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ABSTRACT: Methods of algebraic quantum field theory are used to classify all field- and observable algebras, whose common germ is the $U(1)$-current algebra. An elementary way is desribed to compute characters of such algebras. It exploits the Kubo-Martin-Schwinger condition for Gibbs states.

CONTENTS

I. INTRODUCTION

Two dimensional conformal models have become a major tool for the investigation of surface critical phenomena [1,2,3] and in the construction of string theories [4,5,6]. Adopting the point of view of conformal invariant quantum field theory (QFT) on Minkowski space, we apply in the present paper methods of the algebraic approach to local QFT [7-9] to classify such models.

Starting from certain specific observables, such as the stress energy tensor, currents etc., whose commutation relations are explicitly known in 2-dimensional conformal QFT, we want to determine all local quantum field theories in which these observables appear. The basic ingredient in our construction are the localized morphisms[1] of the given algebra of observables. We will make use of the fundamental fact that the relevant information on the desired fields is encoded in these morphisms and can be deciphered in a systematic manner [9].

It is the aim of the present paper to expound this method in the case of the simplest class of such theories. They involve the chiral $U(1)$-current algebra [10,11] and its QFT-extensions. We recall that conserved chiral currents have left- and right-moving components which live on the circle. We classify the algebras of all ("right movers' or left movers' ") observables on the circle whose germ is the chiral current algebra. We find their superselection sectors and associated fields that are relatively local to the observables. From there we arrive at local 2-dimensional models by using the principle of modular invariance. The application of this principle requires the computation of characters of the associated algebras (or groups of their unitaries). We will demonstrate that this can be achieved in an elementary way by an analysis of the Gibbs equilibrium states with the help of the Kubo-Martin-Schwinger (KMS) condition [12].

Our Theorem 1 (Section 1A) states that the method of localized morphisms is general in *conformal* QFT, and yields fields which intertwine arbitrary positive energy representations of the algebra of observables. So they can generate the whole Hilbert space of physical states from the vacuum. We hope to apply this method in a forthcoming paper to the classification of QFT-representations of the Virasoro algebra with central charge not restricted to the value $c = 1$.

For the convenience of the reader we give in the two subsequent sections of this introduction an account of the algebraic approach to the construction of charged fields and of some basic facts in conformal quantum field theory. The third subsection contains a detailed summary of our results.

1A. Observables, fields and superselection sectors

In two dimensional conformal quantum field theory one starts from Bose fields - currents and/or the stress energy tensor - with explicitly known commutation relations [13,14]. One may regard these as local observables, they generate a *-algebra $\mathcal{A}$. The Hilbert space $\mathcal{H}$ of all physical states will in general decompose into subspaces $\mathcal{H}_g$ called superselection sectors [15] which carry inequivalent irreducible positive energy representations of $\mathcal{A}$. Among them is the vacuum sector $\mathcal{H}_0$ which contains the vacuum Ω.

[1] A morphism of an algebra is a map of the algebra into itself which preserves the algebraic relations, cf. Section 1A.

The observables map each sector $\mathcal{H}_g$ into itself. Our problem is to construct additional field operators which are relatively local to the observables, and which make transitions between superselection sectors such that the whole Hilbert space $\mathcal{H}$ is generated from the vacuum $\Omega \in \mathcal{H}_0$.

The construction of such fields helps to solve another problem. One may find that one has started from an algebra of observables which is too small - because it admits unacceptably many superselection sectors, for instance. One may then search among the new fields for possible additional observables. They should be Bose fields, i.e. commute at spacelike distances.

According to the construction principle of Doplicher, Haag and Roberts [9] the information about such fields is hidden in the algebra of local observables $\mathcal{A}$, because $\mathcal{A}$ determines its localized morphisms γ, and the fields are to be constructed from such localized morphisms. The fields come in the form of a typical "bosonization formula"

$$\psi_g(x) = A(g;x)\Gamma_g, \qquad \text{if} \quad \psi_g\Omega \in \mathcal{H}_g. \tag{1.1a}$$

The factor Γ_g depends only on the quantum numbers g of the field, and all the space time dependence is in the factor $A(g;x)$ which is a function of the observable fields (e.g. currents).

We proceed now to a more precise description of the construction. Given the collection of local algebras of observables $\mathcal{A}(\mathcal{O}) \subset \mathcal{A}$ which are generated by observables A that are localized in bounded regions $\mathcal{O}$ of Minkowski space M one must first determine the localized morphisms γ of $\mathcal{A}$. By definition, a morphism γ of $\mathcal{A}$ which is localized in $\mathcal{O}$ is a **C**-linear map $\mathcal{A} \mapsto \mathcal{A}$ which obeys

$$\begin{aligned} &\gamma(A_1A_2) = \gamma(A_1)\gamma(A_2), \quad \gamma(A_1^*) = \gamma(A_1)^*, \\ &\gamma(A) = A \quad \text{for } A \in \mathcal{A}(\mathcal{O}'), \end{aligned} \tag{1.2}$$

where $\mathcal{O}'$ lies relatively spacelike to $\mathcal{O}$. It is known [9] that the desired outer morphisms can in principle be obtained as limits of *inner* automorphisms of the algebra $\mathcal{A}$ of local observables, which "transport charge to infinity" in Minkowski space.

If π_0 is the representation of the algebra of local observables $\mathcal{A}$ in the vacuum sector, and γ is a localized morphism, then

$$\pi_\gamma(A) = \pi_0(\gamma(A)) \tag{1.3}$$

defines another representation of $\mathcal{A}$ whose unitary equivalence class depends only on the equivalence class $g = [\gamma]$ of the morphism γ modulo inner automorphisms. The label g has the physical meaning of a collection of quantum numbers which distinguish various superselection sectors, i.e. inequivalent representations of $\mathcal{A}$. It will be called "charge" for short. Under general assumptions on the observable algebra (DHR-duality)[1] there will exist morphisms in the same equivalence class which are localized in arbitrary domains $\mathcal{O}$, and two morphisms γ are equivalent modulo inner automorphisms of $\mathcal{A}$ if and only if they generate equivalent representations π_γ.

To every morphism γ localized in some $\mathcal{O}$ one seeks a field operator ψ_γ which implements the automorphism in the sense that

$$A\psi_\gamma = \psi_\gamma\gamma(A) \qquad \text{for all } A \in \mathcal{A}. \tag{1.4}$$

[1] DHR-duality requires that the observable algebra is maximal in the sense that it is impossible to add operators to $\pi_0(\mathcal{A})$ which are also local *and* which map the vacuum sector $\mathcal{H}_0$ into itself. DHR-duality is true for the $U(1)$-current algebra [16].

Because there exist morphisms with arbitrary localization regions $\mathcal{O}$ in the same equivalence class $g = [\gamma]$, one can choose an operator Γ_g which implements some representative of g, and write a "bosonization formula"

$$\psi_\gamma = A(\gamma)\Gamma_g \tag{1.1b}$$

where $A(\gamma)$ is a suitable observable. Γ_g depends only on the charge g while information on the localization region of ψ_γ is in the factor $A(\gamma)$. The field algebras $\mathcal{F}(\mathcal{O})$ which extend the algebras $\mathcal{A}(\mathcal{O})$ are defined as the *-algebras generated by operators ψ_γ, where γ is localized in $\mathcal{O}$. These fields are relatively local to the observables in the sense that

$$[\psi_\gamma, A] = 0 \qquad \text{if } \psi_\gamma \subset \mathcal{F}(\mathcal{O}_1),\ A \subset \mathcal{A}(\mathcal{O}_2), \tag{1.5}$$

and $\mathcal{O}_1$ and $\mathcal{O}_2$ are relatively spacelike to each other. Fields localized at a point are constructed as limits, by choosing in (1.1b) sequences of morphisms γ whose localization regions shrink to a point.

At this point we would like to note that in two space time dimensions it would not be reasonable to assume from the outset that the fields are also local relative to themselves or to other charged fields. Firstly, there exist interesting field theoretic models in which such nonlocal fields appear naturally, so the study of such fields is of interest in its own right [17]. Secondly - and this is important in the present context - some algebras of observables in conformal QFT are obtained as products of two commuting algebras living on 1-dimensional compact manifolds (the circle $\mathbf{S}^1$). Then the corresponding localized morphisms involve products of localized morphisms of these 1-dimensional components. The fields on the circle obtained from them can be used as building blocks of *local* 2-dimensional fields, although they may not be local by themselves.

The general construction of the field algebra $\mathcal{F}$ as an extension of the algebra $\mathcal{A}$ of observables by localized morphisms γ of $\mathcal{A}$ was recently completed by Doplicher and Roberts [9] for $d > 2$ dimensions. Their charge shift operators Γ_g in the bosonization formula (1.1b) are elements of Cuntz algebras. Together with the field algebra $\mathcal{F}$, a gauge group (of the first kind) K is constructed. This is a symmetry group of the quantum field theory which acts trivially on the observables.

The restriction to $d > 2$ in the work of Doplicher and Roberts comes about as follows. Besides the morphisms of the algebra $\mathcal{A}$ one needs to consider intertwiners $\epsilon \in \mathcal{A}$ between morphisms γ and σ. They obey the relation

$$\gamma(A)\epsilon = \epsilon\sigma(A)$$

for all $A \in \mathcal{A}$. Properties of these intertwiners are used in an essential way in the general analysis, and these properties can be established from general principles of quantum field theory only in $d > 2$ dimensions [9], because in two dimensions the spacelike complement of a bounded region in Minkowski space is not connected. For $d = 2$ the right substitute for the Cuntz algebras is not known yet - but see the lectures of J. Fröhlich and B. Schroer in [17,18] for interesting results that are relevant here. *Conformal* quantum field theory in 2 dimensions lives on a space time where the spacelike complement of a neighbourhood of a point *is* connected. But the nontrivial topology of this space time causes complications.

The Doplicher Haag Roberts method works also in the conformal field theories we are interested in, although in two dimensions some proofs need to be reexamined,

and some assertions do not continue to hold. It is known that by this construction one gets all those superselection sectors $\mathcal{H}_g$ which carry representations π_g of $\mathcal{A}$ that are unitarily equivalent to π_0 when restricted to observables localized in $\mathcal{O}'$, where $\mathcal{O}'$ is the spacelike complement of some bounded (and topologically trivial) domain $\mathcal{O}$. It is a remarkable[1] fact that these are indeed *all* sectors in *conformal invariant* quantum field theory, in any number of dimensions $d \geq 2$, at least if the local algebras of observables satisfy the natural condition that weak limits of observables are also observables.

THEOREM 1. *In conformal invariant quantum field theory in $d \geq 2$ dimensions all superselection sectors $\mathcal{H}_g$ with positive energy are generated from the vacuum sector $\mathcal{H}_0$ by localized morphisms, assuming $\mathcal{A}(\mathcal{O})$ are von Neumann algebras and satisfy DHR-duality.*

REMARK: In algebraic field theory, the algebra of observables is regarded as defined by its vacuum representation, so $\mathcal{A}(\mathcal{O})$ and $\pi_0(\mathcal{A}(\mathcal{O}))$ can be identified.

PROOF: The crucial fact behind this theorem is that conformal field theory lives on a space time manifold $\tilde{M}$ (a tube) whose spacelike surfaces are compact (spheres) [19]. It contains Minkowski spaces $M \subset \tilde{M}$ which are determined by the point ζ at their spacelike infinity. We admit bounded domains $\mathcal{O} \subset M$ contained in any one of them and write

$$\mathcal{A}_\zeta = \bigcup_{\mathcal{O}\subset M} \mathcal{A}(\mathcal{O})$$

for the algebra of localized observables on M. The spacelike complement $\mathcal{O}'$ of a bounded region $\mathcal{O} \subset M$ is itself a bounded domain in some Minkowski space. The assertion follows then from the known fact that positive energy representations of nets of von Neumann algebras are locally normal, i.e. when restricted to any $\mathcal{A}(\mathcal{O})$ they are unitarily equivalent [20]. *q.e.d.*

In the example of the current algebra discussed in our paper, the localized morphisms are in fact automorphisms, i.e. they have an inverse. This situation is somewhat simpler and was dealt with long ago [21]. From the general analysis one expects that the physical Hilbert space has the form

$$\mathcal{H} = \sum_g \mathcal{H}_g, \tag{1.6}$$

i.e. it is a direct sum of inequivalent irreducible representation spaces $\mathcal{H}_g$ of $\mathcal{A}$ with multiplicity 1. The gauge group K is abelian. Mathematically speaking, the field algebra $\mathcal{F}_\zeta \subset \mathcal{F}$ on Minkowski space is constructed as a central extension of the group of automorphisms γ of the algebra of observables $\mathcal{A}_\zeta$ by the circle group $\mathbf{T} \subset \mathbf{C}$, whereas its subalgebra $\mathcal{A}_\zeta$ is central extension of the group of its own *inner* automorphisms by $\mathbf{T}$. (Locality implies that this extension always exists [21]). The observables are precisely the K-invariant elements of the field algebra.

[1] The same device (1.3) could be used in the representation theory of compact connected Lie groups, with reflections as outer automorphisms. But this would yield representations of the same dimension, so there is no chance of getting all from one. Example: The three 8-dimensional representations of SO(8) (spinor, conjugate spinor, and vector) are related by outer automorphisms.

1B. The Conformal Current Algebra on the Circle

In the general case of a d-dimensional conformal QFT the point of departure is either a Minkowski (M) space theory with Wightman functions invariant under infinitesimal conformal ($so(d,2)$) transformations, or, equivalently, a Euclidean theory with $Spin(d+1,1)$ invariant Schwinger functions. Such a theory can be extended by analytic continuation to a QFT on the universal cover $\tilde{M}$ of compactified Minkowski space, invariant under global transformations in (the ∞ sheeted universal cover of) the Minkowski space conformal group [19]. In $d>2$ dimensions $\tilde{M}$ is the cylinder space $\mathbf{S}^{d-1}\times\mathbf{R}$, the "Einstein universe" singled out by I. Segal [22] for its conformally invariant global causal structure. We shall consider here a QFT on the 2-dimensional cylinder $\tilde{M}=\mathbf{S}^1\times\mathbf{R}$, although it is not simply connected. Doing this means that we restrict attention to half-integer "helicity" fields; we recall that there is no proper spin or helicity in 1+1 dimensions since there is no rotation group.

If $\xi = x^0 - x^1$ and $\bar{\xi} = x^0 + x^1$ are the light cone coordinates in M then the corresponding compact picture coordinates are

$$z = \frac{1+\frac{i}{2}\xi}{1-\frac{i}{2}\xi} = e^{i(\sigma^0-\sigma^1)} \qquad \bar{z} = \frac{1+\frac{i}{2}\bar{\xi}}{1-\frac{i}{2}\bar{\xi}} = e^{i(\sigma^0+\sigma^1)} \tag{1.7}$$

where σ_1 is the angle parametrizing space $\mathbf{S}^1$ and σ_0 is the variable parametrizing time $\mathbf{R}$.

Free fields and conserved chiral currents can be split into mutually commuting right and left moving components which depend only on one of these light cone variables. For the symmetric traceless stress energy tensor $\Theta^\mu{}_\nu$ and the $u(1)_R\times u(1)_L$ conserved current J^μ (satisfying $\partial_\mu J^\mu = 0 = \partial_\mu\epsilon^{\mu\nu}J_\nu$) these chiral components are $\Theta_\pm = (\Theta^0{}_0 \pm \Theta^1{}_0)/2$ and $J_\pm = (J^0 \pm J^1)/2$.

Upon variable transformation to $z,\bar{z}$ they become

$$T(z) = 2\pi\left(i\frac{d\xi}{dz}\right)^2 \Theta_+(\xi(z)) \tag{1.8}$$

etc. ($i\,d\xi/dz = 4(1+z)^{-2}$). In spite of the anomalous transformation law of the stress energy tensor under general diffeomorphisms there is no extra term proportional to $\mathbf{1}$ because the coordinate transformation (1.7) is fractional linear.

General fields in conformal QFT live on the cylinder $\mathbf{S}^1\times\mathbf{R}$ as was mentioned. However, for observable fields which depend only on one light cone variable, periodicity in the angle σ_1 which parametrizes space $\mathbf{S}^1$ implies periodicity in time $\mathbf{R}$. As a result, observables such as $T(z)$ and $J(z)$ are indeed one-valued functions on the circle $\{|z|=1\}$.

Since Minkowski space M is a subset of the cylinder specified by the point ζ at its spacelike infinity, the projections of the connected and bounded subsets $\mathcal{O}\subset M$ on the circle $\{|z|=1\}$ are intervals

$$I\subset\mathbf{S}^1, \qquad \zeta\notin I.$$

Relatively spacelike domains $\mathcal{O}\subset M$ project onto disjoint intervals on the circle. Thus

relatively spacelike $\equiv$ *disjoint*

on the circle. Consequently, observable fields on $\mathbf{S}^1$ commute at different points.

The Fourier-Laurent expansion of $T(z)$ introduces the Virasoro generators $L_n = L^*_{-n}$

$$T(z) = \sum_n L_n z^{-n-2}, \qquad n\in\mathbf{Z}. \tag{1.9}$$

The commutation relations of the stress tensor with itself follow from its tracelessness, conservation, scale covariance, and association with the generators of translations in two dimensions, and lead [13] to the commutation relations of the Virasoro algebra

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{nm} \tag{1.10}$$

Similarly the commutation relations of the current $J(z)$ are determined, they are Heisenberg commutation relations for a free Bose field

$$[J(z_1), J(z_2)] = -\delta'(z_{12}), \tag{1.12}$$

where the z-picture δ-function is defined by

$$\oint_{\mathbf{S}^1} \delta(z_{12})u(z_2)\frac{dz_2}{2\pi i} = u(z_1), \quad (z_{12}\equiv z_1 - z_2). \tag{1.13}$$

Using the Fourier-Laurent expansion

$$J(z) = \sum_n J_n z^{-n-1}, \qquad J^*_n = J_{-n}, \tag{1.14}$$

these commutation relations and those with the stress tensor read

$$[J_m, J_n] = m\,\delta_{n+m,0}\,, \qquad [J_m, L_n] = m\,J_{m+n}. \tag{1.15}$$

A stress tensor with correct commutation relations (with $c=1$) can be made out of the current J by the Sugawara construction[1]

$$T(z) = \frac{1}{2}:J(z)^2: = \frac{1}{2}\{J_+(z)J(z) + J(z)J_-(z)\} \tag{1.16}$$

where the currents frequency parts are defined by

$$J_+(z) = J(z) - J_-(z) = \sum_{n=1}^{\infty} J_{-n}z^{n-1}.$$

In particular this gives for the *conformal Hamiltonian* $H = L_0$

$$H = \frac{1}{2}J_0^2 + \sum_{n=1}^{\infty} J_{-n}J_n. \tag{1.17}$$

The conformal Hamiltonian generates a time evolution automorphism

$$\alpha_t(A) = e^{iHt}Ae^{-iHt} \tag{1.18}$$

[1] For recent reviews see [23,24].

It acts on the currents according to

$$\alpha_t((J(z)) = e^{it}J(e^{it}z),$$
$$[H, J(z)] = \frac{d}{dz}(zJ(z)) \tag{1.19}$$

1C. SUMMARY OF RESULTS

In this paper we start from the $U(1)$-current algebra (in 2 dimensions) as our original algebra of local observables. We adjoin to it a conformal Hamiltonian which is some (unspecified) function of the currents. The algebra factorizes into current algebras on the circle, denoted by $\hat{U}$, as explained in Section 1B. We regard $\hat{U}$ as the *germ* of all algebras of observables on the circle which extend it. As we shall see, this extension problem is made meaningful and soluble by the requirement that the Hamiltonian H which generates the time evolution automorphism of these algebras is a function of the original current J.

Our Theorem 1 (in Section 1A) asserts that there always exists a sufficiently large class of localized morphisms of the algebra of observables $\mathcal{A}$ in a conformal QFT to generate all superselection sectors (i.e. all inequivalent positive energy representations of $\mathcal{A}$) from the vacuum sector. For the algebra $\hat{U}$ we are able to exhibit them explicitly. They look as follows:

$$J(z) \mapsto \gamma(J(z)) = J(z) + \rho(z)\mathbf{1} \tag{1.20}$$

where ρ is a smooth function on the circle, with $z\rho(z)$ real. $\rho(z)$ is called a charge distribution. Note that these maps do not change the commutation relations (1.12) and they commute with the *-operation (taking adjoints). Moreover, they have an inverse, so they are automorphisms of the current algebra. These automorphisms are localized in I if $\operatorname{supp}\rho \subset I$. The equivalence classes of such automorphisms of $\hat{U}$ modulo inner automorphisms are distinguished by the total charge

$$g = \int \frac{dz}{2\pi i}\rho(z). \tag{1.21}$$

If the Hamiltonian H is given by the Sugawara formula (1.17) then the automorphism corresponding to a constant charge distribution with total charge g acts on it according to

$$H \mapsto H + gJ_0 + \frac{1}{2}g^2. \tag{1.22}$$

This relates the energy spectrum in the vacuum representation and in the superselection sector which contains a state of charge g.

We show that any Hilbert space of physical states $\mathcal{H}$ which can be generated from the vacuum Ω by fields which are local relative to the currents is a multiplicity free sum of inequivalent irreducible representation spaces for $\hat{U}$ which are labeled by the charge g; g equals the eigenvalue of the spatial integral $Q \equiv J_0$ of the current J, and the charge spectrum $\{g\}$ must be of the form $g_0 + \mathbf{Q}$, where $\mathbf{Q}$ is an additive subgroup of the real numbers $\mathbf{R}$.

We examine the algebras generated by these fields which create $\mathcal{H}$ from Ω. It is proven that the net of field algebras $\mathcal{F}(I)$ for subsets I of the punctured circle

is uniquely determined by the charge spectrum $\{g\}$ of the theory, up to a Klein transformation.[1] See Proposition 3.1 and Lemma 3.6.

