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I. Montvay
Deutsches Elektronen-Synchrotron DESY, Hamburg

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Bosonization and a model with Yukawa-coupling*

I. Montvay
Deutsches Elektronen-Synchrotron DESY, D-2000 Hamburg

November 23, 1988

Abstract

The bosonization method for dynamical fermions in 3−1 dimensions and its tests with free fermions are briefly summarized. First numerical results obtained by this method in a simple interacting model with naive fermions and a single component real scalar field are described. Signals of a strong symmetry breaking phase transition are observed on 4^4 and 6^4 lattices. In this model, and probably also in other scalar-fermion models, the linear bosonization algorithm works fine and gives interesting non-perturbative information.

1 Introduction

A well known difficulty in numerical simulations of quantum field theories is that the algorithms for dynamical fermions are slow, in fact very much slower than for purely bosonic fields. The main reason is that the simulation is done in the effective bosonic theory, obtained by integrating out the fermionic variables in the path integral representation. The resulting fermion determinant makes the effective bosonic action non-local and the numerical simulation slow. The use of the Jordan-Wigner representation of the fermion algebra can, in principle, improve the situation. In 1+1 dimensions it is possible to construct local algorithms on the basis of this representation. These algorithms [2] use the low dimensionality of the problem in an essential way, and the extension to higher dimensions seemed very difficult due to the appearance of oscillating phase factors. However, as it was shown recently [3,4], on a strongly asymmetric lattice with smaller lattice spacing in the imaginary time direction the sign oscillations are suppressed and an efficient fermion Monte Carlo simulation is possible. Tests of the method were performed [3,4] in 2−1 and 3−1 dimensions with free staggered [5] and Wilson [6] fermions. The method and these tests will be briefly summarized in Section 2. In Ref. [4] the application to a simple interacting scalar-fermion model with Yukawa-coupling was also described. First results of this simulation will be discussed in Section 3.

The interest in simulating scalar-fermion quantum field theories with Yukawa-couplings is at least twofold:

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- To study the non-perturbative effects of strong Yukawa-couplings on the Higgs sector of the standard model;
- To investigate the question of the non-perturbative formulation of chiral fermion gauge theories.

The first problem involves the question of non-perturbative upper limits for scalar- and fermion-masses in these models. The second problem is particularly interesting in connection with the possible physical existence of mirror fermions at the electroweak scale. (For a review of these non-perturbative aspects of the standard electroweak model see [7].) In general, one can say that our present knowledge about the non-perturbative properties of scalar-fermion models is insufficient. The first exploratory investigations including also numerical simulations were performed only recently [8]-[13].

2 The bosonization method for dynamical fermions

The creation and annihilation operators in a fermionic Hamiltonian obey the anticommutation relations ($m, n = 1, 2, \ldots, N$):

$$\{c_m^\dagger, c_n\} = \delta_{m,n} \qquad \{c_m, c_n\} = \{c_m^\dagger, c_n^\dagger\} = 0 \tag{1}$$

These define the *fermion algebra*, which can be realized in a computer by the Jordan-Wigner matrix representation

$$c_n = \sigma_3^{(1)}\sigma_3^{(2)}\cdots\sigma_3^{(n-1)}\sigma_-^{(n)} \qquad c_n^\dagger = \sigma_3^{(1)}\sigma_3^{(2)}\cdots\sigma_3^{(n-1)}\sigma_+^{(n)} \tag{2}$$

Here $\sigma_j^{(n)}$: $n = 1, \ldots, N$; $j = 1, 2, 3$ denote Pauli-matrices which commute for different n:

$$\left[\sigma_j^{(m)}, \sigma_k^{(n)}\right] = 2i\delta_{m,n}\epsilon_{jkl}\sigma_l^{(m)} \tag{3}$$

and, as usual

$$\sigma_\pm \equiv \frac{1}{2}(\sigma_1 \pm i\sigma_2) \tag{4}$$

For a quadratic Hamiltonian one needs the relation

$$c_m^\dagger c_n = \sigma_+^{(m)}\sigma_-^{(n)}\rho(m,n) \tag{5}$$

where the *phase factor* $\rho(m,n) = \rho(n,m)$ is given by

$$\rho(m,m) = 1 \qquad \rho(m,m+1) = -1 \qquad \rho(m,m+k) = -\sigma_3^{(m+1)}\sigma_3^{(m+2)}\cdots\sigma_3^{(m+k-1)} \tag{6}$$

The basis vectors of this representation $|\{\xi\}\rangle$ can be labelled by $\{\xi\} \equiv \{\xi(i);\ i = 1, \ldots, N\}$, where $\xi(i) = \pm 1$ denotes the eigenvalue of $\sigma_3^{(i)}$. The occupation number of the state with index i is given by $(\xi(i) - 1)/2$. As it can be seen from Eq. (6), the phase factor $\rho(m,n)$ depends on the occupation number of the states between m and n.

Let us now consider a relativistic Dirac-field $\psi_{\alpha x}$, where $\alpha = 1, 2, 3, 4$ is the Dirac index and x the lattice point in 3-dimensional space. The Hamilton operator for free Wilson-fermions with mass m is:

$$H = \sum_x \psi_x^\dagger m\beta\psi_x - \frac{r}{2}\sum_{x,k}\left\{\psi_{x-\hat{k}}^\dagger\beta\psi_x + \psi_x^\dagger\beta\psi_{x+\hat{k}}\right\} - \frac{i}{2}\sum_{x,k}\left\{\psi_{x+\hat{k}}^\dagger\alpha_k\psi_x - \psi_x^\dagger\alpha_k\psi_{x+\hat{k}}\right\} \tag{7}$$

The summation $\sum_k$ goes over the three orthogonal directions. $\hat{k}$ is the unit vector in direction k and in a $2\otimes 2$ block notation the Dirac-matrices β, α_k are

$$\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \alpha_k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix} \tag{8}$$

