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# The Use Of Quantum Electrodynamics To Calculate Radiative Effects In Electron Storage Rings

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**Abstract**

In a seminal paper,Bell and Leinaas have introduced the use of quantum mechanical Heisenberg operators for the orbital variables and vacuum expectation values for bilinear products of electromagnetic field operators as a tool for calculating radiative effects on the orbit and the spin in electron storage rings. We will review these techniques,which give a more systematic method than treating radiation as the emission of "little bullets". This is at the cost of introducing some formalism not usually employed in accelerator theory as overhead.

We will give some examples for the calculation of energy spread,beam size and polarization. The usual results are obtained,but in a different way. The advantages and disadvantages of this technique and its relation to the normal approach will be discussed.

## 1 Motivation;Introduction

–This talk is mainly for people who enjoy doing a calculation for its own sake. It is intended to be pedagogical.

–We will show how to eliminate any explicit reference to photons radiated by electrons stored in a ring. This approach can be applied to cases where the photons are not directly observed, for example calculation of the relative energy spread of the electron beam.

–Both the displacements from the closed orbit and the spin can be treated this way.

–Using this idea, we will do a simple calculation of the equilibrium polarization. It is equivalent to the 1973 theory of Derbenev and Kondratenko.

(We avoid any discussion of the particle physics consequences which might follow from having $\geq 50\%$ longitudinal polarization in the electron beams from SLC,KEK,LEP or HERA.If the electron has an electric dipole moment, even transverse polarization is interesting. )

(Also,in today's lecture,we avoid going beyond the original Russian theory,although this has recently become possible,and in spite of the fact that this theoretical advance will be important for KEK, LEP and HERA "polarizers".)



–We are,inevitably,presenting the subject from a certain viewpoint. Those attending this talk should be aware there are other ways to approach the polarization calculation which may be equally valid. It will be interesting to compare them when the different theories have matured.

We treat an electron in a storage ring as a **quantum mechanical** system. As time flows, both the orbit and the spin evolve according to the Schrödinger equation with an appropriate Hamiltonian. Deviations from the closed orbit are treated as perturbations caused by interactions between the electron and the radiation field. Once the Hamiltonian is known, we apply ordinary perturbation theory to find the possible expectation values and transition probabilities for the spin.

–Thus,in our picture,we have **non-interacting single particles in an external field**. These are described in the usual way by a wave function which is a function of orbital and spin variables. The effects of different photons are **added coherently in the quantum mechanical sense**. There is **no** average over different electron trajectories in the classical sense.

–In our picture a storage ring with a single electron is dynamically equivalent to a hydrogen atom,as it is treated in the theory of quantum electrodynamics!

There is a selected bibliography at the end of these lecture notes. A copy of the notes will be available to anyone who requests it from CAS.



## 2 What Is The System Hamiltonian?

–We outline briefly how it is obtained. The derivation could be done rigorously with more time.

–Starting point: the Dirac equation for a single electron. We introduce the electromagnetic interaction by minimal coupling, $p_\mu \rightarrow p_\mu - eA_\mu$ ,plus an additional term for the anomalous magnetic moment which is proportional to $\sigma_{\mu\nu}F^{\mu\nu}$.

–The extra anomalous moment term would be unnecessary if we would make a complete theory with virtual photons,closed loops,etc. We choose just to avoid this extra baggage by using the phenomenological term in addition to the minimal coupling.

–Then we go to a frame in which the electron is nearly at rest. Using a Foldy-Wouthuysen transformation,we eliminate small components of the wave function,obtaining an effective Hamiltonian. This has terms which involve the spin,and other terms which do not.

–The Hamiltonian for large components (the small ones vanish as the particle velocity approaches zero):

$$H = m + \overbrace{\frac{(\vec{p} - e\vec{A})^2}{2m} - \frac{p^4}{8m^2}}^{\text{operators}} \cdots$$

$$+ e\phi - \underbrace{\frac{1}{2}\vec{\sigma}\cdot\vec{\Omega}}_{\text{spin orbit term}}$$

$$+\vec{\sigma}\cdot(\textit{terms proportional to field gradients}) + \ldots$$

$$+\textit{higher order terms}$$

$\vec{\Omega}$ is linear in the electromagnetic field. It has the form discovered by Thomas,commonly referred to as the "BMT" form. In terms of rest frame spin and laboratory frame fields we have:

$$\vec{\Omega} =$$

$$-\frac{e}{m}[(a + \frac{1}{\gamma})\vec{B} - (a + \frac{1}{1+\gamma})\vec{\beta}\times\vec{E} - \frac{a\gamma}{1+\gamma}\vec{\beta}(\vec{\beta}\cdot\vec{B})]$$

$a = \frac{g-2}{2}$, $g$ is the electron's g-factor. Numerically, $a =$ .001159652.

$\gamma$ is the Lorentz factor. $\vec{E}$ and $\vec{B}$ are **superpositions of accelerator guide fields( classical) fields and quantized radiation fields.**



–Neglecting the gradient terms is an approximation made by everyone so far,including the new theories for the polarization which go beyond the Derbenev-Kondratenko calculation. It needs to be checked that nothing important has been left out,but this hasn't been done.

–Jackson calculated the polarization in the "instantaneous" rest frame using the Hamiltonian above. Jackson did not consider the depolarizing effects of field gradients from focussing magnets. His calculation was equivalent to the original Sokolov-Ternov work. Bell and Leinaas found the polarization using an accelerated reference frame and for a weak focussing machine. They avoided the horrible integrals over Airy functions by an adroit use of vacuum expectation values of radiation fields.

–The calculation of Sokolov and Ternov also showed that the electron trajectory can be treated as a classical trajectory,since one has very large orbital quantum numbers,so

$$\vec{x}_{op}, \vec{p}_{op} \Rightarrow (\vec{x}, \vec{p})_{\text{classical}}$$

This prescription is only true if we "remove" the quantized radiation field and have only the classical accelerator guide fields.

–We will be interested in the next level of approximation,where we treat the radiation effects as a perturbation.

–Based on the reasoning above,we take the Hamiltonian in the **laboratory** frame to be

$$\boxed{H = \underbrace{\sqrt{m^2 + (\vec{p} - e\vec{A})^2}}_{\text{orbital part}} - \underbrace{\frac{1}{2}\vec{\sigma}\cdot\vec{\Omega}}_{\text{spin part}}} \qquad (1)$$

Once again,we emphasize that $\vec{A}$ and $\vec{\Omega}$ are linear superpositions of classical accelerator fields and quantum radiation fields. This is our basic expression. (We have assumed $\phi = 0$.)

–The orbital part of the Hamiltonian can be developed further in order to obtain the form normally used to describe betatron and synchrotron motion of the electrons in the stored beam. This is done in the usual way, which originated with Courant and Snyder.



## 3 The Energy Spread And Beam Size

–We consider here deviations of the electron energy and position from the closed orbit. The source of these deviations is the fluctuating electromagnetic field seen by the electron.

–Let $\delta \equiv \frac{\Delta E}{E_0}$. $\delta$ is the deviation of the relative energy from the average beam energy. We are interested in computing $< \delta^2 >$,which originates from the field fluctuations,but is limited by the classical radiation damping.

–In principle,as discussed by Bell and Leinaas,we really should write

$$\vec{E}_{rad} = \vec{E}_{rad,classical} + \vec{E}_{quantum}$$

The role of $\vec{E}_{rad,classical}$ is to produce a net average energy loss via the Larmor radiation formula,or rather its relativistic generalization.

–Often we will neglect this difference between $\vec{E}_{rad}$ and $\vec{E}_{quantum}$ and treat both the synchrotron oscillations and classical damping phenomenologically.