It should be mentioned that the result asserting uniqueness of the field algebra depends on an extra hypothesis which is natural in quantum field theory on Minkowski space. In a quantum field theory on Minkowski space M in which the Hilbert space $\mathcal{H}$ of all physical states is generated from the vacuum by *local* fields, the vacuum cannot be annihilated by any field $\psi \in \mathcal{F}(\mathcal{O})$ which is localized in a bounded domain $\mathcal{O} \subset M$, and for any open $\mathcal{O}$ the fields $\psi \in \mathcal{F}(\mathcal{O})$ suffice to generate a dense subset of $\mathcal{H}$. This is the Reeh-Schlieder theorem [17]. Our fields are local relatively to the observables, but not necessarily to themselves, therefore the theorem does not apply. We adopt its assertion as a hypothesis. This is natural especially if we want to use our fields as building blocks of local fields.

The field algebras are constructed explicitly. They have the properties listed above as expected for a theory with an abelian gauge group. Explicit forms of the bosonization formulae (1.1) are derived, see Proposition 3.4. Fields at a point are constructed as limits in Section 5A.

The global field algebra $\mathcal{F}$ has a very simple structure. It is spanned by its unitary elements ψ. They have the form

$$\psi = \eta W(u)\Gamma_g. \tag{1.23}$$

Such a field creates charge g. η is a complex phase factor, $W(u)$ are the Weyl operators depending on some real test function u,

$$W(u) = \exp\left(i\int \frac{dz}{2\pi i}J(z)u(z)\right), \tag{1.24}$$

and there appear in addition the unitary charge shift operators Γ_g which were mentioned in Section 1A. The multiplication law in $\mathcal{F}$ is given by the multiplication law for Weyl operators, (cf. Section 2, Eq. (2.6)) together with the relations

$$\Gamma_{g_1}\Gamma_{g_2} = \Gamma_{g_1+g_2}, \qquad \Gamma_0 = \mathbf{1}$$
$$W(u)\Gamma_g = \Gamma_g W(u)e^{ig\tilde{u}_0} \tag{1.25}$$

where $\tilde{u}_0 = \oint u(z)dz/2\pi i z$.

The structure of the subalgebras $\mathcal{F}(I)$ which consist of fields localized on intervals I of the punctured circle is more subtle. $\mathcal{F}(I)$ includes only fields ψ such that the automorphisms γ of $\hat{U}$ which they induce

$$A \mapsto \gamma(A) = \psi^* A\psi \tag{1.26}$$

are localized in I. However, if we were to include all ψ with this property in $\mathcal{F}(I)$ then elements of the center of $\hat{U}$ such as linear combinations of the elements e^{isQ} would be in all the $\mathcal{F}(I)$. Such elements can annihilate the vacuum, so this would violate the Reeh-Schlieder property, which we assumed. Another important constraint on

[1] A Klein transformation multiplies fields ψ_g of charge g with unitary elements $Y(g)$ of the center of the observable algebra, like $e^{i\pi cQ}$. It can change commutation relations in general, but leaves the commutators of two fields with the same quantum numbers g unaffected.

the field algebras comes from the requirement of time translation covariance (see Definition in Section 3A)

$$e^{iHt}\mathcal{F}(I)e^{-iHt} \subset \mathcal{F}(e^{it}I).$$

A net of field algebras with these properties - which is unique up to a Klein transformation as was said above - is obtained by choosing suitable representatives ψ_ρ of the form (1.23). Such a choice of fields ψ_ρ is described in Proposition 3.4 in Section 3B. It depends on the point ζ at spacelike infinity. This choice of fields ψ_ρ has the property that the group theoretical commutator of two such fields (with the same ζ) is a c-number.

These field algebras are attached to the punctured circle which is the projection of a Minkowski space without points at infinity. Upon covering the whole circle with such "Minkowski spaces" (charts) one needs transition functions which are in general not c-numbers but involve multiplication of the fields by unitary elements of the center of the observable algebra, i.e. a Klein transformation, except for observable fields. One may speak of a "bundle of Minkowski space QFT's".

Quantum fields $\psi_g(z)$ at a point can be constructed from the field operators ψ_ρ by a limit procedure (Section 5). Their correlation functions follow from the multiplication law (1.25), (2.6) in the field algebra, and the formula (2.4) for vacuum expectation values $\omega_0(W(u))$ of Weyl operators. The multiplication law in the field algebra $\mathcal{F}$ embodies the information on operator product expansions for the quantum fields $\psi_g(z)$.

It is amusing to change the point of view. Suppose that our experimental colleagues in the two-dimensional world were authorized to buy some new equipment so that other quantities besides currents and energy became observable. What could these larger algebras of observables be? This is exactly our problem of finding all algebras of observables whose germ is the current algebra. We find a complete classification in Section 4. The classification follows from the observation that these algebras have to be among the field algebras constructed before, and are singled out by the fact that the fields must be relatively local to themselves.

The observable algebras $\mathcal{A}_N$ extending $\hat{U}$ are labeled by an integer $N = 1, 2, \ldots$, where $N = \frac{1}{2}g^2$ is given by the value of the smallest positive charge g that is created by a field in $\mathcal{A}_N$. The algebras are maximal, i.e. cannot be further extended, if N is a product of distinct primes

$$N = 1, 2, 3, 5, 6, 7, 10, \ldots.$$

These new algebras of observables and their superselection sectors (positive energy representations) and associated field algebras which make transitions between these sectors are studied in Section 4. They provide further examples of the constructs described in Section 1A with gauge groups $\mathbf{Z}_{2N}$.

Among them is the $\widehat{su(2)}$-current algebra of level 1. In this case our findings agree with known results [25]. There are two superselection sectors (including the vacuum sector) and they are mapped into each other by (localized) outer automorphisms of the $\widehat{su(2)}$ current algebra and the associated loop group. In addition we obtain the field operators which make transitions between the two sectors.

These results show that the celebrated vertex operators of conformal field theory are special cases of the much more general constructs of algebraic field theory which were recalled in Section 1A.[1]

To make effective use of these algebras of observables and fields one desires their characters. They are determined by their restriction to the groups of unitaries in these algebras. In Section 6 we describe an elementary way of computing such characters by exploiting the KMS-condition for Gibbs states on algebras of fields or observables. The characters of $\mathcal{A}_N$ are found. They are expressed in terms of Theta-functions. They are related to the modular forms of weight $\frac{1}{2}$ which were studied by Serre and Stark [31].

There are 2N superselection sectors for $\mathcal{A}_N$. We show that its partition functions transform according to a 2N-dimensional representation of the modular group $SL(2, \mathbf{Z})$. This can be used to construct modular invariant partition functions for the 2-dimensional theory, and to construct local 2-dimensional fields from fields on the circle which are only relatively local to the observables. This is briefly discussed in Section 6, for more details see [32].

The paper concludes with an outlook which discusses possible generalizations.

II. POSITIVE ENERGY REPRESENTATIONS OF THE CURRENT ALGEBRA

2A. Exponential (Weyl) form of $\hat{U}$

Consider the space $\mathcal{S} = \mathcal{S}(\mathbf{S}^1)$ of real test functions u on $\mathbf{S}^1$,

$$u(z) = \sum_n \tilde{u}_n z^{-n}, \qquad \tilde{u}_n^* = \tilde{u}_{-n}. \tag{2.1}$$

We define the smeared current by

$$J(u) = \int \frac{dz}{2\pi i} J(z)u(z) = \sum_n \tilde{u}_n^* J_n \tag{2.2}$$

which in view of the reality of u is a hermitean operator. Equation (1.19) gives rise to an action of the time translations by automorphisms α_t on the currents,

$$\alpha_t(J(u)) = J(u_t), \quad \text{where} \quad u_t(z) = u(e^{-it}z). \tag{2.3}$$

The current commutation relations (1.12) now take the form

$$[J(u), J(v)] = \int \frac{dz}{2\pi i} u'(z)v(z) \equiv A(u, v). \tag{2.4}$$

It will be convenient to work with the exponentials of the smeared currents (Weyl operators)

$$W(u) = e^{iJ(u)}. \tag{2.5}$$

The commutation relations (2.4) are equivalent to the composition law

$$W(u)W(v) = e^{-\frac{1}{2}A(u,v)}W(u+v). \tag{2.6}$$

[1] The vertex operator construction has appeared in various guises [26,27], for early work cf. [28,29,30].

and the adjoints of the Weyl-operators are given by

$$W(u)^* = W(-u). \qquad (2.7)$$

The Weyl operators thus generate a *-algebra of bounded operators with a distinguished action of the time translations

$$\alpha_t(W(u)) = W(u_t), \qquad u_t(z) = u(\epsilon^{-it}z). \qquad (2.8)$$

By an abuse of notation we will denote this algebra by $\hat{U}$. Throughout the remainder of this article we will only make use of the structure incorporated in the relations (2.6) to (2.8).

2B. Positive energy representations of $\hat{U}$

Since $\hat{U}$ describes a system with infinitely many degrees of freedom there exists an abundance of representations and we have to select those which are of interest here by some selection criterion.

The standard class of representations of $\hat{U}$ considered in the literature are representations having a ground state (lowest weight representations). From a general point of view it appears to be more natural only to demand that in the representations of interest there exists a generator H of the "time translations" which is non-negative. The condition that H is bounded below means that we are dealing with elementary systems, but there is no a priori reason to impose further conditions on the spectrum of H (such as the existence of discrete eigenvalues). In the present case of the dynamical system $(\hat{U}, \alpha_t)$ it turns out, however, that any representation admitting a generator $H \geq 0$ of the time translations is a (direct sum of) ground state representations.

Before giving the argument, we introduce some notation: We denote the representations of $\hat{U}$ by $(\pi, \mathcal{H})$. They consist of a representation (Hilbert-) space $\mathcal{H}$ and a mapping π of $\hat{U}$ into $\mathcal{B}(\mathcal{H})$ (the algebra of all bounded operators on $\mathcal{H}$) which respects the algebraic relations in $\hat{U}$, i.e. π is a homomorphism. As already mentioned, we are interested in those representations $(\pi, \mathcal{H})$ for which there exists a positive selfadjoint operator H on some domain in $\mathcal{H}$ such that

$$\pi(\alpha_t(W)) = \epsilon^{iHt}\pi(W)\epsilon^{-iHt}, \qquad W \in \hat{U}. \qquad (2.9)$$

We call such representations "positive energy representations". Note that relation (2.9) does not completely fix the operator H, since one may add to it any selfadjoint operator from the commutant $\pi(\hat{U})'$ of $\pi(\hat{U})$, without affecting this relation.

The ambiguities involved in the choice of a Hamiltonian H implementing the dynamics can be removed by the requirement that[1]

$$\epsilon^{iHt} \in \pi(\hat{U})'' \qquad (2.10)$$

and that 0 is the lower boundary of the spectrum of H in each subspace of $\mathcal{H}$ reducing the representation π. It is a very general fact [33] that such a "minimal generator" H always exists and it is uniquely fixed by these conditions. We will adopt this convention for the time being, but we want to point out already now that the ad hoc choice of the lowest eigenvalue of H as the zero-point of energy in each subrepresentation of $(\pi, \mathcal{H})$ will turn out to be inappropriate in the extension of the time translations to the charged fields. A more appropriate choice of the generator H will follow from the requirement that the fields should transform covariantly under time translations, as will be discussed later.

[1] We recall that the double commutant $\mathcal{B}''$ of a *-algebra $\mathcal{B}$ of bounded operators is equal to its weak closure. Regarding $\hat{U}$ as an algebra of observables, relation (2.10) may thus be understood as the requirement that H is observable.

After this disgression let us turn now to our problem, the determination of all positive energy representations of $\hat{U}$. We shall need the following elementary facts.

Lemma 2.1. *Every positive energy representation $(\pi, \mathcal{H})$ of $\hat{U}$ can be decomposed into a direct sum of ground state representations. More precisely, there exists a family of unit vectors $\Omega_i \in \mathcal{H}, i \in \mathbf{I}$ (some index set) such that the subspaces $\mathcal{H}_i = \pi(\hat{U})\Omega_i \subset \mathcal{H}$ are mutually orthogonal, $\mathcal{H} = \sum_i \mathcal{H}_i$, and $H\Omega_i = 0$ for all $i \in \mathbf{I}$, where H is the minimal generator of the time translations.*

Proof: Since $t \mapsto \pi(\alpha_t W(u))$ is periodic with period 2π, every operator $\pi(W), W \in \hat{U}$ can be decomposed into a series of bounded operators

$$\tilde{W}_n = (2\pi)^{-1}\int_0^{2\pi} ds\, \epsilon^{-ins}\pi(\alpha_s(W)), \qquad n \in \mathbf{Z}.$$

Moreover,

$$\tilde{W}_n\epsilon^{iHt} = \epsilon^{i(H-n)t}\tilde{W}_n,$$

hence $\tilde{W}_n$ transfers the energy n to all vectors. Now let Δ be any closed interval contained in $(-1,1)$ and assume that there exists some vector $\Omega \neq 0$ in the spectral subspace associated with the spectrum of H in Δ. Since $H \geq 0$ it follows that $\tilde{W}_n\Omega = 0$ if $n < 0$. Hence the spectrum of H on the subspace $\pi(\hat{U})\Omega$ is contained in

$$\bigcup_{n \in \mathbf{N}_0} (\Delta + n).$$

Now according to our convention, 0 is the lower boundary point of the spectrum of H on each subspace which reduces π. Thus 0 must be contained in Δ, showing that 0 is the only point within the interval $(-1,1)$ which is in the spectrum of H. Hence $H\Omega = 0$. Proceeding to the orthogonal complement of $\pi(\hat{U})\Omega$ in $\mathcal{H}$ we can apply the same arguments, so the statement follows by iteration. *q.e.d.*

Having thus reduced the problem of finding all positive energy representations of $\hat{U}$ to that of determining all ground state representations, we are essentially done. We recall that, by the GNS reconstruction theorem, it is sufficient to determine all functionals

$$\omega(W) = (\Omega, \pi(W)\Omega), \quad W \in \hat{U} \qquad (2.11)$$

i.e. the ground states. In view of the composition law (2.6) one must in fact only exhibit the generating functionals $\omega(W(u)), u \in \mathcal{S}$. We will restrict here our attention to those states ω for which the functions

$$\lambda \longrightarrow \omega(W(\lambda u)), \quad u \in \mathcal{S} \qquad (2.12)$$

are continuous in $\lambda \in \mathbf{R}$. These states determine exactly those representations of the Weyl operators in which the currents can be defined as (unbounded) operators. Any functional ω satisfying this continuity requirement will be called regular.

LEMMA 2.2. *i) Let ω be a regular state on $\hat{U}$ generating a ground state representation. Then there exists some positive, normalized measure μ on $\mathbf{R}$ such that for $u \in S$*

$$\omega(W(u)) = \int d\mu(g) \exp(ig\tilde{u}_0 - \frac{1}{2}\sum_{n=1}^{\infty} n|\tilde{u}_n|^2). \tag{2.13}$$

Conversely, given any positive normalized measure μ on $\mathbf{R}$, then the functional ω fixed by (2.13) determines a ground state on $\hat{U}$.

ii) Let $\omega_g, g \in \mathbf{R}$ be the ground states given by

$$\omega_g(W(u)) = \exp(ig\tilde{u}_0 - \frac{1}{2}\sum_{n=1}^{\infty} n|\tilde{u}_n|^2). \tag{2.14}$$

These states are pure. They induce disjoint irreducible representations of $\hat{U}$ for different values of g. Relation (2.13) thus gives the central decomposition of the ground states ω. (According to standard terminology, the pure ground states are called lowest weight states and we will use this term in the following.)

In order to make this paper selfcontained we indicate the proof of this well known fact [25] in Appendix A, where we also determine all Gibbs states for $\hat{U}$.

III. CHARGED STATES AND CHARGED FIELD OPERATORS

3A. THE STRUCTURE OF LOCALLY GENERATED RESENTATIONS OF $\hat{U}$

According to the results of the previous section all positive energy representations $(\pi, \mathcal{H})$ of $\hat{U}$ can be decomposed into irreducible lowest weight representations labeled by a "charge" $g \in \mathbf{R}$. But the requirement of energy positivity does not impose any restrictions on the charge spectrum (i.e. on the values of g appearing in a representation) or its multiplicity.

Such restrictions arise if one assumes that the representation space $\mathcal{H}$ is generated from a vacuum vector (ground state) Ω with the help of field operators which are local relative to the current. Since we want to construct such fields we must identify first of all the appropriate representations of $\hat{U}$. This is done in an implicit manner in the following definition (cf. also the subsequent explanations).

DEFINITION. *A positive energy representation $(\pi, \mathcal{H})$ of $\hat{U}$ is said to be locally generated by fields if there exists an irreducible set of (weakly closed) algebras $\mathcal{F}(I) \subset \mathcal{B}(\mathcal{H})$, assigned to the closed subsets I of the punctured circle $\mathbf{S}^1 \setminus \{-1\}$, such that:*

i) The observables localized in I are among the fields in $\mathcal{F}(I)$, and all fields are local relative to the observables. In formulae

$$\pi(\hat{U}(I)) \subset \mathcal{F}(I) \subset \pi(\hat{U}(I'))'.$$

*Here $\hat{U}(I)$ denotes the *-algebra generated by the Weyl operators $W(u)$ with $\operatorname{supp} u \subset I$, and $I' = \mathbf{S}^1 \setminus I$. $\pi(\hat{U}(I'))'$ is the commutant of $\pi(\hat{U}(I'))$ in $\mathcal{B}(\mathcal{H})$.*

ii) There is a representation e^{iHt} of the time translations on $\mathcal{H}$, satisfying the conditions (2.9) and (2.10), such that for any closed set I_1 contained in the interior of some I_2 and for all t in a sufficiently small neighbourhood of 0

$$e^{iHt}\mathcal{F}(I_1)e^{-iHt} \subset \mathcal{F}(I_2).$$

The generator $H \geq 0$ has the simple eigenvalue 0. The corresponding eigenvector will be called the ground state Ω.

iii) The field algebras have the Reeh-Schlieder property (cp. Section 1C), that is the ground state is cyclic and separating for each $\mathcal{F}(I)$ if I has nonempty interior. ("Cyclic" means that the set of vectors $\psi\Omega, \psi \in \mathcal{F}(I)$ is dense in $\mathcal{H}$ for each I. "Separating" means that $\psi\Omega = 0$ is possible for $\psi \in \mathcal{F}(I)$ only if $\psi = 0$.)

iv) The charge operator $Q = \pi(J_0)$ has pure point spectrum.

These conditions require perhaps some comments. The first condition is clearly satisfied if there exists on $\mathcal{H}$ an irreducible set of field operators which are local relative to the current. These fields, smeared with test functions having support in I, will then generate the "field algebras" $\mathcal{F}(I)$. We consider only subsets I of the punctured circle since we do not want to assume from the outset that the fields are single-valued on $\mathbf{S}^1$. We also do not anticipate any commutation relations of the fields (other than that they should be relatively local to the observables).

The second condition relates to covariance properties of the fields and to the existence of a unique ground state vector Ω. Note that the generators H complying with this condition will in general differ from the minimal ones considered in the previous section. The familiar Reeh-Schlieder property iii) [15] was recalled in Section 1C.

The last condition amounts to a mild form of charge quantization. In principle one could also allow for a continuous spectrum of Q, but we do not investigate this possibility here.

Having thus described the class of representations in which we are interested, we can now give a much more explicit characterization.

PROPOSITION 3.1. *Let $(\pi, \mathcal{H})$ be a positive energy representation of $\hat{U}$ which is locally generated by fields. Then:*

i) There exists a $g_0 \in \mathbf{R}$ such that the spectrum of the operator $(Q - g_0 1)$ is an additive subgroup of $\mathbf{R}$.

ii) The restriction of $\pi(\hat{U})$ to any one of the spectral subspaces $\mathcal{H}_g \subset \mathcal{H}$ corresponding to the eigenvalues g of Q is irreducible.

iii) Any locally generated representation $(\pi, \mathcal{H})$ is uniquely fixed (up to unitary equivalence) by the spectrum of Q. Moreover, the underlying system of field algebras $\mathcal{F}(I), I \subset \mathbf{S}^1 \setminus \{-1\}$ is unique (up to Klein transformations).[1]

PROOF: The lengthy proof of this statement is obtainedby an adaptation of the general arguments in [21] to the present model. The reader may omit it in a first reading and proceed to Section 3B.

[1] For a discussion of Klein transformations cf. for example [15] and footnote to Section 1C. We recall that a Klein transformation does not change the commutation relations of fields carrying the same charge, but it may change the commutation relations of fields carrying different charges.

Since 0 is a simple eigenvalue of H and since $e^{iHt} \in \pi(\hat{U})''$ it is clear that Ω must be an eigenvector of Q corresponding to some eigenvalue g_0. From the discussion in Section 2 we also know that there exists only one (up to unitary equivalence) irreducible positive energy representation of $\hat{U}$ for a given charge g_0. Hence making use a second time of the fact that $e^{iHt} \in \pi(\hat{U})''$ and that H has a simple eigenvalue on the sector $\mathcal{H}_{g_0}$ of charge g_0 it is clear that the restriction of π to $\mathcal{H}_{g_0}$ has multiplicity one, thus $\pi(\hat{U})$ is irreducible on this space.

Next we note that in view of the composition law (2.6) for the Weyl operators and the properties of the field algebras $\mathcal{F}(I)$ given in the preceding definition we have

$$e^{i\lambda Q}\mathcal{F}(I_1)e^{-i\lambda Q} \subset \mathcal{F}(I_2), \qquad \lambda \in \mathbf{R}$$

whenever I_1 is contained in the interior of I_2. Thus if $\psi \in \mathcal{F}(I_1)$ we find that all weak limit points $\psi_g, g \in \mathbf{R}$ of the bounded sequences

$$\frac{1}{\Lambda}\int_0^\Lambda d\lambda\, e^{-i\lambda g}e^{i\lambda Q}\psi e^{-i\lambda Q}, \qquad \Lambda \nearrow \infty$$

are elements of $\mathcal{F}(I_2)$, for which the following relation holds

$$\psi_g e^{i\lambda Q} = e^{i\lambda(Q-g)}\psi_g. \tag{3.1}$$

Hence any nonzero operator ψ_g changes the charge of the vectors in $\mathcal{H}$ by g. (We will also say ψ_g "creates" or "carries" the charge g.) Now if g_1 is in the (discrete) spectrum of Q, and if E_{g_1} is the projection onto the corresponding spectral subspace $\mathcal{H}_{g_1}$, then $E_{g_1}\mathcal{F}(I_1)\Omega$ is dense in $\mathcal{H}_{g_1}$ because of the cyclicity of Ω. Thus the operators ψ_g generate from Ω a dense set in $\mathcal{H}_{g_1}$ if $g = g_1 - g_0$.