In the continuum limit the second term in the Hamiltonian, which is proportional to the Wilson parameter $0 < r \leq 1$, removes the additional lattice fermion species from the spectrum. $r = 0$ corresponds to "naive" lattice fermions. Using Eq. (5) one obtains the Hamiltonian in the Jordan-Wigner representation. For instance, in the case of naive lattice fermions we have

$$H = \sum_{\alpha,x} m\lambda_\alpha \sigma_-^{(\alpha x)}\sigma_-^{(\alpha x)} - \frac{1}{2}\sum_{\alpha,x,k}\left\{\lambda(\alpha,k)\sigma_+^{(\gamma(\alpha,k)\ x+\hat{k})}\sigma_-^{(\alpha x)}\rho(\gamma(\alpha,k)\ x+\hat{k},\alpha x) + h.c.\right\} \tag{9}$$

where $\lambda_1 = \lambda_2 = -\lambda_3 = -\lambda_4 = 1$ and $\lambda(\alpha,k)$ and $\gamma(\alpha,k)$ are given in a matrix form by

$$\lambda(\alpha,k) = \begin{pmatrix} i & -1 & +i \\ i & +1 & -i \\ i & -1 & +i \\ i & -1 & -i \end{pmatrix} \qquad \gamma(\alpha,k) = \begin{pmatrix} 4 & 4 & 3 \\ 3 & 3 & 4 \\ 2 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} \tag{10}$$

The fermion variables can be represented in the computer by single bits. The order of these variables is relevant due to the sign factor ρ appearing in Eq. (5). In the case of Wilson-fermions it is reasonable to store the four Dirac-components belonging to a lattice site consecutively. The sites can be stored in the usual lexicographic order. Arbitrary boundary conditions are possible but, of course, on the boundaries the phase factor $\rho(\ldots x+\hat{k},\ \ldots x)$ is sometimes given by a longer product than inside the lattice. These phase factors have to be often computed during the simulation, therefore it is advantageous to keep them short. In the way just described the average work per site needed to compute them everywhere on an L^3 lattice is growing with L^2. Since, however, the phase factor ρ can be obtained by counting the number of up bits between two bit addresses, the computation can be done very quickly.

The aim of the numerical simulation is to compute traces like

$$Z = Tr\ e^{-\beta H} \tag{11}$$

with $\beta \equiv T^{-1}$ the inverse temperature (or "imaginary time"). This is done on $L^3 \cdot L_0$ asymmetric lattices, where $\beta = a_0 L_0$. The lattice is much finer in the time direction, therefore the ratio of the lattice spacings a_0 is much smaller than 1. For small a_0 one can use a linear (or some other low order polynomial) approximation for the exponential $e^{-a_0 H}$, as it was first proposed by Duncan in Ref. [3]. The discussion here will follow Ref. [4], where some improvements of this method were suggested. The calculation of the trace in Eq. (11) starts by the replacement

$$Tr\ e^{-L_0 a_0 H} \to Tr\ \left\{e^{-a_0 H_D}(1 - a_0 H_R)\right\}^{L_0} \tag{12}$$

Here $H = H_D + H_R$ is a decomposition of the Hamiltonian into a diagonal piece H_D and the rest H_R containing the off-diagonal terms in the fermionic variables. This form allows the combination of the fermionic algorithm with the usual Monte Carlo integration procedure for the bosonic variables. Namely, in the case of interacting theories the purely bosonic piece of the Hamiltonian can be added to H_D. Another advantage of Eq. (12) is that it is insensitive to

a constant shift in the Hamiltonian, because the constant is contributing to H_D and therefore drops out from the relative probability defining the heatbath procedure (see below).

In the specific case of free Wilson-fermions the diagonal part is given by

$$H_D|\{\xi\}\rangle = m\Lambda\{\xi\}\,|\{\xi\}\rangle \tag{13}$$

where now $\{\xi\} \equiv \{\xi(\alpha,x);\ \alpha = 1,2,3,4;\ x = 1,\ldots,L^3\}$ and

$$\Lambda\{\xi\} \equiv \sum_{\alpha,x} \lambda_\alpha \delta_{1,\xi(\alpha,x)} \tag{14}$$

In this notation, after inserting in Eq. (12) complete set of states L_0-times, we have

$$Z = \sum_{\{\xi\}_1 \cdots \{\xi\}_{L_0}} \prod_{t=1}^{L_0} \left\{ e^{-a_0 m \Lambda\{\xi\}_t} \langle\{\xi\}_t|1 - a_0 H_R|\{\xi\}_{t+1}\rangle \right\} \quad - \cdots \tag{15}$$

where the dots are small corrections due to the replacement in (12) and, by definition, $\{\xi\}_{L_0+1} \equiv \{\xi\}_1$. This corresponds to periodic boundary conditions in the time direction.

The non-zero contributions in the sum (15) are given by such sequences $\{\xi\}_1 \ldots \{\xi\}_{L_0}$ that the consecutive states $\{\xi\}_t$ and $\{\xi\}_{t-1}$ differ at most by a flip of a single "active fermion link". A *fermion link* is a pair of fermion variables connected by a non-zero off-diagonal matrix element of H_R. A fermion link is active if the fermion variable pair is unequal. The sum in Eq. (15) is performed by sampling with a heatbath Monte Carlo. The probability of a configuration is taken to be proportional to

$$P(\{\xi\}_1 \ldots \{\xi\}_{L_0}) = \prod_{t=1}^{L_0} e^{-a_0 m \Lambda\{\xi\}_t} \left| \langle\{\xi\}_t|1 - a_0 H_R|\{\xi\}_{t+1}\rangle \right| \tag{16}$$

A heatbath step consists of keeping the states $|\{\xi\}_{t-1}\rangle$ and $\{\xi\}_{t+1}$ fixed and choosing $|\{\xi\}_t\rangle$ with relative probability

$$W(\{\xi\}_t) = e^{-a_0 m \Lambda\{\xi\}_t} \left| \langle\{\xi\}_{t-1}|1 - a_0 H_R|\{\xi\}_t\rangle \langle\{\xi\}_t|1 - a_0 H_R\,|\{\xi\}_{t+1}\rangle \right| \tag{17}$$

Since the probability P is defined in Eq. (16) by the absolute value of matrix elements, the partition function in (15) is given by

$$Z = \left\langle \prod_{t=1}^{L_0} \frac{\langle\{\xi\}_t|1 - a_0 H_R|\{\xi\}_{t-1}\rangle}{\left|\langle\{\xi\}_t|1 - a_0 H_R|\{\xi\}_{t+1}\rangle\right|} \right\rangle_P \tag{18}$$

