin{pmatrix} +\partial H_1/\partial p_z \\ -\partial H_1/\partial z \end{pmatrix} = \begin{pmatrix} -\partial H_1/\partial \phi \\ +\partial H_1/\partial I \end{pmatrix} \; . \tag{3.90}$$

Thus by the variation of constants we have transformed away the unperturbed part of the Hamiltonian. The equations of motion for I, ϕ have a Hamilton form with H_1 as the new Hamiltonian. The variable I plays the role of the canonical momentum and ϕ is the new coordinate.

Note, that the left side of (3.89) is just the Poisson bracket $(z, p_z)_{\phi,I}$. Therefore eqn. (3.89) represents a necessary and sufficient condition for ϕ, I to be (new) canonical variables [1].

3.5.3 The Slowly Varying Part of H_1

We now express the Hamiltonian in terms of the new variables:

$$\begin{aligned} H_1 &= N \cdot x(I_x, \phi_x, s)\ y(I_y, \phi_y, s) \\ &= 2N\sqrt{\beta_x\beta_y}\sqrt{I_x I_y}\cos(\Phi_x + \phi_x)\cos(\Phi_y + \phi_y) \end{aligned} \quad (3.91)$$

For the present it will be convenient to work with complex numbers. Therefore we write

$$\cos\phi = \frac{1}{2}(e^{i\phi} + e^{-i\phi}) \quad (3.92)$$

and obtain

$$H_1 = \sum_{j=1,-1} \frac{1}{2} N_0 \sqrt{\beta_x\beta_y}\sqrt{I_x I_y} e^{i(\Phi_x + j\Phi_y)} e^{i(\phi_x + j\phi_y)} + c.c. \quad (3.93)$$

Furthermore, replacing the independent variable s by $\theta = 2\pi s/L$ we may write:

$$\frac{dI}{d\theta} = -\frac{\partial \hat{H}_1}{\partial \phi}\ ;\ \frac{d\phi}{d\theta} = +\frac{\partial \hat{H}_1}{\partial I} \quad (3.94)$$

with

$$\hat{H}_1 = \frac{L}{2\pi} \cdot H_1\ . \quad (3.95)$$

Following our initial idea we look for certain dangerous frequencies in the distortion represented by $\hat{H}_1$. Therefore we separate $\hat{H}_1$ into a periodic and a nonperiodic factor:

$$\hat{H}_1 = \frac{L}{2\pi} \sum_{j=1,-1} \frac{1}{2} N_0 \sqrt{\beta_x\beta_y} e^{i[\Phi_x + j\Phi_y - (Q_x + jQ_y)\theta]} \sqrt{I_x I_y} e^{i[\phi_x + j\phi_y + (Q_x + jQ_y)\theta]} + c.c. \quad (3.96)$$

The periodic part is represented by its Fourier coefficients κ_{qj}:

$$\frac{L}{2\pi} \cdot N_0 \sqrt{\beta_x\beta_y} e^{i[\Phi_x + j\Phi_y - (Q_x + jQ_y)\theta]} = \sum_q \kappa_{qj} e^{i\phi_{qj}} e^{-iq\theta}\ ; \quad (3.97a)$$

$$\kappa_{qj} e^{i\phi_{qj}} = \frac{1}{2\pi} \int_s^{s+L} ds k_s \sqrt{\beta_x\beta_y} e^{i[\Phi_x + j\Phi_y - (Q_x + jQ_y + q)\cdot 2\pi s/L]}\ . \quad (3.97b)$$

The κ_{qj} are called "driving terms". The Hamiltonian then has the form:

$$\hat{H}_1 = \sum_{j,q} \kappa_{qj} \sqrt{I_x I_y} \cos[\phi_x + j\phi_y + (Q_x + jQ_y - q)\theta + \phi_{qj}]\ . \quad (3.98)$$

We see that $\hat{H}_1$ has an explicit dependence on the independent variable θ. This dependence is oscillatory with frequencies $Q_x \pm Q_y - q$. Since N assumed to be small, the driving terms are a small perturbation only. Thus they can induce only a slow change in the variables I, ϕ. Therefore terms with a quickly oscillating θ-dependence cannot have much influence on I, ϕ because they average away before they can make a significant change. Only terms with small $Q_x \pm Q_y - q$ can act for long enough coherently on the variables and cause serious effects. Note that such terms are only present if the tunes satisfy a resonant condition

$$Q_x \pm Q_y \simeq p = \text{integer}\ . \quad (3.99)$$

As a consequence we only consider resonant cases. With the plus sign we have a sum resonance and with the negative sign it is called a difference resonance. Then we can omit all terms of the sum over q except the resonance driving term $\kappa_p^{\pm}$:

$$\acute{H}_1 = \kappa_p^{\pm}\sqrt{I_x I_y}\cos[\phi_x \pm \phi_y + \theta(Q_x \pm Q_y - p) - \varphi_{pj}] \tag{3.100}$$

With only one term remaining in the Hamiltonian we can eliminate the explicit dependence of $\acute{H}_1$ on the independent variable θ by introducing a new phase variable :

$$\psi_x = \phi_x + \frac{1}{2}\Delta\cdot\theta + \varphi_{pj}\ ; \tag{3.101a}$$

$$\psi_y = \phi_y \pm \frac{1}{2}\Delta\cdot\theta \tag{3.101b}$$

with

$$\Delta = Q_x \pm Q_y - p\ . \tag{3.102}$$

Then the new Hamilton function has the form:

$$K_1 = \frac{1}{2}\cdot(\Delta\cdot I_x \pm \Delta\cdot I_y) + \kappa_p^{\pm}\sqrt{I_x I_y}\cos(\psi_x \pm \psi_y) \tag{3.103}$$

and the equations of motions are :

$$\frac{dI_x}{d\theta} = -\frac{\partial K_1}{\partial\psi_x} = \kappa_p^{\pm}\sqrt{I_x I_y}\sin(\psi_x \pm \psi_y)\ ; \tag{3.104a}$$

$$\frac{dI_y}{d\theta} = -\frac{\partial K_1}{\partial\psi_y} = \pm\kappa_p^{\pm}\sqrt{I_x I_y}\sin(\psi_x \pm \psi_y)\ ; \tag{3.104b}$$

$$\frac{d\psi_x}{d\theta} = +\frac{\partial K_1}{\partial I_x} = \frac{1}{2}\cdot\Delta + \frac{1}{2}\kappa_p^{\pm}\sqrt{\frac{I_y}{I_x}}\cos(\psi_x \pm \psi_y)\ ; \tag{3.104c}$$

$$\frac{d\psi_y}{d\theta} = -\frac{\partial K_1}{\partial I_y} = \pm\frac{1}{2}\cdot\Delta + \frac{1}{2}\kappa_p^{\pm}\sqrt{\frac{I_x}{I_y}}\cos(\psi_x \pm \psi_y)\ . \tag{3.104d}$$

