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INTRODUCTION TO CONFORMAL INVARIANT QUANTUM FIELD THEORY IN TWO AND MORE DIMENSIONS[1]

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What is conformal invariant quantum field theory? And why does it live on a tube? The first sections of these lectures will try to answer this. In the literature the discussion is usually purely Euclidean. But it is much simpler and more transparent to start from a quantum field theory in Minkowski space. I will review this approach in sections I-III; the discussion is based on joint work with M. Lüscher. It yields the Virasoro algebra for 2-dimensional theories in a straightforward way. At the same time it makes clear that 2 dimensions are not as special as is often believed. Section V will answer the question what is the substitute for the Virasoro algebra in d dimensions.

The relation between quantum field theory in Minkowski space and Euclidean field theory is of course well known, and the results of the Minkowski space analysis are easily transcribed into Euclidean language (section IV). It is less generally known that there exists also a (semi-)group theoretical version of this correspondence which has produced important results and promises more (section VI).

A great deal is known about conformal invariant quantum field theory in $d \geq 2$ dimensions from work which was done in the seventies [3-22]. Space does not permit

[1]Cargese lectures 1987. To appear in: *Nonperturbative Quantum Field Theory*, G. t'Hooft et al (eds), Plenum N.Y. 1988

to review it all, but I would like to mention very briefly some of the results that are not covered here.

Uniqueness of conformal invariant two and three point functions formed the basis of the Migdal Polyakov bootstrap approach which constructed conformal invariant Greens functions by skeleton graph expanions [10] . These expansions are the iterative solution of the infinite set of coupled Schwinger Dyson (SD) equations for all the Euclidean Greens functions. A group theoretical approach to conformal QFT was started by the author [14] . It used conformal partial wave expansions of Euclidean Greens functions to show that the ∞ many SD-eqs. amount to the requirement of poles in the partial wave amplitudes with factorizable residues. Roughly speaking, operator product expansions are the solution of these constraints. Using these partial wave expansions , the physical principles of QFT (Osterwalder Schrader positivity, invariance and locality) were reduced to a crossing relation for partial wave amplitudes for 4-point functions (and positivity of their real residues).

This crossing relation was in Euclidean space. There exists also a Minkowski space version. They were also studied independently by Ferrara, Gatto, Grillo and Parisi, and by Polyakov. The two crossing relations can be related by use of the semigroup technology (see e.g. [13]).

The mathematical meaning of the Minkowskian crossing relation rests crucially on the convergence of operator product expansions (in Minkowski space) on the vacuum. Ferrara et al. [15] had shown that the contributions of all derivatives $\partial_{\mu_1} \ldots \partial_{\mu_n} \phi^k(z)$ of a nonderivative ("quasiprimary") field ϕ^k to Wilson operator product expansions could be summed up with the result

$$\phi^i(x)\phi^j(y)|\Omega >= \sum_k \int dz \phi^k(z)|\Omega > B^{kij}(z;xy).$$

Integration is over Minkowski space and the kernels $B^{kij}(z;xy)$ are determined by conformal invariance up to certain normalization constants, given the dimension of the fields [19]. This holds as asymptotic expansion for arbitrary states $|\Omega >$ of finite energy. The expansions are convergent if $|\Omega >= vacuum$. Convergence was proven in [19] by pointing out that the expansions are nothing but the decomposition of a unitary representation of the quantum mechanical conformal group into its irreducible components. I lectured at length on the use of convergent operator product expansions in conformal QFT at the 1976 Cargese school [17]; here I will refer the reader to ref.[18].

Also omitted is the study of models. A Virasoro algebra for the Thirring model was constructed as early as 1972 by Ferrara, Gatto and Grillo [3].

I. SCALE AND CONFORMAL INVARIANCE IN d DIMENSIONS

In the last 15 years Quantum Field Theory (QFT) and Classical Statistical Mechanics (CSM) have merged into a single discipline [1]. Nonperturbative studies in quantum field theory are nearly always done in the language of classical statistical mechanics. On the other hand, conformal invariance is important for the study of systems of classical statistical mechanics[1] at a critical point, but the crucial object to study is the stress energy tensor of a quantum field theory in Minkowski space.

[1]It may one day become important again for elementary particle physics if $N = 4$ supersymmetric Yang-Mills theory describes nature, for instance.

According to Wilsons renormalization group theory [2], the long distance behavior of a system at a critical point is goverened by a renormalization group fixed point, and is therefore described by an exactly scale invariant Euclidean Field Theory. If the theory has Osterwalder Schrader positivity (which is true if there is a theory with nearest neighbour interaction and time reversal invariance in the same universality class) then one can analytically continue to obtain a scale invariant quantum field theory in Minkowski space which obeys the usual principles. It can be expected to have a conserved symmetric traceless stress energy tensor $\Theta_{\mu\nu}$ which defines the generators of space time symmetries

$$\Theta_{\mu\nu}(x) = \Theta_{\nu\mu}(x) = \Theta_{\mu\nu}(x)^*, \qquad \partial^\mu \Theta_{\mu\nu}(x) = 0.$$

$$\begin{aligned} P_\mu &= \int d\mathbf{x}\Theta_{\mu 0}(x^0, \mathbf{x}), \\ M_{\mu\nu} &= \int d\mathbf{x}(x_\mu \Theta_{\nu 0} - x_\nu \Theta_{\mu 0}), \\ D &= \int d\mathbf{x}\mathcal{D}_0, \qquad \mathcal{D}_\mu(x) = x^\nu \Theta_{\nu\mu}(x). \end{aligned} \tag{1}$$

If Θ is traceless then D is time independent and scale invariance holds. (The renormalization group picture does not permit spontaneous breaking of scale invariance, because a scale invariant effective potential is a homogeneous function of the background field). In two dimensions the converse is also true. In more dimensions, $\Theta_{\mu\nu}$ may have to be redefined to make it traceless [4]. But if Θ is traceless then also the generator K_μ of conformal transformations is conserved.

$$K_\mu = \int \mathcal{K}_{\mu 0}, \qquad \mathcal{K}_{\mu\rho} = 2x_\mu x^\nu \Theta_{\nu\rho} - x^2 \Theta_{\mu\rho} \tag{2a}$$

More generally we have the relation

$$\partial^\rho \mathcal{K}_{\mu\rho} = 2x_\mu \partial^\nu \mathcal{D}_\nu \tag{2b}$$

This relation was first derived in [5,6] for Lagrangean field theories (under specified conditions on possible derivative couplings which were shown to be fulfilled for Yang Mills theories, for instance).

Scale- and conformal transformations in d-dimensional space time act according to

$$\begin{aligned} &x^\mu \mapsto \rho x^\mu, \quad \rho > 0, && ds^2 \mapsto \rho^2 ds^2 \quad \text{dilatations} \\ &x^\mu \mapsto \sigma(x)^{-1}(x^\mu - c^\mu x^2), && ds^2 \mapsto \sigma(x)^{-2} ds^2 \quad \text{conf. transf.} \\ &\sigma(x) = 1 - 2c \cdot x + c^2 x^2 \end{aligned} \tag{3}$$

The d-vector c^μ parametrizes conformal transformations. The transformation law of fields with dimension of mass l and arbitrary spin is furnished by the theory of induced representations [6] ($\Sigma_{\mu\nu}$ = spin representation matrices of the Lorentz group)

$$\begin{aligned} [\phi(x),\ D] &= i(l + x_\nu \partial^\nu)\phi(x) \\ [\phi(x), K_\mu] &= -i(2lx_\mu + 2x_\mu x_\nu \partial^\nu - x^2 \partial_\mu - 2ix^\nu \Sigma_{\mu\nu})\phi(x) \end{aligned} \tag{4}$$

for fields which are not derivatives of other fields (nowadays called quasiprimary fields). More generally there can be an extra term $\kappa_\mu \phi(x)$ in the second formula.

Conformal transformations leave the light cone invariant but they can take spacelike x into timelike x. Because of this seeming causality problem, conformal symmetry was regarded with suspicion in spite of its long history [8], or was considered consistent only for free field theories which have commutators concentrated on the light cone. This would rule out anomalous dimensions. The final resolution of this causality puzzle was found by Lüscher and the author [9] and will be discussed in the next section.

II. CONFORMAL FIELD THEORY LIVES ON COMPACT SPACE S^{d-1} or WHY CONFORMAL FIELD THEORY IS INTERESTING IN d DIMENSIONS.

There is no problem with global conformal invariance of *Euclidean* Greens functions, it follows from infinitesimal conformal invariance in Minkowski space. It was shown in [9] that vacuum expectation values $< 0|\phi(x_1)..\phi(x_n)|0 >$ of products of field operators (Wightman functions) admit analytic continuation to an infinite sheeted covering space $\tilde{M}$ of Minkowski space M whenever the Euclidean Greens functions of the theory have conformal invariance. The analysis is based on use of a semigroup, see section VI. I. Segal had shown that the quantum mechanical conformal group $\tilde{G}$, which is the infinite sheeted universal covering of $SO(d,2)$ in d dimensions, can act on $\tilde{M}$, and that $\tilde{M}$ admits a $\tilde{G}$-invariant causal ordering [11]. Thus the action of $\tilde{G}$ on $\tilde{M}$ will not interchange timelike and spacelike on $\tilde{M}$. The analytically continued Wightman functions will be globally $\tilde{G}$-invariant and respect locality for relatively spacelike points on $\tilde{M}$. $\tilde{M}$ has the topology of a torus $S^{d-1} \times \mathbf{R}$. Its points can be parametrized by pairs (e, τ), where $e = (e^1, \ldots, e^d) \in S^{d-1}$ is a unit d-vector, and $\tau \in \mathbf{R}$. Minkowski space M is embedded in $\tilde{M}$ as shown in figure 1.