Let $\psi_g, \phi_g \in \mathcal{F}(I)$ be operators as constructed above. Then $\psi_g^*\phi_g \in \mathcal{F}(I)$ commutes with the charge operator Q and therefore maps $\mathcal{H}_{g_0}$ into itself. Moreover $\psi_g^*\phi_g$ commutes with all operators in $\pi(\hat{U}(I'))$, so we can make use of the fact that in every irreducible positive energy representation $(\pi_{g_0}\mathcal{H}_{g_0})$ of $\hat{U}$ there holds the duality relation [16]

$$\pi_{g_0}(\hat{U}(I'))' = \pi_{g_0}(\hat{U}(I))''. \tag{3.2}$$

(This relation can be established either by transferring the general argument in [34] to the present theory on the circle, or by explicit computation as in [35].) For the case at hand it means that the restriction of the operator $\psi_g^*\phi_g$ to $\mathcal{H}_{g_0}$ coincides with some operator $W \in \pi(\hat{U}(I))''$. But $\mathcal{H}_{g_0}$ contains the vector Ω which is separating for $\mathcal{F}(I)$, and consequently

$$\psi_g^*\phi_g = W \in \pi(\hat{U}(I))''$$

on the whole Hilbert space. Similarly one can show that $\phi_g\psi_g^* \in \pi(\hat{U}(I))''$.

We can now proceed in a standard manner. By polar decomposition of ψ_g, i.e.

$$\psi_g = V_g(\psi_g^*\psi_g)^{\frac{1}{2}},$$

we obtain (partial) isometries $V_g \in \mathcal{F}(I)$ which change the charge of any vector by g. Let us assume for a moment that one can always find operators $\psi_g \in \mathcal{F}(I)$ such that the operators V_g appearing in this decomposition are isometries, i.e. $V_g^*V_g = 1$. The general case will be discussed at the end of our argument.

If g_1, g_2 are in the spectrum of Q we pick isometries $\psi_{g'}, \psi_{g''} \in \mathcal{F}(I)$ creating the charges $g' = g_1 - g_0$ and $g'' = g_2 - g_0$. Since $\psi_{g''}^*\psi_{g'}^*\psi_{g'}\psi_{g''} = 1$ it is clear that the vector $\psi_{g'}\psi_{g''}\Omega$ is not zero, showing that there exist vectors in $\mathcal{H}$ carrying the charge $g_1 + g_2 - g_0$. Observing that the operator $\psi_{g'}^*$ creates the charge $-g'$, it is also clear that the vector $\psi_{g'}^*\Omega$ (which is nonzero since Ω is separating for $\mathcal{F}(I)$) carries the charge $2g_0 - g_1$. Hence the spectrum of $(Q - g_0 1)$ is an additive subgroup of $\mathbf{R}$, as claimed.

Now let $\Psi \in \mathcal{H}_{g_1}$ be any nonzero vector and let $\psi_{g'}, \phi_{g'} \in \mathcal{F}(I)$ be arbitrary field operators creating the charge $g' = g_1 - g_0$. Since e^{-iHt} and $e^{iHt}\psi_{g'}e^{-iHt}\phi_{g'}^*$ (t small) are elements of $\pi(\hat{U})''$ we find that any vector $\Phi \in \mathcal{H}_{g_1}$ which is orthogonal to $\pi(\hat{U})''\Psi$ must satisfy

$$(\Phi, \psi_{g'}e^{-iHt}\phi_{g'}^*\Psi) = 0$$

if t is sufficiently small. Since H is a positive operator, this equation extends by analyticity to all $t \in \mathbf{R}$. By taking a mean over t we thus find (since the ground state is unique) that

$$(\Phi, \psi_{g'}\Omega)(\Omega, \phi_{g'}^*\Psi) = 0$$

for arbitrary $\psi_{g'}, \phi_{g'} \in \mathcal{F}(I)$. This is only possible if $\Phi = 0$ since the operators $\psi_{g'}, \phi_{g'}$ generate from Ω a dense set of vectors in $\mathcal{H}_{g_1}$. Hence the restriction of $\pi(\hat{U})$ to $\mathcal{H}_{g_1}$ is irreducible.

Let us now turn to the question of the uniqueness of the representation $(\pi, \mathcal{H})$ of $\hat{U}$ and of the field algebras $\mathcal{F}(I)$ if the spectrum of Q is given. The uniqueness of the representation is obvious: it is the direct sum of the disjoint irreducible representations $(\pi_g, \mathcal{H}_g)$ of charge g, where g runs through the spectrum of Q. So let us assume that there exist two systems of field algebras $\mathcal{F}^{(1)}(I)$, $\mathcal{F}^{(2)}(I)$ on $\mathcal{H}$. Each of these algebras is generated by fields $\psi^{(1)}$ and $\psi^{(2)}$, respectively, creating fixed charges g. Picking any two such fields it is clear that $\psi^{(2)}\psi^{(1)*}$ commutes with Q and with all operators in $\pi(\hat{U}(I'))$. Thus the restriction of $\psi^{(2)}\psi^{(1)*}$ to any sector $\mathcal{H}_{g_i}$ can be represented by some operator $W_i \in \pi(\hat{U}(I))''$ because of the duality relation (3.2), i.e.

$$\psi^{(2)}\psi^{(1)*} = \sum_i E_i W_i$$

where E_i are the central projections onto $\mathcal{H}_{g_i}$. Assuming that $\psi^{(1)}, \psi^{(2)}$ are isometries, we can rewrite this equation in the form

$$\psi^{(2)} = \sum_i E_i \psi_i^{(1)}$$

where $\psi_i^{(1)} \in \mathcal{F}^{(1)}(I)$ are fields carrying the same charge as $\psi^{(2)}$. From this we conclude that for all $W \in \pi(\hat{U})''$ and all i (since $\psi_i^{(1)}W\psi_k^{(1)*} \in \pi(\hat{U})''$)

$$E_i\psi^{(2)}W\psi^{(2)*} = E_i\psi_i^{(1)}W\psi_i^{(1)*}. \tag{3.3}$$

We first choose operators $W \in \pi(\hat{U}(I_1))$ where $I_1 \supset I$ and I_1' has a nonempty interior. Then $\psi_i^{(1)}W\psi_i^{(1)*}$ and $\psi_i^{(2)}W\psi_i^{(2)*}$ are, because of duality, elements of $\pi(\hat{U}(I_1))''$. Hence multiplying relation (3.3) from the left and right with ψ^* and ψ, respectively, where $\psi \in \mathcal{F}^{(1)}(I_2)$, $I_2 \subset I_1'$ is an isometry carrying an arbitrary charge, we see that relation (3.3) also holds if we replace the projections E_i by E_k for arbitrary k. This implies that

$$\psi^{(2)} W \psi^{(2)*} = \psi_i^{(1)} W \psi_i^{(1)*} \tag{3.4}$$

for operators W with the special localization properties. Since the fields in $\mathcal{F}^{(i)}(I)$, $i = 1, 2$ commute with the operators in $\pi(\hat{U}(I'))$, relation (3.4) also holds if we replace W by sums of the form $\sum W_1 W_2$, where $W_1 \in \pi(\hat{U}(I_1))$ and $W_2 \in \pi(\hat{U}(I'))$. In view of the composition law (2.6) for the Weyl operators $W(u)$ it is then clear that relation (3.4) holds for all $W \in \pi(\hat{U})''$. Fixing some i and setting $Z = \psi_i^{(1)*}\psi^{(2)}$ we can rewrite this equation (bearing in mind that $\psi^{(2)}$ is an isometry) according to

$$Z^* W Z = W.$$

From this we see in particular that $Z^*Z = 1$. Setting $W = ZZ^*$ (recall that Z is an element of $\pi(\hat{U})''$) it is also clear that $ZZ^* = 1$, hence Z is a unitary element of the center of $\pi(\hat{U})''$. From this it follows that

$$\psi^{(2)} = \psi^{(2)}\psi^{(2)*}\psi_i^{(1)} Z,$$

i.e. any $\psi^{(2)} \in \mathcal{F}^{(2)}(I)$ is obtained by multiplying some operator in $\mathcal{F}^{(1)}(I)$ with a suitable element of the center of $\pi(\hat{U})''$. It has still to be shown that this central element can be chosen to be the same for all fields $\psi^{(2)}$ carrying the same charge: Let $\psi'^{(2)} \in \mathcal{F}^{(2)}(I)$ be any such field. Then we have (since $\psi^{(2)}$ is an isometry)

$$\psi'^{(2)} = \psi'^{(2)}\psi^{(2)*}\psi^{(2)} = \psi'^{(2)}\psi^{(2)*}\psi_i^{(1)} Z$$

with the same central element Z as before. Thus Z depends only on the charge carried by the fields, which means that it induces a Klein transformation.

It remains to discuss the possibility that the operator V_g appearing in the polar decomposition of $\psi_g \in \mathcal{F}(I)$ are only partial isometries, i.e. that $V_g^* V_g = E$ is a nonzero projection in $\pi(\hat{U}(I))''$. As a consequence of the spectral properties of H and the cyclicity of Ω any such projection can be represented in the form $E = WW^*$, where W is an isometry in an algebra $\pi(\hat{U}(\hat{I}))''$ corresponding to some possibly slightly larger region $\hat{I} \supseteq I$. (For a general argument cf. [36].) Hence one can always proceed from a partial isometry V_g to an isometry with the desired properties by right multiplication with a suitable "observable" W. This, finally, completes the proof of the proposition. *q.e.d.*

The reader will have noticed that only very little specific information about the model under consideration was used in the derivation of this result. The essential input was the information that the duality relation (3.2) holds in all charge sectors of the current algebra. From the general analysis of the superselection structure of quantum field theory [9] we know that this feature is typical for theories with a global *abelian* gauge group. Moreover, we are prepared to find that all superselection sectors can be generated from the vacuum state with the help of localized automorphisms of the algebra $\hat{U}$. (In theories with a *non-abelian* global gauge group one would need localized morphisms which are not invertable.) These automorphisms will be the key to the construction of the charged fields.

3B. From localized automorphisms to charged fields

For the actual construction of the charged fields we proceed as follows. We start from the "largest" representation $(\hat{\pi}, \hat{\mathcal{H}})$ of $\hat{U}$ which is compatible with the constraints found in the preceding proposition: it is the direct sum of all irreducible representations $(\pi_g, \mathcal{H}_g)$ of charge $g \in \mathbf{R}$. The representation space $\hat{\mathcal{H}}$ of this universal representation is non-separable, but this will not cause any problems. The advantage of working with this universal represention is that one can obtain any other locally generated representation of $\hat{U}$ by projection. A concrete realization of $(\hat{\pi}, \hat{\mathcal{H}})$ which will be convenient in the following can be obtained as follows.

To begin with, we introduce a group of charge shift automorphisms γ_g, $g \in \mathbf{R}$ of the current algebra, setting

$$\gamma_g(W(u)) = e^{ig\tilde{u}_0} W(u), \qquad \tilde{u}_0 \equiv \oint u(z)\frac{dz}{2\pi i z}. \tag{3.5}$$

These automorphisms commute with α_t and satisfy

$$\gamma_{g_1}\gamma_{g_2} = \gamma_{g_1+g_2}, \qquad \gamma_g^{-1} = \gamma_{-g}. \tag{3.6}$$

Let ω_0 be the vacuum state (considered as a linear functional on the current algebra), i.e. the lowest weight state of charge $g = 0$. The lowest weight states of charges $g \neq 0$ are obtained by composing ω_0 with γ_g. The GNS representation induced by ω_0 will be denoted by $(\pi_0, \mathcal{H}_0)$ and the cyclic vector corresponding to ω_0 by Ω. The space $\hat{\mathcal{H}}$ of all charged states consists of vectors Φ whose components $\Phi_g, g \in \mathbf{R}$ are elements of $\mathcal{H}_0$ and are different from zero for an at most countable number of g. The norm in $\hat{\mathcal{H}}$ is given by

$$||\Phi||^2 = \sum_{g \in \mathbf{R}} ||\Phi_g||_0^2, \tag{3.7}$$

where $||\cdot||_0$ denotes the Hilbert norm in $\mathcal{H}_0$. We define a representation of $\hat{U}$ on $\hat{\mathcal{H}}$ by

$$(\hat{\pi}(W)\Phi)_g = \pi_0(\gamma_g(W))\Phi_g \qquad \text{for } W \in \hat{U}. \tag{3.8}$$

(Since this representation is faithful we omit the symbol $\hat{\pi}$ in the following.)

On $\hat{\mathcal{H}}$ we introduce unitary charge carrying operators $\Gamma_e, e \in \mathbf{R}$, setting

$$(\Gamma_e \Phi)_g = \Phi_{g-e}. \tag{3.9}$$

One easily verifies that

$$\begin{aligned} \Gamma_{e_1}\Gamma_{e_2} &= \Gamma_{e_1+e_2} \\ \Gamma_e^* W \Gamma_e &= \gamma_e(W) \quad \text{for } W \in \hat{U}. \end{aligned} \tag{3.10}$$

We also introduce a continuous unitary representation $t \mapsto U_1(t)$ of the time translations by setting

$$(U_1(t)\Phi)_g = U_0(t)\Phi_g \tag{3.11}$$

where U_0 are the minimal time translations in $\mathcal{H}_0$. The operators $U_1(t)$ commute with the charge shifts,

$$U_1(t)\Gamma_e U_1(t)^{-1} = \Gamma_e, \tag{3.12}$$

and act on the Weyl algebra according to

$$U_1(t)WU_1(t)^{-1} = \alpha_t(W). \tag{3.13}$$

We also note that $U_1(t) \in \hat{U}''$. Hence U_1 is the minimal representation of the translations introduced in Section 2. We will later proceed from U_1 to a represention U of the translations which complies with the condition of covariance of the charged fields.

Let us now turn to the construction of charged fields, i.e. charge carrying operators with specific localization properties relative to the currents.

We define an arbitrary charge distribution as a 1-form $\rho(z)dz/2\pi i$ on $\mathbf{S}^1$ with $z\rho(z) \in S$. Given any such ρ we can construct a localized automorphism γ_ρ of $\hat{U}$ setting

$$\gamma_\rho(W(u)) = e^{i\rho[u]}W(u), \tag{3.14a}$$

where

$$\rho[u] = \oint \frac{dz}{2\pi i}\rho(z)u(z) \in \mathbf{R} \qquad \text{for } u \in S. \tag{3.14b}$$

Localized here refers to the property that $\gamma_\rho(W(u)) = W(u)$ if ρ and u have disjoint supports, since $\rho[u]$ vanishes in that case. Note that

$$\gamma_{\rho_1}\gamma_{\rho_2} = \gamma_{\rho_1+\rho_2}. \tag{3.14c}$$

Our aim would be to construct for any ρ a unitary field operator ψ_ρ satisfying

$$\psi_\rho^* W \psi_\rho = \gamma_\rho(W) \quad , \quad W \in \hat{U}. \tag{3.15a}$$

Such operators will create from the vacuum Ω states with total charge

$$g = \int \frac{dz}{2\pi i}\rho(z) = \rho[1], \tag{3.15b}$$

and are relatively local to the current in the sense of the previous remark.

It follows from (3.14c) that the operators ψ_ρ must satisfy an equation of the form

$$\psi_{\rho_1}\psi_{\rho_2} = Z(\rho_1,\rho_2)\psi_{\rho_1+\rho_2}, \tag{3.16}$$

where Z belongs to the center of $\hat{U}''$ in the faithful representation under consideration. (Note that the commutant $\hat{U}'$ of $\hat{U}$ is equal to the center of $\hat{U}''$ since the representation of $\hat{U}$ is a direct sum of disjoint irreducible representations.)

The associativity of the multiplication law for the ψ's implies that the Z-factors in (3.16) obey the cocycle relation

$$Z(\rho_1,\rho_2)Z(\rho_1+\rho_2,\rho_3) = \gamma_{-g_1}(Z(\rho_2,\rho_3))Z(\rho_1,\rho_2+\rho_3). \tag{3.17}$$

The trivial solutions of this equation, the *coboundaries* are

$$B(\rho_1,\rho_2) = X(\rho_1)\gamma_{-g_1}(X(\rho_2))X(\rho_1+\rho_2)^{-1} \tag{3.18}$$

where $X(.)$ are arbitrary unitaries in the center of $\hat{U}''$.

If the operators $\psi_{\rho_1}, \psi_{\rho_2}$ correspond to densities ρ_1, ρ_2 with the same total charge, then $\psi_{\rho_1}\psi_{\rho_2}^*$ is an observable (more precisely, $\psi_{\rho_1}\psi_{\rho_2}^* \in \hat{U}''$). Indeed, it follows from Eqs. (3.14) that

$$\psi_{\rho_1}\psi_{\rho_2}^* W(u)\psi_{\rho_2}\psi_{\rho_1}^* = e^{i(\rho_2[u]-\rho_1[u])}W(u). \tag{3.19}$$

The vanishing of the total charge of $\rho_2 - \rho_1$ implies that it is the derivative (up to a factor i) of some test function $v \in S$:

$$\rho_2(z) - \rho_1(z) = iv'(z). \tag{3.20}$$

It then follows from (2.6) and (2.4) that $\psi_{\rho_1}\psi_{\rho_2}^*$ induces the same automorphism of $\hat{U}$ as $W(v)$, hence $W(v)^*\psi_{\rho_1}\psi_{\rho_2}^*$ is an element of the center of $\hat{U}''$.

We shall now show that there exists a privileged choice of fields ψ_ρ so that $Z(\rho_1,\rho_2)$ is a complex number.

It is evident from the previous remark that such fields can be obtained by multiplying the constant charge shift operators $\Gamma_g, g = \rho[1]$ with a suitable element $W(\hat{\rho})$ of $\hat{U}$. We set

$$\psi_\rho = W(\hat{\rho})\Gamma_g, \tag{3.21a}$$

where $\hat{\rho}$ must satisfy

$$\frac{d}{dz}\hat{\rho}(z) = i[\rho(z) - \frac{g}{z}]. \tag{3.21b}$$

Hence

$$\hat{\rho}(z) = i\sum_{n\neq 0}\tilde{\rho}_{-n}\frac{z^n}{n} + l(\rho) \tag{3.21c}$$

where $l(\rho)$ is some constant. The coefficients $\tilde{\rho}_{-n}$ are given by

$$\rho(z) = \sum_n \tilde{\rho}_{-n}z^{n-1}, \qquad (\tilde{\rho}_0 = \rho[1] = g). \tag{3.22}$$

The functional $l(\rho)$ has to be linear, as we shall see shortly. The field (3.21a) indeed induces the desired automorphism (3.14a), since

$$\psi_\rho^* W(u)\psi_\rho = e^{A(\hat{\rho},u)+ig\tilde{u}_0}W(u) = e^{i\rho[u]}W(u).$$

Here we have used (2.6), (3.5), (3.10), and the identity

$$A(\hat{\rho},u) + ig\tilde{u}_0 = \oint (\rho(z) - \frac{g}{z})u(z)\frac{dz}{2\pi} + ig\tilde{u}_0 = i\rho[u].$$

On the other hand,

$$\psi_{\rho_1}\psi_{\rho_2} = W(\hat{\rho}_1)\gamma_{-g_1}(W(\hat{\rho}_2))\Gamma_{g_1+g_2} = Z(\rho_1,\rho_2)\psi_{\rho_1+\rho_2} \tag{3.23a}$$

where

$$Z(\rho_1,\rho_2) = W(\hat{\rho}_1)\gamma_{-g_1}(W(\hat{\rho}_2))W(\rho_1+\rho_2)^{-1}$$
$$= e^{i(l(\rho_1)+l(\rho_2)-l(\rho_1+\rho_2))Q}\exp(\frac{1}{2}A(\rho_1,\hat{\rho}_2) - ig_1 l(\rho_2)). \tag{3.23b}$$

Hence $Z(\rho_1, \rho_2)$ is a constant phase factor if $l(\rho)$ is linear in ρ, which we will assume henceforth.

Next we study the ambiguities in the definition of ψ_ρ. If $\{\psi'_\rho\}$ is any other collection of unitaries inducing the automorphism (3.10) of U, it follows from the previous remarks that

$$\psi'_\rho = X(\rho)\psi_\rho \qquad (3.24)$$

where $X(\rho)$ are unitaries in the center of $\hat{U}''$. These operators satisfy an equation of type (3.12) with $Z(\rho_1, \rho_2)$ replaced by

$$Z'(\rho_1, \rho_2) = X(\rho_1)\gamma_{-g_1}(X(\rho_2))X(\rho_1 + \rho_2)^{-1}Z(\rho_1, \rho_2) \qquad (3.25)$$

where $g_1 = \rho_1[1]$ is the total charge of ρ_1. Here we have used the fact that $\Gamma_{g_1}^{-1}\psi_{\rho_1} \in \hat{U}''$ (cf. (3.21a)). Hence the cocycles Z and Z' differ by a coboundary in $\hat{U}''$, and thus belong to the same cohomology class.

The condition that the operator $Z(\rho_1, \rho_2)$ in the composition law (3.12) is a c-number is clearly necessary if the vacuum Ω is to be separating for the field algebras generated by the operators ψ_ρ (cf. Section 3A). But, as is clear from the preceding discussion, it does not yet fix these fields uniquely. If $\{\psi'_\rho\}$ is another family of field operators complying with this condition it follows from Eq. (3.25) that the corresponding unitary operators must satisfy

$$X(\rho_1)\gamma_{-g_1}(X(\rho_2)) = \xi(\rho_1, \rho_2)X(\rho_1 + \rho_2) \qquad (3.26)$$

where $\xi(.,.)$ is some phase factor. The following lemma gives the (continuous) solution of this equation.