From these equations we see immediately that for the sum resonance

$$Q_x + Q_y \simeq \text{integer}$$

the difference of horizontal and vertical amplitudes is a constant:

$$I_x' = I_y' \longrightarrow I_x - I_y = \text{constant}\ . \tag{3.105}$$

This means that each of the two amplitudes can take large values and the system can be unstable. For the difference resonance

$$Q_x - Q_y \simeq \text{integer}$$

and the sum of horizontal and vertical amplitude is a constant:

$$I_x' = -I_y' \longrightarrow I_x + I_y = \text{constant}\ . \tag{3.106}$$

This means that both values are bounded so that instability is impossible. In the following we wish to investigate the two resonant cases independently.

3.5.4 The Case of the Difference Resonance

We now consider the situation where a horizontal oscillation is excited by kicking the beam and we want to see how this motion developes. The equation of motion can be solved by introducing variables [9]

$$\omega_z = \sqrt{I_z} \cdot e^{i \cdot \psi_z} \tag{3.107}$$

($z \equiv x, y$) in which the equation of motion is (the indices of κ will be omitted)

$$\frac{d}{d\theta}\,\omega_x = \frac{i}{2}\left(\kappa \cdot \omega_y + \Delta \cdot \omega_x\right) ; \tag{3.108a}$$

$$\frac{d}{d\theta}\,\omega_y = \frac{i}{2}\left(\kappa \cdot \omega_x - \Delta \cdot \omega_y\right) . \tag{3.108b}$$

The solution is

$$\begin{aligned} \begin{pmatrix} \omega_x(\theta) \\ \omega_y(\theta) \end{pmatrix} &= \frac{\omega_x(0)}{4\Omega} \cdot \left\{ \begin{pmatrix} 2\Omega + \Delta \\ \kappa \end{pmatrix} e^{i\Omega\theta} + \begin{pmatrix} 2\Omega - \Delta \\ -\kappa \end{pmatrix} e^{-i\Omega\theta} \right\} \\ &+ \frac{\omega_y(0)}{4\Omega} \cdot \left\{ \begin{pmatrix} \kappa \\ 2\Omega - \Delta \end{pmatrix} e^{i\Omega\theta} + \begin{pmatrix} -\kappa \\ 2\Omega + \Delta \end{pmatrix} e^{-i\Omega\theta} \right\} \end{aligned} \tag{3.109}$$

with

$$\Omega = \frac{1}{2}\sqrt{\kappa^2 + \Delta^2} . \tag{3.110}$$

To extract information on the resonance behaviour we consider the case

$$\begin{aligned} \omega_y(0) &= 0 \Longleftrightarrow y = 0 \; ; \; p_y = 0 \; ; \\ \omega_x(0) &\neq 0 \quad . \end{aligned} \tag{3.111}$$

For I_x, I_y the solution is then of the form:

$$I_x = \frac{I_x^0}{4\Omega^2} \cdot \left(\Delta^2 + \kappa^2 \cos^2 \Omega\theta\right) ; \tag{3.112a}$$

$$I_y = \frac{I_x^0}{4\Omega^2} \cdot \kappa^2 \sin^2 \Omega\theta . \tag{3.112b}$$

Thus during the motion we observe an emittance exchange between the x and y-planes with a frequency Ω. The coupling ratio $r = I_y^{max}/I_x^{max}$ is

$$r = \frac{\kappa^2}{\kappa^2 - \Delta^2} . \tag{3.113}$$

As expected from eqn. (3.106), the oscillations for the difference resonance remain stable.

From (3.113) it is clear that the quantity κ can be interpreted as the width of the resonance.

The starting condition

$$\omega_x(0) = 0 : \omega_y(0) \neq 0$$

of course leads to the same result.

This is an example of a coupling resonance (at $\Delta = 0$, $I_x^{max} = I_y^{max}$; see eqn. (3.108)) in which there is a large exchange of energy between both degrees of freedom but in which no instability appears.

Finally we remark that for the vertical phase we obtain, with (3.109)

$$\psi_y = \psi_y(0) = \text{constant}$$

and therefore the original variable ϕ_y becomes

$$\phi_y = \frac{1}{2}\Delta \cdot \theta + \psi_y(0)\ .$$

The development of the horizontal and vertical coordinates x, y

$$\begin{aligned}
x(s) &= \sqrt{2I_x(s)\cdot\beta_x(s)}\cdot Re\left\{e^{i[\Phi_x(s)+\phi_x(s)]}\right\} \\
&= \sqrt{2I_x(0)\cdot\beta_x(s)}\cdot Re\left\{\frac{\omega_x(s)}{\omega_x(0)}\cdot e^{i[\Phi_x(s)+\phi_x(s)-\psi_x(s)+\psi_x(0)]}\right\} \\
&= \sqrt{2I_x(0)\cdot\beta_x(s)}\cdot Re\left\{\frac{\omega_x(s)}{\omega_x(0)}\cdot e^{i[(\Phi_x(s)-\Phi_x(0))-\frac{1}{2}\Delta\cdot\Theta+\Phi_x(0)+\phi_x(0)]}\right\}\ ; \\
y(s) &= \sqrt{2I_y(s)\cdot\beta_y(s)}\cdot Re\left\{e^{i[\Phi_y(s)+\phi_y(s)]}\right\} \\
&= \sqrt{2I_x(0)\cdot\beta_y(s)}\cdot Re\left\{\frac{\omega_y(s)}{\omega_x(0)}\cdot e^{i[\Phi_y(s)+\phi_y(s)-\psi_y(s)+\psi_x(0)]}\right\} \\
&= \sqrt{2I_x(0)\cdot\beta_y(s)}\cdot Re\left\{\frac{\omega_y(s)}{\omega_x(0)}\cdot e^{i[(\Phi_y(s)-\Phi_y(0))+\frac{1}{2}\Delta\cdot\Theta+\Phi_y(0)+\psi_x(0)]}\right\}
\end{aligned}$$