Here $\langle\cdots\rangle_P$ means an expectation value in the Monte Carlo process defined by the probability P. The expectation value of some operator can also be obtained similarly. As a simple example, the expectation value of $\bar{\psi}_x\psi_x$ is

$$\left\langle \bar{\psi}_x\psi_x \right\rangle = Z^{-1} \left\langle \sum_\alpha \lambda_\alpha \delta_{1,\xi(\alpha x)_1} \prod_{t=1}^{L_0} \frac{\langle\{\xi\}_t|1 - a_0 H_R|\{\xi\}_{t+1}\rangle}{\left|\langle\{\xi\}_t|1 - a_0 H_R|\{\xi\}_{t+1}\rangle\right|} \right\rangle_P \tag{19}$$

The Monte Carlo step determined by the relative probability W in (17) does not change the exactly conserved quantum numbers like, for instance, the fermion number. In other words, the simulation is done in the *canonical*, instead of grand canonical ensemble. Nevertheless,

Table I

The comparison of $\epsilon^{-\beta E} = 0.0183156$ for $\beta E = 4$ to the linear approximation $(1 - \beta E/L_0)^{L_0}$ for different L_0. The relative deviation is Δ.

L_0	$(1 - \frac{\beta E}{L_0})^{L_0}$	Δ
100	0.0168704	7.9%
400	0.0179506	2.0%
1000	0.0181686	0.8%
4000	0.0182785	0.2%

the grand canonical ensemble can also be reproduced by choosing the initial configuration randomly and repeat the Monte Carlo summation many times.

The applicability of the linear approximation can be estimated by comparing the value of $\epsilon^{-\beta E}$ to $(1 - \beta E/L_0)^{L_0}$, where E is a relevant eigenvalue of H_R. For $\beta = 4$, $E = 1$ this is done in Table I. The relative deviation Δ is well approximated by

$$\Delta = \frac{(\beta E)^2}{2L_0} = \frac{1}{2} a_0 E^2 \beta \tag{20}$$

The order of magnitude of the maximum of relevant energy eigenvalues is not easy to guess. A conservative estimate could be $E \simeq const.\ (Lm)^3$, which corresponds to a constant particle density. (Here L^3 is the volume and m is the fermion mass in lattice units.) This would lead to $a_0 \simeq \beta^{-1}(Lm)^{-6}$, but for free fermions and in the single component Yukawa model discussed in the next section it turned out that good results could be obtained for much larger values of a_0. The relevant E values for large enough β came out to be of the order of 1. This is not surprising, because at large β the states with higher energy are exponentially suppressed in the trace. (Note that a constant vacuum energy can always be shifted into H_D, therefore E is always understood above the zero vacuum energy.)

The order of magnitude of the relative error of $\langle \bar{\psi}_x \psi_x \rangle$ on a 4^4 lattice (which means *by definition* $L = 4$ and $\beta = 4$) with a mass $m = 1.0$ for naive fermions turned out to be similar to Δ in Table I. This is shown by Fig. 1, taken from Ref. [4]. The statistics corresponds typically to 5000 sweeps per a_0 value after 1000 equilibrating sweeps (a "sweep" means on the average one heatbath change per fermionic variable on the $L^3 \cdot L_0$ lattice). The statistical errors were estimated by the usual binning of the sequence of results (actually into bins of length 2^n). In Ref. [4] it was shown that for free naive or Wilson fermions the expectation value of $\bar{\psi}\psi$ as a function of the mass m, inverse temperature β and fermion number N_f is well reproduced on a 4^4 lattice with an asymmetry parameter $a_0 = 1/320$.

The relative speed of the bosonized fermion algorithm compared to a purely bosonic updating is roughly given by a_0^{-1}. In this rough estimate the problem of the phase factor oscillations is not yet included (see the discussion below) and no extrapolation to $a_0 \to 0$ is considered. It is, however, conceivable that this extrapolation is not necessary. One can also work with a small but finite a_0 which, for instance in the linear approximation, corresponds to an action $H_D - a_0^{-1} \log(1 - a_0 H_R)$. This contains some higher order correction terms in addition to $H_D + H_R$, but in the euclidean invariant continuum limit these corrections probably do not matter.

An estimate similar to Table I also gives that the second order replacement

$$\exp(-L_0 a_0 E) \to (1 - a_0 E + \frac{1}{2} a_0^2 E^2)^{L_0} \tag{21}$$

is by far superior: for $\beta E = 4$ and $L_0 = 100$ one obtains an estimate with only 0.1% deviation. The second order analogue for the replacement in Eq. (12) is

$$Tr\ \epsilon^{-L_0 a_0 H} \to Tr\ \left\{ \epsilon^{-a_0 H_D} (1 - a_0 H_R + \frac{1}{2} a_0^2 H_R^2) \right\}^{L_0} \tag{22}$$

Since a_0^{-1} can be much smaller, a quadratic bosonized fermion algorithm is presumably much faster than the linear one.

In Ref. [4] and for the Yukawa model discussed below, the linear algorithm according to Eq. (12) was used. The initial configurations were chosen randomly out of all possible time-independent configurations. It can be shown that in the $a_0 \to 0$ limit this is a correct procedure, although there are also non-zero configurations which cannot be reached from this by subsequent heatbath steps. A non-zero contribution is obtained from a sequence $\{\xi\}_1 \ldots \{\xi\}_{L_0}$ if the subsequent states are either identical or differ at most by a flip of a single active fermion link. Due to the periodicity the configuration has to be the same after going around once in time direction. Therefore, either the subsequent flips have to compensate each other or otherwise the same state can also be reached if a change is "going around the world" in the space with periodic boundaries. These later sequences have, however, a minimum number of L factors of a_0 in the probability and hence are negligible for $a_0 \to 0$. More generally, all configurations wich cannot be reached by subsequent heatbath steps from a time independent configuration are suppressed by higher powers of a_0 and have smaller entropy. In practice, as it is shown by Fig. 1, the deviations due to the omission of some configurations become small for $a_0 \simeq 10^{-2} - 10^{-3}$. From this point of view the quadratic algorithm based on Eq. (22) is probably better, because there a broader class of configurations can be reached from a time-independent one.