LEMMA 3.2. *Any solution $X(\rho)$ of Eq. (3.26) with values in the unitary operators of the center of $\hat{U}''$, which is continuous with respect to ρ in the sense of distributions has the form*

$$X(\rho) = \eta(\rho)e^{i\lambda(\rho)Q}Y(g), \qquad g = \rho[1] \qquad (3.27)$$

where $\eta(\rho)$ is an arbitrary phase factor, $\lambda(\rho)$ is a real linear continuous functional of ρ, and $Y(g), g \in \mathbf{R}$ are unitary elements of the center of $\hat{U}''$ satisfying

$$Y(g_1)\gamma_{-g_1}(Y(g_2)) = Y(g_1 + g_2).$$

Moreover the phase factor $\xi(.,.)$ appearing in (3.26) is related to the functional $\lambda(.)$ and phase factor $\eta(.)$ by

$$\eta(\rho_1)\eta(\rho_2)\eta(\rho_1 + \rho_2)^{-1}e^{-ig_1\lambda(\rho_2)} = \xi(\rho_1, \rho_2). \qquad (3.28)$$

The proof of this lemma is given in Appendix B. What this statement tells us is that our Ansatz (3.21) for the fields is sufficiently general. Apart from the trivial modifications arising by multiplying the fields ψ_ρ with an arbitrary phase factor η, or with certain central elements Y which depend only on the total charge carried by ψ_ρ ("Klein transformations"), the only freedom left is the choice of the functional $l(.)$ in our definition (3.19) of the fields. Again it is not difficult to see that $l(.)$ cannot be arbitrary if the vacuum Ω is to be separating for the field algebras. The appropriate form of $l(.)$ will be established in the subsequent section.

3C. THE CANONICAL FIELD BUNDLE. COMMUTATION RELATIONS AND COVARIANCE PROPERTIES

The representation (3.19) of ψ_ρ allows to compute the group theoretic commutator of two such fields. According to (3.21), it is a complex phase factor

$$\psi_{\rho_1}\psi_{\rho_2}\psi^*_{\rho_1}\psi^*_{\rho_2} = Z(\rho_1, \rho_2)Z(\rho_2, \rho_1)^* \qquad (3.29)$$

$$= \exp(-A(\hat{\rho}_1, \hat{\rho}_2) - il(g_1\rho_2 - g_2\rho_1)). \qquad (3.30)$$

Our next objective is to find a functional l for which the phase factor in (3.30) simplifies whenever ρ_1 and ρ_2 have support in disjoint intervals I_1 and I_2. Given two such intervals we shall first compute the variation of $A(\hat{\rho}_1, \hat{\rho}_2)$ when one of the charge densities, say ρ_2, changes within the region I_2 while the total charge g_2 is kept fixed. To this end we choose a point

$$\zeta = e^{i\theta_\zeta} \in \mathbf{S}^1, \qquad -\pi \leq \theta_\zeta < \pi$$

in the complement of I_1 and I_2, and define $ln_\zeta z$ as the branch of the logarithm with a cut along $\zeta\mathbf{R}_+$

$$ln_\zeta z = ln|z| + i\theta, \qquad \theta_\zeta \leq \theta < 2\pi + \theta_\zeta \qquad (3.31)$$

for $z = |z|e^{i\theta}$.

Let $\Delta\rho_2$ be the change of ρ_2. Since the support of $\Delta\rho_2$ lies in the contractible region I_2 and

$$\oint \Delta\rho_2(z)dz = \int_{I_2} \Delta\rho_2(z)dz = 0 \qquad (3.32)$$

there exists a primitive $\widehat{\Delta\rho_2}$ with support in I_2 such that

$$\frac{d}{dz}\widehat{\Delta\rho_2}(z) = i\Delta\rho_2(z).$$

Since $\rho_1(z)\widehat{\Delta\rho_2}(z) = 0$, Eq. (3.21b) implies that for $\zeta \notin I_2$

$$A(\hat{\rho}_1, \widehat{\Delta\rho_2}) = \int \left(\rho_1(z) - \frac{g_1}{z}\right)\widehat{\Delta\rho_2}(z)\frac{dz}{2\pi}$$

$$= -\frac{g_1}{2\pi}\int \widehat{\Delta\rho_2}(z)dln_\zeta z = -g_1\oint ln_\zeta z\Delta\rho_2(z)\frac{dz}{2\pi i}. \qquad (3.33)$$

This tells us that the functional

$$T = -A(\hat{\rho}_1, \hat{\rho}_2) - \oint \frac{dz}{2\pi i}[g_1\rho_2(z) - g_2\rho_1(z)]ln_\zeta z \qquad (3.34)$$

does not depend on the precise choice of ρ_2, provided that it is localized in I_2. Due to its skew symmetry in ρ_1, ρ_2, T is also constant under charge preserving deformations of ρ_1 within I_1. It is then obvious that T is also constant under continuous changes of I_1, I_2 and ζ. Thus T depends only on the relative location of I_1, I_2, and ζ, i.e. it is a *topological invariant*. This result could have been established by general arguments as in [9]. But in contrast to the situation in higher dimensions, where T would be 0 or $i\pi$ (Bose-Fermi alternative), this topological invariant has a much richer spectrum in the present case.

Lemma 3.3. *Let ρ_1 and ρ_2 have support in disjoint intervals I_1 and I_2 on $\mathbf{S}^1$. Then the value of the topological invariant $\mathcal{T}$ is*

$$\mathcal{T} = \pm i\pi g_1 g_2 \tag{3.35}$$

where the sign is + if one passes from I_1 to I_2 through ζ in the positive direction, and it is − otherwise.

Proof: $\mathcal{T}$ being a topological invariant unaffected by continuous changes of the point ζ, the densities ρ_1 and ρ_2, and the intervals I_1 and I_2 (without changing their relative positions), one can reduce its computation to the special case of densities ρ_1, ρ_2 which are characteristic functions. We omit the trivial calculations. The result is as given in the lemma. *q.e.d.*

The above analysis motivates the introduction of a 1-parameter family of unitary field operators.

Definition. *Let ζ be an arbitrary complex number of absolute value 1. Then*

$$\psi_\rho^\zeta = \eta_\zeta(\rho) W(\hat{\rho}_\zeta)\Gamma_g, \qquad g = \rho[1] \tag{3.36}$$

where

$$\eta_\zeta(\rho) = \exp\left(\frac{i}{2} g \oint \frac{dz}{2\pi}\rho(z) ln_\zeta z\right) \tag{3.37}$$

is a phase factor (which will turn out to be convenient), and

$$\hat{\rho}_\zeta(z) = i\sum_{n\neq 0} \rho_{-n}\frac{z^n}{n} - \oint \frac{dz}{2\pi}\rho(z) ln_\zeta z \quad \in \mathcal{S}. \tag{3.38}$$

In other words, we have chosen the functional $l(\rho)$ in (3.21c) to be

$$l_\zeta(\rho) = -\oint \frac{dz}{2\pi}\rho(z)\ln_\zeta z. \tag{3.39}$$

We note that Eq. (3.39) defines a real functional since both $z^{-1}dz$ and $ln_\zeta z$ are pure imaginary on the unit circle while $z\rho(z)$ is real.

Proposition 3.4. *The unitary field operators ψ_ρ^ζ defined by Eqs.(3.36) to (3.38) induce the localized automorphisms (3.14a). and they have the properties*

i) There holds the fusion rule

$$\psi_{\rho_1}^\zeta \psi_{\rho_2}^\zeta = e^{\mathcal{T}/2}\cdot \psi_{\rho_1+\rho_2}^\zeta \tag{3.40}$$

where $\mathcal{T}$ is an imaginary number (cf. relation (3.34)).

ii) If the charge densities ρ_1 and ρ_2 have supports in disjoint intervals $I_1, I_2 \subset \mathbf{S}^1\setminus\{\zeta\}$ then

$$\psi_{\rho_1}^\zeta \psi_{\rho_2}^\zeta = e^{\pm i g_1 g_2 \pi}\psi_{\rho_2}^\zeta \psi_{\rho_1}^\zeta. \qquad (g_a = \rho_a[1], \quad a = 1,2) \tag{3.41}$$

where the sign is + (−) if the path from I_1 to I_2 through ζ goes in the positive (negative) direction.

iii) For $\lambda \in \mathbf{R}$ the unitaries $\psi_{\lambda\rho}^\zeta$ satisfy

$$\psi_{\lambda_1\rho}^\zeta \psi_{\lambda_2\rho}^\zeta = \psi_{(\lambda_1+\lambda_2)\rho}^\zeta. \tag{3.42}$$

In particular $(\psi_\rho^\zeta)^* = \psi_{-\rho}^\zeta$.

This statement follows from the definition of ψ_ρ^ζ, Eq. (3.36), and from the properties derived above, in particular from Lemma 3.3. The property (3.42) follows from the choice (3.37) of the phase factor as is verified by a direct computation.

We will now proceed to a study of the ζ-dependence of the ψ's. To this end we calculate the "transition operators" $\psi_\rho^{\zeta_2}(\psi_\rho^{\zeta_1})^*$. A straightforward compution gives the following result.

Lemma 3.5. *Let ρ have support in an interval $I \subset \mathbf{S}^1\setminus\{\zeta_1,\zeta_2\}$, then*

$$\psi_\rho^{\zeta_2}(\psi_\rho^{\zeta_1})^* = e^{-\sigma i\pi g^2} e^{2\pi i g Q} \tag{3.43}$$

where σ takes the values $0, \pm 1$ depending on the relative positions of the points ζ_1, ζ_2 and the interval I : $\sigma = 0$ if the path from ζ_1 to ζ_2 crossing I meets -1, and $\sigma = \pm 1$ if this path does not meet -1 and is positively respectively negatively oriented.

Lemma 3.5 suggests that for fixed ζ, ψ_ρ^ζ ($\operatorname{supp}\rho \subset \mathbf{S}^1\setminus\{\zeta\}$) should be regarded as local sections of a "field bundle" over $\mathbf{S}^1$. The following statement shows that the above choice of sections is essentially unique.

Lemma 3.6. *Let ϕ_ρ^ζ, $\zeta \in \mathbf{S}^1, \operatorname{supp}\rho \subset \mathbf{S}^1\setminus\{\zeta\}$. be fields with the properties listed in Proposition 3.4 and assume that $\phi_\rho^{\zeta_1} = \phi_\rho^{\zeta_2}$ if the support of ρ belongs to the component of $\mathbf{S}^1\setminus\{\zeta_1,\zeta_2\}$ containing the point $z = -1$. Then there exists a unitary operator V in the center of $\mathcal{U}''$ and a phase factor $\eta(\rho)$ satisfying the equation*

$$\eta(\lambda_1\rho)\eta(\lambda_2\rho) = \eta((\lambda_1+\lambda_2)\rho) \tag{3.45}$$

for real λ such that

$$\phi_\rho^\zeta = \eta(\rho) V \psi_\rho^\zeta V^{-1}. \tag{3.46}$$

We omit the straightforward proof of this lemma which is essentially based on the information contained in Lemma 3.2.

We proceed to studying the covariance properties of the unitary charged field operators under time translations. It follows from (3.12), (3.13) and (2.6) that

$$U_1(t)\psi_\rho^\zeta U_1(t)^{-1} = \frac{\eta_\zeta(\rho)}{\eta_\zeta(\rho^t)}\exp\left(-iQ\oint\frac{dz}{2\pi}[\rho(z) - \rho^t(z)] ln_\zeta z\right)\psi_{\rho^t}^\zeta \tag{3.47}$$

where $t \mapsto \rho^t$ is the standard time translation law for a charge density

$$\rho^t(z) = e^{-it}\rho(e^{-it}z) \tag{3.48}$$

and

$$\frac{\eta_\zeta(\rho)}{\eta_\zeta(\rho^t)} = \exp\left(\frac{i}{2}g\oint\frac{dz}{2\pi}[\rho(z) - \rho^t(z)] ln_\zeta z\right). \tag{3.49}$$

For sufficiently small t for which $\operatorname{supp}\rho^t \subset \mathbf{S}^1 \setminus \{\zeta\}$ we have

$$\oint \frac{dz}{2\pi}[\rho(z) - \rho^t(z)] ln_\zeta z = gt \tag{3.50}$$

so that

$$U_1(t)\psi^\zeta_\rho U_1(t)^{-1} = e^{\frac{i}{2}g^2 t} e^{-igtQ}\psi^\zeta_{\rho^t}. \tag{3.51}$$

We must get rid of the charge operator appearing on the right hand side if the vacuum is to be separating for the local field algebras. A covariant time evolution law for the charged fields complying with this requirement can be obtained by redefining the time evolution operator according to

$$U(t) = U_1(t)e^{\frac{i}{2}Q^2 t}. \tag{3.52}$$

Since

$$e^{\frac{i}{2}Q^2 t}\Gamma_g e^{-\frac{i}{2}Q^2 t} = e^{-\frac{i}{2}g^2 t + igQt}\Gamma_g$$

we then have

$$U(t)\psi^\zeta_\rho U(t)^{-1} = \psi^\zeta_{\rho^t}. \tag{3.53}$$

We can summarize our analysis by the following

PROPOSITION 3.7. *The map $t \mapsto U(t)$, where $U(t)$ is defined by (3.52) and (3.11), gives rise to a continuous unitary representation of the time translations on $\mathcal{H}$ such that*

$$U(t)\psi^\zeta_\rho U(t)^{-1} = \psi^\zeta_{\rho^t}$$

$$\textit{if}\ \operatorname{supp}\rho_\tau \subset \mathbf{S}^1 \setminus \{\zeta\} \quad \textit{for}\ |\tau| \le t. \tag{3.54}$$

The selfadjoint operator H on $\hat{\mathcal{H}}$ defined from $U(t) = e^{iHt}$ has the spectrum

$$\frac{1}{2}g^2 + \mathbf{Z}_+ \qquad on\ \ \mathcal{H}_g,\ g \in \mathbf{R}.$$

Moreover, Ω is the unique ground state of H, and $e^{iHt} \in \hat{U}''$.

So we have established within our setting the well known fact that the "natural" energy of a lowest weight state of charge g is $\frac{1}{2}g^2$.

We also record the most general time evolution operator which allows for a c-number phase factor in the covariance law (3.54) (which would be compatible with the requirement that Ω is separating) for the fields. Setting

$$U_\mu(t) = U(t)e^{i\mu Qt} \tag{3.55}$$

we have

$$U_\mu(t)\psi^\zeta_\rho U_\mu(t)^{-1} = e^{i\mu gt}\psi^\zeta_{\rho^t}. \tag{3.56}$$

μ corresponds to a "chemical potential" in the terminology of statistical mechanics.

We note that there arises a further restriction on the ambiguities in the definition of the charged fields coming from time translation covariance. If the fields ϕ^ζ_ρ of

Lemma 3.6 are to satisfy the same covariance law (3.54) as ψ^ζ_ρ then the phase factor $\eta(\rho)$ in (3.46) must be translation invariant

$$\eta(\rho^t) = \eta(\rho). \tag{3.57}$$

The properties of η imply that the product of fields

$$\psi^\zeta_{\epsilon_1\rho^{t_1}}\psi^\zeta_{\epsilon_2\rho^{t_2}}\cdots\psi^\zeta_{\epsilon_n\rho^{t_n}} \quad \text{with}\ \epsilon_1 + \epsilon_2 + \cdots + \epsilon_n = 0 \tag{3.58}$$

is independent of the choice of phase factors η. We shall profit from this fact in the next section where the Wightman functions of pointlike charged fields will be determined from the algebraic properties of ψ^ζ_ρ. Because of relation (3.58) these function do not depend on arbitrary phase factors.

Let us summarize the results of this section. Starting from the notion of locally generated representations we were led to consider the universal representation of $\hat{U}$ on the Hilbert space $\hat{\mathcal{H}}$ of all charged states. This space is big enough to implement all local charge shift automorphisms γ_ρ by unitary charged field operators ψ_ρ. To put it another way: we have embedded the current algebra into a field algebra such that its outer automorphisms γ_ρ extend to the field algebra and become inner automorphisms there.

The ambiguities involved in the definition of the field operators ψ_ρ can be removed by algebraic constraints following in particular from the requirement that the vacuum Ω should be separating for the algebras $\mathcal{F}(I), I \subset \mathbf{S}^1$. That is, elements of the local field algebras $\mathcal{F}(I)$ cannot annihilate the vacuum. We have seen that one can find such coherent families of fields ψ^ζ_ρ only for intervals $I \subset \mathbf{S}^1 \setminus \{\zeta\}$ of the punctured circle (since the transition functions in Lemma 3.5 are center-valued operators). This means that all charged field operators live on an infinite covering of $\mathbf{S}^1$. The situation changes, however, if one restricts fields corresponding to certain specific values of the charge to suitable subspaces of $\hat{\mathcal{H}}$, as we will see in the subsequent section.

We have also determined the form of the Hamiltonian from the requirement of covariance of the fields. The Hamiltonian was found to be unique up to a shift by multiples of the charge operator (inducing a global gauge transformation).

We conclude this section by recalling Proposition 3.1, according to which the algebras $\mathcal{F}(I)$ of charged fields in a locally generated representation of $\hat{U}$ are only unique up to a Klein transformation. In the present universal representation it is not possible to perform such transformations which would change the commutation relations of the fields (cf. Lemma 3.6). This possibility arises, however, in certain subrepresentations of $\hat{U}$ as will be discussed in the subsequent section.

IV. EXTENDED ALGEBRAS OF LOCAL OBSERVABLES

4A. Local extensions of the current algebra

Having complete control on the universal locally generated representation of $\hat{U}$ and the corresponding algebras of charged fields we can now turn to the question of how the current algebra can be embedded into various field theoretic settings. We will adopt the point of view that the current algebra is only a subalgebra of some larger (as yet unknown) algebra of observables. In particular we give up the (implicit) assumption that the charge Q is superselected. That the algebra $\hat{U}$ must be regarded as an insufficient algebra of observables is quite obvious: Q is not quantized and accordingly the physical state space is non-separable. So what we are looking for are extensions $\mathcal{A}$ of $\hat{U}$ which may be regarded as more acceptable algebras of observables.[1]

In order to cast this idea into a well-posed problem we must fix the rules of the game. The extensions $\mathcal{A}$ of $\hat{U}$ we are interested in should still live on the circle $\mathbf{S}^1$ in the sense that for each interval $I \subset \mathbf{S}^1$ there exists a subalgebra $\mathcal{A}(I) \subset \mathcal{A}$ containing $\hat{U}(I)$. The algebras $\mathcal{A}(I)$ should satisfy the obvious condition of isotony

$$\mathcal{A}(I_1) \subseteq \mathcal{A}(I_2) \qquad \text{if } I_1 \subseteq I_2 \tag{4.1}$$

and they should be local in the sense that

$$\mathcal{A}(I_1) \subseteq \mathcal{A}(I_2)' \qquad \text{if } I_1 \cap I_2 = \emptyset. \tag{4.2}$$

($'$ denotes the commutant as usual). It should also be possible to extend the time evolution automorphism α_t of $\hat{U}$ to the algebra $\mathcal{A}$ in such a way that it acts covariantly on $\mathcal{A}$, i.e.

$$\alpha_t(\mathcal{A}(I)) = \mathcal{A}(e^{it}I), \qquad t \in \mathbf{R} \tag{4.3}$$

in an obvious notation. Furthermore there should exist a vacuum representation of $\mathcal{A}$ in which the automorphisms α_t can be implemented by a continuous unitary representation $t \mapsto e^{iHt}$ with a unique invariant ground state Ω which is cyclic and separating[2] for the algebras $\mathcal{A}(I)$. Finally we assume that e^{iHt} is contained in the weak closure of $\hat{U}$ in the vacuum representation, i.e. in passing from $\hat{U}$ to $\mathcal{A}$ *we do not want to change the dynamics*. This assumption that the Hamiltonian is "some function of the current" may be regarded as a weaker substitute for the Sugawara formula.

We will call an algebra $\mathcal{A}$ with these specific properties a local extension of $\hat{U}$. Given such a local extension one can always produce other ones simply by passing to suitable subalgebras of $\mathcal{A}$. So an obvious question is: What are the *maximal* local extensions of $\hat{U}$?

It is clear that the vacuum representation of any local extension $\mathcal{A}$ of $\hat{U}$ gives rise to a locally generated representation of $\hat{U}$ in the sense of the definition given in Section 3A. Hence according to Proposition 3.1 any such $\mathcal{A}$ can be realized as a subalgebra of the algebra of all (possibly Klein-transformed) charged fields ψ_ρ^ζ on a suitable subspace of the universal space $\hat{\mathcal{H}}$.

We must therefore determine those fields ψ_ρ^ζ (ζ fixed for the moment) which are suited to generate local extensions of $\hat{U}$. As was discussed in Section 3B, all fields

[1]More precisely: we are looking for extensions of quotients of $\hat{U}$ with respect to ideals $\hat{\mathcal{Z}}$ generated by suitable subalgebras of the center of $\hat{U}$. In the subsequent discussion $\hat{U}$ is to be understood as some such quotient.

[2]If Ω is cyclic it is of course separating for the vacuum representation of the algebras $\mathcal{A}(I)$ because of locality. So what we exclude by this assumption is the possibility that the local algebras contain a two-sided ideal which is annihilated by Ω. Phrased differently, we require that the vacuum representation is locally faithful. It may thus be regarded as the defining representation of the algebra of observables.

are relatively local to the current, so we will first examine which of these fields are relatively local with respect to themselves. (Recall that a Klein transformation does not change the commutation relations of fields carrying the same charge.)