over successive machine revolutions $\theta_n = 2\pi n$ after we excited an horizontal oscillation (see eqn. (3.111) can be written as:

$$\begin{aligned}
\frac{4\Omega}{\sqrt{2I_x(0)\cdot\beta_x}}\cdot x_n &= (2\Omega+\Delta)\cdot\cos[2\pi n(Q_x-\frac{1}{2}\Delta+\Omega)+\Phi_x(0)+\phi_x(0)] \quad (3.114a)\\
&+ (2\Omega-\Delta)\cdot\cos[2\pi n(Q_x-\frac{1}{2}\Delta-\Omega)+\Phi_x(0)+\phi_x(0)] \\
&= 4\Omega\cdot\cos[2\pi n(Q_x-\frac{1}{2}\Delta)+\Phi_x(0)+\phi_x(0)]\cdot\cos[2\pi n\Omega] \\
&- 2\Delta\cdot\sin[2\pi n(Q_x-\frac{1}{2}\Delta)+\Phi_x(0)+\phi_x(0)]\cdot\sin[2\pi n\Omega]\ ; \\
\frac{4\Omega}{\sqrt{2I_x(0)\cdot\beta_y\cdot\kappa}}\cdot y_n &= \cos[2\pi n(Q_y+\frac{1}{2}\Delta+\Omega)+\psi_x(0)+\Phi_y(0)] \quad (3.114b)\\
&- \cos[2\pi n(Q_y+\frac{1}{2}\Delta-\Omega)+\psi_x(0)+\Phi_y(0)]\ . \\
&= -2\cdot\sin[2\pi n(Q_y+\frac{1}{2}\Delta)+\psi_x(0)+\Phi_y(0)]\cdot\sin[2\pi n\Omega)\ .
\end{aligned}$$

We see that the motion of the y coordinate contains two frequencies:

$$Q_{I,II} = Q_y - \frac{1}{2}\Delta \pm \Omega \quad (3.115)$$

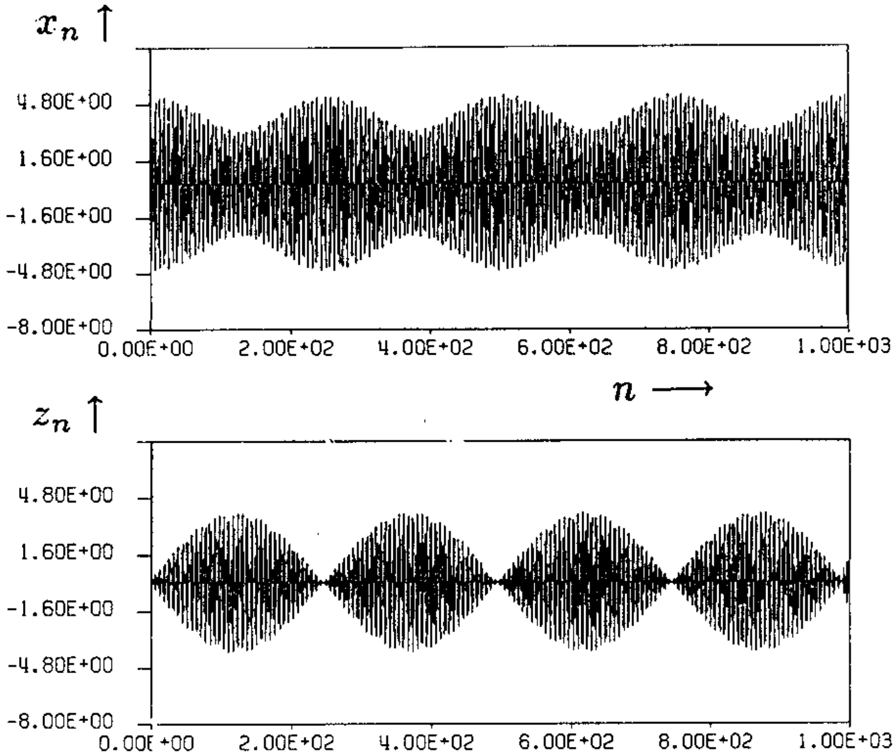

Figure 9: Coupled motion observed on successive machine turns

or, using $\Delta = Q_x - Q_y - p$:

$$Q_{I,II} = \frac{1}{2}(Q_x + Q_y - p) \pm \frac{1}{2}\sqrt{\kappa^2 + \Delta^2}\ . \tag{3.116}$$

We obtain a similar result for the motion of the x-coordinate (see Fig. 11).

Thus the obvervable tunes are now Q_I, Q_{II} and we find these in both horizontal and vertical motion. This allows us to measure the driving term κ of a difference resonance by varying the distance of the unperturbed tunes from the resonance Δ and measuring the machine tunes. Q_I, Q_{II} lie on a symmetric pair of hyperbolas which at $\Delta = 0$ are separated by the driving term κ (see Fig. 12).

3.5.5 The Sum Resonance

As mentioned above, near a sum resonance we expect unstable motion. The condition for stability is the next point we wish to examine. The equations of motion are conveniently solved by introducing the variables

$$\omega = \sqrt{I_x}\cos\psi_x + i\sqrt{I_y}\cos\psi_y \tag{3.117a}$$

$$\bar{\omega} = \sqrt{I_x}\sin\psi_x + i\sqrt{I_y}\sin\psi_y \tag{3.117b}$$

for which the equations of motion are (the indices of κ will be omitted)

$$\omega'' = \frac{1}{4}\cdot(\kappa^2 - \Delta^2)\omega \tag{3.118a}$$

$$\bar{\omega}'' = \frac{1}{4}\cdot(\kappa^2 - \Delta^2)\bar{\omega} \tag{3.118b}$$