–From the Hamiltonian we obtained above we can derive the relativistically correct equation(via the Lorentz force):

$$\frac{d\Delta E}{dt} = e\vec{E}\cdot\vec{\beta}$$

Since we are interested only in the fluctuations,we can also write:

$$\frac{d\delta}{dt} = e\vec{E}_{\text{quantum}}\cdot\vec{\beta}$$

(If the meaning is obvious,we will drop the subscript "quantum")

At the time $t_0$

$$\delta(t_0) = \frac{e}{m\gamma}\int_{-\infty}^{t_0} \vec{E}\cdot\vec{\beta}\, dt$$

This doesn't lead to the correct result for $<\delta^2>$ because we have neglected the effect of synchrotron oscillations and classical radiation damping! We save time by just writing down the correct expression when these effects are included:

$$\delta(t_0) = \frac{e}{m\gamma}\int_{-\frac{T}{2}}^{t_0} e^{-\Gamma_s(t_0-t)}\cos\omega_s(t_0-t)\ \vec{E}(t)\cdot\vec{\beta}(t)\ dt$$

$\omega_s$ is the synchrotron oscillation frequency and $\Gamma_s$ is a damping constant. T is any time longer than "many" damping times.



–We have replaced $-\infty$ by the finite time $-\frac{T}{2}$,with the understanding we will take the limit $T \to +\infty$ at the end.

–Calculate $<\delta^2>$ at some asymptotic time $\frac{T}{2}$ in the future,after the interaction with the radiation field is turned on for some finite time T.

$$<\delta^2(\frac{T}{2})> = (\frac{e}{m\gamma})^2 \int_{-\frac{T}{2}}^{\frac{T}{2}} dt \int_{-\frac{T}{2}}^{\frac{T}{2}} dt'$$
$$\times\ e^{-\Gamma_s(\frac{T}{2}-t)}\cos\omega_s(\frac{T}{2}-t)\ e^{-\Gamma_s(\frac{T}{2}-t')}\cos\omega_s(\frac{T}{2}-t')$$
$$\times\ <0|\vec{E}(t)\cdot\vec{\beta}(t)\vec{E}(t')\cdot\vec{\beta}(t')|0>$$

( $<0|\ldots|0>$ means take the vacuum expectation value.Strictly speaking , there should be a density matrix in the states of the radiation field,but doing this simply amounts to multiplying and dividing by the same numerical constant.)

–To go any further, we must find the vacuum expectation value and do the integrals explicitly. The vacuum expectation value is a well- defined function, familiar to all students of quantum electrodynamics. As a function of $|t-t'|$ it is sharply peaked at zero and vanishes rapidly if the time difference exceeds a correlation length. (The correlation length is of order $\rho/\gamma$, where $\rho$ is the bending radius.)

(Appendix B of these notes contains a detailed account of how to get these vacuum expectation values. It would take up too much time to go through it here.)

It will be sufficient to use the approximate formula:

$$< 0|\vec{E}(t)\cdot\vec{\beta}\vec{E}(t')\cdot\vec{\beta}|0 > \approx \frac{55\hbar\ \gamma^7}{24\sqrt{3}\,\rho^3}\ \underbrace{\delta(t-t')}_{\text{Dirac delta}}$$

Using this result we can complete the calculation since

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} dt\ \rho^{-3}\ e^{-2\Gamma_s(\frac{T}{2}-t)}\ \cos^2\omega_s t$$

$$\simeq \frac{1}{4\Gamma_s} < \rho^{-3} >$$

(< ... > means take the average around the ring. We have used here the approximation $\Gamma_s << \omega_s < \omega_0$)

Putting it all together:

$$< \delta^2 > = \frac{55\hbar}{96\sqrt{3}}(\frac{e}{m})^2\gamma^5\frac{< \rho^{-3} >}{\Gamma_s}$$

This is not the form one usually sees. From Renieri's article in the 1976 CERN Summer School in Erice, or from Helm, Lee, Morton, and Sands :

$$\Gamma_s = \frac{r_\epsilon}{3C_0}\gamma^3 \oint \frac{ds}{\rho^2}$$

($\oint ds$ is an integral once around the machine, $C_0$ is the ring circumference, the velocity of light = 1, and $r_\epsilon$ is the classical electron radius $\frac{e^2}{m}$.)



Our final result is the standard one:

$$\boxed{< \delta^2 > = \frac{55}{32\sqrt{3}}\bar{\lambda}_C\ \gamma^2\ \frac{\oint\frac{ds}{\rho^3}}{\oint\frac{ds}{\rho^2}}} \qquad (2)$$

($\bar{\lambda}_C \equiv \hbar/m$. We have neglected a usually small effect due to the shift of the closed orbit as the energy varies.)

–To calculate the r.m.s. size(horizontal) of the beam we proceed in exactly the same way as above. Using our Hamiltonian and defining an **operator** $\hat{x}$,which is the quantum mechanical operator for deviations from the closed orbit,we can find the operator equation of motion:

$$\frac{d^2\hat{x}}{ds^2} + (\frac{2\Gamma_x}{R})\frac{d\hat{x}}{ds} + K(s)\hat{x} =$$
$$\frac{e}{E_0}(F_x^{rad} + \frac{R}{\rho_x}\int_{-\infty}^{s}\vec{E}\cdot\vec{\beta}\ ds)$$

($\Gamma_x$ is the damping constant for horizontal betatron oscillations in an uncoupled machine,$R$ is the machine average radius, $K(s)$ is a function proportional to the magnetic field gradient,usually located in the quadrupoles,$F_x$ is the component of the Lorentz force for the radiation field.)

–A similiar equation can be written for $\hat{z}$, vertical displacement. Essentially,$\hat{x}$, $\hat{z}$ are linear combinations of electromagnetic field operators,using the appropriate Green's functions from accelerator linear orbit theory.

Explicit expressions for these Green's functions can be found in the DESY Red Report 87-126 by us,in more detail in the talk by Skuja at the Minneapolis Spin Conference last September and in lots of other places.

By carrying out a very similiar procedure as that above,we obtain:

$$\boxed{< x^2 >= \frac{55}{64\sqrt{3}}\ \gamma^2\ \bar{\lambda}_C \frac{<\frac{1}{\rho^3}>}{<\frac{1}{\rho^2}>}\{D_x^2\ +\ 2\frac{<\frac{\mathcal{H}_x}{\rho^3}>}{<\frac{1}{\rho^3}>}\}} \qquad (3)$$

($D_x \equiv$ horizontal dispersion function,$\alpha_x, \beta_x, \gamma_x$ are the usual Twiss parameters.)

$$\mathcal{H}_x = (\gamma_x D_x^2 + 2\alpha_x D_x D_x' + \beta_x (D_x')^2)$$
$$-\frac{64\sqrt{3}}{55\gamma}(\beta_x D_x' + \alpha_x D_x) + \frac{58}{55\gamma^2}\beta_x$$

($D' \equiv \frac{dD}{ds}$.) The last two terms are recoil corrections which are very small in the case of high electron energy. They vanish if we neglect $F_x$ in the driving term for the equation for $\hat{x}$. If we ignore them,we get the standard formula for the beam size.

–To summarize,we have shown how the r.m.s. spread of orbital variables for the beam in a storage ring can be calculated without explicitly introducing a naive version of the photon concept.



# 4 Demystifying The Polarization Formula

–In 1973 Derbenev and Kondatenko showed that a storage ring with inhomogeneous fields would have depolarizing spin orbit resonances. The polarization could be computed if one knew a single vector function of s: $\vec{d}(s)$. Later Chao and others developed methods for computation of $\vec{d}$ in practical cases. However many people had considerable difficulty understanding how the basic formula was derived.

–We will show in this section that the formula given by Derbenev and Kondratenko is obtained if one calculates to the lowest order in the fine structure constant,i.e. if one neglects multiple photon emission. The reader should be warned that the data even at SPEAR energies show a much richer structure of spin orbit resonances than are predicted by the DK formula,and that,at the higher energies of KEK,HERA and LEP the situation is expected to be even more complex. Some support for this conjecture comes from Monte Carlo calculations done by Limberg at CERN and DESY.