Let $g \in \mathbf{R}$ be fixed and let ρ_1, ρ_2 be any two charge densities with support in disjoint intervals I_1, I_2 of $\mathbf{S}^1 \setminus \{\zeta\}$ and $\rho_1[1] = \rho_2[1] = g$. According to Proposition 3.4 we have the commutation relations

$$\begin{aligned} \psi_{\rho_1}^\zeta \psi_{\rho_2}^\zeta &= e^{\pm ig^2\pi} \psi_{\rho_2}^\zeta \psi_{\rho_1}^\zeta \\ \psi_{\rho_1}^{\zeta *} \psi_{\rho_2}^\zeta &= e^{\mp ig^2\pi} \psi_{\rho_2}^\zeta \psi_{\rho_1}^{\zeta *} \end{aligned} \tag{4.4}$$

where the sign in the exponent depends on the relative location of I_1, I_2, and ζ. Hence the fields of a given charge are local (on the punctured circle) if and only if

$$g^2 = 2N, \qquad N \in \mathbf{N}.$$

Since we are interested in maximal local extensions of $\hat{U}$ we next ask ourselves whether the fields of charge g with $g^2 = 2N$ can be complemented by fields of charge g' with $g'^2 = 2N'$, $N' \in \mathbf{N}$ such that the resulting algebra still contains only local fields. By taking products there will appear in this algebra fields carrying the charge $g + g'$, and if these fields are also to be local with respect to themselves we must have that $gg' \in \mathbf{N}$. From this it follows that $NN' = (N'')^2$ for some $N'' \in \mathbf{N}$, and assuming without loss of generality that $N' < N$ we conclude that N contains as a factor the square of some natural number M. (Since $N > N'$, the number N must contain in its decomposition into primes some factor which is not contained in N'.) Under these circumstances the fields $\psi_{M^{-1}\rho}^\zeta$ with $\rho[1] = g$ are indeed local, and since there holds the "fusion rule"

$$(\psi_{M^{-1}\rho}^\zeta)^M = \psi_\rho^\zeta \tag{4.5}$$

(cf. Proposition 3.4) they generate the fields of charge g. The existence of such extra local fields implies that our algebra of observables was not maximal yet. Thus the fields generating maximal local extensions of $\hat{U}$ carry charges g for which $N = \frac{1}{2}g^2$ contains in its decomposition into primes each prime factor at most once,

$$N = 1, 2, 3, 5, 6, 7, 10, \ldots \tag{4.6}$$

Up to now we have only considered the fields generating local extensions of $\hat{U}$ on the punctured circle. We will now remove this defect by proceeding to a quotient of $\hat{U}$ with respect to the ideal generated by a certain specific subalgebra of the center of $\hat{U}$. To this end let us have a look at the transition operators

$$\psi_\rho^{\zeta_1} \psi_\rho^{\zeta_2 *} = e^{-\sigma i\pi g^2} e^{\sigma 2\pi i g Q} \tag{4.7}$$

considered in Lemma 3.5. If $g^2 = 2N$, N a natural number, the phase factor on the right hand side disappears. Moreover, if we take the quotient of the $*$-algebra generated by all fields ψ_ρ^ζ, $\zeta \in \mathbf{S}^1$ and $\rho[1] = g$, with respect to the two sided ideal generated by $[e^{2\pi i g Q} - 1]$, the right hand side of (4.7) simply becomes the unit operator in that quotient. (Equivalently we may restrict the field operators and the current algebra to the subspace of $\hat{\mathcal{H}}$ where the right hand side of (4.7) is equal to 1, cf. the subsequent subsection.) This quotient algeba clearly lives on the circle $\mathbf{S}^1$ and not on its covering.

We have thus reached our goal: given any natural number N we consider the $*$-algebra on $\hat{\mathcal{H}}$ generated by all fields of charge $g = (2N)^{\frac{1}{2}}$ and proceed to the quotient algebra with respect to the ideal $\hat{\mathcal{Z}}_N$ generated by $[-1 + \exp(2\pi i(2N)^{\frac{1}{2}}Q)]$. We then define for each interval $I \subset \mathbf{S}^1$

$$\mathcal{A}_N(I) = *\text{-algebra}\ \{\psi_\rho : \operatorname{supp}\rho \subset I,\ \rho[1] = (2N)^{\frac{1}{2}}\}, \tag{4.8a}$$

where ψ_ρ denotes the class of ψ_ρ^ζ with $\zeta \notin I$. The algebra $\mathcal{A}_N$ is defined as the smallest algebra containing all local algebras $\mathcal{A}_N(I)$. It is spanned by its unitary elements ψ. These are of the form

$$\psi = \eta W(u)\Gamma_g, \qquad g \in \sqrt{2N}\mathbf{Z} \tag{4.8b}$$

where η is a complex phase factor, $W(u)$ is a Weyl operator, and Γ_g are the charge shift operators, cp. Eqs. (3.19)f, (3.21a).

The time evolution on $\mathcal{A}_N$ acts on the generating elements ψ_ρ by

$$\alpha_t(\psi_\rho) = \psi_{\rho^t}. \tag{4.9}$$

Since we have taken in the above construction a quotient with respect to $\hat{\mathcal{Z}}_N$, the algebra $\mathcal{A}_N$ is an extension of $\hat{\mathcal{U}}_N = \hat{\mathcal{U}}/\hat{\mathcal{Z}}_N$. It is obviously also a *local* extension of $\hat{\mathcal{U}}_N$ as may be seen from the discussion in the preceding section. In particular the algebras $\mathcal{A}_N$ have a vacuum representation with the desired properties which is induced by the vector $\Omega \in \hat{\mathcal{H}}$.

It remains to discuss the question in which sense the algebras $\mathcal{A}_N$ are maximal local extensions of $\hat{\mathcal{U}}_N$ if N contains each prime as a factor at most once. So given such N, let $\mathcal{B}_N$ be another local extension of $\hat{\mathcal{U}}_N$ such that $\mathcal{B}_N(I) \supset \mathcal{A}_N(I)$, $\quad I \subset \mathbf{S}^1$. It is then clear that in the vacuum representation of $\mathcal{B}_N$ there will exist states with charge $g = (2N)^{\frac{1}{2}}$. Moreover, by the above discussion there cannot exist states carrying charges $g' \notin g\mathbf{Z}$ because one cannot accomodate fields carrying such incommensurable charges in a system of local algebras. Thus the algebras $\mathcal{A}_N$ and $\mathcal{B}_N$ generate from the vacuum the same representation of $\hat{\mathcal{U}}$. It then follows from the third part of Proposition 3.1 that the weak closures of the algebras $\mathcal{A}_N(I)$ and $\mathcal{B}_N(I), I \subset \mathbf{S}^1$, in the vacuum representation are related by a Klein transformation; thus they must be equal since $\mathcal{B}_N(I) \supset \mathcal{A}_N(I)$. Hence $\mathcal{A}_N$ and $\mathcal{B}_N$ generate the same system of local algebras in the (locally faithful) vacuum representation, and in this sense $\mathcal{A}_N$ is maximal.

Proposition 4.1. *The local extensions of the current algebra can be distinguished by the smallest nonzero eigenvalue of* $\frac{1}{2}Q^2$ *in their respective vacuum representation. Any such eigenvalue is a natural number* N, *and the corresponding extension is generated by the dynamical system* $(\mathcal{A}_N, \alpha_t)$. $\mathcal{A}_N$ *is the* $*$*-algebra spanned by the fields* ψ_ρ *of charge* $g \in \sqrt{2N}\mathbf{Z}$ *of Proposition 3.4 as described in Eqs. (4.8), and the time evolution automorphism is given by Eq. (4.9), viz.* $\alpha_t(\psi_\rho) = \psi_{\rho^t}$.

If N *contains each primefactor at most once then this extension is maximal.*

We conclude this subsection with two remarks. Firstly we note that we could proceed further with the above construction and consider fermionic (half-integer spin) extensions of the current algebra corresponding to charges g for which g^2 is an odd integer, etc. But these algebras will appear anyway in our discussion of the superselection structure of the algebras $\mathcal{A}_N$ in the subsequent section.

Secondly we would like to point out that there exists an abundance of local extensions of the current algebra if one relaxes the assumption that the Hamiltonian H is a function of the currents (in the sense made precise at the beginning of this section). It is then no longer reasonable to look for the most general extension.

To give a trivial example: the tensor product of $\hat{\mathcal{U}}$ with the algebra generated by the fields in an *arbitrary* local quantum field theory on the circle would then appear as a local extension of $\hat{\mathcal{U}}$. Less trivial examples are obtained by embedding $\hat{\mathcal{U}}$ into certain Kac-Moody algebras for groups of rank > 1. On the other hand, for the Kac-Moody algebra corresponding to the simply laced Lie group SU(2) there holds a Sugawara formula for the energy density which involves only the $U(1)$-current. Hence the latter example provides a proper local extension of the current algebra. We will return to this example in Section 5.

4B. Superselection structure of local extensions

We were led to the dynamical systems $(\mathcal{A}_N, \alpha_t)$ by searching for extensions of the current algebra which can be regarded as algebras of local observables in some quantum field theory. In accord with this view we discuss now the superselection structure of $(\mathcal{A}_N, \alpha_t)$, i.e. we analyze all its locally generated representations (in analogy to the corresponding problem for $\hat{\mathcal{U}}$) and the fields which make transitions between these representation spaces (= superselection sectors for $\mathcal{A}_N$).

We begin by noting that the algebra $\mathcal{A}_N$ still has a center. It is generated by the $\mathbf{Z}_{2N}$-charge

$$V = \exp(2\pi i Q(2N)^{-\frac{1}{2}}). \tag{4.10}$$

(Recall that $V^{2N} = \exp(2\pi i Q(2N)^{\frac{1}{2}}) = 1$ in $\mathcal{A}_N$.) As we will see shortly, there exist locally generated representations of $\mathcal{A}_N$ which can be distinguished by this (multiplicative) charge. In fact, the eigenvalues of V distinguish all superselection sectors of the theory. Thus, since V commutes with all observables, it generates a *gauge group* $\mathbf{Z}_{2N}$ of the theory and different superselection sectors transform according to different representations of this gauge group.

Since $\hat{\mathcal{U}}_N \subset \mathcal{A}_N$ and since in the (locally faithful) vacuum representation of $\mathcal{A}_N$ the dynamics α_t can be implemented by time translations in the weak closure of $\hat{\mathcal{U}}_N$ any locally generated representation of $\mathcal{A}_N$ gives also rise to a locally generated representation of $\hat{\mathcal{U}}$ in which the $*$-algebra $\hat{\mathcal{Z}}_N$ generated by $[-1 + \exp(2\pi i(2N)^{\frac{1}{2}}Q)]$ is represented by 0. Thus we can rely once again on Proposition 3.1 telling us that each locally generated representation of $\mathcal{A}_N$ can be realized on some subspace of our universal representation of $\hat{\mathcal{U}}$. Since the operator $\exp(2\pi i(2N)^{\frac{1}{2}}Q)$ must be represented by 1 the maximal representation space for $\mathcal{A}_N$ is thus the subspace of $\hat{\mathcal{H}}$ on which Q has the spectrum $(2N)^{-\frac{1}{2}}\mathbf{Z}$. Bearing in mind that the algebra $\mathcal{A}_N$ contains operators which change the charge of a state by $\pm(2N)^{\frac{1}{2}}$ we conclude that there exist exactly $2N$ superselection sectors for $\mathcal{A}_N$ (including the vacuum sector) which can be distinguished by the eigenvalues

$$e^{i\pi n/N}, \qquad n = 0, 1, \ldots, 2N-1 \tag{4.11}$$

of the operator V defined in (4.10).

It remains to show that these sectors are locally generated, i.e. we have to exhibit field operators carrying the "elementary" charge $(2N)^{-\frac{1}{2}}$ which are relatively local to the fields generating the algebras $\mathcal{A}_N$. It is clear from the commutation relations (3.41) that the fields ψ_ρ^ζ, with $\rho[1] = (2N)^{-\frac{1}{2}}$ in our canonical field bundle do not have this property. But if we perform a Klein transformation and define

$$\phi_\rho^\zeta = e^{i\pi Q(2N)^{-\frac{1}{2}}}\psi_\rho^\zeta, \qquad \rho[1] = (2N)^{-\frac{1}{2}} \tag{4.12}$$

it is easily seen that these new fields are relatively local to the fields generating $\mathcal{A}_N$. This proves that the above representations are locally generated. We remark that conjugation with these fields yields outer automorphisms of the algebras $\mathcal{A}_N$, and the fields could be recovered from these automorphisms in the manner described in Section 1B.

The commutation relations of fields ϕ_ρ^ζ of the same charge $\rho[1]$ are unaffected by the Klein transformation and are therefore the same as those of the original fields ψ_ρ^ζ. Thus if $\rho_1[1] = \rho_2[1] = (2N)^{-\frac{1}{2}}$ and ρ_1, ρ_2 have disjoint support on $\mathbf{S}^1 \setminus \{\zeta\}$ we have

$$\begin{aligned} \phi_{\rho_1}^\zeta \phi_{\rho_2}^\zeta &= e^{\pm i\pi/2N}\phi_{\rho_2}^\zeta\phi_{\rho_1}^\zeta \\ \phi_{\rho_1}^{\zeta *}\phi_{\rho_2}^\zeta &= e^{\mp i\pi/2N}\phi_{\rho_2}^\zeta\phi_{\rho_1}^{\zeta *} \end{aligned} \tag{4.13}$$

where the sign in the exponent depends on the relative location of the supports of ρ_1, ρ_2 and of ζ.

It is also noteworthy that, using the time translations $U(t)$ defined in (3.52), (3.11), one can extend the fields ϕ_ρ^ζ to the $4N$-fold covering of $\mathbf{S}^1$ by setting for arbitrary $t \in \mathbf{R}$

$$\phi_\rho(t) = \alpha_t(\phi_\rho) = U(t)\phi_\rho^\zeta U(t)^{-1}. \tag{4.14}$$

According to Lemma 3.5 this definition of ϕ_ρ does not depend on ζ. Moreover it follows from that lemma that

$$\alpha_{2\pi}(\phi_\rho) = e^{-i\pi/2N}V\phi_\rho, \tag{4.15}$$

where V is the $\mathbf{Z}_{2N}$-charge defined in (4.10). From this relation we see in particular that the univalence automorphism $\alpha_{2\pi}$ is generated by

$$e^{2\pi iH} = e^{i\pi Q^2}. \tag{4.16}$$

Of course this operator can also be expressed as a linear combination of the operators $V^k, k = 1, \ldots 2N$ which generate the center of $\mathcal{A}_N$.

We conclude this section with the remark that the algebras $\mathcal{A}_N$ corresponding to even N admit an alternative interpretation as even part of a fermionic (half-integer spin) field algebra $\mathcal{F}_N$. Indeed, if $N/2$ is odd we can define $\mathcal{F}_N$ as the *-algebra generated by the canonical fields ψ_ρ^ζ, $\rho[1] = (N/2)^{\frac{1}{2}}$ of Section 3 which live on a two-fold covering of $\mathbf{S}^1$. The lowest weight states (ground states) of $\mathcal{A}_N$ corresponding to the values of $e^{i\pi n/N}, n = 0, 1, \ldots N-1$ of the multiplicative charge then induce inequivalent representations of the field algebra $\mathcal{F}_N$. In representations corresponding to even n the univalence automorphism $\alpha_{2\pi}$ maps ψ_ρ into $-\psi_\rho$ ("Neveu Schwarz sectors") whereas in representations corresponding to odd n this automorphism acts trivially ("Ramond sectors"). It distinguishes between spin structures [37].

V. QUANTUM FIELDS AFFILIATED WITH THE EXTENDED ALGEBRAS

5A. Construction of quantum fields

The algebraic approach to quantum field theory is based on the insight that for the physical interpretation of a theory it is not really necessary to have any a priori information about the physical significance of individual local operators (fields, currents, stress energy tensor etc.). All this information is already encoded in the map assigning to the regions of configuration space certain specific "local" subalgebras of the algebra of all observables, and in the action of the dynamics.

So from this point of view we have already achieved a complete classification of quantum field theories which are related to the current algebra: Given the local observables in such a theory in the vacuum representation one simply has to determine the lowest nonzero eigenvalue of $Q^2/2$, which is some natural number N. The associated map

$$I \mapsto \mathcal{A}_N(I)$$

which fixes the corresponding system of local algebras, is then given by relation (4.8), and the dynamics by the relation (4.9).

Yet in order to make contact with the more conventional treatments of local quantum field theory which are based on quantum fields localized at a point we want to discuss now how such fields appear in the present setting. The basic idea is very simple: starting from the unitary charged field operators ψ_ρ^ζ of Section 3 we will go to the limit of charge densities ρ which have support at a point. In doing so we must of course pay tribute to the fact that there do not exist bounded field operators which are localized at a point. We must therefore "renormalize" the operators ψ_ρ^ζ by a factor R_ρ which tends to infinity, and the limits can then be defined as operator-valued distributions. (This procedure is of course not new, cp. for instance [38] and references quoted there.)

We proceed as follows. First we put $\zeta = -1$ and choose a sequence of functions $\delta_\lambda \in S$ localized in an interval of length λ about $1 \in \mathbf{S}^1$, which goes in the limit $\lambda \searrow 0$ to the Dirac measure $\delta(z-1)$. In order to simplify the notation we set for given charge $g \in \mathbf{R}$

$$\phi_{g,\lambda}(t) = R_{g,\lambda}\alpha_t(\psi_{g\delta_\lambda}^{\zeta=-1}), \qquad t \in \mathbf{R} \tag{5.1}$$

where $R_{g,\lambda}$ is a renormalization constant which we fix by the condition

$$(\Gamma_g\Omega, \phi_{g,\lambda}(0)\Omega) = 1. \tag{5.2}$$

Here Γ_g are the charge shift operators introduced in Section 3. An easy calculation, based on the relation (3.10) and Lemma 2.2 gives

$$R_{g,\lambda} = \exp\left(\frac{g^2}{2}\left[\sum_{m=1}^{\infty}\frac{1}{m}|(\tilde\delta_\lambda)_m|^2 - \int\frac{dz}{2\pi i}\delta_\lambda(z)\ln z\right]\right). \tag{5.3}$$

Since $(\tilde\delta_\lambda)_m$ converges to 1 if δ_λ approaches $\delta(z-1)$ this expression clearly diverges if λ tends to 0.

We could now go on to show that for any test function h the limit

$$\phi_g(h) = \lim_{\lambda\to 0}\int dt\, h(t)\phi_{g,\lambda}(t) \tag{5.4}$$

exists on a suitable domain in $\mathcal{H}$ and defines an operator-valued distribution $\varphi_g(.)$. It is however more instructive to compute the vacuum correlation functions of the fields $\phi_g(.)$, and then appeal to the GNS-construction for the reconstruction of the field operators.

Let us fix charges $g_1, \dots g_n \in \mathbf{R}$ and calculate

$$\mathcal{W}_\lambda(g_1,t_1;\dots;g_n,t_n) = (\Omega, \phi_{g_1,\lambda}(t_1)\dots\phi_{g_n,\lambda}(t_n)\Omega) \tag{5.5}$$

for non-coinciding arguments t_i, $-\pi < t_i < \pi$, $i = 1,\dots,n$. For these special configurations the limit $\lambda \searrow 0$ exists in the sense of equicontinuous functions; the result can then be extended to arbitrary points as a tempered distribution by analytic continuation, making use of energy positivity.

From the definition of the charged fields ψ_ρ^ζ, the composition law of the Weyl operators, and the action of the group of charge shift operators Γ_g on the Weyl operators $W(f)$ it is obvious that

$$\phi_{g_1,\lambda}(t_1)\dots\varphi_{g_n,\lambda}(t_n) = \xi \prod_{j=1}^{n} R_{g_j,\lambda} W(g_1\widehat{\delta_\lambda^{t_1}} + \dots + g_n\widehat{\delta_\lambda^{t_n}})\Gamma_{g_1+\dots+g_n}, \tag{5.6}$$

where ξ is some phase factor. The actual calculation of ξ simplifies if one assumes that $g_1 + \dots + g_n = 0$ (otherwise, $\mathcal{W}_\lambda$ vanishes anyway) and that λ is so small that the densities $\delta_\lambda^{t_j}, j = 1\dots n$ have disjoint supports. Under the latter assumption we can apply Lemma 3.3 which, together with relation (3.50) yields

$$A(\widehat{\delta_\lambda^{t_j}}, \widehat{\delta_\lambda^{t_k}}) = -i\pi \operatorname{sign} t_{jk} + it_{jk} \tag{5.7}$$

where we have put

$$t_{jk} = t_j - t_k.$$

After these preparations it is straightforward to establish by induction in n the form of ξ:

$$\xi = \exp\left(i\frac{\pi}{2}\sum_{j<k} g_j g_k \operatorname{sign} t_{jk}\right). \tag{5.8}$$

By taking matrix elements of the expressions in relation (5.6) in the vacuum state we thus find that

$$\mathcal{W}_\lambda(g_1,t_1;\dots;g_n,t_n) = \prod_{j=1}^{n} R_{g_j,\lambda} \exp\left(i\frac{\pi}{2}\sum_{j<k} g_j g_k \operatorname{sign} t_{jk} - \frac{1}{2}\|\sum_j g_j\widehat{\delta_\lambda^{t_j}}\|^2\right), \tag{5.9}$$

where the square of the norm is given by (cf. Lemma 2.2)

$$\|\sum_j g_j\widehat{\delta_\lambda^{t_j}}\|^2 = \sum_{j,k} g_j g_k \sum_{m=1}^{\infty}\frac{1}{m}|(\tilde{\delta}_\lambda)_m|^2 \cos m t_{jk}. \tag{5.10}$$

Plugging the specific form of the renormalization constants $R_{g,\lambda}$ into relation (5.9) we can now proceed to the limit $\lambda \searrow 0$, giving

$$\mathcal{W}_0(g_1,t_1;\dots;g_n,t_n) = \exp\left(\sum_{j<k} g_j g_k [i\frac{\pi}{2}\operatorname{sign} t_{jk} - 2\sum_{m=1}^{\infty}\frac{\cos m t_{jk}}{m}]\right). \tag{5.11}$$

Summing up the series of cosines in (5.11)

$$2\sum_{m=1}^{\infty}\frac{\cos m\tau}{m} = -\ln(4\sin^2\frac{\tau}{2}), \tag{5.12}$$

we are finally led to the result

$$\mathcal{W}_0(g_1,t_1;\dots;g_n,t_n) = \prod_{j<k}(-4\sin^2\frac{t_{jk}-i\epsilon}{2})^{g_j g_k/2}. \tag{5.13}$$

Here we have added $-i\epsilon$ to the time differences, thereby defining the integration rules for coinciding arguments in accord with energy positivity.