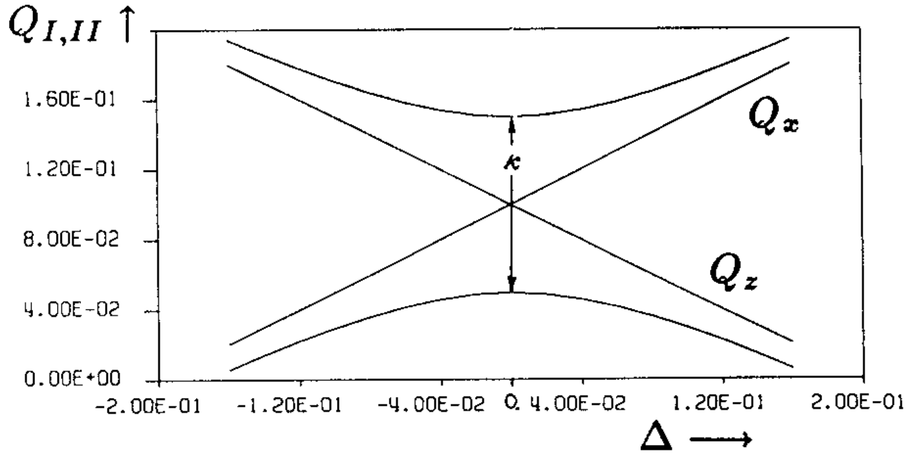

Figure 10: Coupled tunes as a function of the distance from the coupling resonance. The straight lines are the expected values for the horizontal and vertical tunes without coupling

with the solution

$$\omega = \omega_0^+ \cdot e^{i\Omega\theta} + \omega_0^- \cdot e^{-i\Omega\theta} \tag{3.119}$$

with $\Omega = \frac{1}{2}\sqrt{\Delta^2 - \kappa^2}$. We see that for $\Delta \leq \kappa$ the motion becomes unstable. Therefore we can again interpret the driving term κ as the width of the sum resonance.

4 Off-Momentum Motion. Chromaticity

4.1 Introductory Remark

Small deviations of the momentum from its nominal value change the focussing of the particles and lead to distortions of the beam optics. A crucial aspect of these distortions is a tune shift due to a momentum deviation. The linear part of the tune shift

$$\xi_z = \partial Q_z / \partial\delta ||_{\delta=0}$$

is called chromaticity. [1] In the sections which follow we want to focus on this linear tune shift.

In large machines in particular, uncompensated chromatic effects cause intolerable impairment of the machine performance. The most obvious effect is that the tunes will be forced to periodically cross unstable resonances when the particles perform synchrotron oscillations. There is also a collective instability in which the chromaticity plays an important role, the head tail instability. Consider a bunched beam of particles. The particles in the head of the beam induce transverse fields which excite the particles in the tail. Because the particles in head and tail exchange their position in half a synchrotron period there is a feedback effect on the driving particles which may lead to instability. The condition for instability is that the

[1] Sometimes, the chromaticity is alternatively defined as $\frac{1}{Q_z}\partial Q_z/\partial\delta$.

particles in the head which have a negative momentum deviation oscillate somewhat faster than the ones in the tail. This can be the case if the chromaticity is not compensated.

As a consequence, the chromaticity has to be compensated. This is achieved by introducing sextupole magnets into the lattice. (This is also the reason why we included these nonlinear magnetic elements in the linear machine theory.) They are placed at positions where the dispersion function is different from zero. Thus the off-momentum closed orbit (in linear approximation, the dispersion orbit as introduced in section 2.7) passes off centre through these magnets. The sextupole field expanded about this orbit provides a linear focussing term proportional to the momentum deviation δ as shown in section 2.8 (eqn. (2.58-60)). One makes use of this effect to compensate the chromaticity.

In order to determine the required strength of sextupolar correction elements, one needs to calculate the chromaticity in the quadrupoles and dipoles. Although this can be done exactly by solving the equations of motion as derived from a Hamiltonian including terms propotional to δ as given by eqn. (2.57), one usually prefers to treat the chromaticity by perturbation theory. The results are then expressed in terms of the on-momentum optical functions. This formulation of the chromaticity has the advantage that the beam optics need not to be recalculated for every value of the momentum deviation. The procedure results in a compact expression for the chromaticity which contains all the effects proportional to δ. After an introduction to the method which follows the procedure already given by Courant and Snyder [2], the exact expression for the chromaticity is derived in section 4.2.2.

4.2 Calculation of Chromaticity

4.2.1 Demonstration of the Method on a Simplified Example

In order to demonstrate the perturbation procedure on a simple example, we calculate the linear tune shift with momentum for uncoupled betatron oscillations. Let us first consider how a single thin lens quadrupole at position s modifies the revolution matrix $\underline{M}$:

$$\underline{M} + \delta\underline{M} = \begin{pmatrix} 1 & 0 \\ \Delta k \cdot ds & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos 2\pi Q + \alpha(s)\sin 2\pi Q & \beta(s)\sin 2\pi Q \\ -\gamma(s)\sin 2\pi Q & \cos 2\pi Q - \alpha(s)\sin 2\pi Q \end{pmatrix} .$$

The elements of the modified revolution matrix $\underline{M} + \delta\underline{M}$ are composed of the modified lattice functions and tunes:

$$\underline{M} + \delta\underline{M} =$$

$$\begin{pmatrix} \cos 2\pi(Q+\delta Q) + (\alpha+\delta\alpha)\sin 2\pi(Q+\delta Q) & (\beta+\delta\beta)\sin 2\pi(Q+\delta Q) \\ -(\gamma+\delta\gamma)\sin 2\pi(Q+\delta Q) & \cos 2\pi(Q+\delta Q) - (\alpha+\delta\alpha)\sin 2\pi(Q+\delta Q) \end{pmatrix}$$

Thus by comparing the last two equations for the trace of $\underline{M} + \delta\underline{M}$ we obtain:

$$\begin{aligned} tr\{\underline{M} + \delta\underline{M}\} &= 2\cos 2\pi(Q+\delta Q) \\ &= 2\cos 2\pi Q + \beta(s)\sin 2\pi Q \cdot \Delta k(s) \cdot ds \end{aligned}$$

or

$$\frac{\cos 2\pi(Q+\delta Q) - \cos 2\pi Q}{\sin 2\pi(Q+\delta Q)} = \frac{1}{2}\beta(s)\Delta k(s) \cdot ds \; . \tag{4.1}$$

We consider now several successive modifications of $\underline{M}$ by thin lens quadrupoles at different locations and we work to first order in Δk. Thus we neglect small changes to β at one position caused by Δk at another position. This means that the contributions to the tune change appear as a sum on the right hand side of eqn. (4.1):

$$\frac{\cos 2\pi(Q+\delta Q) - \cos 2\pi Q}{\sin 2\pi Q} = \frac{1}{2}\sum_i \beta(s_i)\Delta k(s_i) \cdot ds \ . \tag{4.2}$$

In the case of a continuous distortion $\Delta k(s)$ (which is the case for chromatic focussing errors), the sum transforms into an integral around the lattice:

$$\frac{\cos 2\pi(Q+\delta Q) - \cos 2\pi Q}{\sin 2\pi Q} = \frac{1}{2}\int_0^L \beta(s)\Delta k(s) \cdot ds \ . \tag{4.3}$$

Expanding $cos2\pi(Q+\delta Q)$ about Q eventually leads to:

$$\Delta Q = -\frac{1}{4\pi}\int_0^L \beta(s)\Delta k(s) \cdot ds \ . \tag{4.4}$$

For $\Delta k(s)$, we insert the contributions to the momentum dependent focussing error as given in chapter 2. In a strong focussing machine, the main contribution usually comes from the quadrupoles and sextupoles:

$$\Delta k_x(s) \simeq +\delta \cdot [k(s) - D_x\lambda(s)] \ .$$

In small circumference machines in particular, the contributions from the dipole magnets are not negligible. A rigorous derivation of the chromaticity in the general case is given in the next section.