–Our starting point is again to find a closed orbit solution, this time for the spin. We use the ordinary quantum mechanics of two component spinors to describe the spin in the zero th order,that is, with the radiation field turned off. The Schrödinger equation is:

$$\frac{1}{i}\frac{\partial \psi}{\partial s} = H_{spin}\psi$$

($H_{spin}$ is the same BMT Hamiltonian derived earlier, see eq. 1 above. )

–It is equivalent to finding a solution to the Schrödinger equation if one knows the (unique) periodic solution to the BMT equation:

$$\frac{d\vec{n}}{ds} = \vec{\Omega} \times \vec{n}$$

($\vec{\Omega}$ is a periodic function of s. It is the same as defined for eq. 1 above. The detailed notation and formulas are given in Appendix A. We assume both $\vec{n}(s)$ and the complex solution(see the appendix) $\vec{\Sigma}(s)$ are known.)

–Again we have to turn on the radiation field and compute what we want in lowest order. What we want is the probability for a state quantized along the $\vec{n}$ axis(which together with $\vec{\Sigma}$ defines a natural coordinate system for the spin)to flip either up or down. The spin flip probability will turn out to be proportional to the interaction time T,so the spin flip rate is a constant provided T is sufficiently long.



–The basic approach is to find the final state wave function at the time $\frac{T}{2}$ when the radiation field is turned off,given the state at time $-\frac{T}{2}$ when it is turned on. This has,in lowest order, a well-known solution in quantum mechanics:

$$\psi_{final} = \psi_{initial} + i \int_{-\frac{T}{2}}^{\frac{T}{2}} H_{rad}(t)\ dt\ \psi_{initial}$$

Notice that we have removed the zero th order time dependence by going into the interaction representation. The sole effect of doing this is to move the spinors from $-\frac{T}{2}$ to time t.)

–We then find the amplitude for finding a one photon state with the electron having spin up(down) after starting with zero photons (vacuum state) and an electron having spin down(up). One squares the amplitude to find the probability. It can easily be shown that:

$$\boxed{\mathcal{P} = \int_{-\frac{T}{2}}^{\frac{T}{2}} \int_{-\frac{T}{2}}^{\frac{T}{2}} da\ db\ < 0|S^{\dagger}(b)S(a)|0 >} \qquad (4)$$

$\mathcal{P}$ is the flip (up in this case) probability, "a" and "b" are times we integrate over and $S(t)$ is the operator equaling the expression below, while $S^{\dagger}$ is its hermitean conjugate)

$$S(t) \equiv \chi_{+}^{\dagger}(t) H_{spin,rad} \chi_{-}(t) = -\frac{1}{2}\vec{\Sigma}(t) \cdot \vec{\Omega}_{rad}(t)$$

($\chi_{\pm}$ are the spinors defined in Appendix A. $H_{spin,rad}$ is evaluated in the interaction representation.)

–The physics is really contained in the definition of $\vec{\Omega}_{rad}$. The central idea of Bell and Leinaas, upon which we base our work, is that we must consider **all** of the fields experienced by the spin as a result of turning on the radiation fields. These not only include the direct experience with the vacuum fields, but the extra magnetic fields of the accelerator which are seen because the orbit is displaced due to radiation. **Any variation of the Thomas-BMT spin Hamiltonian caused by quantum fluctuations must be included.** In practice, some terms turn out to be more important than others.

–In this spirit, we divide $\vec{\Omega}_{rad}$ into three terms:

$$\vec{\Omega}_{rad} = \vec{\Omega}_{BMT} + \vec{\Omega}_{EXT} + \vec{\Omega}_{\delta\beta}$$

The first term comes from the vacuum fluctuations of the fields in the Thomas-BMT expression for $\vec{\Omega}$. The second term comes from the additional fields seen because the betatron and synchrotron oscillations are excited by the radiation. The third term is due to the variation of $\gamma$ and $\beta$, also due to radiation- excited synchrotron oscillations. If we neglect recoil effects, both the second and third terms are proportional to an integral over $\vec{E} \cdot \vec{\beta}$ multiplied by a Green's function which has two arguments and obeys a causality condition, since cause must precede effect.



We can write a formula which makes explicit the dependence on radiation field operators:

$$\boxed{\vec{\Omega}_{rad} = \vec{\Omega}_{BMT} + \frac{c}{m\gamma} \int_{-\frac{T}{2}}^{t} \vec{G}(t,t') \, \vec{E} \cdot \vec{\beta} \, dt} \qquad (5)$$

(Note $\vec{G}(t,t') = 0$ if $t' > t$. We have implicitly made the assumption of linear orbit theory here. )

–We have now only a simple exercise in evaluating vacuum expectation values and performing various integrals. It is useful to define a **scalar** function which is periodic with the period of the machine, which we call $d(\theta)$ (We use $\theta$ as the independent variable to indicate this periodicity, instead of using t, the time.)

$$\boxed{d(\theta) \equiv \int_{\theta}^{\infty} \vec{\Sigma}(\theta') \cdot \vec{G}(\theta', \theta) \, d\theta'} \qquad (6)$$

–To distinguish the previously mentioned $\vec{d}$ of Derbenev and Kondratenko from our $d$, we add a subscript to $\vec{d}$, viz. $\vec{d}_{DK}$. Then it can be shown by just calculating the probability that

$$\boxed{\vec{d}_{DK} = \textbf{Imaginary}\{\vec{\Sigma}^* \, d\}} \qquad (7)$$

and that the local rate for spin flip $\Gamma_{\pm}(\theta)$ is given by the formula first published by Derbenev and Kondratenko:

$$\boxed{\Gamma_{\pm}(\theta) = \tfrac{1}{2}\Gamma_0\{1 - \tfrac{2}{9}n_s^2 \mp \tfrac{5\sqrt{3}}{8}[\vec{n} - \vec{d}_{DK}]_z + \tfrac{11}{18}|\vec{d}_{DK}|^2\}} \qquad (8)$$

($\Gamma_+$ is the rate for flip up, $\Gamma_0 \equiv \frac{5\sqrt{3}}{8}\hbar(\frac{e}{m})^2\gamma^5\omega_0^3$.)

–By integrating around the machine,we can calculate the total rate for either direction of spin flip and the equilibrium level of polarization from a detailed balancing argument.

–We turn now to a brief comparison with the best existing data, from SPEAR, published in 1982. Figure 1 shows this data,together with a curve calculated with reasonable assumptions about machine errors . For a "flat" machine without spin rotators, the errors determine the coupling to the spin orbit resonances. The function $d(\theta)$ contains only "linear" spin orbit resonances. These have the form

$$\nu_s \pm q_x = m \quad \text{where} \quad \mathrm{m} = \text{ integer}$$

or

$$\nu_s \pm q_z = n \quad \text{where} \quad \mathrm{n} = \text{ integer}$$

( $\nu_s$ is the spin tune and $q_{x,z}$ are the horizontal and vertical betatron tunes respectively.)

–There is a satisfactory fit to both linear resonances. Detailed fitting is impossible,since the machine conditions were not well enough known.