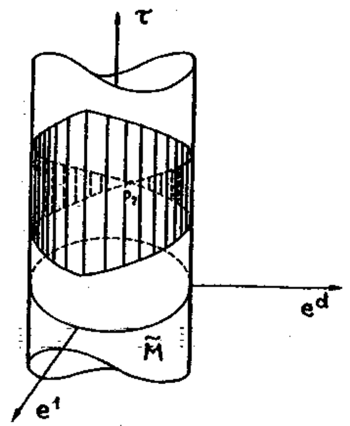

Figure 1. Manifold $\tilde{M}$.
Shaded part is M.
p_2 is the unique point at spatial infinity of $\tilde{M}$.
Drawing for 2 space time dimensions

$$M = \{(e, \tau) \in \tilde{M}; -\pi < \tau < \pi,\ e^d > -\cos\tau\} \tag{1}$$

The relation with the usual Minkowski coordinates x^μ is given by

$$x^0 = \frac{\sin\tau}{\cos\tau + e^d}, \qquad x^i = \frac{e^i}{\cos\tau + e^d}, \quad i = 1, \ldots, d-1 \tag{2}$$

A point (e_1, τ_1) is positive timelike relative to (e_2, τ_2) if

$$\tau_2 - \tau_1 > Arccos\ e_1 \cdot e_2 \qquad (3)$$

Arccos is the principal value of arccos.

We see that conformal field theory lives on a compact space. In this assertion, space really means space, not space time. From a spacelike surface Σ in Minkowski space, a compact spacelike surface in $\tilde{M}$ is obtained by adding one point at spacelike infinity. The assertion that the theory lives on a compact space means that wave functions etc. have a particular behavior at spacelike infinity, i.e. it imposes boundary conditions.

Time evolution in a general sense fixes the state (Schrödinger wave function) on a spacelike surface Σ, given the state on a neighbouring surface. It is governed by the stress energy tensor. In a conformal theory, it is natural to consider spacelike surfaces τ =const.. The operator

$$H = P^0 + K^0$$

of τ-translations is called the conformal Hamiltonian. H is a positive operator because P^0 is. This follows from

$$H = P^0 + K^0 = P^0 + \mathcal{R}P^0\mathcal{R} \geq 0$$

$\mathcal{R}$ is inverse radius transformation times space reflection in even dimensions d (see below); it was discovered by Kastrup long ago that this is an important special element of the identity component of the conformal group. A similar argument for the positivity of H was first given by I. Segal [11].

On a compact space, the Hamiltonian is expected to have a discrete spectrum, and there should be only a finite number of independent eigenvectors of H with energy below any finite value E. And indeed, the conformal Hamiltonian H has these properties if the theory admits Wilson operator product expansions and if there is only a finite number of (quasiprimary) fields with dimension $l \leq E$. In this case the spectrum of H consists of the values [9]

$$l + n, \quad n = 0, 1, 2, \ldots, \quad l = \text{dimension of a field.}$$

In addition, the vacuum $|0>$ is eigenvector of H to eigenvalue 0.

In conclusion there will be only *a finite number of degrees of freedom effective below a finite energy* E in a conformal theory. Such a theory should be tractable in any number of dimensions d. The use of conformal symmetry is based on the same philosophy as the renormalization group: To treat exactly critical systems is difficult, because for infinite correlation length infinitely many degrees of freedom appear coupled in an essential way. In the renormalization group approach, the treatment of the problem is decomposed into infinitely many similar steps, each of them involves treatment of an auxiliary statistical system which has both an infrared and an ultraviolet cutoff and therefore also finite correlation length , and which one can hope to handle by making approximations with a finite number of degrees of freedom [2,47].

Solving a conformal invariant field theory means constructing field operators ϕ and a Hilbert space of states $\mathcal{H}$ such that all the principles of local relativistic

quantum field theory are fulfilled, plus conformal invariance [14].[1] The most difficult is locality [14,15], all other principles can be implemented by use of convergent [19] operator product expansions [16] on the vacuum.[2] This leaves as free parameters the Lorentz spins s_i and dimensions l_i of the fields ϕ^i , and coupling constants g^{α}_{ijk} that multiply the 3-point functions. (In the presence of spin there can be a finite number of linearly independent conformal invariant 3-point functions labelled by α) [10,19]. They would have to be determined from locality.

Let P_E be the projector to eigenvalues of H below E and let $\phi_E \equiv P_E \phi P_E$. Consider only fields of dimension $l \leq E$ (since only for these $\phi_E|0> \neq 0$); there will be finitely many of them. One could try to find approximate solutions by minimizing the violation of locality in the presence of such an energy cutoff, e.g.

$$\sum_{i,j} \int_{x \text{ spacelike}} tr\Big([\phi_E^i(x), \phi_E^j(0)]_{\pm} ([\phi_E^j(0)^*, \phi_E^i(x)^*]_{\pm} \Big) = Min \qquad (4)$$

for fields on the tube, $\int_x = \int ded\tau$.

Let us return to the quantum mechanical conformal group $\tilde{G}$ = covering group of $SO(d,2)$, and its action on the tube $\tilde{M}$. $\tilde{G}$ has a discrete center [20] Γ,

$$\Gamma = \Gamma_1 \Gamma_2 \approx \mathbf{Z}_2 \times \mathbf{Z} \qquad (5)$$

The nontrivial element of Γ_1 is γ_1 = rotation of e by 2π. It acts trivially on $\tilde{M}$. The transformation law under Γ_1 distinguishes bosons and fermions (in $d > 2$ dimensions). Let $\mathcal{R}$ be "conformal inversion", i.e. a rotation of e by π for even dimension d. Then Γ_2 consists of elements γ_2^n, $n \in \mathbf{Z}$ with

$$\gamma_2 = \mathcal{R} e^{i\pi H}$$

The tube $\tilde{M}$ consists of Minkowski space M and its translates under Γ_2 .(In mathematical terminology, M is a "fundamental domain" for Γ_2). We see that

$$e^{2\pi i n H} \in \Gamma \qquad \text{for} \quad n \in \mathbf{Z}.$$

In an irreducible representation of $\tilde{G}$, elements of its center are represented by multiples of the identity (by Schurs lemma). Therefore the spectrum of H in an irreducible representation contains only eigenvalues of the form (l = lowest eigenvalue)

$$l + n, \qquad n = 0, 1, 2, \ldots$$

It turns out that for any (quasiprimary) field ϕ^i , vectors $\phi^i(\cdot)|0>$ span an irreducible representation space for $\tilde{G}$, and the lowest eigenvalue of H is given by the dimension l_i of ϕ^i [19]. ($|0>= vacuum$). Since H generates τ-translations, it follows that

$$\phi(e, \tau + 2\pi)|0> = e^{2\pi i l} \phi(e, \tau)|0>$$

[1] The existence of a "fundamental field" (which satisfies a field equation in the original nonasymptotic theory) can be read off the list of fields by looking for the shadow of a missing member in a tower of fields, see [14] .

[2] I discussed the use of convergent operator product expansions at length in my Cargèse lectures on conformal field theory in 1976, therefore I don't want to repeat it and refer the reader to the article [18].

However, one can not expect in general that the fields themselves are periodic functions of τ up to a phase factor [21,22]. If it were the case, then the operator product expansion of fields ϕ_1 and ϕ_2 of dimensions l_1 and l_2 could only contain fields of dimension $l_1 + l_2 + integer$, which is usually not the case in models. For instance [22] , conformal invariant ϕ^3-theory in $d = 6+\epsilon$ dimensions has a basic field ϕ of dimension $l = \frac{1}{2}(d-2) + \frac{1}{18}\epsilon + ...$, but the operator product expansion of two such fields contains a tower of symmetric tensor fields O_s of rank s with dimensions l_s given by

$$l_s = d-2+s+\sigma_s, \qquad \frac{1}{2}\sigma_s = \left[\frac{1}{18} - \frac{2}{3(s+2)(s+1)}\right]\epsilon + \ldots$$

We see that $l_s - s \mapsto 2l$ as $s \mapsto \infty$ but $l_s - s \not\equiv 2l$. Similar results hold for ϕ^4-theory in $4-\epsilon$ dimensions .

There is an important exception, cf section III. Currents and stress energy tensor in a *2-dimensional* conformal field theory are periodic functions of τ of period 2π. They can be regarded as living on compactified Minkowski space $\bar{M} = \tilde{M}/\Gamma_2$. This is consistent with causality because these fields have commutators which are concentrated on the light cone. Not all local fields in the 2-dimensional conformal models share these properties, though.

Above we considered general dimension d. In two dimensions the group $\tilde{G}$ with center $\mathbf{Z}_2 \times \mathbf{Z}$ is not the universal covering of $SO(2,2)$, and there could exist theories which live not on $\bar{M} = S^1 \times \mathbf{R}$ but on its simply connected covering. But one chooses not to consider them.

III. THE STRESS ENERGY TENSOR IN 2-DIMENSIONAL SCALE INVARIANT FIELD THEORY

The content of this section is taken from an unpublished old manuscript by Lüscher and the author [23].

In two dimensional Minkowski space it is convenient to use light cone coordinates

$$x_\pm = x^0 \pm x^1 \equiv t \pm r \tag{1}$$

In two dimensions, scale invariance implies tracelessness of the stress energy tensor (see Appendix)

$$\Theta_{\mu\nu} = \Theta_{\nu\mu}, \qquad \Theta^\mu{}_\mu = 0, \qquad \partial^\mu \Theta_{\mu\nu} = 0. \tag{2}$$

A traceless stress energy tensor has only two independent components

$$\Theta_+ = \Theta_{00} + \Theta_{01} \quad \text{and} \quad \Theta_- = \Theta_{00} - \Theta_{01}. \tag{3}$$

The conservation equation $\partial^\mu \Theta_{\mu\nu} = 0$ reads

$$\frac{\partial}{\partial x_-}\Theta_+(x_+,x_-) = 0, \qquad \frac{\partial}{\partial x_+}\Theta_-(x_+,x_-) = 0. \tag{4}$$

Therefore Θ_+ depends only on x_+ , and Θ_- depends only on x_-.