4.2.2 Exact Expression of the Chromaticity

The first step in a rigorous calculation of the Q-shift for off energy particles which also takes into account coupling is to again write the (4-dimensional) transfer matrix as

$$\underline{M}(s,s_0) + \delta\underline{M}(s,s_0)$$

where $\underline{M}(s,s_0)$ refers to the unperturbed motion of the on-energy particles. We achieve that by decomposing the ring into small intervals:

$$s_\mu \le s \le s_\mu + \Delta s_\mu \ ; \ (\mu = 1,2,...,p)$$

and by considering the perturbation

$$\Delta\underline{A}(s) = \delta \cdot \underline{B}(s)$$

appearing in the equations of motion (2.58) in each interval separately.

Then we can write:

$$\begin{aligned} & \underline{M}(s_0+L,s_0) + \delta\underline{M}(s_0+L,s_0) \\ = \ & \underline{M}(s_0+L,s_p) \cdot \underline{M}^{-1}(s_p+\Delta s_p,s_p) \cdot [\underline{M}(s_p+\Delta s_p,s_p) + \delta\underline{M}(s_p+\Delta s_p,s_p)] \\ \times \ & \underline{M}(s_p,s_{p-1}) \cdot \underline{M}^{-1}(s_{p-1}+\Delta s_{p-1},s_{p-1}) \end{aligned}$$

$$\times\ [\underline{M}(s_{p-1} + \Delta s_{p-1}, s_{p-1}) + \delta\underline{M}(s_{p-1} + \Delta s_{p-1}, s_{p-1})]$$

$$*$$

$$*$$

$$*$$

$$\times\ \underline{M}(s_{\mu+1}, s_\mu) \cdot \underline{M}^{-1}(s_\mu + \Delta s_\mu, s_\mu) \cdot [\underline{M}(s_\mu + \Delta s_\mu, s_\mu) + \delta\underline{M}(s_\mu + \Delta s_\mu, s_\mu)]$$

$$*$$

$$*$$

$$*$$

$$\begin{aligned} \times\ & \underline{M}(s_2, s_1) \cdot \underline{M}^{-1}(s_1 + \Delta s_1, s_1) \cdot [\underline{M}(s_1 + \Delta s_1, s_1) + \delta\underline{M}(s_1 + \Delta s_1, s_1)] \\ \times\ & \underline{M}(s_1, s_0)\ . \end{aligned} \tag{4.5}$$

According to eqn. (2.58) we have:

$$\underline{M}(s + \Delta s, s) = \underline{1} + \Delta s \cdot \underline{A}_0(s)\ ; \tag{4.6a}$$

$$\underline{M}(s + \Delta s, s) + \delta\underline{M}(s + \Delta s, s) = \underline{1} + \Delta s \cdot [\underline{A}_0(s) + \Delta\underline{A}(s)]\ , \tag{4.6b}$$

so that the factor

$$\underline{M}^{-1}(s_\mu + \Delta s_\mu, s_\mu) \cdot [\underline{M}(s_\mu + \Delta s_\mu, s_\mu) + \delta\underline{M}(s_\mu + \Delta s_\mu, s_\mu)] \tag{4.7}$$

can be written as

$$\begin{aligned} & \underline{M}^{-1}(s + \Delta s, s) \cdot [\underline{M}(s + \Delta s, s) + \delta\underline{M}(s + \Delta s, s)] \\ = \ & [\underline{1} - \Delta s \cdot \underline{A}_0(s)] \cdot [\underline{1} + \Delta s \cdot \underline{A}_0(s) + \Delta s \cdot \Delta\underline{A}(s)] \\ = \ & \underline{1} + \Delta s \cdot \Delta\underline{A}(s)\ . \end{aligned} \tag{4.8}$$

Then eqn. (4.5) becomes:

$$\begin{aligned} & \underline{M}(s_0 + L, s_0) + \delta\underline{M}(s_0 + L, s_0) \\ = \ & \underline{M}(s_0 + L, s_p) \cdot [\underline{1} + \Delta s_p \cdot \Delta\underline{A}(s_p)] \\ \times\ & \underline{M}(s_p, s_{p-1}) \cdot [\underline{1} + \Delta s_{p-1} \cdot \Delta\underline{A}(s_{p-1})] \\ & * \\ & * \\ & * \\ \times\ & \underline{M}(s_{\mu+1}, s_\mu) \cdot [\underline{1} + \Delta s_\mu \cdot \Delta\underline{A}(s_\mu)] \\ & * \\ & * \\ & * \\ \times\ & \underline{M}(s_2, s_1) \cdot [\underline{1} + \Delta s_1 \cdot \Delta\underline{A}(s_1)] \\ \times\ & \underline{M}(s_1, s_0) \end{aligned} \tag{4.9}$$

and the expression for $\delta\underline{M}$ is in first approximation:

$$\begin{aligned} & \delta\underline{M}(s_0 + L, s_0) \\ = \ & \int_{s_0}^{s_0+L} d\tilde{s} \cdot \underline{M}(s_0 + L, \tilde{s}) \cdot \Delta\underline{A}(\tilde{s}) \cdot \underline{M}(\tilde{s}, s_0) \\ = \ & \underline{M}(s_0 + L, s_0) \cdot \int_{s_0}^{s_0+L} d\tilde{s} \cdot \underline{M}^{-1}(\tilde{s}, s_0) \cdot \Delta\underline{A}(\tilde{s}) \cdot \underline{M}(\tilde{s}, s_0)\ . \end{aligned} \tag{4.10}$$

The Q-shift is then calculated by considering the eigenvalues of the perturbed one turn matrix $\underline{M} + \delta\underline{M}$[5]:

$$(\underline{M} + \delta\underline{M})(\vec{v}_k + \delta\vec{v}_k) = (\lambda_k + \delta\lambda_k)(\vec{v}_k + \delta\vec{v}_k) \ .$$