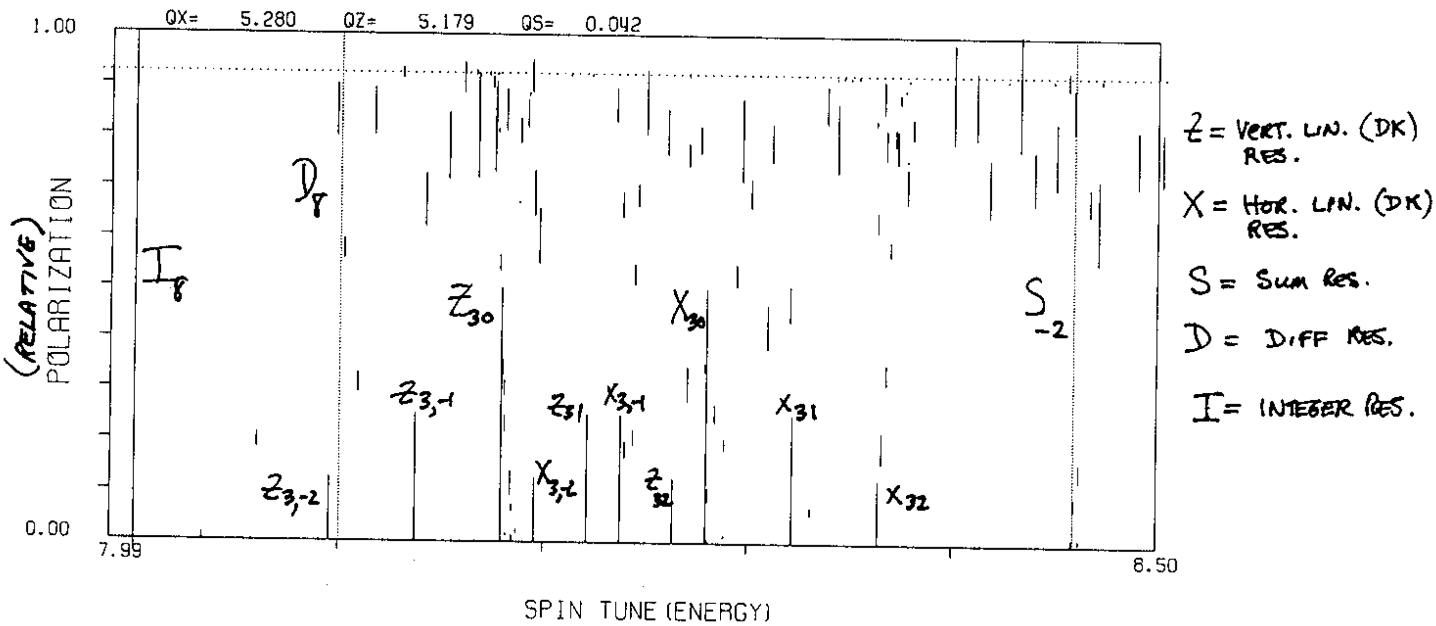
–Figure 2 shows an attempt to fit the same data,with no additional parameters,if the modulation of the spin tune by the synchrotron oscillations is taken into account phenomenologically.Sidebands to the linear resonances are the result.

–The curves in these figures were calculated using the program PAULI developed at DESY by one of the authors. No absolute measurements of polarization were made,so we assumed the maximum measured polarization was 92.4%. The calculations depend on the modelling of the machine errors with one adjustable scale factor.

In conclusion: –We have succeeded recently in deriving a new formulation of this theory which includes the effect of arbitrarily many non-flip photons. Since there are $> 10^{10}$ of these for every photon which flips the spin,we feel this may represent an improvement! Higher order resonances are predicted to exist.There is a clear prediction of the energy dependence of the strengths. No numerical work has been done, so we do not know if our theory can represent the SPEAR data. Unfortunately we are not able to complete the numerical work now or in the near future. Our priority will go into finishing the new theory, which could perhaps be the subject of another CAS lecture someday.All theories,by us or by others,such as Mane or Buon,would benefit greatly from a serious experimental effort to understand the spin,at CERN or elsewhere.

FIGURE "O" DATA + RESONANCE POSITIONS

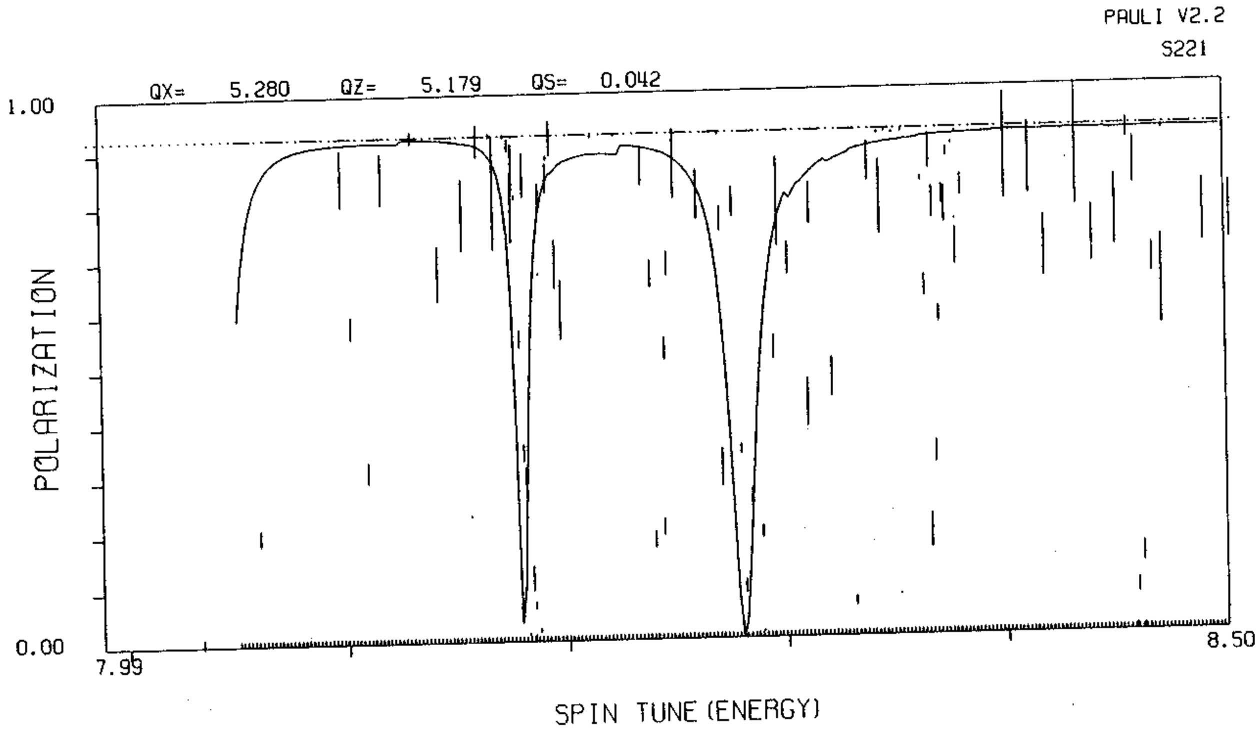
SPEAR POLARIZATION VS. SPIN TUNE




We thank the Directorate of DESY for their support over portions of the past six years,and our colleagues there for their warm hospitality. Our sincerest thanks to John Bell,who patiently devoted two days to explaining his paper with Leinaas to one of us in December 1986.We also thank Dr.P.J.Bryant for inviting us to give this lecture at CERN.