The stress energy tensor is a local field which commutes at spacelike distances

$$[\Theta_\pm(x_+,x_-), \Theta_\pm(y_+,y_-)] = 0 \quad \text{when} \quad (y_+ - x_+)(y_- - x_-) < 0 \tag{5}$$

Given that Θ_+ (Θ_-) depends only on x_+ (x_-), this is only possible if

$$[\Theta_+(x_+), \Theta_-(y_-)] \equiv 0, \tag{6}$$

and the commutator of two + components is localized at $x_+ = y_+$. It follows that

$$[\Theta_+(x_+), \Theta_+(y_+)] = \sum_{l=0}^{3} \delta^{(l)}(x_+ - y_+) O_l(y_+). \tag{7}$$

This is seen as follows. Let us simplify the notation by writing $\Theta(x), (x \in \mathbf{R})$ for $\Theta_+(x_+)$. Consider the bilocal operator

$$F(\xi, y) = [\Theta(x), \Theta(y)], \qquad \xi = x - y \tag{8}$$

$F(\xi, y)$ is an operator valued distribution which vanishes unless $\xi = 0$. Let $f(x)$ be a test function which equals 1 in a neighborhood of zero. Then the quantities

$$O_k(y) = \frac{(-1)^k}{k!} \int d\xi \xi^k f(\xi) F(\xi, y), \qquad k = 0,1,2,... \tag{9}$$

are local operator valued distributions. They do not depend on the particular shape of $f(\xi)$. Let $U(\lambda)$ be the operator which implements scale transformations by λ,

$$U(\lambda)\Theta(x)U(\lambda)^{-1} = \lambda^2 \Theta(\lambda x) \tag{10}$$

since Θ has dimension 2. Therefore

$$U(\lambda) O_k(y) U(\lambda)^{-1} = \lambda^{3-k} O_k(\lambda y). \tag{11}$$

Thus, O_k is a local field of dimension $3-k$. The dimension cannot be negative. Otherwise, the unique scale- and translation invariant two-point function which satisfies the spectrum condition

$$< 0|O_k^*(x) O_k(y)|0 >= B_k(x - y - i\epsilon)^{2k-6}, \qquad B_k \in \mathbf{C} \tag{12}$$

would not be positive for any B_k.

The matrix elements $< \psi|F(\xi,y)|\chi >$ are tempered distributions localized at $y=0$, therefore of the form $\sum_{l\geq 0} \delta^{(l)}(\xi) H_l(y)$ [24]. By definition of O_k, $H_l(y) =< \psi|O_l(y)|\chi >$. Since $O_l = 0$ for $l > 3$, because it would have negative dimension, eq.(7) follows as an operator equation.

Next we investigate the consequences of $[\Theta(x), \Theta(y)] = -[\Theta(y), \Theta(x)]$ when inserted in eq.(7). It requires

$$\sum_{l=0}^{3} \delta^{(l)}(x-y) O_l(y) = -\sum_{l=0}^{3} \delta^{(l)}(y-x) O_l(x).$$

Now

$$\delta^{(l)}(y-x) O_l(x) = \sum_{k=0}^{l} (-)^k \binom{l}{k} \delta^{(k)}(x-y) \frac{\partial^{l-k}}{\partial y^{l-k}} O_l(y)$$

Equating coefficients of $\delta^{(k)}(x-y)$ we obtain

$$O_0(y) = -\sum_{l=0}^{3} \frac{\partial^l}{\partial y^l} O_l(y), \tag{13a}$$

$$O_1(y) = \sum_{l=1}^{3} l \frac{\partial^{l-1}}{\partial y^{l-1}} O_l(y), \tag{13b}$$

$$O_2(y) = -\frac{1}{2}\sum_{l=2}^{3} l(l-1) \frac{\partial^{l-2}}{\partial y^{l-2}} O_l(y). \tag{13c}$$

From eq.(12) we observe that O_3 is a constant which can be written as

$$O_3(y) = c\frac{i^3}{6\pi}.$$

Eq.(13c) implies that $O_2(y) = 0$, and (13a) reduces to

$$2O_0(y) = -\frac{\partial}{\partial y} O_1(y).$$

Putting these results together, we have

$$[\Theta(x), \Theta(y)] = c\frac{i^3}{6\pi}\delta'''(x-y) + \delta'(x-y)O_1(y) - \delta(x-y)\frac{1}{2}\frac{\partial}{\partial y}O_1(y).$$

Using translation covariance, eq. (I-1a) requires that

$$\int dx[\Theta(x), \Theta(y)] = -2i\frac{\partial}{\partial y}\Theta(y).$$

Therefore $\frac{1}{2}(\partial/\partial y)O_1(y) = 2i(\partial/\partial y)\Theta(y)$. The requirement of scale invariance, eq.(11), fixes the integration constant, so that $O_1 = 4i\Theta$. Thus we find as a final result for the commutation relations in Minkowski space

$$\begin{aligned}
[\Theta_+(x_+), \Theta_+(y_+)] =& c_+\frac{i^3}{6\pi}\delta'''(x_+ - y_+) \\
&+ 4i\delta'(x_+ - y_+)\Theta_+(y_+) - 2i\delta(x_+ - y_+)\frac{\partial}{\partial y_+}\Theta_+(y_+), \\
[\Theta_-(x_-), \Theta_-(y_-)] =& c_-\frac{i^3}{6\pi}\delta'''(x_- - y_-) \\
&+ 4i\delta'(x_- - y_-)\Theta_-(y_+) - 2i\delta(x_- - y_-)\frac{\partial}{\partial y_+}\Theta_-(y_-), \\
[\Theta_-(x_-), \Theta_+(y_+)] =& 0.
\end{aligned} \tag{14}$$

If parity invariance holds, then

$$c_+ = c_- = c$$

In conclusion, the fact that the components $\Theta_\pm$ of the stress energy tensor depend only on a single variable $x_\pm$ each leads to the result that the commutation relations of the stress tensor with itself are determined (not only at equal times), with only

one free constant. In short, $\Theta_{\mu\nu}$ is a Lie field, i.e. it generates a Lie algebra in which $c \cdot identity$ appears as only other generator.

We remember now that conformal field theory lives on a tube, whose points may be parametrized by (e, τ), e = unit d-vector. In two dimensions we may set

$$(e^1, e^2) = (\sin\sigma, \cos\sigma) \tag{15a}$$

Fields on the tube must be periodic functions of σ. This parametrization gives for the Minkowski space coordinates

$$x_\pm = \tan\frac{\tau \pm \sigma}{2}. \tag{15b}$$

Since the stress tensor on $\tilde{M}$ must be a periodic function of σ, but Θ_+ depends only on $\tau + \sigma$, it must be periodic in τ also. As we noted before, this means that the stress tensor can be thought to live on compactified Minkowski space. Thus, if we define the stress energy tensor on the tube[1] by

$$\mathit{\Theta}_\pm(\vartheta) = (\cos\frac{\vartheta}{2})^{-4}\Theta_\pm(\tan\frac{\vartheta}{2}) \qquad (\vartheta = \frac{\tau \pm \sigma}{2}) \tag{16}$$

then $\mathit{\Theta}_\pm$ are periodic functions of ϑ. We may therefore consider the Fourier components

$$L_k = \frac{1}{8}\int_{-\pi}^{\pi} d\vartheta e^{ik\vartheta}\mathit{\Theta}_+(\vartheta) \tag{17}$$

The commutation relations for the stress energy tensor translate into

$$[L_l, L_k] = (l-k)L_{l+k} + \frac{c}{12}l(l^2-1)\delta_{k+l,0} \tag{18}$$

These are the commutation relations of the Virasoro algebra Vir [25]. Hermiticity of the stress energy tensor translates into

$$L_{-k} = L_k^* \tag{19}$$

The other component Θ_- of the stress tensor gives rise to another Virasoro algebra with generators $\bar{L}_k$ wich commute with the first. The hermitean linear combinations of

$$L_0, L_1, L_{-1} \quad \text{and} \quad \bar{L}_0, \bar{L}_1, \bar{L}_{-1} \tag{20}$$

generate the conformal algebra

$$so(2,2) \approx so(1,2) \times so(1,2).$$

These generators leave the vacuum invariant (because $L_m|0>$ has zero norm for $m = 0, \pm1$ by by eqs.(24),(26) below). Therefore we have conformal symmetry as expected. The conformal Hamiltonian, which generates τ-translations, is

$$H = L_0 + \bar{L}_0 \tag{21}$$

while $L_0 - \bar{L}_0$ generates rotations of e, i.e. translations of σ.

[1] The general formula for transforming arbitrary quasiprimary fields to the tube (in any dimension d) can be found in [9]

Let us now imagine that we live in a 2-dimensional conformal invariant world in which the stress energy tensor is the only observable field [7,18]. This means that the algebra $\mathcal{A}$ of observables is generated by the stress energy tensor, and may be taken as universal envelopping algebra of the Virasoro algebras, for instance. The Hilbert space of physical states will then split into superselection sectors[3] [26,27] which carry irreducible *-representations of the algebra of observables, hence unitary irreducible representations of the sum of the two Virasoro algebras. From the fact that $H \geq 0$ in these representations one can deduce that

$$L_0 \geq 0 \qquad \text{and} \qquad \bar{L}_0 \geq 0. \tag{22}$$

This reflects the fact that in 2 dimensions there are two positive generators of translations $P^0 \pm P^1$. (Remember that $H \geq 0$ follows from $P^0 \geq 0$).

Let us find the irreducible representations of the Virasoro algebra with positive energy $L_0 \geq 0$. From the commutation relations

$$[L_0, L_m] = -mL_m \tag{23}$$

we conclude that L_m will take an eigenvector of L_0 to eigenvalue E into an eigenvector to eigenvalue $E-m$. Since L_0 is positive, there must be a lowest eigenvalue. It follows that there is an eigenvector $|h>$ of L_0 such that

$$L_0|h>=h|h>, \qquad L_k|h>=0 \quad \text{for} \quad k>0. \tag{24}$$

This is called a lowest weight vector. h is the lowest eigenvalue of L_0 in the irreducible representation of Vir. The vacuum state $|0>$ has weight $h=0$.