Using the fact that

$$\underline{M}\vec{v}_k = \lambda_k\vec{v}_k$$

we obtain in first order:

$$\underline{M}\delta\vec{v}_k + \delta\underline{M}\vec{v}_k = \lambda_k \cdot \delta\vec{v}_k + \delta\lambda_k \cdot \vec{v}_k \ . \tag{4.11}$$

The tune shift is defined by

$$\Delta Q_k = -\frac{i}{2\pi \cdot \lambda_k} \cdot \delta\lambda_k \ . \tag{4.12}$$

$\delta\vec{v}_k$ can be written in terms of the eigenvectors $\vec{v}_I, \vec{v}_{-I}, \vec{v}_{II}, \vec{v}_{-II}$:

$$\delta\vec{v}_k = \sum_l A_{kl} \cdot \vec{v}_l \ .$$

Then by multiplying eqn. (4.11) from the left by $(\vec{v}_k^+\underline{S})$ and by using the orthogonality relations (3.37) we get:

$$\vec{v}_k^{\ +}\underline{S} \cdot \delta\underline{M} \cdot \vec{v}_k = \delta\lambda_k \cdot i \ .$$

Recalling that

$$\underline{M}^T(s_1, s_2) \cdot \underline{S} \cdot \underline{M}(s_1, s_2) = \underline{S}$$

so that

$$\begin{aligned}
\vec{v}_k^{\ +}(s_0) \cdot \underline{S} \cdot \underline{M}(s_0 + L, s_0) &= \vec{v}_k^{\ +}(s_0) \cdot \left[\underline{M}^{-1}(s_0 + L, s_0)\right]^T \cdot \underline{S} \\
&= \left[\underline{M}^{-1}(s_0 + L, s_0) \cdot \vec{v}_k(s_0)\right]^+ \cdot \underline{S} \\
&= \left[\lambda_k^{-1} \cdot \vec{v}_k(s_0)\right]^+ \cdot \underline{S} \\
&= \lambda_k \cdot \vec{v}_k^{\ +}(s_0) \cdot \underline{S}
\end{aligned}$$

we find that eqns. (4.10) and (4.12) give :

$$\begin{aligned}
\Delta Q_k &= -\frac{1}{2\pi \cdot \lambda_k} \cdot \vec{v}_k^{\ +}(s_0) \cdot \underline{S} \cdot \underline{M}(s_0 + L, s_0) \\
&\quad \times \int_{s_0}^{s_0+L} d\tilde{s} \cdot \underline{M}^{-1}(\tilde{s}, s_0) \cdot \Delta\underline{A}(\tilde{s}) \cdot \underline{M}(\tilde{s}, s_0) \cdot \vec{v}_k(s_0) \\
&= -\frac{1}{2\pi} \cdot \vec{v}_k^{\ +}(s_0) \cdot \int_{s_0}^{s_0+L} d\tilde{s} \cdot \underline{M}^T(\tilde{s}, s_0) \cdot \underline{S} \cdot \Delta\underline{A}(\tilde{s}) \cdot \underline{M}(\tilde{s}, s_0) \cdot \vec{v}_k(s_0) \\
&= -\frac{1}{2\pi} \cdot \int_{s_0}^{s_0+L} d\tilde{s} \cdot \vec{v}_k^{\ +}(\tilde{s}) \cdot \underline{S} \cdot \Delta\underline{A}(\tilde{s}) \cdot \vec{v}_k(\tilde{s}) \\
&= -\delta \cdot \frac{1}{2\pi} \cdot \int_{s_0}^{s_0+L} d\tilde{s} \cdot \vec{v}_k^{\ +}(\tilde{s}) \cdot \underline{S} \cdot \underline{B}(\tilde{s}) \cdot \vec{v}_k(\tilde{s})
\end{aligned} \tag{4.13}$$

where $\underline{B}$ is given by eqn. (2.60).

Since in chap. 2 we have been careful to keep our treatment of the equations of motion symplectic, $\Delta\underline{A}$ is free from damping terms. Thus

$$\underline{S} \cdot \Delta\underline{A} = -\Delta\underline{A}^{+} \cdot \underline{S} \tag{4.14}$$

(see eqn.(2.65) and (2.58b)). Therefore by eqn. (4.13), ΔQ_k is real:

$$\Delta Q_k = \Delta Q_k^{*} \; . \tag{4.15}$$

With eqn. (4.13) we are in a position to calculate the tune shift, resulting from an energy deviation, for an arbitrary distribution of (pointlike) sextupoles:

$$\lambda(s) = \sum_{\mu} \lambda^{(\mu)} \cdot \delta(s - s_{\mu}) \; . \tag{4.16}$$

The aim of the chromaticity correction is to find a distribution where ΔQ_I and ΔQ_{II} vanish. Then resonance crossing due to the synchrotron oscillations can be avoided.

Eqn. (4.13) is valid for an arbitrarily coupled machine. In the special case of an uncoupled machine (no skew quadrupoles, no solenoids) we have (see eqns. (3.41,45,46,67)):

$$\vec{v}_I(s) = \frac{1}{\sqrt{2\beta_x(s)}} \cdot \begin{pmatrix} \beta_x(s) \\ -[\alpha_x(s) - i] \\ 0 \\ 0 \end{pmatrix} \cdot e^{i\Phi_x(s)} \, , \tag{4.17a}$$

$$\vec{v}_{II}(s) = \frac{1}{\sqrt{2\beta_z(s)}} \cdot \begin{pmatrix} 0 \\ 0 \\ \beta_z(s) \\ -[\alpha_z(s) - i] \end{pmatrix} \cdot e^{i\Phi_z(s)} \tag{4.17b}$$

and the matrix $\underline{B}$ takes the form (eqn. (2.60) with $N = 0$; $R = 0$):

$$\underline{B} = \begin{pmatrix} K_x D_2 & (K_x D_1 + K_y D_3) & K_y D_2 & 0 \\ (K_x^2 + k) - \lambda \cdot D_1 & -K_x D_2 & \lambda \cdot D_3 & -K_x D_4 \\ K_x D_4 & 0 & K_y D_4 & (K_x D_1 + K_y D_3) \\ \lambda \cdot D_3 & -K_y D_2 & (K_y^2 - k) + \lambda \cdot D_1 & -K_y D_4 \end{pmatrix} .$$

In this case eqn. (4.13) simplifies to:

$$\Delta Q_x = -\frac{\delta}{4\pi} \int_{s_0}^{s_0+L} d\tilde{s} \left\{ \beta_x [K_x^2 + k - \lambda \cdot D_1] + 2\alpha_x K_x D_2 - \gamma_x [K_x D_1 + K_y D_3] \right\} \tag{4.18a}$$

$$\Delta Q_y = -\frac{\delta}{4\pi} \int_{s_0}^{s_0+L} d\tilde{s} \left\{ \beta_z [K_y^2 - k + \lambda \cdot D_1] + 2\alpha_y K_y D_4 - \gamma_y [K_x D_1 + K_y D_3] \right\} \tag{4.18b}$$

Note that the matrix elements B_{13}, B_{14}, B_{23} and B_{31}, B_{32}, B_{41} which are a source of coupling for off- energy particles give no contribution to the tune shift in linear order.