FIGURE 1 LINEAR RESONANCE (D-K) CALCULATION

SPEAR POLARIZATION VS. SPIN TUNE

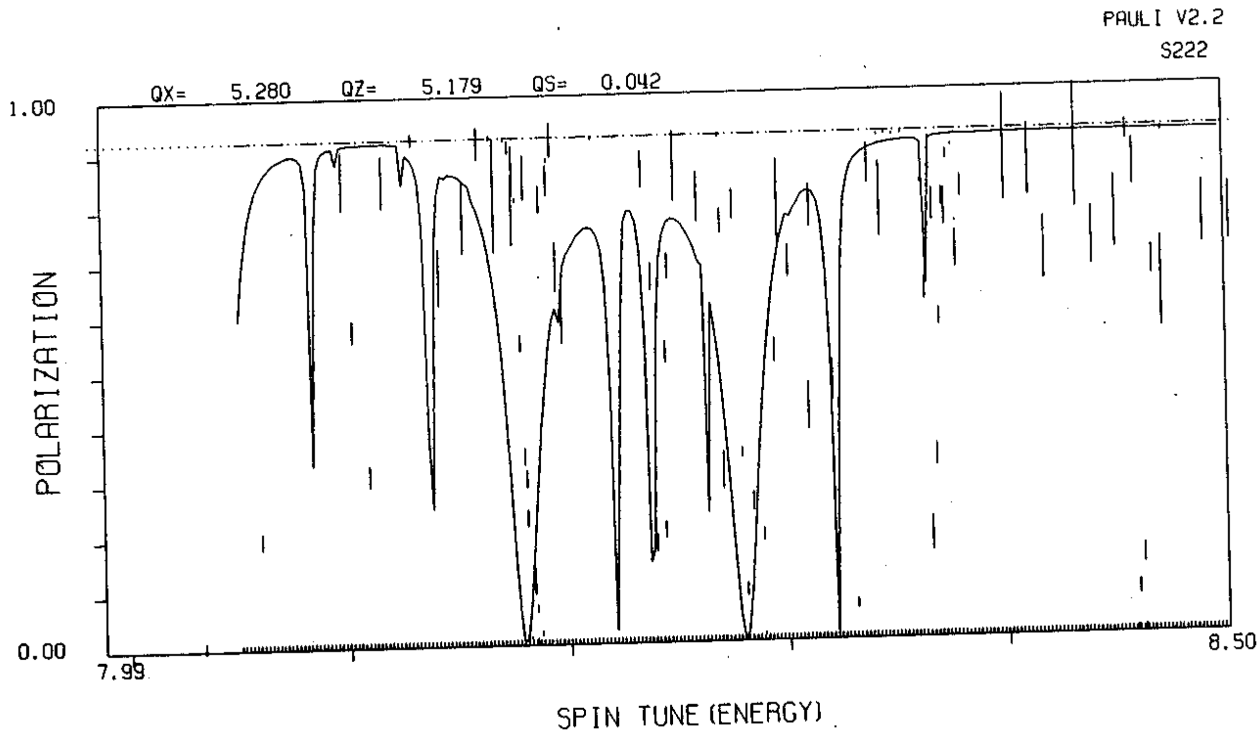



1/4 ERRORS
ERRORS: BEND, QUAD ROLL; QUAD DISPLACEMENT; SEXTUPOLES ON
LINEAR RESONANCES (DERBENEV_KONDRATENKO) ONLY

preliminary

FIGURE 2 - EFFECT OF ENERGY SPREAD ON LINEAR RESONANCES

SPEAR POLARIZATION VS. SPIN TUNE




1/4 ERRORS
ERRORS: BEND, QUAD ROLL; QUAD DISPLACEMENT; SEXTUPOLES ON
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- *References pertinent only to the appendices are found at the end*

## Appendix A
## Properties of Closed Orbit Spin Solution

The periodic solution to the BMT equation containing only fields of the accelerator evaluated on the closed orbit is $\vec{n}$. The direction cosines of $\vec{n}$ are $(u, v, w)$. These are functions of azimuth around the machine.

The so-called "spin phase" is a function of azimuth which increases by $2\pi\nu$, for each increase of $2\pi$ in azimuth. It is called $\xi$. The equation which is obeyed by $\xi$ is

$$\frac{d\xi}{d\theta} = \frac{\vec{\Omega}_0 \cdot (\vec{n} + \vec{n}_0)}{1 + \vec{n} \cdot \vec{n}_0}$$

$\vec{\Omega}_0$ is the precession frequency in the BMT equation

$$\frac{d\vec{\Sigma}}{d\theta} = \vec{\Omega}_0 \times \vec{\Sigma}$$

and the same equation for $\vec{n}$ replacing $\vec{\Sigma}$.

$$\xi(\theta + 2\pi) = \xi(\theta) + 2\pi\nu$$

This definition of the spin tune,which can be found by integrating the differential equation for $\xi$, does not depend on the starting point of the one turn integral around the machine,since $\frac{d\xi}{d\theta}$ is a periodic function of azimuth.In the differential equation, $\vec{n}_0$ stands for the $\vec{n}$ at an arbitrary starting point in the machine,at which point we define $\theta = 0$.

The nonperiodic solutions to the BMT equation can be written in terms of a vector $\vec{\Sigma}$ which is a complex function of the direction cosines of the (real) periodic solution $\vec{n}$. Explicitly the coordinates of $\vec{\Sigma}$ are:

$$\Sigma_x = \frac{wu + iv}{\sqrt{1 - w^2}}\, e^{-i\xi}$$

$$\Sigma_s = \frac{wv - iu}{\sqrt{1 - w^2}}\, e^{-i\xi}$$

$$\Sigma_z = -\sqrt{1 - w^2}\, e^{-i\xi}$$

Since the precession frequency in the BMT equation is real,both $\vec{\Sigma}$ and $\vec{\Sigma}^*$ are solutions,and together with $\vec{n}$ they form a complete set of basic vectors for all solutions of this equation. There are thus three solutions,one periodic and two which are



quasiperiodic. These give a natural coordinate system for the closed orbit spin states.

Relations obeyed by the closed orbit spin solutions are:

$$\vec{n} \cdot \vec{n} = 1 \qquad \vec{n} \cdot \vec{\Sigma} = 0 \qquad \vec{\Sigma}^* \cdot \vec{\Sigma} = 2$$

$$\vec{n} \times \vec{\Sigma} = i\vec{\Sigma} \qquad \vec{\Sigma}^* \times \vec{\Sigma} = -2i\vec{n}$$

$$\Sigma_s^* \Sigma_s = 1 - n_s^2$$

If we let $\vec{n}(\theta)$ be the quantization axis,the corresponding spinors $\chi$ will satisfy the Schrödinger equation

$$-i\frac{d\chi}{ds} = H_0 \chi$$

with $H_0 = -\frac{1}{2}\vec{\sigma} \cdot \vec{\Omega}_0(s)$. An explicit solution for these spinors can be given in terms of the polar and azimuthal angles $\Theta_s, \phi_s$ of $\vec{n}$ . $\chi_+(\theta)$ is the spin up solution and $\chi_-(\theta)$ the spin down solution

$$\chi_+(\theta) = e^{-i\xi(\theta)\frac{\vec{\sigma}\cdot\vec{n}(\theta)}{2}} \begin{pmatrix} e^{-\frac{i}{2}\phi_s} \cos\frac{\Theta_s}{2} \\ e^{\frac{i}{2}\phi_s} \sin\frac{\Theta_s}{2} \end{pmatrix}$$

$$\chi_-(\theta) = e^{-i\xi(\theta)\frac{\vec{\sigma}\cdot\vec{n}(\theta)}{2}} \begin{pmatrix} -e^{-\frac{i}{2}\phi_s} \sin\frac{\Theta_s}{2} \\ e^{\frac{i}{2}\phi_s} \cos\frac{\Theta_s}{2} \end{pmatrix}$$

For reference we give the BMT spin precession frequency $\vec{\Omega}_0$

$$\vec{\Omega}_0 =$$

$$-\frac{e}{m}\Big[(a + \frac{1}{\gamma})\vec{B} - (a + \frac{1}{1+\gamma})\vec{\beta} \times \vec{E} - \frac{a\gamma}{1+\gamma}\vec{\beta}(\vec{\beta} \cdot \vec{B})\Big]$$

$a = \frac{g-2}{2}$, $g$ is the electron's g-factor. Numerically, $a = .001159652$.

## Appendix B
## Derivation of Vacuum Expectation Values

We will be discussing the vacuum expectation value of a bilinear product of electromagnetic field operators as a function of the time separation between the times corresponding to each operator, assuming that the spatial distances also vary along a circular arc corresponding to uniform motion on a circle at highly relativistic speeds. Everything which is done is a completely standard technique, well-known to quantum field theorists. We will find that these vacuum expectation values, which can be thought of as correlation functions, peak strongly near small time separations between the two points. They remain large for time separations $\sim \frac{\rho}{\gamma}$ $(c = 1)$, where $\rho$ is the radius of curvature for the circular arc. We often must perform integrals over combinations of these vacuum expectation values. The various cases are tabulated at the end of this appendix.