The representation space $\mathcal{H}_h$ will be spanned by vectors of the form

$$L_{k_n}...L_{k_2}L_{k_1}|h> \equiv |h; k_1, ..., k_n> \tag{25}$$

Their scalar products can be calculated from the Virasoro commutation relations, the action (24) of generators L_k with $k \geq 0$ on the lowest weight vector $|h>$, and the relation $L_{-n} = L_n^*$, as follows. Generators L_n with $n>0$ are moved to the right until they annihilate $|h>$, and generators $L_n = L_{-n}^*$ with $n<0$ are moved to the left until they annihilate $<h|$. The remaining generators L_0 give factors h. Thus, for instance

$$\begin{aligned} <h|L_{-m}^*L_{-m}|h> &= <h|[L_m, L_{-m}]|h> \\ &= 2mh + \frac{c}{12}m(m^2-1) \qquad \text{for} \quad m>0 \end{aligned} \tag{26}$$

if $<h|h>=1$. In a unitary representation, one must have $<\psi|\psi>\geq 0$; the zero norm vectors are divided out to get $\mathcal{H}_h$. Thus one must have

$$<h|A^*A|h> \geq 0 \tag{27}$$

for any monomial A in the generators L_m. For instance, eq. (26) tells us that $c \geq 0$. By computing the norm $<\psi|\psi>$ of the vector

$$|\psi> = \left[8L_{-9} + 6L_{-7}L_{-2} + 12L_{-6}L_{-3} - 8L_{-2}L_{-5}L_{-2} + 12L_{-3}L_{-4}L_{-2} - 5L_{-3}^3\right]|0>$$

[3] By definition, observables do not make transitions between different superselection sectors. As a result, relative phases between vectors in different superselection sectors are unobservable.

one obtains the restriction

$$c \geq \tfrac{1}{2}$$

or $c=0$. The theory of one hermitean anticommuting massless free field $\varphi(x)$ yields a model with $c=\frac{1}{2}$. Such a field depends on a single variable $x = x_+$ or x_-.

From (25) and the commutation relations it follows that the spectrum of H in an irreducible positive energy representation of $Vir \oplus Vir$ with lowest weight $(h, \bar{h})$ consists of eigenvalues of the form

$$E = h + \bar{h} + n, \qquad n = 0, 1, 2, ...$$

Now we remember that application of a field operator of dimension l to the vacuum yields a representation space of the conformal algebra with eigenvalues of H of the form $l+m$, $m = 0, 1, 2, ...$ Therefore, classification of the unitary irreducible representations of Vir yields information on the dimension l of fields in the theory , i.e. on the critical indices of the statistical mechanical system in Euclidean space.

All this was known to Lüscher and the author in 1976 [23]. We tried to evaluate the restriction on c, h from unitarity, with little success. Working out norms of various vectors yielded strange results.[1] The problem was solved later by other authors with the help of the Kac determinant [28] which was not known in 1976. Friedan, Quiu and Shenker [29] showed that either

$$\begin{aligned} c &= 1 - \frac{6}{(m+2)(m+3)}, \qquad 0 \leq m, \\ h &= \frac{[(m+3)r - (m+2)s]^2 - 1}{4(m+2)(m+3)}, \qquad 1 \leq s \leq r \leq m+1, \end{aligned}$$

with m, r, s integer, or

$$c \geq 1, \quad h \geq 0 \quad \text{arbitrary.}$$

Goddard and Olive showed that to each of these pairs there exists a unitary irreducible positive energy representation [30].

It has become costumary to use complex variables $z_\pm$ on the unit circle in place of the angles

$$z_\pm = e^{i(\tau \pm \sigma)} = \frac{1 + ix_\pm}{1 - ix_\pm}.$$

One defines the stress tensor in these coordinates

$$T(z) = 2\pi \left(i\frac{dx}{dz}\right)^2 \Theta(x(z)),$$

where $T = T_\pm$, $z = z_\pm$, $x = x_\pm$ etc. The Fourier expansion (17) reads now

$$T(z) = \sum_{n \in \mathbf{Z}} L_n z^{-n-2}.$$

Consider smeared fields

$$T(f) = \frac{1}{2\pi i}\oint_{S^1} dz T(z) f(z), \qquad z^{-1}f(z) \quad \text{real.}$$

[1] By contrast, this method combined with the theory of induced representations worked beautifully for positive energy representations of the conformal group $\tilde{G}$ in $d=4$ dimensions. It yielded both a complete classification and explicit construction of all of them [20]

The commutation relations read in this language

$$[T(f),T(g)] = T(\mathcal{D}_f g) + \frac{c}{12}\omega(f,g)$$
$$\mathcal{D}_f g = fg' - gf'; \qquad (' = \frac{d}{dz})$$
$$\omega(f,g) = \oint \frac{dz}{2\pi i} f''' g.$$

Thus the adjoint action of the Virasoro algebra defines an action $g \mapsto \mathcal{D}_f g$ of $T(f)$ on functions g on the circle.
$g \mapsto g + \delta t \mathcal{D}_f g$ is of the form of the action of an infinitesimal diffeomorphism [31]

$$z \mapsto z' = z + \delta z, \qquad \delta \ln z = i z^{-1} f(z) \delta t$$

on functions of weight -1 on the circle. By definition, such functions transform under diffeomorphisms $z \mapsto z'(z)$ according to

$$g(z) \mapsto \left(\frac{\partial z'}{\partial z}\right)^{-1} g(z')$$

The extra term $\frac{c}{12}\omega(f,g)$ signals that Vir is a *central extension* of the Lie algebra of the diffeomorphism group $Diff(S^1)$ on the circle. $Diff(S^1)$ has an infinite sheeted covering group [32] whose center is isomorphic to **Z** and coincides with the center of one factor $\widetilde{SO}(1,2)$ of the universal covering group of the conformal group in two dimensions. This gives another way of understanding why the stress energy tensor is periodic in τ, i.e. is invariant under the action of the center of the conformal group. It is because the adjoint action of the center of a Lie group on its Lie algebra is trivial.

The results presented above can be generalized to currents j_μ which are conserved together with their axial brothers $j^5_\mu = \epsilon_{\mu\nu} j^\nu$.

$$\partial^\mu j_\mu = 0, \quad \partial^\mu j^5_\mu = 0.$$

In light cone coordinates this gives

$$\frac{\partial}{\partial x_-} j_+(x) = 0, \qquad \frac{\partial}{\partial x_+} j_-(x) = 0.$$

so that $j_\pm$ depend on only one variable $x_\pm$ each. (This observation has been a cornerstone in the construction of 2-dimensional models for decades). The commutation relations can be worked out as for the Virasoro algebra. In the language of complex variables on the circle one has a Fourier expansion for $J \propto j_\pm(x_\pm)$.

$$J(z) = \sum_{n \in \mathbf{Z}} J_n z^{-1-n}.$$

The commutation relations take the form of a Kac Moody algebra, which is the Lie algebra of the central extension of a loop group [32-36]. For instance

$$[J_n, J_m] = n\delta_{n+m,0}, \qquad [J_n, L_m] = nJ_{n+m}$$

for a single current that is associated with a chiral $U(1)$-symmetry. This larger algebra is called the conformal current algebra. It contains Vir. A lowest weight vector for the conformal current algebra is a simultaneous eigenvector of L_0 and J_0 which obeys

$$L_0|q,h> = h|q,h>$$
$$J_0|q,h> = q|q,h>$$
$$L_n|q,h> = J_n|q,h> = 0 \quad \text{for } n > 0.$$

q is the charge. The Sugawara prescription permits to construct a stress energy tensor with the right commutation relations from the current [33]

$$T(z) = \tfrac{1}{2} : J(z)^2 :$$

or, in modes

$$L_n = \tfrac{1}{2} \sum_{l,m: l+m=n} : J_l J_m :$$

Normal ordering :: means that generators J_l with larger l stand to the right. Validity of the Sugawara relation means that the algebra of observables is not simply the universal envelopping algebra of the Lie algebra of current and stress energy tensor, but is obtained from it by factoring out an ideal. (Equating an ideal to zero is the algebraic variant of field equations). Sandwiching the Sugawara relation between lowest weight vectors gives a relation between charge q and h [34],

$$h = \tfrac{1}{2} q^2.$$

IV. ANALYTIC CONTINUATION TO EUCLIDEAN SPACE

In the last section I explained how to compute expectation values $< h|L_{k_1} ... L_{k_n}|h >$ of products of Virasoro generators in the vacuum $|0>$ or other lowest weight state $|h>$. Since L_k are the Fourier components of the stress tensor, this yields the vacuum expectation values $< 0|T(z_1)...T(z_n)|0 >$ etc. of products of stress tensors in a two dimensional theory. We moved the generators L_n with $n \geq 1$ to the right, using the Virasoro commutation relations, until they annihilate the lowest weight vector $|h>$. For the vacuum $|0>$ we could do the same for $n \geq -1$, because the vacuum is conformal invariant. The procedure is therefore equivalent to the following. Let $T^{(-)}$ be the annihilation part of T, viz.

$$T^{(-)}(z) = \sum_{n \geq -1} L_n z^{-2-n}, \qquad (1)$$
$$T^{(-)}(z)|0> = 0.$$

Its commutation relations with the stress tensor are

$$[T^{(-)}(z_0), T(z)] = \frac{c}{2z_{01}^4} + \frac{2}{z_{01}^2} T(z_1) + \frac{1}{z_{01}} T'(z_1), \qquad (2)$$

where $z_{01} = z_0 e^\epsilon - z_1$, $\epsilon \mapsto +0$. (The familiar $i\epsilon$-prescription for time τ turns into a factor $e^\epsilon > 1$ here.) Proceeding as above, i.e. moving the annihilation part of T to the right, yields the recursion relation ($\widehat{\ }$ means "omit" .)

$$< T(z_0)T(z_1)...T(z_n) > = \frac{c}{2} \sum_{k=1}^n \frac{1}{z_{0k}^4} < T(z_1)...\widehat{T(z_k)}...T(z_n)|0 >$$
$$+ \sum_{k=1}^n \left(\frac{2}{z_{0k}^2} + \frac{\partial}{\partial z_k}\right) < T(z_1)...T(z_n) > \qquad (3)$$

Here $<\quad>$ is short for vacuum expectation values $<0|...|0>$ for now.

Vacuum expectation values of local field operators (like the stress tensor) may be analytically continued to Euclidean space. This results in Euclidean Greens functions (=Schwinger functions) which are symmetrical in their arguments. Analytic continuation is also possible for fields which are not local, but pseudolocal as is natural in two dimensions. For a discussion of this case I refer to J. Fröhlichs lectures at this school [37].