As it was realized already in the early attempts in higher dimensional bosonization [2], a potential source of problems in the calculation of expectation values is that the phase factors in the nominator and denominator (like in Eqs. (18,19)) can strongly oscillate and make the convergence to the average very slow. A rough estimate implies that in the exponential algorithms of Ref. [2] for $a_0 = O(1)$ this could be catastrophic because the relative fluctuations may grow exponentially with the lattice volume. In the limit $a_0 \to 0$ the situation is, however, completely different, because the phase factor fluctuations tend to zero. This can be shown, for instance, for free fermions and for the single component Yukawa model discussed below. Since the problem is of kinematical nature, it is presumably true also in other interacting models in the sense that for small a_0 the *ratio* of the nominator and denominator as a function of the volume behaves as a normal expectation value. (This actually turned out to be true in every model I tried up to now.) This statement is weaker than the corresponding one for the nominator and denominator separately, because the correlation between them can be strong.

In the case of free fermions or for the single component Yukawa model, it can also be shown that in the linear heatbath procedure the expectation value in Eq. (18) determining Z is exactly 1 (i.e. the denominator in the expectation values is 1). The configurations which can have a contribution -1 to Z cannot be reached in the linear approximation from a time-independent configuration. In second or higher order procedures Z is not exactly 1, it

only tends to 1 for $a_0 \to 0$. This may be a slight disadvantage of the higher order algorithms, but the advatages are presumably much more important.

3 One-component scalar fermion model

The partition function of a model containing both fermionic and bosonic degrees of freedom can be written in the linearized case as

$$Z = \int [d\phi] e^{-S[\phi]} \sum_{\{\xi\}_1 \cdots \{\xi\}_{L_0}} \prod_{t=1}^{L_0} \left\{ e^{-a_0 H_D[\phi,\{\xi\}_t]} \langle \{\xi\}_t | 1 - a_0 H_R[\phi] | \{\xi\}_{t-1} \rangle \right\} \tag{23}$$

Here the euclidean path integral $\int [d\phi]$ has to be performed over the bosonic variables $\{\phi_{x,t}\}$. The purely bosonic part of the euclidean action is denoted by $S[\phi]$. The Hamiltonian is $H[\phi] \equiv H_D[\phi] + H_R[\phi]$, which depends on the fermionic variables $\{\xi\}_1, \ldots, \{\xi\}_{L_0}$ and contains also the interaction piece between fermions and bosons. The number of timeslices for the boson fields can be equal to L_0, but it can also be smaller, for instance only L. In the second case the bosonic variables are repeated on several neighbouring fermionic timeslices for the calculation of the trace. Since the time direction is treated differently, euclidean rotation invariance (or Lorentz-invariance) is not automatic. For its restoration in the continuum limit it may be necessary to choose different bosonic couplings on spacelike and timelike links and tune their ratio appropriately.

In the scalar-fermion model considered here $\phi_{x,t}$ is the single component scalar field on the space point x in the timeslice t. The naive- or Wilson-fermion field is $\psi_{x,t}$. The scalar action is in general

$$S = \sum_{x,t} \left\{ \mu \phi_{x,t}^2 + \lambda \phi_{x,t}^4 - \kappa_s \sum_k \left(\phi_{x+\hat{k},t} \phi_{x,t} + \phi_{x-\hat{k},t} \phi_{x,t} \right) - \kappa_t \left(\phi_{x,t+1} \phi_{x,t} + \phi_{x,t-1} \phi_{x,t} \right) \right\} \tag{24}$$

κ_s and κ_t are the hopping parameters in the spacelike, respectively, timelike directions. Here we shall consider only the limit of infinitely strong quartic self-coupling $\lambda \to \infty$. In this case the first two terms of S are absent and the values of $\phi_{x,t}$ are restricted to $\phi_{x,t} = \pm 1$ ("Ising limit"). The Hamiltonian $H = H_D + H_R$ on the timeslice t is given by

$$H_D[\phi] = \sum_x (m + G\phi_{x,t}) \psi_{x,t}^\dagger \beta \psi_{x,t}$$

$$H_R = -\frac{r}{2} \sum_{x,k} \left\{ \psi_{x+\hat{k},t}^\dagger \beta \psi_{x,t} - \psi_{x,t}^\dagger \beta \psi_{x+\hat{k},t} \right\} + \frac{i}{2} \sum_{x,k} \left\{ \psi_{x-\hat{k},t}^\dagger \alpha_k \psi_{x,t} - \psi_{x,t}^\dagger \alpha_k \psi_{x-\hat{k},t} \right\} \tag{25}$$

Here only the case of massless naive fermions with $m = r = 0$ will be considered. The simplicity of this model is due to the fact that the Yukawa interaction proportional to G can be included in the diagonal part of the Hamiltonian H_D.

An important limit of this model is $\kappa_s = \kappa_t = 0$. In this case the scalar field has no kinetic term, and therefore its random fluctuations are independent on different sites. The consequence of the interaction with the fermion field is, however, that the scalar kinetic term is reproduced dynamically, and the physical content of the model is similar to the case with non-zero scalar hopping parameters. (This can be seen, for instance, in the hopping

parameter expansion, similarly to Ref. [10].) The scalar boson can be considered in this limit to be a bound state of a fermion-antifermion pair.

In the numerical simulation on 4^4 and 6^4 lattices the $\lambda = \infty$, $m = r = 0$ case was considered. The number of timeslices for the fermion and scalar field was equal (L_0), and the asymmetry parameter was in most case $a_0 = 1/160$. For vanishing scalar hopping parameters $\kappa_s = \kappa_t = 0$ some global quantities obtained on a 4^4 lattice with $a_0 = 1/160$ (i.e. $L_0 = 640$) are shown as a function of the Yukawa-coupling G in Table II. The time independent initial configuration was chosen randomly in such a way that it had always a fermion number $N_f = 2$. (In the expectation values in Table II the difference to zero fermion number is smaller than the statistical errors.) The statistics corresponds typically to 10000 sweeps after 3000 equilibrating sweeps. The measurements of the expectation values were performed after every 10th sweep. (A "sweep" corresponds to one update per field variable.) The expectation values of ϕ_x and $\bar{\psi}_x\psi_x$ were defined by the absolute value of lattice averages. This is necessary, because on these small lattices the averages of both ϕ_x and $\bar{\psi}_x\psi_x$ change sign during the run, even if the expectation values are already quite large. The sign of ϕ_x and $\bar{\psi}_x\psi_x$ is in most cases opposite, therefore $\langle\phi_x\bar{\psi}_x\psi_x\rangle$ is negative.