Our starting point is the usual technique for quantizing the electromagnetic field (see the book by Berestetskii, Lifschitz and Pitaevskii, page 5ff). The vector potential operator is

$$\hat{\mathbf{A}} = \sum_{k,\alpha}\left(\hat{c}_{k\alpha}\mathbf{A}_{k\alpha} + \hat{c}^{\dagger}_{k\alpha}\mathbf{A}^{*}_{k\alpha}\right)$$

where **boldface** capital letters denote vector fields. The $\hat{c}$'s are the standard creation and annhilation operators obeying

$$\hat{c}\hat{c}^{\dagger} - \hat{c}^{\dagger}\hat{c} = 1$$

$\hat{c}$'s with differing values of the subscript $k\alpha$ commute. The subscript $k$ refers to the photon momentum and $\alpha$ refers to the photon polarization. The coefficients of the creation and annihilation operators are the classical fields

$$\mathbf{A} = \sqrt{\frac{2\pi}{\omega}}\mathbf{e}^{(\alpha)}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$$

A time-like metric is used: $ab \equiv a_0 b_0 - \mathbf{a}\cdot\mathbf{b}$. All operators will be treated in the Heisenberg representation, which means the operators have explicit time dependence, and the states are fixed. The symbol $\mathbf{e}^{(\alpha)}$ denotes the unit vectors associated with the polarization. There are two independent polarizations, both of which are perpendicular to the vector $\mathbf{k}$ (We do not use our usual form $\vec{k}$, in order to conform with Berestetskii et al's notation for the purposes of this appendix only).

The vacuum expectation value for the bilinear product at two independent space time points $x$ and $x'$ is denoted

$$< 0|\hat{\mathbf{A}}(x)\hat{\mathbf{A}}(x')|0 >$$



We wish to calculate this function explicitly. From it, the vacuum expectation values of all possible bilinear products of the electric and magnetic fields can then be calculated by differentiating according to the standard recipe for calculating the fields from the vector potential in the radiation gauge.

We evaluate the vacuum expectation value by using the Fourier expansion of $\hat{\mathbf{A}}$. Only $\hat{c}\hat{c}^{\dagger}$ contributes, and only then if the photon variables are the same for both operators. Our desired vacuum expectation value is then the *single sum* over photons

$$< 0|\hat{\mathbf{A}}_i(x)\hat{\mathbf{A}}_j(x')|0 >=$$

$$\sum_{\mathbf{k},\alpha}\frac{2\pi}{\omega}\mathbf{e}^{(\alpha)}_i\mathbf{e}^{*(\alpha)}_j e^{-ik(x-x')}$$

$i, j$ can take on the values 1,2,3 to indicate the different components of $\hat{\mathbf{A}}$.

After passing from "box quantization" where $\mathbf{k}$ is only allowed to have a countable number of discrete values over to the continuum limit, where any photon momentum is possible, it is shown in most quantum mechanics texts that this amounts to a replacement

$$\sum_{\mathbf{k}} \rightarrow \int\int\int \frac{d^3k}{(2\pi)^3}$$

We will define a Lorentz invariant function $D$ of the space time interval $x$

$$D(x) \equiv (x_0 - i0)^2 - \mathbf{x}^2$$

The time interval $x_0$ gets a small negative imaginary part for reasons of causality. We take equation (90.29) on page 385 of Berestetskii et al :

$$\int f(k_\mu)e^{-ikx}\frac{d^3k}{\omega} = -f(i\partial_\mu)\frac{4\pi}{D}$$

Using this relation, which is not difficult to prove, we have

$$< 0|\hat{\mathbf{A}}_i(x)\hat{\mathbf{A}}_j(x')|0 >= -\frac{1}{\pi}\sum_{\alpha}\mathbf{e}^{(\alpha)}_i\mathbf{e}^{*(\alpha)}_j\frac{1}{D(x-x')}$$

We see immediately that reversing the sign of the time difference is equivalent to taking the complex conjugate.

Next we have to find expression for the bilinear products of electric and magnetic fields and explicitly form the sum over polarizations. We have the relations

$$\mathbf{E} = -\dot{\mathbf{A}} \qquad\qquad \mathbf{B} = \nabla\times\mathbf{A}$$

We will convert the various bilinear products of the fields into various second partial derivatives of $\frac{1}{D}$. As an example of how this is done, we calculate $< 0|\mathbf{E}_z(x)\mathbf{E}_z(x')|0 >$ in some detail. The other relationships are listed in Tables B-I,II,III below.

$$< 0|\mathbf{E}_z(x)\mathbf{E}_z(x')|0 >= -\frac{1}{\pi}\partial_t\partial_{t'}\sum_\alpha \mathbf{e}_z^{(\alpha)}\mathbf{e}_z^{*(\alpha)}\frac{1}{D} = \frac{1}{\pi}\partial^2_{\Delta t}\sum_\alpha \mathbf{e}_z^{(\alpha)}\mathbf{e}_z^{*(\alpha)}\frac{1}{D}$$

In the last equation we use the identity $\partial_{t'} = -\partial_{\Delta t}$ where $\Delta t \equiv x_0 - x_0'$,etc. The unit vector along the photon direction $\mathbf{n}_k = \mathbf{k}/\omega$, together with the two unit vectors for the polarization,form a complete orthonormal basis set,so we have

$$\sum_\alpha \mathbf{e}_i^{(\alpha)}\mathbf{e}_j^{*(\alpha)} + \mathbf{n}_{ki}\mathbf{n}_{kj} = \delta_{ij}$$

Making use of the Fourier transform relation we can write

$$\mathbf{n}_k = -\frac{\partial}{\partial_{\Delta t}}$$

This comes from the formula (90.29) in Berestetskii et al,because we have the integral involving

$$e^{-ik(x-x')} = e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')-i\omega(t-t')}$$

So

$$< 0|\mathbf{E}_z(x)\mathbf{E}_z(x')|0 >= \frac{1}{\pi}[\partial^2_{\Delta t} - \partial^2_{\Delta z}]\frac{1}{D}$$

We can tabulate all of the possible bilinear relations involving only the electric field in the Table B-I. Blank entries indicate the answer may be found by reflection around the main diagonal of the array. For Table B-II,the minus sign indicates antisymmetry about the main diagonal.All of the operators in the table are supposed to operate on $\frac{1}{\pi D}$. If we carry out the indicated double differentiation,we get the entries in Table B-VII divided by $\pi D^3$.



**Table B-I Operators for $< 0|\mathbf{E}(x)\mathbf{E}(x')|0 >$**

| | $\mathbf{E}_x(x')$ | $\mathbf{E}_y(x')$ | $\mathbf{E}_z(x')$ |
|---|---|---|---|
| $\mathbf{E}_x(x)$ | $\partial^2_{\Delta t} - \partial^2_{\Delta x}$ | $-\partial_{\Delta x}\partial_{\Delta y}$ | $-\partial_{\Delta x}\partial_{\Delta z}$ |
| $\mathbf{E}_y(x)$ | | $\partial^2_{\Delta t} - \partial^2_{\Delta y}$ | $-\partial_{\Delta y}\partial_{\Delta z}$ |
| $\mathbf{E}_z(x)$ | | | $\partial^2_{\Delta t} - \partial^2_{\Delta z}$ |

**Table B-II Operators for $< 0|\mathbf{E}(x)\mathbf{B}(x')|0 >$**

| | $\mathbf{B}_x(x')$ | $\mathbf{B}_y(x')$ | $\mathbf{B}_z(x')$ |
|---|---|---|---|
| $\mathbf{E}_x(x)$ | 0 | $\partial_{\Delta t}\partial_{\Delta z}$ | $-\partial_{\Delta t}\partial_{\Delta y}$ |
| $\mathbf{E}_y(x)$ | – | 0 | $\partial_{\Delta t}\partial_{\Delta x}$ |
| $\mathbf{E}_z(x)$ | – | – | 0 |