Recursion relations like (3) remain true for the Schwinger functions $< T(z_1)...T(z_n) >$ by uniqueness of analytic continuation [31,38] . In the models of interest, Schwinger functions have an interpretation as expectation values of classical statistical mechanical systems, and we may thus interpret the notation $<\quad>$ as such expectation values.

Since the conformal Hamiltonian H (which is generator of translations of the conformal time τ which parametrizes the tube) is positive, we may continue in the variable τ:

$$\tau \rightarrow i\tau$$

The complex variables become

$$z_+ = e^{-\tau+i\sigma} \equiv z,$$
$$z_- = e^{-\tau-i\sigma} \equiv \bar{z}.$$

It is costumary to write

$$T_+(z_+) = T(z)$$
$$T_-(z_-) = \bar{T}(z)$$

We see that in the Euclidean domain, z_+ and z_- are complex conjugate of each other. The fact that $T_+(z_+)$ is independent of z_-, viz

$$\frac{\partial}{\partial \bar{z}} T(z) = 0 \qquad \text{etc.,}$$

translates into the property that Schwinger functions are holomorphic in arguments z of $T(z)$, and antiholomorphic in arguments z of $\bar{T}(z)$.

Belavin, Polyakov and Zamolodchikov introduced the concept of a primary field [31]. This is a field ψ in whose operator product expansion appears no field of lower dimension than the dimension of ψ. Its commutation relations with the stress tensor read, in the z-picture

$$[L_n, \psi(z)] = z^n \left(z \frac{d}{dz} + (n+1)h \right) \psi(z)$$

$\psi(z)$ leads from the vacuum to a superselection sector which carries a representation with lowest weight h of the Virasoro algebra with generators L_n. If ψ is independent of $\bar{z}$ then h is the dimension. (In general the dimension proper is $l = h + \bar{h}$, while $h - \bar{h}$ is the Lorentz spin. $\bar{h}$ is the lowest weight for the other Virasoro algebra). The possibility of finding the commutation relations not only at equal times follows again from the fact that Θ_+ is independent of of x_-. These commutation relations can be used again to relate expectation values $< 0|T(z)A|0 >$ to expectation values

$< 0|A|0 >$, for monomials A in the fields. These relations remain valid under analytic continuation to Euclidean space. For instance [31]

$$< T(z_0)\psi_1(z_1)...\psi_n(z_n) > = \sum_i \left(\frac{h_i}{z_{0i}^2} + \frac{1}{z_{0i}} \frac{\partial}{\partial z_i} \right) < \psi_1(z_1)...\psi_n(z_n) > .$$

In general, a local field ψ in two dimensions does not depend on $z_+ \equiv z$ or $z_- \equiv \bar{z}$ only, but on both. This implies that the Schwinger functions involving $\psi(z)$ will not be holomorphic or antiholomorphic in z.

Schroer and Rehren [39] have presented arguments that it is always possible to build up local fields ϕ from nonlocal components which depend only on one variable z_+ or z_-.

The first step is the decomposition of the local field $\phi(e, \tau)$ on the tube into pieces which transform according to an irreducible representation of the center Γ of the conformal group, following Schroer and Swieca [21]. These pieces are sums of the form

$$\sum_n e^{i\Delta n} \phi((-)^n e, \tau + \pi n) = \phi_\Delta .$$

These Schroer Swieca fields are natural objects to consider ***in 2 dimensions***, because they are still relatively local to the observables (stress tensor and currents), although generally not to themselves. Relative locality to the observables follows from the fact that the field ϕ was (relatively) local, and stress tensor and currents are 2π-periodic functions of τ which transform trivially under the center Γ. In more than 2 dimensions this would be no longer true.

The second step is an argument, based on operator product expansions, that these Schroer Swieca fields can be chosen to depend on only one variable. There could be a multiplicity problem; I am not prepared to discuss that. I refer the reader to B. Schroers lectures at this school for further results of this approach.

V. QUANTUM FIELD THEORY AS REPRESENTATION THEORY OF A GROUP OR LIE ALGEBRA

It is sometimes said that conformal invariance cannot be very useful in more than two dimensions because then one does not have the infinite dimensional Lie algebra. I have already tried to argue against this pessimistic point of view in section 2. Moreover, the Virasoro algebra is not Lie algebra of a symmetry group, because it does not leave the vacuum invariant (for $c \neq 0$) - only the generators of the conformal algebra $so(1,2) \times so(1,2)$ do. The Virasoro algebra should properly be regarded as a spectrum generating algebra, or as a Lie algebra associated with an algebra of observables [27]. Such an algebra exists in any local relativistic quantum field theory, and the Lie algebra can be described in a general way.

Borchers proposed to regard local relativistic quantum field theory as *-representation of an algebra of test functions (Borchers algebra) [42]. This idea can be specialized to conformal QFT as follows. The algebra consists of equivalence classes of finite sequences of test functions, with product $\otimes$:

$$f = f^0, f^1_{\mu_1,\nu_1}(x_1), f^2_{\mu_1\nu_1\mu_2\nu_2}(x_1, x_2), ..., f^N_{\mu_1\nu_1...\mu_N\nu_N}(x_1, \dots, x_N). \qquad (1)$$

For simplicity of notation, let us adopt the obvious identification of constants f^0, functions f^1 of a single variable, etc. with sequences f of test functions with only one nonvanishing term. The product $\otimes$ in the algebra is defined by

$$\begin{aligned}(f^0 \otimes g^1)_{\mu_1\nu_1}(x_1) &= f^0 g^1_{\mu_1\nu_1}(x_1) \\ (f^1 \otimes g^1)_{\mu_1\nu_1\mu_2\nu_2}(x_1,x_2) &= f^1_{\mu_1\nu_1}(x_1) g^1_{\mu_2\nu_2}(x_2), \\ &\text{etc.}\end{aligned} \tag{2}$$

A complex *-algebra is obtained by admitting complex test functions, with complex conjugation as *-operation. The requirement of a *-representation means that the representation operator $\mathbf{\Theta}(f)$ of f obeys

$$\mathbf{\Theta}(f)^* = \mathbf{\Theta}(f^*)$$

Until further notice we shall restrict attention to real f. The hermitean representation operators $\mathbf{\Theta}(f)$ are to be interpreted as smeared products of stress tensors.

$$\mathbf{\Theta}(f) = f^0 \mathbf{1} + \sum_{k=1}^{N} \int dx_1 ... dx_k f^k_{\mu_1\nu_1...\mu_k\nu_k}(x_1,...,x_k) \Theta^{\mu_1\nu_1}(x_1)...\Theta^{\mu_k\nu_k}(x_k) \tag{3}$$

The algebra $\mathcal{A}$ is obtained from the free algebra generated by constants f^0 and functions f^1 of a single variable and $\otimes$-products of such functions, by imposing the following equivalence relations.

TRACELESSNESS, SYMMETRY AND CONSERVATION OF STRESS TENSOR.

$$f^1_{\mu\nu}(x) \equiv f^1_{\mu\nu}(x) + \eta_{\mu\nu} h(x) + [k_{\mu\nu}(x) - k_{\nu\mu}(x)] + \partial_\mu l_\nu(x) \tag{E1}$$

for arbitrary functions $h, k_{\mu\nu}, l_\mu$ of x. $\eta_{\mu\nu}$ is the metric tensor.

LOCALITY.

$$f^1 \otimes g^1 = g^1 \otimes f^1 \text{ if the supports of } f^1 \text{ and } f^2 \text{ are relatively spacelike.} \tag{E2}$$

CONFORMAL SYMMETRY. *Eq.(E3) below.*

The generators J^{AB} of the conformal algebra $so(d,2)$ are linear combinations of the generators $P_\mu, M_{\mu\nu}, D, K_\mu$ introduced in section I ($A,B = 0...d-1, d+1, d+2$). [6]. They are constructed from the stress tensor according to eqs.(I.1)f. Integration is over space, but because the result is time independent, we may also smear over time. Therefore there exist special test functions $f^{AB}(x)$ of a single variable such that the generators take the form

$$J^{AB} = \Theta(f^{AB}).$$

The conformal Hamiltonian is $H = J^{0d+2}$. The transformation law of the stress tensor under conformal transformations takes the form

$$[J^{AB}, \Theta(g^1)] = \Theta(\partial^{AB} g^1)$$

where ∂^{AB} are known differential operators. This translates into an equivalence relation in the Borchers algebra,

$$f^{AB} \otimes g^1 - g^1 \otimes f^{AB} \equiv \partial^{AB} g^1. \tag{E3}$$

This should be read as an equivalence relation between sequences of test functions in which there is only a single nonvanishing term. Note that it is an equivalence between functions with different numbers of arguments (2 on the left, 1 on the right), i.e. *inhomogeneous*. Our additional requirement that the stress tensor generates some symmetries leads thus to a profound change in the structure of the Borchers algebra. It provides for an inhomogeneous relation.

The algebra constructed in this way shall be called the Θ-Borchers algebra. One is interested in positive energy representations of this algebra. The positive energy requirement reads

$$H = \mathbf{\Theta}(f^{0d+2}) \geq 0. \tag{4}$$

The result of the analysis of the stress tensor of a scale invariant theory in 2 dimensions (section III) may be rephrased as follows:

The Θ-Borchers algebra in 2 dimensions is isomorphic to the envelopping algebra of the Lie algebra $Vir \oplus Vir$ (two copies of Virasoro algebra).

More generally, one may regard the Θ-Borchers algebra as envelopping algebra of a Lie algebra $\mathcal{V}$ which generalizes $Vir \oplus Vir$. The Lie algebra $\mathcal{V}$ is made of equivalence classes of commutators $f \otimes g - g \otimes f$. Its representation operators are linear combinations of

$$\Theta(f^1), \quad [\Theta(f^1), \Theta(g^1)], \quad [[\Theta(f^1), \Theta(g^1)], \Theta(h_1)], \quad ...$$

where f^1, g^1, h^1 are functions of a single variable; $\Theta(f^1) = \mathbf{\Theta}(f^1)$ is the smeared stress tensor.