The expectation values of fields in lowest weight states Ω_e of charge e can be obtained from (5.13) with the help of the action of the automorphism

$$\Gamma_e^*\phi_g(t)\Gamma_e = e^{iegt}\phi_g(t) \tag{5.14}$$

and the formula $\Omega_e = \Gamma_e\Omega$.

Since $\phi_g(t)^* = \phi_{-g}(t)$, as can be seen from relation (5.1) and (3.42) we thus have completed our computation of the n-point functions of the charged quantum fields which were generated by the unitary field operators of Section 3.

The expert will recognize that we have recovered the known explicit formula for the n-point functions of the $U(1)$-vertex operators that generalize the n-point functions of the Thirring model to systems with an arbitrary number of different charges. A better known version of (5.13) is obtained by introducing the "z-picture fields"

$$\psi_g(z) = e^{-ig^2t/2}\phi_g(t), \qquad z = e^{it}. \tag{5.15a}$$

The n-point functions of these fields are given by

$$(\Omega, \psi_{g_1}(z_1)\dots\psi_{g_n}(z_n)\Omega) = \prod_{j<k} z_{jk}^{g_j g_k}, \tag{5.15b}$$

where $z_{jk} = z_j - z_k$ and the rules for approaching the singularities are summarized by

$$|z_1| > |z_2| > \dots > |z_n|. \tag{5.16}$$

Of course, the operators $\psi_g(z)$ and their expectation values are to be understood as multivalued functions of z. We also note that the adjoints of the z-picture fields are given by

$$\psi_g^*(z) = z^{-g^2}\psi_{-g}(z^{-1}). \tag{5.17}$$

It follows from these results that ψ_g is a primary field with respect to both the Virasoro and the current algebra, with all its well known consequences (cf. for example [39]). We have thus rediscovered the structure which one normally anticipates in the investigation of conformal quantum field theories. It appears in the present setting as a byproduct of our systematic investigation of all locally generated representations of the current algebra.

Besides the above quantum fields $\psi_g(z)$, further ones can be constructed in a systematic fashion. For instance, the current $J(z)$ is limit of the renormalized expression $R[e^{iJ(u)}-1]=R[W(u)-1]$ in an obvious way, and so on. The multiplication law (3.40) in the algebra will determine the operator product expansions for the quantum fields, for instance

$$\psi_g(z_1)\psi_{-g}(z_2) = z_{12}^{-g^2}\{1 + g z_{12} J(z) + \dots\}, \; z = \sqrt{z_1 z_2}.$$

Operator product expansions do not converge as operators, but only when applied to the vacuum. Therefore they are not really a multiplication law of an algebra in manifest form (although they determine it indirectly, because they determine all Wightman functions [40]). In this respect the multiplication law in $\mathcal{F}$ is superior.

5B. Examples

We conclude our discussion of quantum fields by considering a few examples of models giving rise to specific local extensions $\mathcal{A}_N$ of the current algebra.

N=1: Our first example, already mentioned, is the $\widehat{su(2)}$ Kac-Moody algebra with Kac-Moody central charge 1 ("level 1"):

$$[J_a(z_1), J_b(z_2)] = i\epsilon_{abc}J_c(z_1)\delta(z_{12}) - \frac{1}{2}\delta_{ab}\delta'(z_{12}) \tag{5.18}$$

where $a, b, c = 1, 2, 3$. With the identification

$$J(z) = \sqrt{2}J_3(z) \tag{5.19}$$

we see that this algebra is an extension of $\hat{U}$. It is in fact a local extension fitting into our setting since the stress energy tensor $T(z)$ can be expressed in terms of $J(z)$

$$T(z) = \tfrac{1}{2} : J(z)^2 : . \tag{5.20}$$

It is easy to see that this algebra generates $\mathcal{A}_1$. We must only show that the smallest nonzero eigenvalue of $Q^2/2$ is 1. But this is clear from the isospin commutation relations of the charges Q_a which are obtained by integrating the currents $J_a(z)$.

In the present case one sees also directly that the isospin currents $J_a(z)$ generate $\mathcal{A}_1$: setting

$$\psi(z) = J_1(z) + iJ_2(z) \tag{5.21}$$

it follows from the commutation relations (5.18) that ψ is a local field carrying the charge $g = \sqrt{2}$. One can thus construct from ψ, ψ^*, and J the system of local algebras making up $\mathcal{A}_1$.

From the discussion in Section 4 we know that in the present model there does exist besides the vacuum sector another superselection sector in which the multiplicative charge $e^{2\pi i Q/\sqrt{2}}$ takes the value -1. This is of course the set of states with half-integer isospin which can be generated from the vacuum by the (Klein transformed) quantum field $\psi_{1/\sqrt{2}}(z)$ constructed in the preceding subsection.

N=2: The theory of a free complex Fermi field ψ is perhaps the simplest example providing an extension of the current algebra: from its field-current commutation relations

$$[J(z_1), \psi(z_2)] = \delta(z_{12})\psi(z_1) \tag{5.22}$$

one sees that the even polynomials in the smeared Fermi fields ψ, ψ^* create from the vacuum states with charges in $2\mathbf{Z}$. Hence they generate the local extension $\mathcal{A}_2$.

According to the discussion in Section 4 there exist 4 different superselection sectors for $\mathcal{A}_2$ which may be distinguished by the eigenvalues $\pm 1, \pm i$ of the multiplicative charge $e^{i\pi Q}$. Of special interest are the sectors corresponding to the imaginary eigenvalues. It is well known, and also clear from our results, that these states can be created from the vacuum with the help of a field of charge $\frac{1}{2}$ (conformal weight $\frac{1}{8}$), and they induce representations of the Fermi algebra in which the fields are single-valued on the circle (Ramond sector).

N=3: The algebra $\mathcal{A}_3$ appears for example in the $\mathbf{Z}_4$-parafermion current algebra analyzed by Zamolodchikov and Fateev [41]. This model is generated by a complex primary field ψ of weight $\frac{3}{4}$ which obeys anticommutation relations with the current

$$[J(z_1), \psi(z_2)]_+ = \sqrt{\tfrac{3}{2}}\delta(z_{12})\psi(z_1)^*, \tag{5.23}$$

and there holds a Sugawara formula for the stress tensor involving only J. By applying all even polynomials in the smeared fields ψ, ψ^* to the vacuum we thus obtain a representation of the current algebra which is locally generated. From (5.23) we see furthermore that the spectrum of the charge operator Q in this representation is $\sqrt{6}\mathbf{Z}$. Hence the even polynomials in ψ, ψ^* generate the local extension $\mathcal{A}_3$ of the current algebra.

It is noteworthy that the fields ψ in this model can also be accomodated in the present setting by applying to the basic fields constructed in Section 3A a generalized Klein transformation[1]. Denoting by $C = C^* = C^{-1}$ the charge conjugation operator which acts on the Weyl operators $W(u)$ and the basic fields ψ_g according to

$$CW(u)C^{-1} = W(-u), \qquad C\psi_g C^{-1} = \psi_{-g}, \tag{5.24}$$

one can represent the Zamolodchikov-Fateev-field ψ by

$$\psi = \frac{1}{\sqrt{2}}(\psi_g + i\psi_{-g})C, \qquad g = \sqrt{\frac{3}{2}}. \tag{5.25}$$

From this equation one easily recovers the $\mathbf{Z}_4$ symmetry of the model by first noting that there exists a unitary operator V (in the vacuum representation of $\psi_{\pm g}$) such that

$$V\psi_{\pm g}CV^{-1} = \pm\psi_{\pm g}C. \tag{5.26}$$

It is then obvious that

$$VC\psi CV^{-1} = i\psi, \tag{5.27}$$

hence the map $\gamma(\psi) = i\psi$ defines an automorphism of the *-algebra generated by the field ψ. Since γ^4 is the identity, this automorphism induces the symmetry group $\mathbf{Z}_4$.

N=4: The algebra $\mathcal{A}_4$ is the first example of a local extension of the current algebra which is not maximal. It can be identified with a subalgebra of the algebra $\mathcal{A}_1$. In the concrete example of the $\widehat{su(2)}$ Kac-Moody algebra discussed above it is the algebra generated by all even polynomials of the currents J_1, J_2, which in view of the commutation relations

$$[J_1(z_1), J_2(z_2)] = i\delta(z_{12})J_3(z_1) \tag{5.28}$$

[1] We owe this remark to R. Paunov.

provide also a local extension of the current algebra.

According to our general results there exist 8 superselection sectors for this algebra which can be labeled by the eigenvaluesof $e^{i\pi Q_3\sqrt{2}}$. Thus for $\mathcal{A}_4$ there exist besides the superselection sectors with (half-) integer isospin also sectors in which the component Q_3 of the isospin has the eigenvalues $\frac{1}{4}, \frac{3}{4}$, etc. (giving rise to the so-called "twisted sector" of $\widetilde{su(2)}$ [14]). This fact can be understood if one notices that the algebra $\mathcal{A}_4$ consists of the fixed points in $\mathcal{A}_1$ under rotations by π around the 3-axis in isospin space.

At this point we terminate our discussion of examples since we think it has become clear now how our results can be used for a quick survey of the local algebraic structure and the superselection rules in concrete models.

We would like to point out that all extensions $\mathcal{A}_N$ can be realized in a unified manner [32]: they appear as coset space models (in the sense of Goddard, Kent and Olive [42]) for the homogeneous spaces $SO(4N)_1/\ SO(2N)_2$, where the subscript indicates the Kac-Moody level. For other approaches to the classification of $c = 1$ models cf. [43,44,45]. So from the point of view of "model building" we have discovered nothing new (see also [46]). But we have seen how these algebraic structures emerge in a systematic manner from the underlying germ: the current algebra.

VI. CHARACTERS OF LOWEST WEIGHT REPRESENTATIONSAND THE KMS-CONDITION

A popular way of merging chiral (right or left moving) field theories on the circle to a 2-dimensional field theory is based on the principle of modular invariance. Roughly speaking, this principle amounts to the requirement that the resulting (Euclidean) field theory, which lives on a torus, should be invariant under reparametrizations. As a consequence, the corresponding (Minkowski) quantum field theory is local.

For the actual application of this method one needs to know the characters of the underlying chiral theories. These characters are then combined into a partition function $Z(\tau)$, τ being the ratio of the periods of the lattice defining the torus of the Euclidean theory, which is invariant under modular transformations. The latter constraint restricts the admissible weight factors of the various characters contributing to the partition function and thus fixes the Hilbert spaces of the 2-dimensional theories. (For an alternative approach to the construction of 2-dimensional theories from one-dimensional constituents see the recent work of Fröhlich, and Schroer and Rehren [17,18].)

In this section we want to point out how the character of the chiral theories can easily be computed within our algebraic setting. We shall exploit the correspondence between finite size Euclidean theories (in the imaginary time direction) and finite temperature Minkowski space quantum field theories, to write down

$$\tau = \frac{i}{2\pi}\beta \tag{6.1}$$

where β is the inverse temperature (which will ultimately be treated as a complex variable with a positive real part). This leads us to consider Gibbs-states (KMS-states) on the algebras which we have constructed.

6A. KMS-STATES ON $\mathcal{A}_N$

Given a representation of the algebra $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ a Gibbs states of inverse temperature β is given by the following formula (if it makes sense)

$$\omega_\beta(A) = Z^{-1} Tr_{\mathcal{H}} e^{-\beta H} A, \qquad A \in \mathcal{A} \tag{6.2}$$

where H is the Hamiltonian, Z is the partition function, and the trace has to be taken over the representation space $\mathcal{H}$. It is of interest here that for the actual computation of the state ω_β it is not necessary to determine the spectral properties of the operator $e^{-\beta H}$ and then to evaluate the trace. The crucial observation [33] is that the Gibbs states can also be characterized by the Kubo-Martin-Schwinger (KMS) condition: it says that for all $A, B \in \mathcal{A}$ the functions $\omega_\beta(A\alpha_t(B))$ and $\omega_\beta(\alpha_t(B)A)$ $\;t \in \mathbf{R}$ are boundary values at the lower and upper rim of the strip

$$S_\beta = \{z : 0 \leq Im z \leq \beta\},$$

respectively, of a function F_{AB} which is analytic in the interior of that strip and continuous at the boundaries. In a somewhat sloppy but suggestive notation we thus have that

$$\omega_\beta(A\alpha_{t+i\beta}(B)) = \omega_\beta(\alpha_t(B)A). \tag{6.3}$$

This condition is an immediate consequence of the relation (6.2) since H is bounded below and the trace is cyclic. Note that it also makes sense in situations where the individual terms on the right hand side of relation (6.2) are ill-defined (i.e. if the operator $e^{-\beta H}$ is not trace class).

Relation (6.3) can be used for the calculation of the possible finite temperature states ω_β if the algebraic structure of $\mathcal{A}$ is sufficiently simple, as is the case for the Kac-Moody algebras and the Virasoro algebra, for instance. The basic strategy is as follows.

Assume that there is a simple algebraic relation between the operators $\alpha_t(B)A$ and $A\alpha_t(B)$. To give an example: let the commutator of these operators be a multiple of the identity $C(t)\mathbf{1}$, as is the case for the currents in the present model. Then one rewrites relation (6.3) according to

$$\begin{aligned}\omega_\beta(A\alpha_{t+i\beta}(B)) &= \omega_\beta([\alpha_t(B), A]) + \omega_\beta(A\alpha_t(B)) \\ &= C(t) + \omega_\beta(A\alpha_t(B)).\end{aligned} \tag{6.4}$$

Here we have used the fact that $\omega_\beta(\mathbf{1}) = 1$. So one obtains an inhomogeneous functional equation for the function $t \mapsto \omega_\beta(A\alpha_t(B))$, and this equation can be solved by Fourier transformation.

In the case of the Kac-Moody algebras or the Virasoro algebra the commutator of the basic fields is not a c-number, but it is linear in these fields. So the first equation in (6.4) provides a recursive relation between the n-point functions and the (n-1)-point functions which can likewise be solved in these models.

This method of computation is well known in statistical mechanics [47]. We apply it here to the algebras $\mathcal{A}_N$ and begin by determining the KMS-states of $\tilde{U}$. Since we are dealing with the exponentials of the currents we will use as our starting point a multiplicative version of (6.4) : if A and B are Weyl operators, then it follows from the composition law (2.6) that

$$\alpha_t(B)A = \eta(t)A\alpha_t(B) \tag{6.5}$$

where $\eta(t)$ is some explicitly known phase factor. Hence the multiplicative analogue of relation (6.4) is

$$\omega_\beta(A\alpha_{t+i\beta}(B)) = \eta(t)\omega_\beta(A\alpha_t(B)). \tag{6.6}$$

We will determine in Appendix A the solutions of this equation. The result is the following counterpart of Lemma 2.2.

LEMMA 6.1. *Let ω_β be a regular KMS-state on $(\hat{U}, \alpha_t)$ at inverse temperature $\beta > 0$. Then there exists some positive normalized measure σ on $\mathbf{R}$ such that for $u \in \mathcal{S}$ (with Fourier components $\tilde{u}_n$)*

$$\omega_\beta(W(u)) = \int d\sigma(g')\exp\left(ig'\tilde{u}_0 - \frac{1}{2}\sum_{n=1}^{\infty} n\coth\frac{n\beta}{2}|\tilde{u}_n|^2\right). \tag{6.7a}$$

Conversely, given any positive normalized measure σ on $\mathbf{R}$, then the functional ω_β fixed by Eq. (6.7a) determines a KMS-state on $\hat{U}$.

For future use we record

COROLLARY. *Let $u_\pm(z) = \sum_{n=1}^{\infty}\tilde{u}_{\mp n}z^{\pm n}$. Then $\omega_\beta(W(u_-)W(u_+))$ is the same for all KMS-states of inverse temperature β on $\hat{U}$ and is given by*

$$\omega_\beta(W(u_-)W(u_+)) = \exp\left(-\sum_{m=1}^{\infty}\frac{m}{e^{m\beta}-1}\tilde{u}_{-m}\tilde{u}_m\right). \tag{6.7b}$$

As in the case of ground states we can read off relation (6.7a) the central decomposition of the states ω_β into states with a definite value g' of the charge Q. (Of course these latter states are no longer pure states on $\hat{U}$ if $\beta < \infty$).

We will use this result for the computation of all KMS-states on the algebra $\mathcal{A}_N$. To this end let us show that these states are "gauge invariant": let $\psi_\rho \in \mathcal{A}_N$ be any unitary field operator[1] with charge $g = \rho[1] \neq 0$, and let ω_β be any KMS-state on $\mathcal{A}_N$. Since the operators $e^{i\lambda Q}$ (being elements of the center of $\hat{U}$) are invariant under time translations α_t and since

$$e^{i\lambda Q}\psi_\rho = e^{i\lambda g}\psi_\rho e^{i\lambda Q} \tag{6.8}$$

we obtain from the KMS-condition (6.3) the relation

$$\omega_\beta(\psi_\rho e^{i\lambda Q}) = \omega_\beta(e^{i\lambda Q}\psi_\rho) = e^{i\lambda g}\omega_\beta(\psi_\rho e^{i\lambda Q}) \tag{6.9}$$

for arbitrary $\lambda \in \mathbf{R}$. But $g \neq 0$, so we see that for small $\lambda \neq 0$ the left hand side of Eq. (6.9) must vanish. Hence, by continuity in λ

$$\omega_\beta(\psi_\rho) = 0 \qquad \text{if } g \neq 0. \tag{6.10}$$

[1] We will work again with the unitary field operators since we do not want to worry about domain problems. The computations are as simple as with the unbounded quantum fields at a point.

The gauge invariant (zero charge) operators in $\mathcal{A}_N$ are just the elements of $\hat{U}$, so the calculation of the KMS-states on $\mathcal{A}_N$ boils down to the determination of the measure $d\sigma(g)$ in relation (6.7).

Let us now remember that $\mathcal{A}_N$ has a nontrivial center whose unitary elements are powers of $V = e^{2\pi iQ/\sqrt{2N}}$. They form the group $\mathbf{Z}_{2N}$. A general KMS-state on $\mathcal{A}_N$ is a convex combination of "primary" states which transform according to an irreducible representation of $\mathbf{Z}_{2N}$, viz.

$$\omega_\beta(e^{2\pi iQ/\sqrt{2N}}A) = e^{2\pi ie/\sqrt{2N}}\omega_\beta(A). \tag{6.10a}$$

The representations of $\mathbf{Z}_{2N}$ are labeled by

$$e = n/\sqrt{2N}, \quad n = 0, 1, \ldots, 2N-1.$$

It suffices thus to consider the primary states which obey Eq. (6.10a). We will call them "states of charge e" for short. The existence of such a central decomposition into primary states can be derived purely from the KMS-condition, similarly as in Section 4B. If we assume that the KMS-state is given by the trace formula (6.2) then the decomposition amounts to a decomposition of $\mathcal{H}$ into subspaces which carry irreducible representations of $\mathcal{A}_N$.

The covariance property (6.10a) for a state of charge e is consistent with representation (6.7a) of this state only if the measure $d\sigma(g')$ has the form

$$d\sigma(g') = \sum_n w_n\delta(g' - ng - e)dg'. \qquad g = \sqrt{2N}. \tag{6.10b}$$

So it remains to determine the relative weights w_n in this sum.

To this end it suffices to exploit the KMS-condition for the operators $\psi_\rho^*\alpha_t(\psi_\rho)$, $\rho[1] = g$. From the definition (3.36) of the operators ψ_ρ and of the dynamics α_t in (3.52) we obtain

$$\begin{aligned}\psi_\rho^*\alpha_t(\psi_\rho) &= e^{igQt+ig^2t/2}W(\hat{\rho})^*\alpha_t(W(\hat{\rho}))\\ \alpha_t(\psi_\rho)\psi_\rho^* &= e^{igQt-ig^2t/2}\alpha_t(W(\hat{\rho}))W(\hat{\rho})^*.\end{aligned} \tag{6.11}$$

The crucial observation is that the operators $W(\hat{\rho})^*\alpha_t(W(\hat{\rho}))$ and $\alpha_t(W(\hat{\rho}))W(\hat{\rho})^*$ commute with the charge shift operator $\Gamma_{g'}$. This is true because the exponentials of Q in the two Weyl operators just cancel since the time translations act trivially on the center of $\hat{U}$. Hence if ω_β is any KMS-state on $\mathcal{A}_N$ we have in view of Lemma 6.1 that

$$\begin{aligned}\omega_\beta(\psi_\rho^*\alpha_t(\psi_\rho)) &= e^{ig^2t/2}F(t)\omega_\beta(W(\hat{\rho})^*\alpha_t(W(\hat{\rho}))\\ \omega_\beta(\alpha_t(\psi_\rho)\psi_\rho^*) &= e^{-ig^2t/2}F(t)\omega_\beta(\alpha_t(W(\hat{\rho}))W(\hat{\rho})^*)\end{aligned} \tag{6.11a}$$

where we have put

$$F(t) = \omega_\beta(e^{igQt}) = \int d\sigma(g')e^{ig'gt}. \tag{6.12}$$

From the assumption that ω_β satisfies the KMS-condition on $\mathcal{A}_N$ and the fact that the analytic continuation of the function $t \mapsto \omega_\beta(W(\hat{\rho})^*\alpha_t(W(\hat{\rho}))$ vanishes nowhere on the strip S_β (as can be seen from relation (6.7a)) it thus follows that $F(t)$

can be analytically continued to S_β. Moreover, from the KMS-boundary condition we see that

$$F(t+i\beta) = e^{-ig^2 t - g^2\beta/2} F(t). \tag{6.13}$$

Proceeding to the Fourier transformation of this equation (in the sense of distributions) and plugging in the specific form of the measure σ, Eq. (6.10b), we find that the weights w_n in this measure must satisfy the recursion relation

$$w_{n+1} = w_n e^{-\beta(g(n+\frac{1}{2})+e)g}.$$

Its solution is

$$w_n = w^0 e^{-\beta(ng-e)^2/2} \tag{6.14}$$

and w^0 is determined by the requirement that the measure σ is normalized, viz.

$$\sum_n w_n = 1. \tag{6.10c}$$

Having determined the measure σ one can now compute the expectation values of arbitrary products of the unitary field operators in the state ω_β. The result is zero unless the sum of the charges of the fields is 0. For zero charge the product is equal to a Weyl operator by the multiplication law in $\mathcal{A}_N$. The explicit form of the KMS-state on Weyl operators can then be used. As in the case of lowest weight states one can obtain the correlation functions of the quantum fields (fields at a point) by proceeding to charge densities which are localized at a point. For the purpose of this calculation it is convenient to determine the above mentioned Weyl operator by making repeated use of the relation (6.11). We omit the trivial details of the calculation.