We note that Courant-Snyder and other authors obtain only the first terms in brackets in eqn. (4.18a,b). The second and third terms which we find in addition result from the use of a second order expansion of the dispersion trajectories: Some of these terms survive after linearisation.

5 Summary

In this lecture we have given a survey of linear machine theory taking into account in a general way the coupling of betatron oscillations.

The equations of motion for on- and off-momentum particles were derived in a strictly canonical manner. Because all transfer matrices are therefore symplectic, well known eigenvector methods can be used to investigate the stability of motion and to estimate the tune schifts due to an momentum deviation. To study the influence of coupling, generalized lattice functions were introduced and canonical perturbation techniques were applied.

Here we only have considered the betatron oscillations of a coasting beam. But it is possible to generalize the 4-dimensional formalism (in terms of the variables x, p_x, y, p_y) described here to include synchrotron oscillations. To achieve that, additional coordinates $\sigma = s - v_0 \cdot t$ and $\eta = \Delta E/E_0$ (v_0 =average velocity of the particles) must be introduced. With the complete set $x, p_x, y, p_y, \sigma, \eta$ one then is in a position to provide, in the framework of this 6-dimensional formalism, an analytical technique which includes consistently and canonically the synchrotron oscillations in the electric fields of the accelerating cavities. Details can be found in [10,11,12,13].

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Appendix A: The Symplecticity Condition

The canonical equations of motion can be written as (see eqns. (2.25) and (2.55)):

$$\frac{d}{ds}\,\vec{z} = \underline{S} \cdot \frac{\partial H}{\partial \vec{z}} \tag{A.1}$$

or in component form as

$$z_i' = S_{ik} \cdot \frac{\partial H}{\partial z_k} \tag{A.2}$$

with the notation

$$\vec{z}^{\,T} = (z_1, z_2, z_3, z_4) \ .$$

We now introduce the Jacobian matrix:

$$\begin{aligned} \underline{J} &= ((J_{ik})) \ ; \\ J_{ik}(s, s_0) &= \frac{\partial z_i(s)}{\partial z_k(s_0)} \ . \end{aligned} \tag{A.3}$$

Then it follows that:

$$\frac{d}{ds}\,J_{ik}(s, s_0) = \frac{\partial}{\partial z_k(s_0)}\,z_i'(s)$$

$$\begin{aligned}
&= \frac{\partial}{\partial z_k(s_0)}\left[S_{in}\cdot\frac{\partial H(s)}{\partial z_n(s)}\right] \\
&= \frac{\partial z_l(s)}{\partial z_k(s)_0}\cdot\frac{\partial}{\partial z_l(s)}\left[S_{in}\cdot\frac{\partial H(s)}{\partial z_n(s)}\right] \\
&= J_{lk}(s,s_0)\cdot S_{in}\cdot\frac{\partial^2 H(s)}{\partial z_l(s)\partial z_l(n)} \\
&= S_{in}\cdot H_{nl}\cdot J_{lk} \qquad (A.4)
\end{aligned}$$

with

$$H_{nl} = \frac{\partial^2 H(s)}{\partial z_l(s)\partial z_n(s)} \qquad (A.5)$$

or that

$$\underline{J}'(s,s_0) = \underline{S}\cdot\underline{H}\cdot\underline{J}(s,s_0) \qquad (A.6)$$

with

$$\underline{H} = ((H_{ik}))\ .$$

Thus we have:

$$\begin{aligned}
\frac{d}{ds}\left\{\underline{J}^T(s,s_0)\cdot\underline{S}\cdot\underline{J}(s,s_0)\right\} &= \{\underline{S}\cdot\underline{H}\cdot\underline{J}(s,s_0)\}^T\cdot\underline{S}\cdot\underline{J} + \underline{J}^T(s,s_0)\cdot\underline{S}\cdot\{\underline{S}\cdot\underline{H}\cdot\underline{J}(s,s_0)\} \\
&= \underline{J}^T(s,s_0)\cdot\underline{H}^T\cdot\underline{S}^T\cdot\underline{S}\cdot\underline{J} + \underline{J}^T(s,s_0)\cdot\underline{S}^2\cdot\underline{H}\cdot\underline{J} \\
&= \underline{J}^T(s,s_0)\cdot\underline{H}\cdot\underline{J}(s,s_0) - \underline{J}^T(s,s_0)\cdot\underline{H}\cdot\underline{J}(s,s_0) \\
&= 0 \qquad (A.7)
\end{aligned}$$

where we have used the relations

$$\begin{aligned}
\underline{S}^T &= -\underline{S}\ ; \\
\underline{S}^2 &= -\underline{1}\ ; \\
\underline{H}^T &= \underline{H}\ .
\end{aligned}$$

From (A.7) we obtain:

$$\begin{aligned}
\underline{J}^T(s,s_0)\cdot\underline{S}\cdot\underline{J}(s,s_0) &= \text{const.}\ . \\
&= \underline{J}^T(s_0,s_0)\cdot\underline{S}\cdot\underline{J}(s_0,s_0) \\
&= \underline{S}\ . \qquad (A.8)
\end{aligned}$$

If the Hamiltonian is quadratic in z_1, z_2, z_3, z_4 as in (2.56) one has according to (A.4):

$$\underline{J}(s,s_0) = \underline{M}(s,s_0)\ . \qquad (A.9)$$

In this case eqn. (A.8) reads as

$$\underline{M}^T(s,s_0)\cdot\underline{S}\cdot\underline{M}(s,s_0) = \underline{S}\ . \qquad (A.10)$$

Equation (A.10) represents the "symplecticity-condition" for the transfer matrix $\underline{M}(s,s_0)$.