Since conformal field theory lives on a tube $\tilde{M} = S^{d-1} \times \mathbf{R}$, it is appropriate to regard x as points on $\tilde{M}$, viz $x = (\epsilon, \tau)$. The requirement that we want theories with a discrete spectrum of the generator H of τ-translations, as is required by validity of operator product expansions, may be incorporated by admitting as test functions all elements of the dual of the space of quasiperiodic generalized functions of τ. (Quasiperiodic functions are those which admit a Fourier series representation, rather than requiring a Fourier integral.)

The complex Virasoro algebra contains generators L_n, $(n > 0)$ which annihilate the vacuum. (They are not hermitian, therefore only contained in the complex Lie algebra, but they are in the real *Euclidean* algebra - see section VI). The same is true in general for the complex Lie algebra $\mathcal{V}$ which is generated by the stress tensor (see above) smeared with complex test functions f. Let $\mathcal{S}$ be the set of real numbers in the spectrum of H, and let $\mathcal{S}_\Delta$ be the set of differences of numbers in $\mathcal{S}$. Let E_γ be the projector on the eigenvalue γ of H. Decompose

$$\begin{aligned}\mathcal{V} = \sum_{\alpha \in \mathcal{S}_\Delta} \mathcal{V}_\alpha, \qquad X = \sum_{\alpha \in \mathcal{S}_\Delta} X_\alpha, \quad (X_\alpha \in \mathcal{V}_\alpha) \\ \text{where} \quad X_\alpha = \sum_{\beta,\gamma \in \mathcal{S}: \gamma = \beta + \alpha} E_\gamma X E_\beta\end{aligned} \tag{5}$$

Elements of $\mathcal{V}_\alpha$ transfer energy α. $\mathcal{V}_\alpha$ contains $\Theta(f^1)$ for $f^1(e,\tau) \propto \tilde{f}^1(e)e^{i\alpha\tau}$ etc. More precisely it contains $\lim_{T\to\infty}(2T)^{-1}\Theta(\chi_T f^1)$ where $\chi_T(\tau) = 1$ for $|\tau| < T$, and $= 0$ otherwise. Clearly the spectrum condition implies that

$$X|0>=0 \quad \text{for } X \in \mathcal{V}_\alpha \quad \text{with } \alpha > 0.$$

The same is true of elements X in $\mathcal{V}_0$ with $<0|X|0>=0$. They form a subalgebra if the vacuum is unique. Clearly

$$[\mathcal{V}_\alpha, \mathcal{V}_\beta] \subseteq \mathcal{V}_{\alpha+\beta}, \quad \text{in particular } [\mathcal{V}_0, \mathcal{V}_\alpha] \subseteq \mathcal{V}_\alpha.$$

It is interesting that in the ϕ^3-model in $6+\epsilon$ dimensions anomalous parts of of the dimensions of all fields are relatively rational, to first order in ϵ. They determine the spectrum of H as we know.

The conformal generators are in $\mathcal{V}_0$ and $\mathcal{V}_{\pm 1}$. A further decomposition of $\mathcal{V}$ can be effected by harmonic analysis [19] on the conformal group $\tilde{G}$.

In conclusion, the situation in d dimensions is much as in two dimensions, except that the representation theory of $\mathcal{V}$ looks less tractable . But remember that the situation in two dimensions was just a little better than this in 1976. We knew the algebra and how to construct positive energy representations by the method of lowest weight, but we were unable to find out which were unitary, except for free field examples, see section III.

There is also a *group theoretical* formulation of the constructions described above. Consider the group **G** which is obtained from the free group on objects $W(f)$ ($f = f_{\mu\nu}(x)real$) by imposing equivalence relations. The representation operator for $W(f)$ is to be interpreted as unitary operator

$$W(f) = e^{i\Theta(f)}, \qquad \Theta(f) = \int d^d x f_{\mu\nu}(x)\Theta^{\mu\nu}(x). \tag{6}$$

The equivalence relations are

$$\begin{aligned} &W(f) = W(-f)^{-1} \\ &W(f)W(g) = W(g)W(f) \quad \text{if } f,g \text{ have relatively spacelike supports} \\ &W(tf^{AB})W(g)W(-tf^{AB}) = W(e^{it\theta^{AB}} g) \quad (t \text{ real}) \end{aligned} \tag{7}$$

and dependence on f should only be through equivalence classes (E1).

The group **G** is the generalization to d dimensions of the central extension of $Diff(S^1) \times Diff(S^1)$ which we encountered in 2 dimensions.

VI. SEMIGROUPS, FINITE AND INFINITE DIMENSIONAL

The results of Lüscher and the author on global conformal invariance, which were reviewed in section II, were derived by use of semigroups [9]. Starting from Greens functions in d-dimensional Euclidean space which are invariant under the Euclidean conformal group G, a contractive representation T of a real subsemigroup S of G (with the same dimension as G) was constructed by appeal to Osterwalder Schrader (OS) positivity [OS-positivity is the physical positivity for quantum field theory [13]]. A nonabelian Hille Yosida theorem was proven [43] which implied that

this contractive representation T of S could be analytically continued (through a complex semigroup S_c) to a unitary positive energy representation of the quantum mechanical conformal group in Minkowski space.

The (finite dimensional) conformal group in d-dimensional Minkowski space is the infinite sheeted universal covering G^* of $SO(d,2)$, its Euclidean brother is the twofold covering G of $SO(d+1,1)$. Both groups share a common subgroup

$$G_+ = \widetilde{SO(d,1)} = Spin(d,1).$$

The semigroup S is mapped into itself by the adjoint action of G_+. And

$$\Lambda \in S \quad \text{implies} \quad \theta(\Lambda^{-1}) \in S.$$

θ is Euclidean time reversal, see below. Schematically, the relation between representations is (T contractive means norms $||T(\Lambda)|| \leq 1$)

pseudounitary representation of G with OS-positivity

$\Downarrow$

contractive representation T of S with $T(\theta(\Lambda^{-1})) = T(\Lambda)^*$

$\Downarrow$

unitary positive energy representation of G^*

Pseudo-unitary means that there exists an invariant but not necessarily positive semidefinite scalar product. For irreducible representations it is furnished by the Knapp-Stein intertwining operator [45], it is the same that is used for the complementary series [46] of unitary representations of G. The semigroup can act on an invariant proper subspace ("half") of the representation space of G, with scalar product modified by action of the time reversal operator [9,13].

The real Lie algebras[1] $\mathcal{G}$ and $\mathcal{G}^*$ of the two groups (Euclidean and Minkowski conformal group in our application- but the construction is more general) are related as follows. There is an involution θ of the Lie algebras $\mathcal{G}^*$ and $\mathcal{G}$ - i.e. an automorphism with $\theta^2 = 1$. This furnishes a split

$$\mathcal{G}^* = \mathcal{G}^*_+ + \mathcal{G}^*_-, \quad \text{with } \theta(X) = \pm X \quad \text{for } X \in \mathcal{G}^*_\pm \tag{1a}$$

The real Lie algebra of the Euclidean group is then

$$\mathcal{G} = \mathcal{G}^*_+ + i\mathcal{G}^*_- \equiv \mathcal{G}_+ + \mathcal{G}_- \tag{1b}$$

$\mathcal{G}^*_+$ generates the common (noncompact) subgroup G_+ of G and G^*. If $\mathcal{G}^*$ admits positive energy representations then $\mathcal{G}^*_-$ contains a nontrivial cone V which is invariant under the adjoint action of G_+ [44]. The elements of V are conjugate under G_+ to $-tH$, $t > 0$, where H is hermitean generator of a factor SO(2) of the maximal compact subgroup of (the adjoint representation of) G^* if G^* is simple [44]. In d-dimensional conformal field theory H is the conformal Hamiltonian. A semigroup S^o is generated by $\mathcal{G}^*_+ - iV \subset \mathcal{G}$, it consists of finite products

$$ue^{X_1}\ldots e^{X_n}, \qquad u \in G_+, \quad X_1,\ldots,X_n \in V. \tag{1c}$$

[1] I use physicists conventions for real Lie algebras: Their generators are hermitean in a unitary representation of the group, i.e. $e^{iX} \in G$ if $X \in \mathcal{G}$.

S is obtained from S° by taking closures.

In two dimensional conformal quantum field theory one has two positive generators $P_\pm$ of translations. Therefore one has the possibility of performing analytic continuation in two variables $t \pm r$. Suitable components of the stress tensor depend on only one of them, as we have seen. In this way the problem is reduced to the study of a 1-dimensional theory. The special case $d = 1$ is therefore of particular interest:

$$G = \widetilde{SO(2,1)} \supset S \Rightarrow G^* = \widetilde{SO(1,2)}.$$

In this case, the two groups are locally isomorphic to each other and to $SL(2,\mathbf{R})$. By carrying out the above construction one gets positive energy representations (i.e. representations of the interpolated discrete series) of the infinite sheeted universal covering group of $SL(2,\mathbf{R})$ from representations of $SL(2,\mathbf{R})$ in the complementary series. (A similar analysis was done for other groups by Schrader [48]). This is of interest because Vir contains infinitely many subalgebras $sl(2,\mathbf{R})$.

Here I wish to generalize these constructions to the infinite dimensional case which is of interest in 2-dimensional conformal field theory.

The *real* Lie algebra $Vir \oplus Vir$ is generated by the stress energy tensor in Minkowski space, smeared with real test functions. I will describe the Euclidean semigroup S, and its Lie algebra $\mathcal{G}$, and initiate study of an associated group. They turn out to be interesting objects. Restricting attention to one factor, which comes from one of the two algebras Vir, one finds an algebra $\mathcal{G}$ which behaves very much like an infinite dimensional brother of $sl(2,\mathbf{R})$.