Let us briefly discuss how these results have to be modified if one changes the dynamics α_t by composing it with a gauge transformation parametrized by a "chemical potential" μ. This new dynamics α_t^μ is given by

$$\alpha_t^\mu(\psi_\rho) = e^{it\mu g}\alpha_t(\psi_\rho). \tag{6.15}$$

In the relations corresponding to (6.11) and (6.11a) there appears then on the right hand side an additional factor $e^{it\mu g}$. Its only effect is that in the weights w_n of the measure σ characterizing the KMS-states for the dynamics α_t^μ one must replace e by $e+\mu$. We summarize these results in

Proposition 6.2. *Any KMS-state on the dynamical system $(\mathcal{A}_N, \alpha_t^\mu)$ for inverse temperature β and charge $e = n/\sqrt{2N}$, $n = 0, 1, \ldots, 2N-1$ is gauge invariant, i.e. vanishes on operators with nonzero total charge. Its restriction to the gauge invariant part $\hat{U}$ of $\mathcal{A}_N$ (the "zero charge operators") is determined by Eq. (6.7a), where the measure is given by ($g = \sqrt{2N}$).*

$$d\sigma(g') = w^0 \sum_n e^{-\beta g'^2/2}\delta(g' - ng - e - \mu) \tag{6.16}$$

w^0 being a normalization constant.

We mention as an aside that one can also find the KMS-states ω_β associated with our universal representation of $\hat{U}$ by the above method. In this case the individual

terms on the right hand side of relation (6.2) make no sense (since the representation space is not separable). But the KMS-condition can be evaluated as before and the KMS-states are again fixed by relation (6.7a), where the measure σ is now given by

$$d\sigma(g') = \sqrt{\beta/2\pi}\, e^{-\beta(g'+\mu)^2/2} dg', \tag{6.17}$$

μ being the chemical potential. These states appear in fact in the string theoretic interpretation of 2-dimensionl models, where Q is viewed as the center of mass momentum which for an uncompactified dimension has a continuous spectrum.

6B. Partition functions

Next we wish to compute the partition function in the presence of a chemical potential μ. As we know, the irreducible representations of $\mathcal{A}_N$ can be labeled by the charge $e = n/g$, $g = \sqrt{2N}, n = 0, \ldots, 2N-1$. Let us denote the corresponding representation spaces by $\mathcal{H}_n$. The corresponding partition function is

$$Z_n(\beta,\mu) = tr_{\mathcal{H}_n} e^{-\beta(H+\mu Q)}. \tag{6.18}$$

Assuming that the trace exists, the logarithmic derivative of Z_n must be expectation value of $H+\mu Q$ in a KMS state $\omega_{\beta,\mu}$ of inverse temperature β, with chemical potential μ and charge e. The one and only such KMS-state was found in the last section. (For a discussion of how the existence of the trace can be deduced from the KMS-condition see end of Appendix A.) Thus

$$-\frac{\partial}{\partial\beta}\ln Z_n(\beta,\mu) = \omega_{\beta,\mu}(H+\mu Q). \tag{6.20}$$

To compute this quantity, we must have H as a concrete element of our algebra of observables. We assume the Sugawara formula

$$H = \sum_{m\geq 1} J_{-m}J_m + \tfrac{1}{2}Q^2 \tag{6.21}$$

(which can be established in our setting by showing that the right hand side of (6.21) induces the time translations given in relation (3.53)). For later convenience, the contribution involving the charge $Q = J_0$ has been singled out. Thus

$$\omega_{\beta,\mu}(H+\mu Q) = \omega_{\beta,\mu}(\frac{1}{2}Q^2 + \mu Q) + \sum_{n\geq 1}\omega_{\beta,\mu}(J_{-n}J_n). \tag{6.22}$$

The KMS-state $\omega_{\beta,\mu}$ on $\mathcal{A}_N$ is also a KMS state on $\hat{U}$. Since the Weyl operators $W(u) = \exp(i\sum J_n\tilde{u}_{-n})$ determine J_n we see from the Corollary to Lemma 6.1 that the expectation value of $J_{-m}J_m$ for $m\geq 1$ is the same for any KMS-state of inverse temperature β on $\hat{U}$, and is given by

$$\omega_{\beta,\mu}(J_{-m}J_m) = \frac{m}{e^{\beta m}-1} = \frac{\partial}{\partial\beta}\ln(1-e^{-\beta m}),\ (m\geq 1). \tag{6.23}$$

So the sum in (6.22) yields

$$\sum_{m\geq 1}\omega_{\beta,\mu}(J_{-m}J_m) = \frac{\partial}{\partial\beta}\ln\prod_{m=1}^{\infty}(1-e^{-\beta m}). \tag{6.24}$$

The remaining terms can be written as

$$\omega_{\beta,\mu}(\frac{1}{2}Q^2 + \mu Q) = \left(-\frac{1}{2}\frac{\partial^2}{\partial t^2} - i\mu\frac{\partial}{\partial t}\right)F(t/g)|_{t=0} \tag{6.25}$$

where $F(t)$ is the function defined in Eq. (6.12). According to the results of Section 6A, $F(t)$ has a Fourier expansion $F(t) = \sum_m w_m e^{itg(mg+\epsilon)}$ whose coefficients w_m are given by Eq. (6.14), with $\epsilon + \mu$ substituted for ϵ in the presence of a chemical potential μ. Thus

$$\omega_{\beta,\mu}(\frac{1}{2}Q^2 + \mu Q) = \sum_m w_m[\frac{1}{2}(mg+\epsilon)^2 + \mu(mg+\epsilon)] \tag{6.26}$$

with

$$w_m = \frac{e^{-\beta(mg+\epsilon+\mu)^2/2}}{\sum_m e^{-\beta(mg+\epsilon+\mu)^2/2}}.$$

The sum in (6.26) can thus be rewritten in the form

$$-\frac{\partial}{\partial\beta}\ln\sum_m e^{-\beta(mg+\epsilon)^2/2}e^{-\beta\mu(mg+\epsilon)},$$

and putting these results together we obtain the derivative of $\ln Z_n$ with respect to β. This shows that

$$Z_n(\beta,\mu) = \frac{\sum_m e^{-\beta(mg+\epsilon)^2/2}e^{-\beta\mu(mg+\epsilon)}}{\prod_{m=1}^{\infty}(1-e^{-\beta m})} \tag{6.27}$$

up to a possible normalization factor $\mathcal{N}(\mu)$. We show that the normalization factor is 1 by examining the asymptotic behaviour as $\beta \to \infty$ of the right hand side of (6.27) and of the trace defining Z_n. In this limit the asymptotic behaviour of the sum in (6.27) is dominated by those terms where $\frac{1}{2}(mg+\epsilon)^2 + \mu(mg+\epsilon)$ assumes its minimum. On the other hand, the trace in Eq. (6.18) is dominated by the ground states of those irreducible representations for $\hat{U}$ with charge g' where $E_{g'} + \mu g'$ is minimal. But the ground state energy $E_{g'}$ equals $\frac{1}{2}(g')^2$ as we know, and the possible values of g' are $ng+\epsilon$, $n \in \mathbf{Z}$. As a result, both sides of (6.27) agree asymptotically for $\beta \to \infty$.

The result (6.27) for the partition function is proportional to a ratio of a classical Θ-function and a Dedekind η-function. see below.

6C. Modular properties of characters of $\mathcal{A}_N$

The characters of the algebras $\mathcal{A}_N$ are the analytic continuations of the partition functions Z_N to imaginary β. For their analysis it is convenient to introduce the new variables

$$e^{2\pi i\tau} = e^{-\beta}, \qquad e^{2\pi i\zeta} = e^{-\beta\mu}.$$

We also note that the functions appearing in the numerator and denominator of Eq. (6.27) are well known.

$$\eta(\tau) = e^{2\pi i\tau/24}\prod_{m=1}^{\infty}(1-e^{2\pi i\tau m})$$

is the Dedekind η-function and

$$\Theta_{l,g^2}(\tau,\zeta,u) = e^{2\pi iu}\sum_{m\in\mathbf{Z}} e^{\pi i\tau(gm+l/2g)^2}e^{2\pi i\zeta(gm+l/2g)}$$

is the classical Θ-function. Recalling that $g^2 = 2N$ we thus can rewrite the partition function in the form (using the new variables)

$$Z_n(\beta,\mu) = e^{2\pi i\tau/24}\frac{1}{\eta(\tau)}\Theta_{2n,2N}(\tau,\zeta,0).$$

The factor $e^{2\pi i\tau/24}$ can be eliminated by changing the zero point of energy H by a constant $-\frac{1}{24}$ so that the vacuum has energy $-\frac{1}{24}$. This appears naturally if we regard the stress tensor not as a function of z but of the angle θ, $e^{i\theta} = z$. In the notation of Section 1B

$$T(\theta) = 2\pi\left(\frac{d\xi}{d\theta}\right)^2\Theta_+(\xi(z)) - \frac{c}{12}\{\xi,\theta\}$$

where c is the central charge (i.e. $c = 1$ in the present case), and $\{\xi,\theta\}$ is the Schwarz derivative

$$\{\xi,\theta\} = \frac{\xi'''}{\xi'} - \frac{3}{2}\left(\frac{\xi'''}{\xi'}\right)^2 = \frac{1}{2}.$$

Its mode expansion reads

$$T(\theta) = L_0 - \frac{c}{24} + \sum_{n\neq 0} L_n e^{-in\theta}.$$

The characters of $\mathcal{A}_N$ in the new convention

vacuum energy $= -\frac{1}{24}$

are denoted by $K_n(\tau,\zeta,N)$. The results of the above discussion can be summarized in

Proposition 6.3. *The characters of $\mathcal{A}_N$ are given by*

$$\begin{aligned} K_n(\tau,\zeta,N) &\equiv Tr_{\mathcal{H}_n}e^{2\pi i(\tau H+\zeta Q)} \\ &= \frac{1}{\eta(\tau)}\Theta_{2n,2N}(\tau,\zeta,0) \end{aligned} \tag{6.28}$$

where the trace runs over the irreducible representation space of $\mathcal{A}_N$ whose states have charge ϵ modulo g, $\epsilon = n/g$, $\frac{1}{2}g^2 = N$.

We note that the characters obey the periodicity condition

$$K_{n+2N} = K_n. \tag{6.29}$$

This property reflects the fact that the representation space $\mathcal{H}_n$ is determined by the representation of the center $\mathbf{Z}_{2N}$ of $\mathcal{A}_N$, i.e. by n modulo $2N$. It can also be verified from the explicit formula for K_n. We proceed to show

PROPOSITION 6.4. *The 2N characters K_n, $n = 1-N, \dots N$ (multiplied with $e^{2\pi i u}$) span a 2N-dimensional representation of the modular group $SL(2,\mathbf{Z})$, which acts on the variables (τ,ζ,u) in Eq. (6.26) according to*

$$(\tau,\zeta,u) \mapsto (\tau_g,\zeta_g,u_g)$$
$$\tau_g = \frac{a\tau+b}{c\tau+d}, \quad \zeta_g = \frac{\zeta}{c\tau+d}, \quad u_g = u - \frac{1}{2}\frac{c\zeta^2}{c\tau+d}, \tag{6.30}$$

g^{-1} is the 2×2 matrix with integer entries $a,b,c,d \in \mathbf{Z}, ad-bc=1$.

The group law $T_{g_1}T_{g_2} = T_{g_1 g_2}$ for the operators $T_g\Theta(\tau,\zeta,u) = \Theta(\tau_g,\zeta_g,u_g)$ can be verified (see Appendix C). It is then sufficient to study the transformation properties of K_n for the two generators of $SL(2,\mathbf{Z})$.

$$T:\ \tau\mapsto\tau+1, \qquad S:\ \tau\mapsto -\frac{1}{\tau}. \tag{6.31}$$

We have from the result (6.28) for K_n

$$K_n(\tau+1,\zeta;N) = \exp\left(i\pi\frac{n^2}{2N} - \frac{1}{12}\right)K_n(\tau,\zeta;N) \tag{6.32}$$

and, according to Appendix C

$$e^{-\pi i\zeta^2/\tau}K_n\left(-\frac{1}{\tau},\frac{\zeta}{\tau};N\right) = \sum_{p=1-N}^{N} S_{np}K_p(\tau,\zeta;N) \tag{6.33}$$

where

$$S_{np} = \frac{1}{\sqrt{2N}}e^{-inp}, \quad n,p = 1-N,\dots N \tag{6.34}$$

so that the $2N\times 2N$ matrix $S=(S_{np})$ satisfies

$$S^2 = C = S^{*2}, \qquad C^2 = 1. \tag{6.35}$$

Here C is the $\mathbf{Z}_{2N}$-charge conjugation matrix characterized by (cp. (C.9))

$$\sum_{p=1-N}^{N} C_{np}K_p(\tau,\zeta;N) = K_{-n}(\tau,\zeta;N). \tag{6.36}$$

6D. MODULAR INVARIANT PARTITION FUNCTIONS. LOCAL 2-DIMENSIONAL QUANTUM FIELD THEORY MODELS

Let $\mathcal{A}_N$ be a maximal algebra of local right moving observables, and let $\bar{\mathcal{A}}_N$ be its left moving counterpart. We shall identify the left movers' modular parameters with $-\bar\tau$ and $-\bar\zeta$ so that the corresponding characters will appear as complex conjugate to those of the right movers. We shall look for partition functions of 2-dimensional conformal models of the form

$$Z \equiv Z(\tau,\zeta;N) = \sum_{n,\bar n} m_{n,\bar n}K_n\bar K_{\bar n}. \tag{6.37}$$

An expression Z of this form will be the trace of $q^H y^Q \bar q^{\bar H}\bar y^{\bar Q}$ over a Hilbert space which is direct sum of tensor products $\mathcal{H}_n \otimes \mathcal{H}_{\bar n}$ of representation spaces for the left- and right movers' algebras $\mathcal{A}_N$ if the coefficients $m_{n,\bar n}$ are non-negative integers. ($q = e^{2\pi i\tau}$, $y = e^{2\pi i\zeta}$). The vacuum vector has charge 0 and appears only in $\mathcal{H}_0\otimes\mathcal{H}_{\bar 0}$. So it has multiplicity 1 if $m_{0,\bar 0} = 1$. We therefore seek solutions Z of this particular form which in addition are modular invariant.

CONDITION:

(a) the coefficients $m_{n,\bar n}$ are non-negative integers, and $m_{0,\bar 0} = 1$.

(b) the function Z is modular invariant.

The partition function carries information about the operator content of the 2-dimensional theory (cf. [3]). Its modular invariance guarantees the crossing symmetry of the correlation functions of all 2-dimensional primary fields (although the chiral fields of Section 4B with charges $\epsilon = n/g$ are not local among themselves). We note that the permutation group S_3, the crossing symmetry group of the 4-point function, appears as a factor group of the modular group (see [24]).

The results of Di Francesco, Saleur and Zuber [48] allow to establish the following

PROPOSITION 6.5. *Suppose that N is a product of distinct primes (so that $\mathcal{A}_N$ is maximal). Then the number of independent modular invariant partition functions is*

$$I(N) = \begin{cases} 2 & \text{for } N=1 \\ 2^k & \text{for } N = p_1\dots p_k\,. \end{cases} \tag{6.38}$$

In more detail, to each splitting $N = pp'$ of N into a product of two coprimes there is a pair of partition functions satisfying the conditions (a) and (b). They are given by

$$Z^{(n_0,\pm)} = \sum_{n=1-N}^{N} K_{n_0 n}\bar K_{\pm n} \tag{6.39}$$

where $n_0 = n_0(p,p')$ is a positive integer which is uniquely determined by the requirement that the following inequality is satisfied for a pair of integers (r_0,s_0) with $(r_0p' - s_0p)^2 = 1$

$$1 \le n_0 = r_0p' + s_0p \le pp'(=N). \tag{6.40}$$

REMARK: For $\zeta = 0$, the case considered in [48], $K_n = K_{-n}$, so that the two invariants coincide.

SKETCH OF PROOF: The existence of a pair of integers (r_0,s_0) satisfying $r_0p' - s_0p = \pm1$ is a consequence of the assumption that p and p' are coprimes. Such a pair is constructed by the Euclid algorithm and is determined up to adding to it (mp, mp'). The inequalities (6.40) for $n_0 = r_0p' + s_0p$ fix it uniquely. The modular invariance of the partition function (6.39) is a straightforward consequence of (6.32)-(6.34), (C.8) and of the identity

$$K_{n_0^2 n} = K_n \qquad \text{for } n_0^2 - 1 = 4r_0s_0N \tag{6.41}$$

which follows from the periodicity condition (6.29). The number of splittings of k objects into two sets of l and $k-l$ objects being 2^{k-1}, we obtain the number (6.38) of modular invariant partition functions. The proof that there are no other hermitean forms of type (6.37) satisfying (a) and (b) follows arguments of [49] and [50].

VII. OUTLOOK AND DISCUSSION

7A. On the construction of "quark fields" for non-abelian current algebras

The construction of fields from outer automorphisms can also be carried out in the case of non-abelian current algebras. We shall briefly discuss the procedure for the case of a Kac-Moody current algebra $d\hat{G}$, or its associated loop group $\tilde{L}G$, where G is a simple simply connected compact Lie group with a finite (nonempty) center $\mathcal{Z}$. The center $\mathcal{Z}$ of G acts trivially on the observables. (This can be taken as a defining property, we conjecture that it is automatic if the observable algebra $\mathcal{A}$ is defined as a maximal local extension of $d\hat{G}$ for a fixed level k.) It will therefore give rise to a superselection rule. In the vacuum sector $\mathcal{Z}$ is represented trivially. In analogy with Section 3 one should like to find localized outer automorphisms of $d\hat{G}$ which intertwine between different superselection sectors. Such automorphisms are known [25]. In loop group language they are given by conjugation with elements in the adjoint loop group $LG/\mathcal{Z}$ which correspond to non-contractible loops $\gamma : \mathbf{S}^1 \mapsto G/\mathcal{Z}$. This defines automorphisms, also denoted by γ, of the loop group LG which lift to its central extension $\tilde{L}G$.

An automorphism γ is called localized on I, if it acts trivially on loops $u : \mathbf{S}^1 \mapsto G$ with $u(z) = 1 \in G$ for $z \notin I$. One verifies that the above automorphisms are localized on I if $\gamma(z) = 1 \in G/\mathcal{Z}$ for $z \notin I$.

It is known that these automorphisms suffice to intertwine all the positive energy representations (superselection sectors) for simply laced G and level 1. The resulting field algebras appear in the mathematical literature ([25], Proposition 4.6.9).

The localized outer automorphisms are equivalent to constant outer automorphisms γ_g of $\tilde{L}G$ (the counterpart of the automorphisms of $\hat{U}$ with constant charge distribution). They are labeled by elements $g \in \mathcal{Z}$ (because the fundamental group of $G/\mathcal{Z}$ is $\mathcal{Z}$) and form a representation of $\mathcal{Z}$. Following the general construction principle of Section 1A, one associates unitary charge shift operators Γ_g with these constant automorphisms. They change the representation of the center of $\mathcal{A}$. One then constructs a field algebra $\mathcal{F}$ as in the abelian case. Its unitary elements are

$$\psi = \eta W \Gamma_g, \qquad W \in \tilde{L}G$$

and the multiplication law is given by

$$\Gamma_{g_1}\Gamma_{g_2} = \Gamma_{g_1+g_2}, \qquad \Gamma_0 = 1$$
$$W\Gamma_g = \Gamma_g \gamma_g(W)$$

Here η can be complex phase factors, and $+$ is group multiplication in $\mathcal{Z}$.

There are, in general, several "natural choices" of γ_g and the associated intertwining map Γ_g between the vacuum sector and a sector of "charge" g. They correspond, typically, to the different components of a field which transforms covariantly under a "fundamental representation" of G (in which the center of G is

represented nontrivially). We shall illustrate the situation with the example of the level 1 representation of $\widehat{su(2)}$ which already appeared as the local extension $\mathcal{A}_1$ of the $\hat{U}(1)$-current algebra.

According to the discussion in Section 5B the $\widehat{su(2)}$ current algebra is reproduced from the algebra $\mathcal{A}_1$ which is characterized by the fact that the minimal charge g carried by a field is given by $\frac{1}{2}g^2 = N = 1$. The charged currents are quantum fields (fields at a point) with components

$$\psi_{\pm\sqrt{(2)}}(z) = \sum \tilde{\psi}_{n,\pm} z^{-n-1} \tag{7.1}$$

Their Fourier transforms satisfy the commutation relations

$$\begin{aligned} [J_n, \tilde{\psi}_{m,\pm}] &= \pm\sqrt{2}\,\tilde{\psi}_{m+n,\pm}, \\ [\tilde{\psi}_{n,+}, \tilde{\psi}_{m,-}] &= \sqrt{2}\,J_{n+m} + n\,\delta_{n+m,0}\,. \end{aligned} \tag{7.2}$$

The charge shift operators

$$\Gamma_\pm \equiv \Gamma_{\pm\frac{1}{2}\sqrt{2}}, \qquad (\Gamma_- = \Gamma_+^* = \Gamma_+^{-1}) \tag{7.3}$$

are used to construct charged fields of charge $\frac{1}{2}\sqrt{2}$. They induce the automorphisms

$$\begin{aligned} J_n &\mapsto \Gamma_-^* J_n \Gamma_- = J_n - \frac{1}{\sqrt{2}}\delta_{n0} \\ \psi_{n,\pm} &\mapsto \Gamma_-^* \psi_{n,\pm} \Gamma_- = \psi_{n\mp1,\pm} \end{aligned} \tag{7.4}$$

and similarly for Γ_+. It is a simple exercise to verify that the map (7.4) does respect the current commutation relations (7.2). The pair of fields $\psi_{\pm\frac{1}{2}\sqrt{2}}(z)$ span the defining representation of $SU(2)$.