Table II

The results of the numerical simulation on a 4^4 lattice with $a_0 = 1/160$ for $\kappa_s = \kappa_t = 0$ and $N_f = 2$. Statistical error estimates in the last numerals are given in parentheses.

G	$\langle\bar{\psi}_x\psi_x\rangle$	$\langle\phi_x\rangle$	$\langle\phi_x\phi_{x+\hat{k}}\rangle$
0.125	0.08(1)	0.015(2)	0.0078(2)
0.250	0.11(2)	0.039(6)	0.0309(5)
0.375	0.12(2)	0.057(5)	0.0611(8)
0.500	0.16(2)	0.09(2)	0.101(2)
0.625	0.35(7)	0.23(5)	0.144(7)
0.750	0.47(9)	0.31(6)	0.18(1)
0.875	0.99(7)	0.60(5)	0.14(2)
1.000	1.32(2)	0.81(3)	0.05(2)
1.125	1.43(6)	0.85(3)	0.05(2)
1.250	1.56(2)	0.93(1)	0.009(2)
1.375	1.55(2)	0.93(1)	0.010(2)
1.500	1.62(2)	0.96(1)	0.006(2)

In order to check the a_0-dependence a high statistics run with 45000 sweeps was also performed at $G = 0.75$ with $a_0 = 1/320$ (i.e. $L_0 = 1280$). The results are:

$$\langle\bar{\psi}_x\psi_x\rangle = 0.48(6) \qquad \langle\phi_x\rangle = 0.31(3) \qquad \langle\phi_x\phi_{x+\hat{k}}\rangle = 0.170(8) \tag{26}$$

These are consistent with Table II, in other words, the a_0-dependence is smaller than the statistical errors.

The statistical errors were always estimated in the usual way by forming bins of length 2^n out of the data sequence. These error estimates become roughly independent from the

bin size for large bins, although the number of sweeps is not large enough to make a definite statement about this. In the case of the longer run corresponding to Eq. (26) the n-dependence of the relative error estimates is roughly similar to the n-dependence seen in a high statistics Metropolis simulation of the 4-dimensional Ising model [14]. Therefore, the efficiency of the Markov chain in the linear bosonized fermion algorithm does not seem to be much worse than in the Metropolis algorithm for the scalar field alone.

In order to obtain information on the volume dependence, I also did a few runs at $\kappa_s = \kappa_t = 0$ on a 6^4 lattice with $a_0 = 1/160$ (i.e. $L_0 = 960$). A comparison to the 4^4 results is shown in Fig. 2a-2c. The increase of the vacuum expectation values $\langle \phi_x \rangle$ and $\langle \bar{\psi}_x \psi_x \rangle$ happens on the larger lattice at smaller values of the Yukawa-coupling G, and it is more abrupt. The behaviour of $\langle \phi_x \rangle$ and $\langle \bar{\psi}_x \psi_x \rangle$ together with the peak in $\langle \varphi_x \varphi_{x+\hat{k}} \rangle$ can be interpreted as signals of a strong symmetry breaking phase transition between a symmetric phase for small G and a broken phase at larger G, where both $\langle \phi_x \rangle \neq 0$ and $\langle \bar{\psi}_x \psi_x \rangle \neq 0$. No sign of a second symmetric phase at very large G was observed, in contrast to Ref. [12], where two phase transitions were seen in another simple scalar-fermion model with Yukawa-coupling. A single symmetry breaking phase transition was also seen in another related model with staggered fermions in Ref. [11]. The order of the symmetry breaking phase transition can only be determined in runs with high statistics on larger lattices.

In addition to the numerical simulation at $\kappa_s = \kappa_t = 0$, I also performed similar runs on a 4^4 lattice for several non-vanishing scalar hopping parameter values. The increase of $\langle \phi_x \rangle$ and $\langle \bar{\psi}_x \psi_x \rangle$ occurs for non-zero κ's at smaller Yukawa-coupling G. If the scalar hopping parameter is larger than the critical value in the Ising-model, where the scalar field obtains a non-zero vacuum expectation value, the expectation value of $\bar{\psi}_x \psi_x$ becomes non-zero very soon after the inclusion of a small Yukawa-coupling. These findings can be interpreted as hints for a simple phase structure of the single component Yukawa model with naive fermions, as it is shown in Fig. 3.

In summary: the linear bosonization algorithm for fermions works fine in the 4-dimensional single component Yukawa model. An even better performance is expected in the case of quadratic bosonization.

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Figure captions

Fig. 1. The dependence of $\langle\bar{\psi}_x\psi_x\rangle$ on the ratio of the timelike to spacelike lattice spacing a_0 for free naive fermions with mass in lattice units $m = 1.0$. The spacelike lattice is 4^3 and the inverse temperature $\beta = 4$. (This is referred to as a 4^4 lattice, although the calculation is actually done on a $4^3 \cdot (4/a_0)$ lattice.) The dot at $a_0 = 0$ is the exact result.

Fig. 2a. The expectation value of the scalar field $\langle\phi_x\rangle$ at vanishing scalar hopping parameters as a function of the Yukawa-coupling G. Open symbols correspond to a 4^4 lattice, the full symbols to a 6^4 lattice.

Fig. 2b. The same as Fig. 2a for $\langle\bar{\psi}_x\psi_x\rangle$.

Fig. 2c. The same as Fig. 2a for $\langle\phi_x\phi_{x+\hat{k}}\rangle$.

Fig. 3. The suggested qualitative phase structure of the single component Yukawa model with naive fermions at infinitely strong scalar quartic self-coupling $\lambda = \infty$.