It has both a noncompact and a compact Cartan subalgebra, both 1-dimensional (not counting the center) , as is the case for $sl(2,\mathbf{R})$. It admits an Iwasawa decomposition $\mathcal{G} = \mathcal{K} + \mathcal{A} + \mathcal{N}$ and a Bruhat decomposition $\mathcal{G} = \mathcal{X} + \mathcal{M} + \mathcal{A} + \mathcal{N}$, as are familiar from the theory of finitedimensional noncompact groups.[1] Half of the generators $\bar{K}_n \in \mathcal{K}$ of the "maximal compact subgroup " K with Lie algebra $\mathcal{K}$ generate compact 1-parameter groups (in the adjoint representation). Together with their commutators they span the Lie algebra $\mathcal{K}$. The Lie algebra $\mathcal{N}$ is nilpotent in the sense that there exists a decreasing sequence of subalgebras $\mathcal{N} = \mathcal{N}_1 \supset \mathcal{N}_2 \supset \dots$ such that

$$[\mathcal{N}_k, \mathcal{N}_m] \subseteq \mathcal{N}_{k+m}, \qquad \text{and} \bigcap_{m=1}^{\infty} \mathcal{N}_m = \emptyset. \tag{2}$$

Let us start with the real Virasoro algebra $\mathcal{G}^* = Vir$. It is generated by

$$T(f) = \frac{1}{2\pi i}\oint dz\; zT(z)f(z), \qquad f(z) \text{ real}, z = ie^{i\tau} \in S^1. \tag{3a}$$

I have changed conventions compared to section III because reality properties will be important here. $T(f)$ is a hermitean linear combination of generators L_n of the Virasoro algebra.

$$T(f) = \sum_n f_n L_n, \quad \text{if } f(z) = \sum f_n z^n. \tag{3b}$$

[1] The Iwasawa decomposition is a cornerstone of modern representation theory of (finite dimensional) noncompact Lie groups, because Harish Chandra's subquotient theorem [49] asserts that every unitary irreducible representation of such a group is subquotient of an induced representation on the homogeneous space $G/KAN = K/M$ (K = maximal compact subgroup, A abelian noncompact, M = centralizer of A in K). The inducing representation of MAN is trivial on the nilpotent subgroup N [46].

For real f we have $\bar{f}_{-n} = f_n$. The Virasoro generators obey the commutation relations and hermiticity condition which were derived in section III

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0} \tag{4a}$$

$$L_n = L^*_{-n} \tag{4b}$$

Note that L_n are not in the real Lie algebra $\mathcal{G}^*$, except for $n = 0$, because they are not hermitean. The hermitean generators of the finite dimensional conformal group $\widetilde{SL(2,\mathbf{R})} \subset G^*$ in Minkowski space are

$$\frac{1}{2}(L_1 + L_{-1}), \quad \frac{i}{2}(L_1 - L_{-1}) \quad \text{(noncompact)} \qquad L_0 \quad \text{(compact)} \tag{5}$$

The involution is now defined by

$$\theta(L_n) = -(-)^n L_{-n}, \qquad \theta(c) = -c \tag{6}$$

This extends the above mentioned involution for the finite dimensional conformal algebra $sl(2,\mathbf{R})$. It is easily verified that θ is an automorphism. It gives

$$\theta(T(f)) = T(\theta f) \quad \text{with } \theta f(z) = -f(-z^{-1}) \tag{7}$$

This yields a decomposition (1) of the Virasoro algebra $\mathcal{G}^*$ into even and odd parts $\mathcal{G}^*_+ + \mathcal{G}^*_-$,

$$T(f) \in \mathcal{G}^*_\pm \quad \text{if } \theta f = \pm f, \qquad c \in \mathcal{G}^*_- \tag{8}$$

The Euclidean algebra is defined by $\mathcal{G} = \mathcal{G}^*_+ + i\mathcal{G}^*_-$, as in eq.(1). Thus

$$T(f_+ + if_-) \in \mathcal{G} \quad \text{if } f_\pm(z) \text{ are real, and } f_\pm(-z^{-1}) = \mp f(z) \tag{9}$$

Equivalently

$$T(f) \in \mathcal{G} \quad \text{if } f(-z^{-1}) = -\overline{f(z)}. \tag{9'}$$

In particular, if we set

$$f^n(z) = i\left(\frac{z}{i}\right)^n \tag{10}$$

then

$$T(f^{-n}) = ii^n L_n \equiv \bar{L}_n \in \mathcal{G}. \tag{11a}$$

In addition, the central charge

$$\bar{c} \equiv ic \in \mathcal{G}.$$

We see that the generators L_n of the Virasoro algebra, multiplied with i if n is odd, are elements of the real Euclidean Lie algebra $\mathcal{G}$, so they would be hermitean in a unitary representation of $\mathcal{G}$. The commutation relations of the hermitean generators are

$$[\bar{L}_n, \bar{L}_m] = i\{(n-m)\bar{L}_{n+m} + \frac{\bar{c}}{12}n(n^2-1)\delta_{n+m,0}\} \qquad \bar{L}_n = \bar{L}^*_n, \qquad \bar{c} = \bar{c}^*. \tag{12}$$

$\mathcal{G}$ contains the finite dimensional Euclidean conformal algebra $\mathcal{H}_1 \approx sl(2,\mathbf{R})$. It is the first of an infinite family of algebras $\mathcal{H}_n \approx sl(2,\mathbf{R})$ $(n = 1,2,3...)$. The generators of $\mathcal{H}_n$ are

$$\begin{aligned} &\frac{1}{2n}(\tilde{L}_n + \tilde{L}_{-n}) \quad \text{(compact)} \\ \frac{1}{n}\tilde{L}_0 + \frac{\tilde{c}}{24}(n - \frac{1}{n}), \quad &\frac{1}{2n}(\tilde{L}_n - \tilde{L}_{-n}) \quad \text{(noncompact)} \end{aligned} \tag{13}$$

Note that, compared to the Minkowski space conformal algebra $sl(2,\mathbf{R})$, one compact and one noncompact generator have switched place. The subgroups $\mathcal{H}_n$ are images of $\mathcal{H}_1$ under an endomorphism ρ_n of Lie algebras $\mathcal{G}$ and $\mathcal{G}^*$ which is induced by the map

$$\begin{aligned} \rho_n(L_n) &= \frac{1}{n}L_{nk} + \frac{c}{24}\left(n - \frac{1}{n}\right)\delta_{k,0}, \\ \rho_n(c) &= nc. \end{aligned} \tag{14}$$

The endomorphisms ρ_n intertwine inequivalent positive energy representations of Vir with different central charge.

The Cartan decomposition of the Lie algebra $\mathcal{G}$ of a finite dimensional noncompact group exhibits $\mathcal{G}$ as a sum of a compact subalgebra $\mathcal{K}$ and a noncompact part $\mathcal{P}$ [46].

$$\begin{aligned} &\mathcal{G} = \mathcal{K} + \mathcal{P} \\ [\mathcal{K},\mathcal{K}] \subseteq \mathcal{K}, \quad &[\mathcal{K},\mathcal{P}] \subseteq \mathcal{P}, \quad [\mathcal{P},\mathcal{P}] \subseteq \mathcal{K} \end{aligned} \tag{15}$$

The generators of $\mathcal{K}$ generate compact 1-parameter groups in the adjoint representation. The same is not necessarily true in the group itself. In particular, the compact generator of $sl(2,\mathbf{R})$ generates a noncompact 1-parameter subgroup of the infinite sheeted universal covering of $SL(2,\mathbf{R})$.

A Cartan decomposition exists also for our infinite dimensional Euclidean algebra $\mathcal{G}$. The meaning of "compact" is somewhat different, though. This is not so unexpected. Infinite dimensional Lie groups are not locally compact, therefore it is not a priori clear what the meaning of a "maximal compact subgroup" K should be.

The $sl(2,\mathbf{R})$-subalgebras $\mathcal{H}_n$ span all of $\mathcal{G}$. So one might expect that the generators of $\mathcal{K}$ should be the compact generators of the $sl(2,\mathbf{R})$-subalgebras. But this is inconsistent with $[\mathcal{K},\mathcal{K}] \subseteq \mathcal{K}$. These generators are all odd under the involution θ. So they cannot form a nonabelian proper subalgebra, since their commutators would be even.

Instead I select for $\mathcal{K}$ (resp. $\mathcal{P}$) those generators of $\mathcal{G}$ that are even (resp. odd) under the involution $L_n \mapsto -L_{-n}$, $c \mapsto -c$, that is $\tilde{L}_n \mapsto -(-)^n\tilde{L}_n$, $\tilde{c} \mapsto -\tilde{c}$. The generators of $\mathcal{K}$ are then

$$\tilde{K}_n = \frac{1}{2n}(\tilde{L}_n - (-)^n\tilde{L}_{-n}) \in \mathcal{K}, \quad n = 1,2,... \tag{16}$$

Half of these generators - those with n odd - are compact generators of some $sl(2,\mathbf{R})$-subalgebra. Among them is the compact generator $\tilde{K}_1$ of the finite dimensional Euclidean conformal algebra.

Let us introduce some more noncompact subalgebras of $\mathcal{G}$:

$\mathcal{A}$	generators $\tilde{L}_0$, $\tilde{c}$	commutative
$\mathcal{N}$	generators $\tilde{L}_n$, $n = 1,2,...$	nilpotent
$\mathcal{X}$	generators $\tilde{L}_{-n}, n = -1,-2,...$	nilpotent,

and the centralizer $\mathcal{M}$ of $\mathcal{A}$ in $\mathcal{K}$. One finds

$$\mathcal{M} = 0. \tag{17}$$

Looking at the list of generators, we see that $\mathcal{G}$ admits an Iwasawa decomposition

$$\mathcal{G} = \mathcal{K} + \mathcal{A} + \mathcal{N}. \tag{18}$$

and a Bruhat decomposition

$$\mathcal{G} = \mathcal{X} + \mathcal{M} + \mathcal{A} + \mathcal{N}. \tag{19}$$

The algebras $\mathcal{N}$ and $\mathcal{X}$ are nilpoptent in the sense described earlier in eq. (2), and $\mathcal{X} = \theta(\mathcal{N})$.

Let us finally note that $\mathcal{G}$ admits both compact and noncompact Cartan subalgebras, such as (disregarding the center $\tilde{c}$)

$$\begin{aligned} \mathcal{T}_1 : &\quad \text{generator } \tilde{K}_1 \quad \text{compact} \\ \mathcal{T}_2 : &\quad \text{generator } \tilde{L}_0 \quad \text{noncompact} \end{aligned} \tag{20}$$

These are at the same time compact and noncompact Cartan subalgebras of the finite dimensional Euclidean conformal algebra $\mathcal{H}_1$.