We hope to return to a more systematic study of the conformal QFT models based on a non-abelian current algebra in a separate publication. For simply laced Lie groups G and level 1 a straightforward generalization of the approach of this paper is possible, because a stress tensor can be constructed from currents associated with the maximal torus T of G. Therefore the non-abelian current algebra $d\hat{G}$ is a local extension of the abelian current algebra $d\hat{T}$ in the terminology of this paper.

7B. Virasoro algebra as a germ of a local field theory. Concluding remarks

The results of this paper (summed up in Section 1C) suggest the following more general program.

Consider the minimal algebra of right movers' observables in a 2-dimensional conformal QFT, the Virasoro algebra Vir of the stress energy tensor (or, rather, the algebra generated by exponentials of the smeared field $T(f)$ for real f). The problem is to find all (maximal) local extensions of Vir, for a given central charge c, and their QFT-representations, i.e. their positive energy representations and fields that intertwine them. We know of one superselected "charge" for all such theories, the unitary operator $\exp(2\pi i L_0)$ which is in the center of the algebra and generates the univalence automorphism $\alpha_{2\pi}$. According to our Theorem 1 one should therefore be able to find localized morphisms of Vir which change the eigenvalues of this univalence generator.

One moral of the present investigation is the uncovering of the role of the maximal local extension of the given "germ" of the (chiral) algebra of observables. Only by studying the extended algebras $\mathcal{A}_N$ we came to a charge quantization, and, as a result, to a class of *rational conformal quantum field theories*. cf. [45].

It may be desirable to generalize the notion of local extension to include algebras generated by locally anticommuting Fermi fields. The separate introduction of such algebras was not needed when the germ of the algebra of observables was taken to be the chiral current algebra $\hat{U}$, since the canonical Fermi fields of charge g_1 with g_1^2 odd are contained in the QFT-representation of the bosonic algebra $\mathcal{A}_{\frac{1}{2}g^2}$ for $g = 2g_1$ (see Section 4B). If Vir is substituted for $\hat{U}$, it may be necessary or convenient to introduce fermionic local extensions in their own right. This is suggested by a study of the critical Ising model [51] (corresponding to $c = \frac{1}{2}$) whose right moving part is most naturally constructed as a QFT-representation of the algebra of canonical anticommutation relations; both double and single-valued representations are admitted. It was further noted in [51] that in all unitary *"minimal theories"* [14,51,42] with Virasoro central charge

$$c_m = 1 - \frac{6}{(m+2)(m+3)}, \quad m = 1, 2, \ldots$$

there exists a (unique) primary field ψ_m of canonical (integer or half-integer) dimension (or spin)

$$s_m = \frac{1}{4} m(m+1).$$

The algebra generated by ψ_m (which includes Vir) seems to be a good candidate for such a local field algebra whose single- and double-valued QFT-representa- tions give rise to the c_m-models (classified in [49]). We hope to come back to these problems in a later paper.

APPENDIX A. GROUND STATES AND KMS-STATES OF $\hat{U}$

In this appendix we want to determine all regular ground states and KMS-states on the current algebra $\hat{U}$. We will outline the calculation for the KMS-states at inverse temperature β. The form of the ground states can be determined by the same method, or simply by going to the limit $\beta \to \infty$ in the subsequent expressions for the KMS-states.

We recall that a linear functional ω on the Weyl algebra $\hat{U}$ is a state if it is normalized, i.e. $\omega(\mathbf{1}) = 1$, and if it satisfies the positivity condition

$$\sum_{i,j=1}^{n} \bar{c}_i c_j \, \omega(W(u_i)^* W(u_j)) \geq 0 \tag{A.1}$$

for any finite sequence of functions $u_1, \ldots u_n \in \mathcal{S}$ and coefficients $c_1, \ldots c_n \in \mathbf{C}$. Note that in view of the composition law (2.6) any state ω on $\hat{U}$ is completely fixed by specifying all expectation values $\omega(W(u)), u \in \mathcal{S}$. A state is called *quasifree* if

$$\omega(W(u)) = e^{ig(u) - \frac{1}{2}||u||^2}, \quad u \in \mathcal{S} \tag{A.2}$$

where $g(.)$ is a real linear functional and $||.||$ a suitable Hilbert seminorm on the *real* space $\mathcal{S}$. From the positivity condition (A.2) one can deduce that $||.||$ must satisfy the inequality [53]

$$||u||^2 + ||v||^2 + iA(u,v) \geq 0 \tag{A.3}$$

for arbitrary $u, v \in \mathcal{S}$. Conversely, if this inequality is satisfied, then the functional ω in (A.2) extends to a state on $\hat{U}$. We recall that the n-point correlation functions of the current J in a quasifree state can easily be recovered from the generating functional $\omega(W(u))$: they are sums and products of the (truncated) 1-point and 2-point functions given by

$$\begin{aligned} &< J[u+iv] >_\omega = g(u) + ig(v) \\ &< J[u-iv]^* J[u+iv] >_\omega = ||u||^2 + ||v||^2 + iA(u,v). \end{aligned} \tag{A.4}$$

Hence condition (A.3) amounts to the familiar positivity condition for the correlation functions. The relevance of the notion of quasifree states for the present investigation comes from the fact that all KMS- and ground states on $\hat{U}$ are convex combinations of quasifree states.

In our calculation of KMS-states on $\hat{U}$ we follow the argumentation in [54].

Let ω_β be any regular KMS state on $\hat{U}$ at inverse temperature $\beta > 0$ and consider the function

$$f(t) = \omega_\beta(W(v)\alpha_t(W(u))), \quad t \in \mathbf{R}. \tag{A.5}$$

According to the KMS-condition, f can be analytically continued into the strip $S_\beta = \{z : 0 \leq Imz \leq \beta\}$ and it is continuous at the boundary. Moreover, f satisfies the boundary condition

$$f(t+i\beta) = \omega(\alpha_t(W(u))W(v)), \quad t \in \mathbf{R}. \tag{A.6}$$

Making use of the fact that according to the composition law (2.6)

$$\alpha_t(W(u))W(v) = e^{-A(u_t,v)} W(v)\alpha_t(W(u)) \tag{A.7}$$

we thus obtain for f the equation

$$f(t+i\beta) = f(t)\exp\left(i\oint \frac{dz}{2\pi} u_t'(z)v(z)\right). \tag{A.8}$$

Note that if there exists a solution f_0 of this equation which is analytic in the interior of S_β, continuous at the boundary, and nowhere zero, i.e. $\inf\{|f_0(z)| : z \in S_\beta\} > 0$, then $g(z) = f(z)/f_0(z)$ is analytic and bounded on S_β and satifies $g(t+i\beta) = g(t)$. So g must be a constant function by Liouvilles theorem, i.e. the solution of (A.8) is unique up to a constant factor.

By taking the logarithm on both sides of Eq. (A.8) and performing a Fourier transformation it is easy to exhibit a function f_0 with the required properties:

$$f_0(z) = \exp\left(-\sum_{n\neq 0} n \frac{e^{-inz}}{e^{n\beta}-1}\tilde{v}_n^*\tilde{u}_n\right) \quad z \in S_\beta. \tag{A.9}$$

It thus follows after an elementary computation that

$$\begin{aligned}\omega_\beta(W(u_t+v)) &= \omega_\beta(W(u_t)W(f))e^{\frac{1}{2}A(u_t,v)} \\ &= C(u,v)e^{-\frac{1}{2}||u_t+v||_\beta^2}\end{aligned} \tag{A.10}$$

where $C(u,v)$ does not depend on t, and we have introduced the seminorm on S

$$||u||_\beta^2 = \sum_{n=1}^{\infty} n\coth(\frac{1}{2}n\beta)|\tilde{u}_n|^2. \tag{A.11}$$

It remains to determine the form of the factor $C(u,v)$ in (A.10). From the first and the third member in (A.10) we see that $C(u,v)$ can only depend on u_t+v, but it must not depend on t. Hence putting $v = -u+g\mathbf{1}$, $g \in \mathbf{R}$ and $\mathbf{1}(z) = 1$ we obtain

$$\omega_\beta(W(u_t - u + g\mathbf{1})) = C'(g)e^{-\frac{1}{2}||u_t-u||_\beta^2} \tag{A.12}$$

where $C'(g)$ is independent of u. Setting in (A.12) $u=0$ we obtain

$$C'(g) = \omega_\beta(W(g\mathbf{1})) = \int d\sigma(g')e^{igg'}. \tag{A.13}$$

The existence of the normalized measure σ in the second inequality follows from Bochners theorem, since ω_β is a regular state for the unitary representation $g \mapsto W(g\mathbf{1})$ of $\mathbf{R}$. Noticing that the set of functions $u_t - u$, $u \in S$ exhausts the set of all functions $u \in S$ whose Fourier component $\tilde{u}_0$ vanishes, we have thus established by relation (A.12) and (A.13) the form of the functional ω_β given in Lemma 6.1. Proceeding to the limit $\beta \to \infty$ we also obtain the form of the ground states on $\hat{U}$ as given in Lemma 2.2.

If ω_β is a primary state[1] then

$$\omega_\beta(W(u)) = e^{ig\tilde{u}_0 - \frac{1}{2}||u||_\beta^2} \tag{A.14}$$

[1] A primary state on $\hat{U}$ is a state which leads to a factorial representation of $\hat{U}$ in which the center consists of multiples of the identity.

for some "charge" $g \in \mathbf{R}$, i.e. the measure σ in relation (A.13) is concentrated at a point. Since the seminorm $||.||_\beta$ satisfies the condition (A.3) we see that Eq. (A.14) defines a quasifree KMS-state for any $g \in \mathbf{R}$. But convex combinations of KMS-states of the same temperature are again KMS-states, hence any normalized measure σ in (A.13) gives rise to a particular KMS-state on $\hat{U}$.

If one goes in relation (A.14) to the limit $\beta \to \infty$ one obtains a ground state ω of charge g which is a pure state (giving rise to an irreducible representation of $\hat{U}$). This follows, as in the standard framework of quantum field theory [15], from the fact that the state ω is weakly clustering, i.e.

$$\lim_{T\mapsto\infty}\frac{1}{2T}\int_{-T}^{T} dt\,\omega(W(u)\alpha_t(W(v))) = \omega(W(u))\omega(W(v)), \tag{A.15}$$

as can be shown by explicit computation.

We conclude this appendix with some remarks concerning the relation between the representations induced by KMS-states for different temperatures and charge spectra. It is clear from the above discussion that two KMS-states will lead to disjoint representations of $\hat{U}$ if the corresponding measures σ in (A.13) (giving the central decomposition) are disjoint. For example the quasifree states in relation (A.14) give rise to disjoint representations for different values of the charge g. On the other hand, if for two KMS-states the measures σ coincide (or, more generally, belong to the same measure class) then one can show that the corresponding representations of $\hat{U}$ are quasi-equivalent (equivalent up to multiplicity). This result remains true also if these states have different temperatures, or if one of them is a ground state. (A proof of this assertion can be obtained e.g. by applying the general result in [55].) So, roughly speaking, one can distinguish the (quasi-) equivalence classes of representations of $\hat{U}$ by the charge spectrum. As expected for theories living on a compact space, the temperature is not a superselection rule.

From the latter fact it follows for example that the (primary) KMS-states of charge g can be represented in the form

$$\omega_\beta(W) = \frac{1}{Z}Tr\, e^{-\beta H}\pi(W),$$

where the trace is to be taken over the representation space of the (irreducible) representation $(\pi, \mathcal{H})$ induced by the ground state of charge g, and H is the Hamiltonian in that representation. Hence one can rediscover the well known fact that $e^{-\beta H}$ is a trace class operator in all lowest weight representations of $\hat{U}$ by an analysis of the relations between the various KMS-states, which we have computed here by making use of the KMS-condition.

APPENDIX B. Solution of a cocycle equation

We give here the proof of Lemma 3.2, i.e. we determine all continuous solutions of the equation

$$X(\rho_1)\gamma_{-g_1}(X(\rho_2)) = \xi(\rho_1,\rho_2)X(\rho_1+\rho_2) \tag{B.1}$$

where ξ is some phase factor and $X(\rho)$ are unitary operators in the center of $\hat{U}''$.

We begin by noting that $\zeta(\rho) = \omega_0(X(\rho))$ is a phase factor (since ω_0 is a pure state), so by multiplying $X(\rho)$ with $\zeta(\rho)^{-1}$ we may assume without loss of generality that $\omega_0(X(\rho)) = 1$ for all ρ. Let us now consider Eq. (B.1) for the cases where $\rho_1[1] = g_1 = 0$, i.e.

$$X(\rho_0)X(\rho) = \xi(\rho_0,\rho)X(\rho+\rho_0), \qquad \rho_0[1] = 0. \tag{B.2}$$

Taking matrix elements of this equation in the state ω_0 we thus see that

$$\xi(\rho_0,\rho) = 1 \qquad \text{if } \rho_0[1] = 0. \tag{B.3}$$

Moreover, setting $Y(g) = X(\tilde{\rho})$, where $\tilde{\rho}(z) = g/z, g = \rho[1]$ and $\Delta\rho = \rho - \tilde{\rho}$ we obtain from (B.2) and (B.3), since $\Delta\rho[1] = 0$

$$X(\rho) = X(\Delta\rho)Y(g). \tag{B.4}$$

Plugging this into (B.1) we get

$$Y(g_1)\gamma_{-g_1}(Y(g_2))Y(g_1+g_2)^{-1} = \xi(\rho_1,\rho_2)X(\Delta\rho_1)^{-1}\gamma_{-g_1}(X(\Delta\rho_2))^{-1}X(\Delta\rho_1+\Delta\rho_2)$$

and making use of the fact that $X(\Delta\rho_1+\Delta\rho_2) = X(\Delta\rho_1)X(\Delta\rho_2)$ which also follows from (B.2) and (B.3) we arrive at

$$Y(g_1)\gamma_{-g_1}(Y(g_2))Y(g_1+g_2)^{-1} = \xi(\rho_1,\rho_2)X(\Delta\rho_2)\gamma_{-g_1}(X(\Delta\rho_2))^{-1}. \tag{B.5}$$

The left hand side of this equation does not depend on the particular choice of ρ_1,ρ_2 (for fixed g_1,g_2). Hence setting $\rho_1 = \tilde{\rho}_1, \rho_2 = \tilde{\rho}_2$ and taking into account that $X(0) = 1$ we see that for $g_1,g_2 \in \mathbf{R}$

$$Y(g_1)\gamma_{-g_1}(Y(g_2))Y(g_1+g_2)^{-1} = \eta(g_1,g_2)\mathbf{1} \tag{B.6}$$

where $\eta(.,.)$ is a phase factor. It is then clear that $\eta(g_1,g_2)$ satifies the 2-cocycle equation

$$\eta(g_1,g_2)\eta(g_1+g_2,g_3) = \eta(g_1,g_2+g_3)\eta(g_2,g_3). \tag{B.7}$$

As is well known, all (continuous) solutions of this equation are of the form

$$\eta(g_1,g_2) = \zeta(g_1)\zeta(g_2)\zeta(g_1+g_2)^{-1}$$

where $\zeta(.)$ is a phase factor. Hence multiplying $Y(g)$ with $\zeta(g)^{-1}$ we may assume without restriction of generality that $\eta(g_1,g_2) = 1$, i.e. that $Y(.)$ satisfies the equation given in the statement of the lemma. We are thus left with the equation

$$\gamma_{-g}(X(\Delta\rho)) = \sigma(g,\Delta\rho)X(\Delta\rho) \quad \text{if } \Delta\rho[1] = 0, \tag{B.8}$$

where $\sigma(.,.)$ is again some phase factor.

From the composition law $\gamma_{g_1}\gamma_{g_2} = \gamma_{g_1+g_2}$ and from the fact that $X(\Delta\rho)X(\Delta\rho') = X(\Delta\rho+\Delta\rho')$ it follows that $\sigma(.,.)$ must satisfy

$$\begin{aligned}\sigma(g_1,\Delta\rho)\sigma(g_2,\Delta\rho) &= \sigma(g_1+g_2,\Delta\rho)\\ \sigma(g,\Delta\rho)\sigma(g,\Delta\rho') &= \sigma(g,\Delta\rho+\Delta\rho').\end{aligned} \tag{B.9}$$

Bearing in mind the anticipated continuity properties of $X(\rho)$ it is then clear that

$$\sigma(g,\Delta\rho) = e^{-igl(\Delta\rho)}. \tag{B.10}$$

where $l(.)$ is a real functional. An obvious special solution of the resulting equation for $X(.)$, viz.

$$\gamma_{-g}(X(\Delta\rho)) = e^{-igl(\Delta\rho)}X(\Delta\rho) \tag{B.11}$$

is $X_0(\Delta\rho) = e^{il(\Delta\rho)Q}$. Expressing the general solution in the form $X(\Delta\rho) = X_1(\Delta\rho)X_0(\Delta\rho)$ we see that $X_1(\Delta\rho)$ must be a fixed point under the action of $\gamma_g, g \in \mathbf{R}$. But apart from multiples of the identity there is no such fixed point (the automorphisms $\gamma_g, g \in \mathbf{R}$ act ergodically on the center of $\hat{U}''$), hence $X_1(\Delta\rho)$ must be a multiple of the identity. So we conclude that $X(\rho) = X(\Delta\rho)Y(g)$ has the specific form given in the statement, and plugging this information into (B.1) we also get the structure of the phase factor $\xi(.,.)$ in this lemma. *q.e.d.*

APPENDIX C. MODULAR PROPERTIES OF Θ-FUNCTIONS AND CHARACTERS K_n.

First we verify that Eq. (6.30) does define a group law. The only non-obvious point is checking the superposition rule for the third variable, which follows from the equation

$$\frac{c_2}{c_2\tau+d_2} + \frac{c_1}{(c_2\tau+d_2)[(c_1a_2+d_1c_2)\tau+c_1b_2+d_1d_2]} = \frac{c_1a_2+d_1c_2}{(c_1a_2+d_1c_2)\tau+c_1b_2+d_1d_2}$$

valid for $a_2d_2 - c_2b_2 = 1$.

The basic tool in deriving the transformation law (6.33) for the A_N-characters under modular inversion is the Poisson formula

$$\sum_m f(m) = \sum_m \hat{f}(m) \text{ for } \hat{f}(y) = \int_{-\infty}^{\infty} e^{-2\pi ixy}f(x)dx. \tag{C.1}$$

It is valid whenever both f and its Fourier transform $\hat{f}$ are absolutely integrable.

It is applied to $(Im\,\tau > 0)$

$$\begin{aligned}f(x) &= \exp\left(\pi i\tau(xg+\frac{n}{g})^2 + 2\pi i\zeta(xg+\frac{n}{g})\right)\\ \hat{f}(m) &= \frac{1}{g\sqrt{-i\tau}}\exp\left(i\pi[\frac{2nm}{g^2} - \frac{1}{\tau}(\zeta-\frac{n}{g})^2]\right)\end{aligned} \tag{C.2}$$

with the result

$$g\sqrt{-i\tau}\,\Theta_{2n,2N}(\tau,\zeta,0) = \sum_{p=1-N}^{N} e^{i\pi np/N}\Theta_{2p,2N}(-\frac{1}{\tau},\frac{\zeta}{\tau},-\frac{1}{2}\frac{\zeta}{\tau}). \tag{C.3}$$

We have used the identity

$$\sum_m \hat{f}(m) = \sum_{p=1-N}^{N}\sum_m \hat{f}(2Nm+p)$$

in which the sum over p can be replaced by a sum over any other interval of length $2N$. Using the identity

$$\eta(-\tau^{-1}) = (-i\tau)^{\frac{1}{2}}\eta(\tau) \tag{C.4}$$

which also follows from the Poisson formula applied to the Θ-series expansion

$$\eta(\tau) = \sum_m (-1)^m e^{2\pi i\tau\frac{3}{2}(n+\frac{1}{6})^2} \tag{C.5}$$

of the Dedekind η-function, we find

$$K_n(\tau,\zeta;N) = e^{-\zeta^2/2\tau} \sum_{p=1-N}^{N} S^*_{np} K_p(-\frac{1}{\tau},\frac{\zeta}{\tau};N) \tag{C.6}$$

where

$$S^*_{np} = \frac{1}{\sqrt{2N}} e^{i\pi np/N}. \tag{C.7.}$$

This proves Eqs. (6.33), (6,36) since

$$\frac{1}{2N}\sum_{p=1-N}^{N} e^{i\pi(n-k)p/N} = \delta_{kn}. \tag{C.8}$$

The property (6.35), (6.36) of S is then obtained by a repeated application of (6.33), which gives for $C = S^2$

$$\sum_{p=1-N}^{N} C_{np}K_p(\tau,\zeta;N) = K_n(\tau,-\zeta;N) = K_{-n}(\tau,\zeta;N). \tag{C.9}$$

We note that the representations of $SL(2;\mathbf{Z})$ thus constructed are never faithful. For any given N the infinite subgroup

$$\mathbf{\Gamma}_{24N} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbf{Z});\quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} mod(24N) \right\} \tag{C.10}$$

of $SL(2,\mathbf{Z})$ is represented trivially.

Indeed, $\Theta_{l,g^2}(\tau,\zeta,u)$ are modular forms of weight $\frac{1}{2}$ with respect to the subgroup $\mathbf{\Gamma}_{24g^2}$ for any positive integer g^2. It is a deep result of Serre and Stark [31] that these Θ-functions actually span the space of all modular forms of weight $\frac{1}{2}$. For related work see [56,57,58].

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