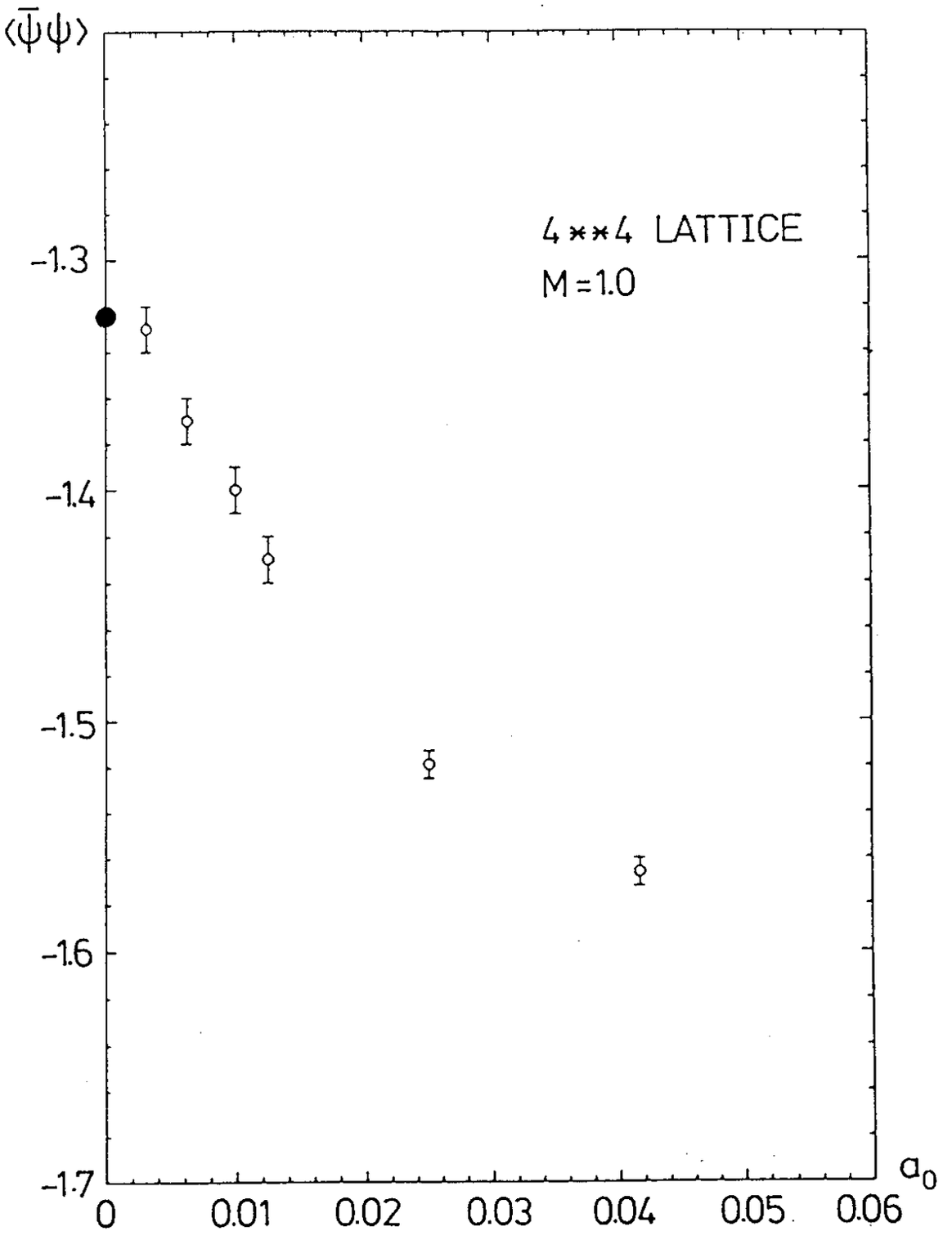

Fig.1

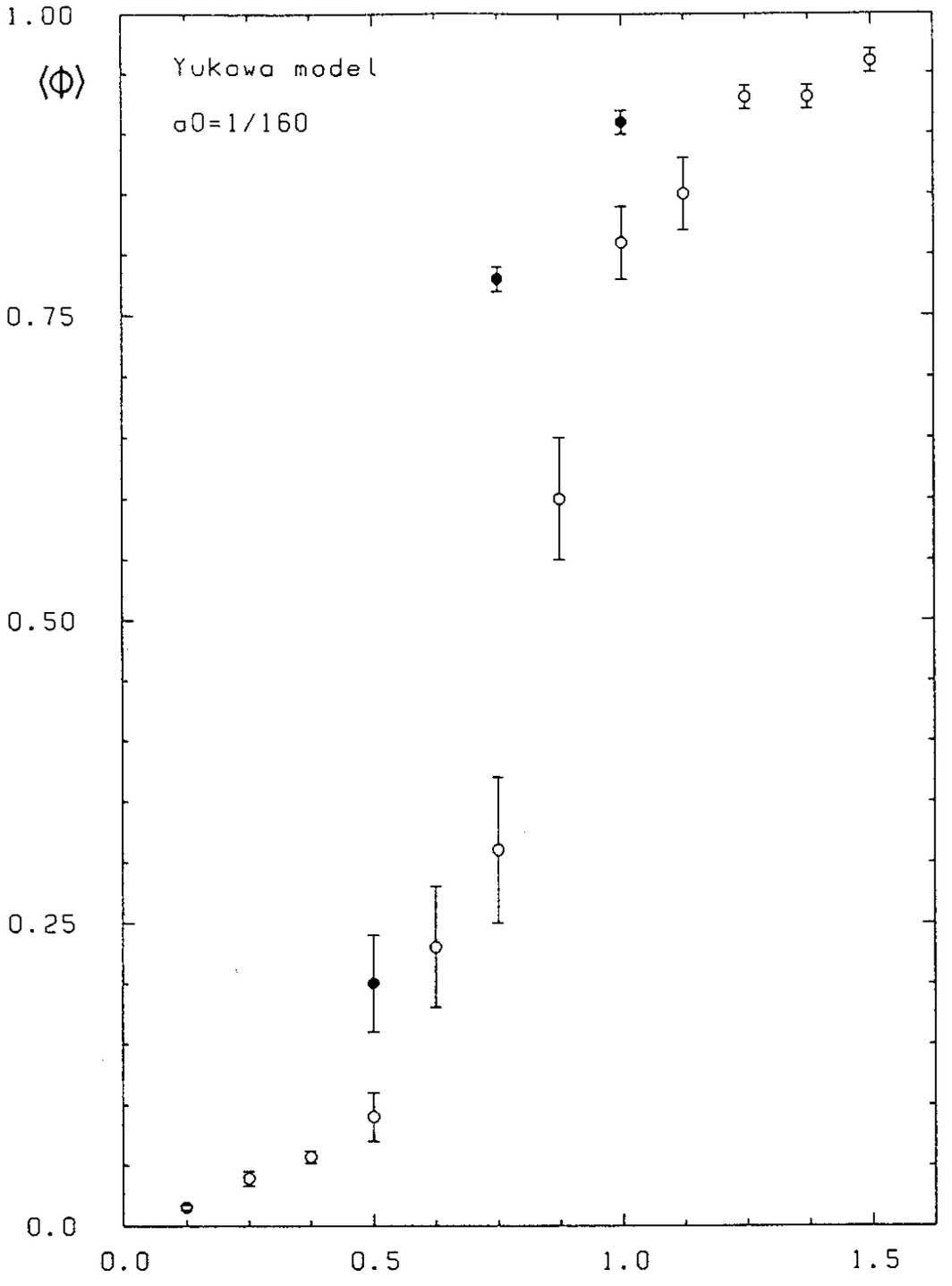

Fig. 2A

G