Next we turn to the description of the Euclidean semigroup S and the invariant cone $V \subset \mathcal{G}^*_-$ that is associated with it.

The real Virasoro algebra $\mathcal{G}^*$ is the Lie algebra of a central extension of the group $Diff(S^1)$ of diffeomorphisms of the circle (and of its infinite sheeted universal covering group which consists [32] of diffeomorphisms F of $\mathbf{R}$ with $F(x + 2\pi) = F(x)+2\pi$). $T(f) \in \mathcal{G}^*$ for real f; it generates a 1-parameter group of diffeomorphisms of S^1, with infinitesimal transformation

$$\delta \ln z = if(z)\delta t$$

From this we see that the subgroup $\mathcal{G}^*_+$ which is generated by

$$T(f_+) \in \mathcal{G}^*_+, \qquad f_+(z) = -f_+(-z^{-1})$$

is associated with diffeomorphisms $F: \quad S^1 \mapsto S^1$ of the circle which are symmetric with respect to the imaginary axis in the sense that

$$F(-z^{-1}) = -F(z)^{-1}.$$

These diffeomorphisms have (at least) two fixpoints $\pm i$, since necessarily $f_+(\pm i) = 0$. From this it is clear that

$$V = \{T(f_-) \in \mathcal{G}^*_-, \quad f_-(z) < 0\} \tag{21}$$

is a cone in $\mathcal{G}^*_-$ which is invariant under the adjoint action (conjugation) by elements of G_+. If $T(if_-) \in iV$ then

$$\delta \ln z = i(if_-(z))\delta t > 0$$

and this property is invariant under conjugation by diffeomorphisms in $\mathcal{G}_+$.

The elements $T(f_+ - if_-)$, $(f_-(z) < 0)$, of $\mathcal{G}_+^* - iV$ generate a semigroup S as in the finite dimensional case, eq.(1c). In particular

$$e^{-T(f)} \in S \quad \text{if } f(z) = f(-z^{-1}) > 0.$$

This real semigroup S is subsemigroup of the complex semigroup S_c which is generated by $T(f)$ with complex f obeying $Imf(z) < 0$. It was pointed out by G. Segal [50] that S_c is a Lie semigroup, and unitary positive energy representations of the central extension $\widetilde{Diff}(S^1)$ of the diffeomorphism group define contractive representations of S_c. It follows from the first assertion that S is also a Lie semigroup. Assuming that the nonabelian Hille Yosida theorem proven in [43] for the finite dimensional case remains valid in our infinite dimensional setting, it furnishes a converse of Segals second assertion: Contractive representations of the real Lie-semigroup S can be continued to unitary positive energy representations of the universal covering group of $\widetilde{Diff}(S^1)$. At the Lie algebra level this result is elementary.

The complex semigroup S_c cannot be embedded into a Lie group, because neither $Diff(S^1)$ nor its universal covering possess a complexification. In the finite dimensional case, S_c cannot be embedded into a Lie group either, because also the universal covering of $SO(d,2)$ possesses no complexification, but the real semigroup S can be embedded into a Lie group $Spin(d+1,1)$. This motivates the search for a Lie group G with Lie algebra $\mathcal{G}$ in the infinite dimensional case.

I do not know whether such a Lie group exists. Heuristic arguments suggest that a nonvanishing central charge $\bar{c} \neq 0$ presents an obstruction. One could begin by constructing a group which is not necessarily a Lie group, by taking finite products of elements of S and the inverse semigroup S^{-1}. With $\Lambda \in S$ also $\theta(\Lambda^{-1}) \in S$. Therefore S^{-1} is image of S under time reflection.

There are reasonable groups N, A, X associated with the subalgebras $\mathcal{N}, \mathcal{A}, \mathcal{X}$ of $\mathcal{G}$. Consider the 1-parameter subgroups of N.

$$n_l(t) = \exp\frac{it}{l}\bar{L}_l, \qquad l = 1,2,3,...$$

In terms of n-fold commutators

$$n_l(t)\bar{L}_k n_l(t)^{-1} = \sum_{n\geq 0}\frac{(it)^n}{n!l^n}[\bar{L}_l,[\bar{L}_l,...[\bar{L}_l,\bar{L}_k]...]]. \tag{22}$$

Remember now that $\bar{L}_k = T(f^k)$, with f^k defined in eq.(10). Evaluating the multiple commutators one finds that the following equation holds for $f(z) = f^k(z) = i(-iz)^k$, and therefore also for finite sums f of such f^k's.

$$\begin{aligned} n_l(t)T(f)n_l(t)^{-1} &= T(f_t) \qquad \text{where} \\ f_t(i\zeta)/i\zeta = f(i\zeta_t)/i\zeta_t, &\qquad \zeta_t = \zeta(1+t\zeta^l)^{-1/l} \end{aligned} \tag{23}$$

We see that the 1-parameter groups $t \mapsto n_l(t)$ with $l > 0$, which are generated by elements of $\mathcal{N}$, can be regarded as groups of (algebraic) transformations $f \mapsto f_t$ of germs of holomorphic functions at $z = 0$ which are pure imaginary on the imaginary axis:

$$f(z) = \sum_{n\geq 0} f_n z^n, \qquad \sum_{n\geq 0}|f_n|t^n < \infty \quad \text{for some } t > 0. \tag{24}$$

They may be regarded as the elements of the dual of the Lie algebra $\mathcal{N}$. We may thus regard N as a group of analytic transformations of germs of real analytic functions on the imaginary axis at $z = 0$.

Similarly the 1-parameter subgroup $t \mapsto n_l(t)$ with $l < 0$, which are generated by $\mathcal{X}$, act on germs of holomorphic functions at $z = \infty$ which are pure imaginary on the imaginary axis.

Finally, A is a 1-parameter subgroup of the finite dimensional conformal group, it acts by dilation.

$$\begin{aligned} a(t)T(f)a(t)^{-1} &= T(f_t), \qquad \text{where} \\ f_t(i\zeta)/i\zeta = f(i\zeta_t)/i\zeta_t, &\qquad \zeta_t = e^t\zeta. \end{aligned} \tag{25}$$

APPENDIX (taken from Lüscher and Mack, ref. 23)

Here it is proven that $\Theta_{\mu\nu}$ is traceless in a scale invariant theory in two dimensions: $\Theta^\mu{}_\mu = 0$. To this end consider the Schwinger two-point function $S_{\mu\nu\rho\sigma}$ of $\Theta_{\mu\nu}$:

$$S_{\mu\nu\rho\sigma}(x) = \text{ analytic continuation of } < 0|\Theta_{\mu\nu}(x)\Theta_{\rho\sigma}(0)|0 > . \tag{1}$$

This function is real analytic for $x \in \mathbf{R}^2, x \neq 0$, and is covariant under rotations and dilations:

$$S_{\mu\nu\rho\sigma}(\lambda x) = \lambda^{-4}S_{\mu\nu\rho\sigma}(x). \tag{2}$$

Moreover, by locality , symmetry, and conservation of $\Theta_{\mu\nu}$ we have:

$$\begin{aligned} S_{\mu\nu\rho\sigma}(x) &= S_{\nu\mu\rho\sigma}(x) = S_{\mu\nu\sigma\rho}(x). \\ S_{\mu\nu\rho\sigma}(x) &= S_{\rho\sigma\mu\nu}(-x) = S_{\rho\sigma\mu\nu}(x). \end{aligned} \tag{3}$$

Thus, $S_{\mu\nu\rho\sigma}$ has six independent components and can be written as follows:

$$S_{\mu\nu\rho\sigma} = (x^2)^{-4}\sum_{i=1}^{6} A_i T^i_{\mu\nu\rho\sigma}, \qquad A_i \in \mathbf{C}, \tag{4}$$

where ($\epsilon_{\mu\nu} = -\epsilon_{\nu\mu}$, $\epsilon_{01} = +1$, $g_{00} = g_{11} = 1$)

$$\begin{aligned} T^1_{\mu\nu\rho\sigma} &= (x^2)^2 g_{\mu\nu}g_{\rho\sigma} \\ T^2_{\mu\nu\rho\sigma} &= (x^2)^2(g_{\mu\rho}g_{\nu\sigma} + g_{\mu\sigma}g_{\nu\rho}) \\ T^3_{\mu\nu\rho\sigma} &= x^2(g_{\mu\nu}x_\rho x_\sigma + g_{\rho\sigma}x_\mu x_\nu) \\ T^4_{\mu\nu\rho\sigma} &= x_\mu x_\nu x_\rho x_\sigma \\ T^5_{\mu\nu\rho\sigma} &= x^2\{g_{\mu\nu}(x_\rho\epsilon_{\sigma\delta}x^\delta + x_\sigma\epsilon_{\rho\delta}x^\delta) + g_{\rho\sigma}(x_\mu\epsilon_{\nu\delta}x^\delta + x_\nu\epsilon_{\mu\delta}x^\delta)\} \\ T^6_{\mu\nu\rho\sigma} &= (x_\mu\epsilon_{\nu\delta}x^\delta + x_\nu\epsilon_{\mu\delta}x^\delta)x_\rho x_\sigma + (x_\rho\epsilon_{\sigma\delta}x^\delta + x_\sigma\epsilon_{\rho\delta}x^\delta)x_\mu x_\nu. \end{aligned}$$

T^5 and T^6 are odd under parity and are absent in (4) if parity is conserved. The continuity equation $\partial^\mu S_{\mu\nu\rho\sigma} = 0$ now fixes the numbers A_i up to two arbitrary constants A_+, A_-:

$$\begin{aligned} A_1 = 3A_+, \quad A_2 = -A_+, \quad A_3 &= -4A_+, \quad A_4 = 8A_+, \\ A_5 = A_-, \quad A_6 &= -2A_- \end{aligned}$$

Upon inserting these values into eq.(4) we find

$$S^{\mu}{}_{\mu\rho\sigma}(x) = S^{\mu}{}_{\mu}{}^{\rho}{}_{\rho}(x) = 0.$$

Hence $< 0|\theta^{\mu}{}_{\mu}(x)\theta^{\rho}{}_{\rho}(0)|0 >$ vanishes, which, by the Reeh-Schlieder theorem [51], implies that $\theta^{\mu}{}_{\mu}(x) = 0$.

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