**Table B-III Operators for $< 0|\mathbf{B}(x)\mathbf{B}(x')|0 >$**

| | $\mathbf{B}_x(x')$ | $\mathbf{B}_y(x')$ | $\mathbf{B}_z(x')$ |
|---|---|---|---|
| $\mathbf{B}_x(x)$ | $\partial^2_{\Delta y} + \partial^2_{\Delta z}$ | $-\partial_{\Delta x}\partial_{\Delta y}$ | $-\partial_{\Delta x}\partial_{\Delta z}$ |
| $\mathbf{B}_y(x)$ | | $\partial^2_{\Delta x} + \partial^2_{\Delta z}$ | $-\partial_{\Delta y}\partial_{\Delta z}$ |
| $\mathbf{B}_z(x)$ | | | $\partial^2_{\Delta x} + \partial^2_{\Delta y}$ |

**Table B-IV** $< 0|\mathbf{E}(2)\mathbf{E}(1)\,|0>$

| | $\mathbf{E}_x(1)$ | $\mathbf{E}_y(1)$ | $\mathbf{E}_z(1)$ |
|---|---|---|---|
| $\mathbf{E}_x(2)$ | $v(1)$ | $v(2)$ | $v(3)$ |
| $\mathbf{E}_y(2)$ | $v(2)$ | $v(4)$ | $v(5)$ |
| $\mathbf{E}_z(2)$ | $v(3)$ | $v(5)$ | $v(6)$ |

**Table B-V** $< 0|\mathbf{E}(2)\mathbf{B}(1)\,|0>, < 0|\mathbf{B}(2)\mathbf{E}(1)|0>$

| | $\mathbf{B}_x(1)$ | $\mathbf{B}_y(1)$ | $\mathbf{B}_z(1)$ |
|---|---|---|---|
| $\mathbf{E}_x(2)$ | 0 | $v(7)$ | $v(8)$ |
| $\mathbf{E}_y(2)$ | $-v(7)$ | 0 | $v(9)$ |
| $\mathbf{E}_z(2)$ | $-v(8)$ | $-v(9)$ | 0 |

| | $\mathbf{E}_x(1)$ | $\mathbf{E}_y(1)$ | $\mathbf{E}_z(1)$ |
|---|---|---|---|
| $\mathbf{B}_x(2)$ | 0 | $-v^T(7)$ | $-v^T(8)$ |
| $\mathbf{B}_y(2)$ | $v^T(7)$ | 0 | $-v^T(9)$ |
| $\mathbf{B}_z(2)$ | $v^T(8)$ | $v^T(9)$ | 0 |

**Table B-VI** $< 0|\mathbf{B}(2)\mathbf{B}(1)\,|0>$

| | $\mathbf{B}_x(1)$ | $\mathbf{B}_y(1)$ | $\mathbf{B}_z(1)$ |
|---|---|---|---|
| $\mathbf{B}_x(2)$ | $v(10)$ | $v(2)$ | $v(3)$ |
| $\mathbf{B}_y(2)$ | $v(2)$ | $v(11)$ | $v(5)$ |
| $\mathbf{B}_z(2)$ | $v(3)$ | $v(5)$ | $v(12)$ |



**Table B-VII** Functions Generated

| Index<br>i | Operator | Function<br>v(i) |
|---|---|---|
| 1 | $\partial^2_{\Delta t} - \partial^2_{\Delta x}$ | $4(\Delta t^2 - \Delta x^2 + \Delta y^2 + \Delta z^2)$ |
| 2 | $-\partial_{\Delta x}\partial_{\Delta y}$ | $-8\Delta x\Delta y$ |
| 3 | $-\partial_{\Delta x}\partial_{\Delta z}$ | $-8\Delta x\Delta z$ |
| 4 | $\partial^2_{\Delta t} - \partial^2_{\Delta y}$ | $4(\Delta t^2 + \Delta x^2 - \Delta y^2 + \Delta z^2)$ |
| 5 | $-\partial_{\Delta y}\partial_{\Delta z}$ | $-8\Delta y\Delta z$ |
| 6 | $\partial^2_{\Delta t} - \partial^2_{\Delta z}$ | $4(\Delta t^2 + \Delta x^2 + \Delta y^2 - \Delta z^2)$ |
| 7 | $-\partial_{\Delta t}\partial_{\Delta z}$ | $8\Delta t\Delta z$ |
| 8 | $\partial_{\Delta t}\partial_{\Delta y}$ | $-8\Delta t\Delta y$ |
| 9 | $-\partial_{\Delta t}\partial_{\Delta x}$ | $8\Delta t\Delta x$ |
| 10 | $\partial^2_{\Delta y} + \partial^2_{\Delta z}$ | $4(\Delta t^2 - \Delta x^2 + \Delta y^2 + \Delta z^2)$ |
| 11 | $\partial^2_{\Delta x} + \partial^2_{\Delta z}$ | $4(\Delta t^2 + \Delta x^2 - \Delta y^2 + \Delta z^2)$ |
| 12 | $\partial^2_{\Delta x} + \partial^2_{\Delta y}$ | $4(\Delta t^2 + \Delta x^2 + \Delta y^2 - \Delta z^2)$ |

One can already see from table B-VII above that,out of the 12 functions, $v(12) = v(6)$,$v(11) = v(4)$ and $v(10) = v(1)$ . Only nine of them are really independent.

We enumerate the 12 independent functions generated by operating with the operators in the tables above on the function $\frac{1}{\pi D}$. We name these $v(i)$ ,where $i = 1...12$. Tables B-IV,V,VI serve to define the $v(i)$ in terms of the fields appearing in the corresponding place in tables B-I,II,III. In the Table B- VII, after the operator, we give the function generated multiplied by $\pi D^3$.

In Table B-V $v^T(i) \equiv v^*_{t\to -t}(i) = v(i)$. The last equality is due to time reversal invarience.

Next we have to put in the details of the actual trajectory for each of the 9 possible functions which determine the 36 possible vacuum expectation values.

Because the motion remains in a horizontal plane (we assume),$\Delta z = 0$ always. We will choose the two points in the vacuum expectation value symmetrically about $t = 0$, at angles of $\pm\frac{\theta}{2}$ on either side of the x axis. We treat only the case of a horizontal bend. This implies that we have the further relations $v(2) = v(3) = v(5) = v(7) = v(9) = 0$. Also, $v(10) = v(1), v(11) = v(4)$, and $v(12) = v(1)$. For planar motion we are reduced to three different functions, which we will call types $A_1, A_2$ and $B$. .

We assume the trajectory is a circle of radius $\rho$ which passes through the point $\Delta x = \Delta y = 0$ . The equation of motion for a circle being traversed with uniform velocity $\beta$ gives the relations,for two points located symmetrically with respect to $y = 0$

$$\Delta x = 0 \qquad \Delta y = 2\rho \sin\frac{\theta}{2} \qquad \Delta z = 0$$

For positive $\rho$ the electron curves to the left as it moves around the circle. We have the relations $\beta = \frac{v}{c}$ ,$\omega = \frac{\beta c}{\rho}$ We make a power series expansion of both numerator and denominator for the functions in Table B-VII as a function of the angle $\theta$ between the two points. It is useful to express the time in units of the correlation time

$$\Delta t \equiv \frac{z\rho}{\beta^2\gamma} \qquad \text{with } c = 1$$

$z$ is then a dimensionless time. $z \sim 1$ for time separations of order of one correlation time.