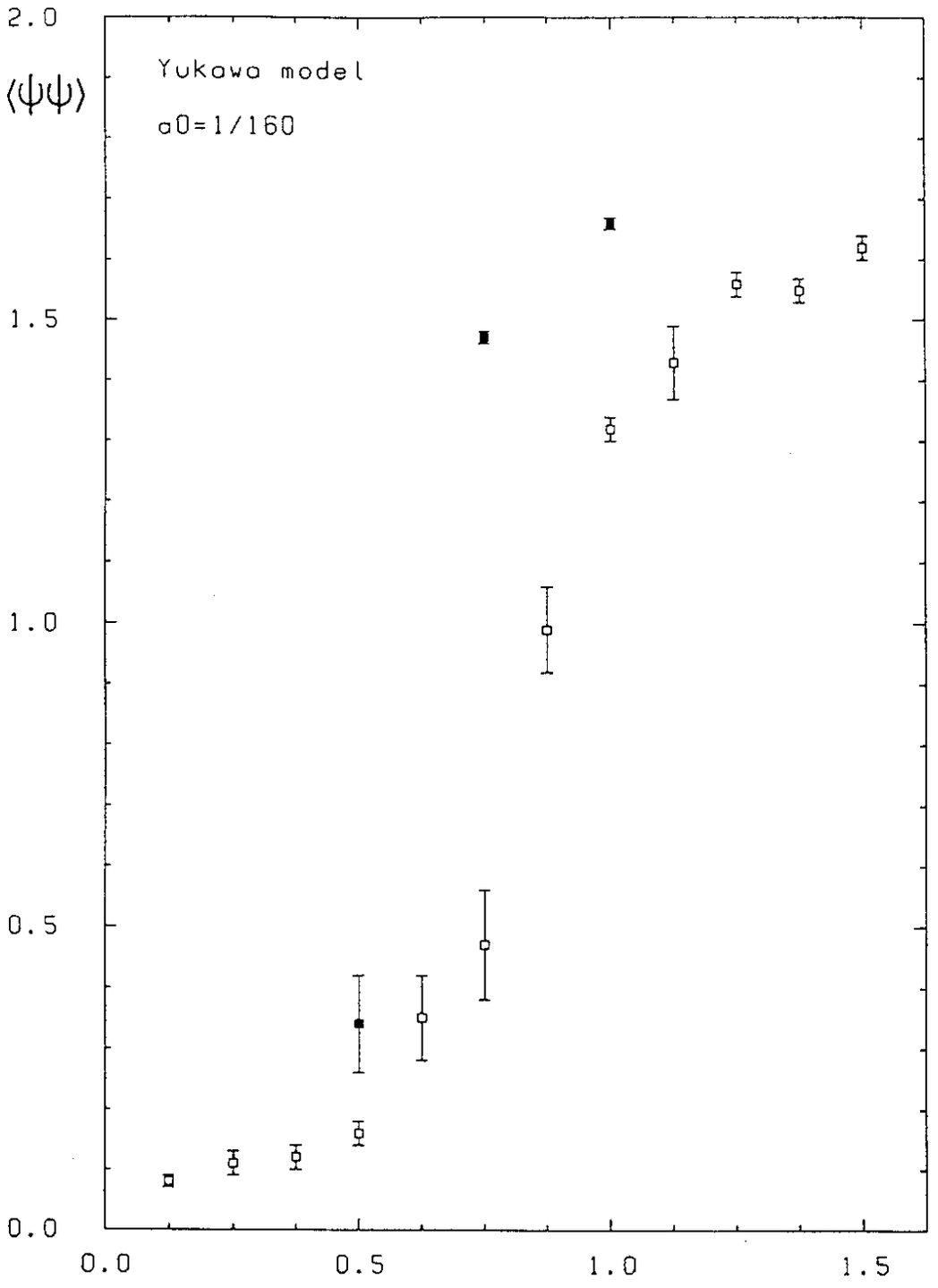

Fig. 2B

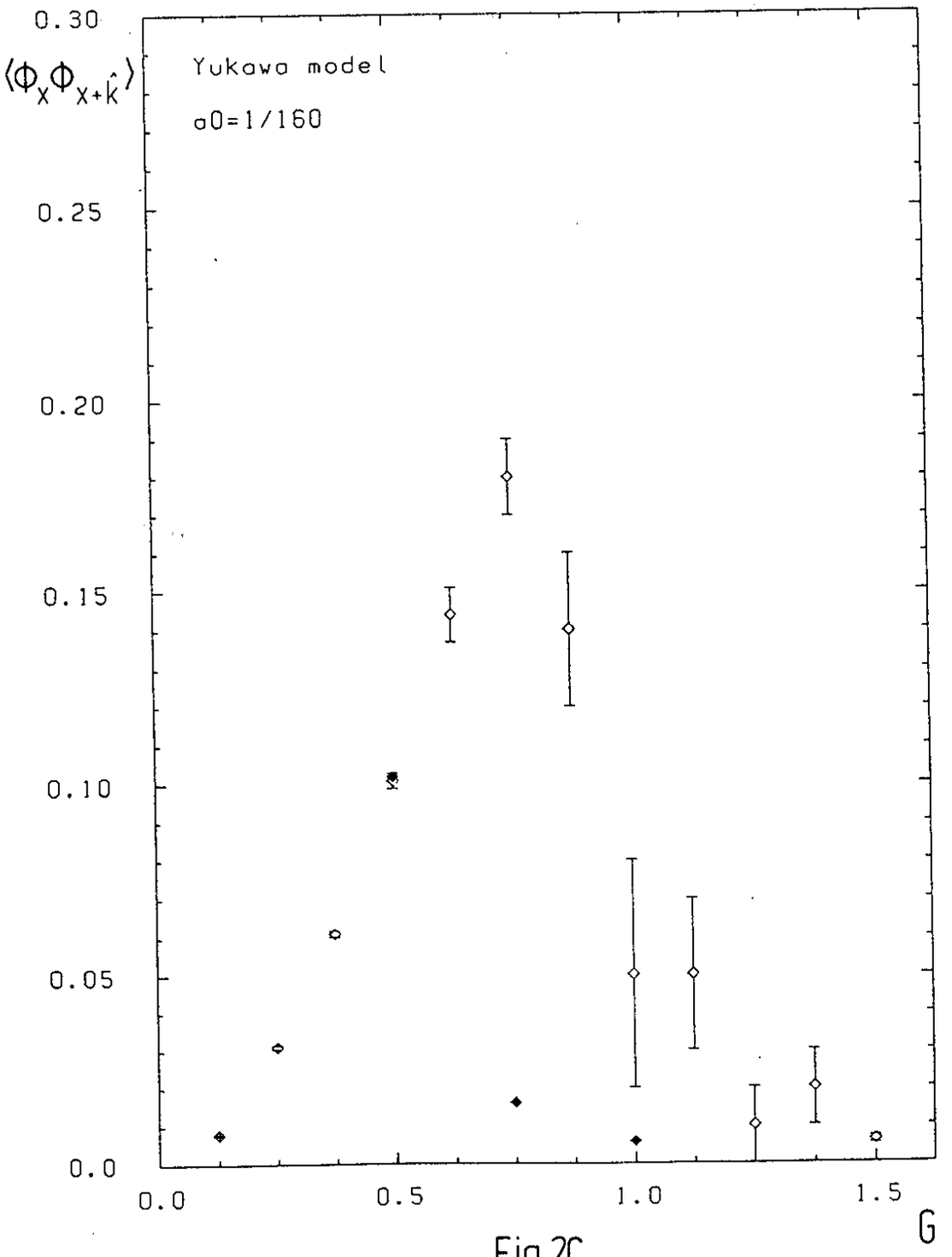
0.30
$\langle \Phi_x \Phi_{x+\hat{k}} \rangle$