Appendix B: Thin Lens Approximation

The equations of motion (2.58) have the general form:

$$\frac{d}{ds}\,\vec{\bar{z}} = \underline{A}(s)\cdot\vec{\bar{z}} \tag{B.1}$$

with

$$\begin{aligned}\underline{A}(s) &= \underline{A}_0(s) + \delta\cdot\underline{B}(s) \\ &= [1+\delta\cdot(K_x\cdot D_1 + K_y\cdot D_3)]\cdot\underline{C}_0(s) + \\ &\quad \underline{C}_1(s)+\underline{C}_2(s)+\underline{C}_3(s)+\underline{C}_4(s)\end{aligned} \tag{B.2}$$

and

$$\underline{C}_0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \tag{B.3a}$$

$$\underline{C}_1(s) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ G_1 & 0 & -\hat{N} & 0 \\ 0 & 0 & 0 & 0 \\ -\hat{N} & 0 & G_2 & 0 \end{pmatrix}; \tag{B.3b}$$

$$\underline{C}_2(s) = \frac{1}{2}\hat{R}\cdot\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \tag{B.3c}$$

$$\underline{C}_3(s) = \delta\cdot\begin{pmatrix} K_xD_2 & 0 & 0 & 0 \\ 0 & -K_xD_2 & 0 & 0 \\ 0 & 0 & K_yD_4 & 0 \\ 0 & 0 & 0 & -K_yD_4 \end{pmatrix}; \tag{B.3d}$$

$$\underline{C}_4(s) = \delta\cdot\begin{pmatrix} 0 & 0 & K_yD_2 & 0 \\ 0 & 0 & 0 & -K_xD_4 \\ K_xD_4 & 0 & 0 & 0 \\ 0 & -K_yD_2 & 0 & 0 \end{pmatrix} \tag{B.3e}$$

where G_1, G_2, $\hat{N}$ and $\hat{R}$ are defined by:

$$G_1 = -\frac{1}{4}R^2 - \left(K_x^2 + k\right) + \delta\cdot\left[\frac{1}{2}R^2 + \left(K_x^2 + k\right) - \lambda\cdot D_1\right]; \tag{B.4a}$$

$$G_2 = -\frac{1}{4}R^2 - \left(K_y^2 - k\right) + \delta\cdot\left[\frac{1}{2}R^2 + \left(K_y^2 - k\right) + \lambda\cdot D_1\right]; \tag{B.4b}$$

$$\hat{N} = N\cdot(1-\delta) - \delta\cdot\lambda\cdot D_3; \tag{B.4c}$$

$$\hat{R} = R\cdot(1-\delta). \tag{B.4d}$$

The solution of eqn. (B.1) can now be obtained using the method of the thin lens approximation .

For this, we divide a lens into a sufficient number of thin lenses with length Δs for which, in the power series expansion of the matrix $\underline{M}(s + \Delta s/2, s - \Delta s/2)$, only the linear terms in Δs are needed. Then for the transfer matrix $\underline{M}$ we obtain in first approximation :

$$\underline{M}\left(s + \frac{\Delta s}{2}, s - \frac{\Delta s}{2}\right) = \underline{1} + \Delta s \cdot \underline{A}(s) \; . \tag{B.5}$$

However, $\underline{\mathrm{M}}$ so calculated is not symplectic (see eqn. (2.64)) for finite Δs.

To ensure that the symplecticity of the matrix $\underline{M}(s + \Delta s/2, s - \Delta s/2)$ is rigorously maintained by the linearisation we write

$$\begin{aligned} \underline{M}\left(s + \frac{\Delta s}{2}, s - \frac{\Delta s}{2}\right) &= \underline{M}_D\left(s + \frac{\Delta s}{2}, s\right) \cdot \underline{C}(s) \\ &\quad \times \underline{\hat{C}}_1(s) \cdot \underline{\hat{C}}_2(s) \cdot \underline{\hat{C}}_3(s) \cdot \underline{\hat{C}}_4(s) \cdot \underline{M}_D\left(s, s - \frac{\Delta s}{2}\right) \end{aligned} \tag{B.6}$$

with

$$\underline{M}_D(s + l, s) = \underline{1} + l \cdot \underline{C}_0 \; ; \tag{B.7}$$

(transfer matrix for a simple drift space of length l)

and

$$\underline{C}(s) = \underline{1} + \delta \cdot (K_x \cdot D_1 + K_y \cdot D_3) \cdot \underline{C}_0 \cdot \Delta s \; ; \tag{B.8a}$$

$$\underline{\hat{C}}_1(s) = \underline{1} + \underline{C}_1 \cdot \Delta s \; ; \tag{B.8b}$$

$$\underline{\hat{C}}_2(s) = \begin{pmatrix} cos\Delta\Theta & 0 & +sin\Delta\Theta & 0 \\ 0 & cos\Delta\Theta & 0 & +sin\Delta\Theta \\ -sin\Delta\Theta & 0 & cos\Delta\Theta & 0 \\ 0 & -sin\Delta\Theta & 0 & cos\Delta\Theta \end{pmatrix} ; \tag{B.8c}$$

$$(\Delta\Theta = \frac{1}{2} R \cdot \Delta s) \; ;$$

$$\underline{\hat{C}}_3(s) = \begin{pmatrix} (1 + A_x) & 0 & 0 & 0 \\ 0 & (1 + A_x)^{-1} & 0 & 0 \\ 0 & 0 & (1 + A_y) & 0 \\ 0 & 0 & 0 & (1 + A_y)^{-1} \end{pmatrix} ; \tag{B.8d}$$

$$A_x = \delta \cdot K_x D_2 \cdot \Delta s \; ;$$
$$A_y = \delta \cdot K_y D_4 \cdot \Delta s \; ;$$

$$\underline{\hat{C}}_4(s) = \underline{1} + \underline{C}_4 \cdot \Delta s \; . \tag{B.8e}$$

In linear order, the right hand side of (B.6) agrees with the r.h.s. of eqn. (B.5). Furthermore, all factor matrices on the r.h.s. of (B.6) and therefore $\underline{M}(s + \Delta s/2, s - \Delta s/2)$ itself are symplectic.

In this way, the linear approximation (B.5) for $\underline{M}(s + \Delta s/2, s - \Delta s/2)$ can be made symplectic by adding terms of higher order in Δs.

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