Yukawa model
a0=1/160
0.25
0.20
0.15
0.10
0.05
0.0
0.0
0.5
1.0
1.5
G

Fig.2C

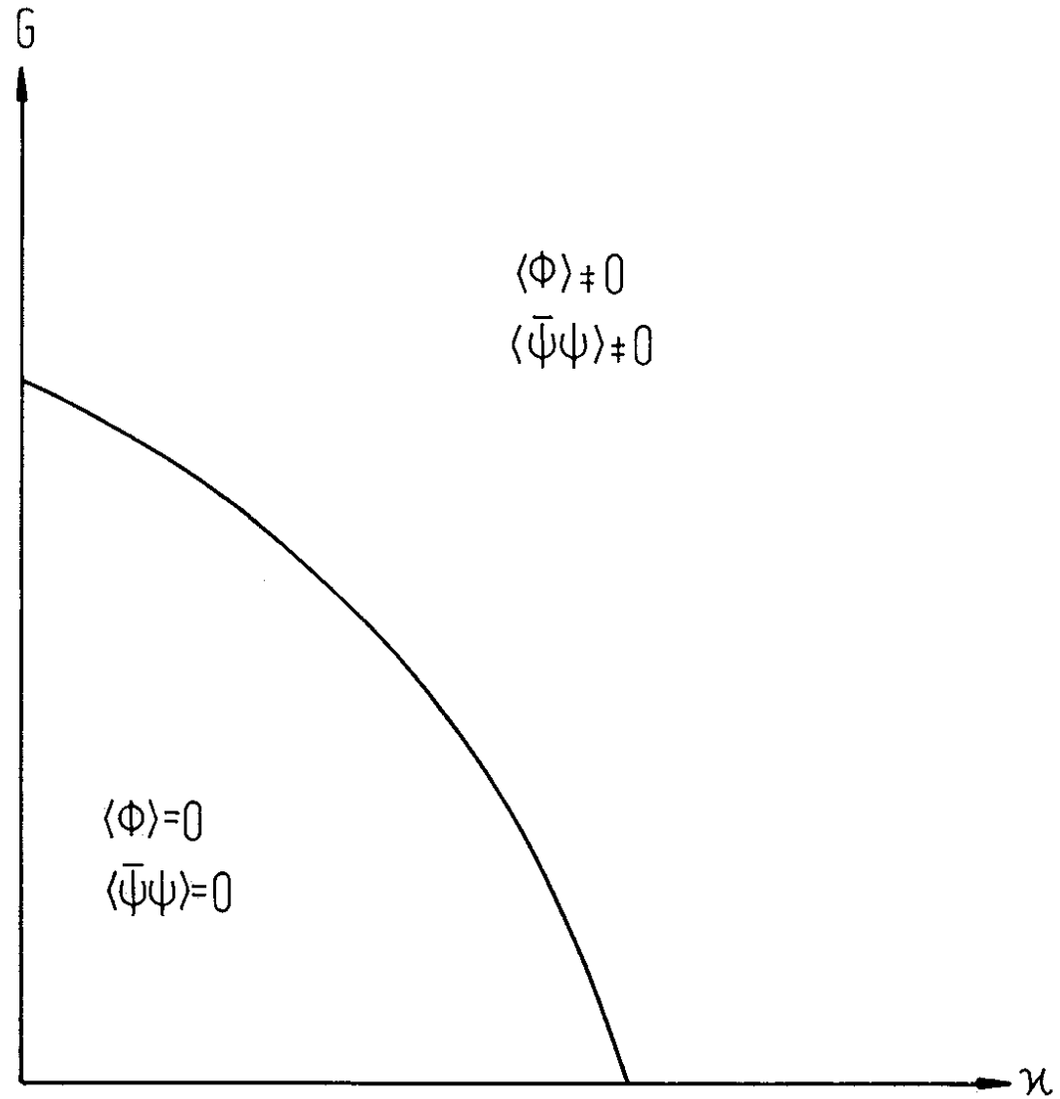

Fig. 3