Also

$$\theta = \omega \Delta t = \frac{z}{\beta\gamma}$$



As is well known,$\theta \sim \frac{1}{\gamma}$. The denominator $D$ will have a simple form in terms of $z$ and $\gamma$. We need only to consider the leading term in the gamma dependence of $D$ ,since there is a factor of $\frac{1}{\pi D^3}$ in front of all the $v(i)$. This is

$$D = \frac{z^2}{\gamma^4}\rho^2\left(1 + \frac{z^2}{12}\right)$$

Table B-VII lists the numerators $v(i)$. Terms with powers of $\theta$ higher than $\theta^4$ can be dropped,since they will not have a pole at $z = 0$ when we divide them by $\pi D^3$.

**Table B-VIII Functions v(i) vs. $\theta$**

| Index<br>i | Function<br>$v(i)/\theta^2\rho^2$ | Type |
|---|---|---|
| 1 | $4\left[(1+\frac{1}{\beta^2}) - \frac{\theta^2}{12} + \frac{\theta^4}{360}\right]$ | $A_1$ |
| 2 | 0 | |
| 3 | 0 | |
| 4 | $4\left[(\frac{1}{\beta^2} - 1) + \frac{\theta^2}{12} - \frac{\theta^4}{360}\right]$ | $B$ |
| 5 | 0 | |
| 6 | | $A_1$ |
| 7 | 0 | |
| 8 | $-\frac{8}{\beta}\left[1 - \frac{\theta^2}{24} + \frac{\theta^4}{1920}\right]$ | $A_2$ |
| 9 | 0 | |
| 10 | | $A_1$ |
| 11 | | $B$ |
| 12 | | $A_1$ |



We could now deduce the behavior of the vacuum expectation values of bilinear products of the electromagnetic fields as a function of the time difference, specialized to the case of a circular orbit. But these are not the physical quantities of direct importance here. Three types of vacuum expectation values appear in the theory.(If recoil were included,the number of different types would increase to six.) Neglecting recoil,we will need to know

$$< 0|\mathbf{a}\cdot\mathbf{\Omega}(\frac{\theta}{2})\mathbf{b}\cdot\mathbf{\Omega}(-\frac{\theta}{2})|0 >$$

where $\mathbf{a},\mathbf{b}$ are arbitrary vectors and $\mathbf{\Omega}$ is the spin precession vector in the radiation field. This is the BMT Hamiltonian with radiation fields. Also we must calculate

$$< 0|\mathbf{a}\cdot\mathbf{\Omega}(\frac{\theta}{2})\mathbf{E}\cdot\beta(-\frac{\theta}{2})|0 >$$

and

$$< 0|\mathbf{E}\cdot\beta(\frac{\theta}{2})\mathbf{E}\cdot\beta(-\frac{\theta}{2})|0 >$$

All of these operators have an additional time dependence because they depend on the direction of $\beta$. We have this quantity for motion in a circle by differentiation of the coordinate equation above

$$\beta_x = \mp\beta\sin\frac{\theta}{2} \qquad \beta_y = \beta\cos\frac{\theta}{2}$$

We expand in powers of $\theta$ and convert it to an expression in terms of the dimensionless correlation time $z$. Carrying out these operations gives

$$< 0|\mathbf{E}\cdot\beta(\frac{\theta}{2})\mathbf{E}\cdot\beta(-\frac{\theta}{2})|0 >=$$

$$\gamma^8\omega^4\frac{\left(4-\frac{5z^2}{3}\right)}{\pi z^4 d^3}$$

where $d$ stands for $(1+\frac{z^2}{12})$. We put it in this form to make it easy to evaluate $\int dt$ over the singularity at $t=0$. We will do this below.

Similiarly,after a lengthy calculation with SMP (see references),

$$< 0|\mathbf{a}\cdot\mathbf{\Omega}(\frac{\theta}{2})\mathbf{E}\cdot\beta(-\frac{\theta}{2})|0 >=$$

$$\frac{e}{m}\gamma^7\omega^4\frac{\left(4z+a\frac{z^3}{3}\right)}{\pi z^4 d^3}\mathbf{a}_z$$

If instead the order of the operators is reversed in the vacuum expectation value,we get the same expression with $i \to -i$. The above expressions are valid for an arbitrary anomalous magnetic moment $a \neq 0$. If we set $a = 0$,then

$$< 0|\mathbf{a}\cdot\mathbf{\Omega}(\frac{\theta}{2})\mathbf{b}\cdot\mathbf{\Omega}(-\frac{\theta}{2})|0 >=$$

$$\left(\frac{e}{m}\right)^2\gamma^6\omega^4\frac{\left(8[\mathbf{a}_x\mathbf{b}_x+\mathbf{a}_z\mathbf{b}_z]+[4-\frac{5z^2}{3}]\mathbf{a}_s\mathbf{b}_s+4z(\mathbf{a}\times\mathbf{b})_z\right)}{\pi z^4 d^3}$$

The additional corrections to the numerator of the last formula if $a \neq 0$ are

$$a(1+a)\left(8\mathbf{a}\cdot\mathbf{b}+4z(\mathbf{a}\times\mathbf{b})_z\right)+\frac{a^2}{3}z^2\left(\mathbf{a}_z\mathbf{b}_z-5\mathbf{a}_x\mathbf{b}_x\right)-\frac{a^2}{12}z^2\mathbf{a}_z\mathbf{b}_z$$

Combining the terms proportional to $z^2$,this can also be written as

$$a(1+a)\left(8\mathbf{a}\cdot\mathbf{b}+4z(\mathbf{a}\times\mathbf{b})_z\right)+\frac{a^2}{12}z^2\left(3\mathbf{a}_z\mathbf{b}_z-20\mathbf{a}_x\mathbf{b}_x\right)$$

Unless it is explicitly stated otherwise,we will assume $a = 0$ in the other parts of this report in order to keep the formulas as simple as possible. The corrections due to $a \neq 0$ will,in every case, be of negligible importance.

It should be clear that the results of this section are only asymptotic formulas.These are correct to the leading power of $\gamma$ as $\gamma \to \infty$. This should be an excellent approximation in all of the cases of current interest,where $\gamma > 5 \times 10^4$.

### Method of Integrating Over Singularities

In this section we collect all information about integrating the singular functions derived above for the various types of integrals needed. There are three functions: the "square" of the Thomas-BMT combination of fields,the cross term between this and the $\vec{E}\cdot\vec{\beta}$ operator and the square of the latter.

In the first order calculation,we need only to know $\int dt$ over the three types of singular functions. Changing to the dimensionless time $z$ this becomes $\int dz/\omega\beta\gamma$ and we have (using either Gutbrod's tables or the one on page 151 of the book by Sokolov and Ternov),

$$< 0|\vec{E}\cdot\vec{\beta}(\frac{z}{2})\vec{E}\cdot\vec{\beta}(-\frac{z}{2})|0 >= \frac{55}{24\sqrt{3}}\gamma^7\omega^3\,\delta(t)$$

$$< 0|\vec{\Omega}\cdot\vec{a}(\frac{z}{2})\vec{E}\cdot\vec{\beta}(-\frac{z}{2})|0 >= -i\frac{e}{m}a_z\gamma^6\omega^3\delta(t)$$



$$< 0|\vec{\Omega}\cdot\vec{a}(\frac{z}{2})\vec{\Omega}\cdot\vec{b}(-\frac{z}{2})|0 >= \left(\frac{e}{m}\right)^2\gamma^5\omega^3\left[\frac{35}{24\sqrt{3}}\vec{a}\cdot\vec{b}+\frac{5}{6\sqrt{3}}a_sb_s-i(\vec{a}\times\vec{b})_z\right]\delta(t)$$

These formulas are to be understood in the sense that we convert the time integral to an integral over $z$,which then gives the same result as integrating over the $\delta$ functions above.

In all cases when we integrate with respect to time, we will neglect the variation of any other factors within a correlation time,considering only the rapidly varying vacuum expectation values and treating other functions multiplying them as constants.

The vacuum expectation values should be multiplied by $\hbar$ if we use a system of units where $\hbar \neq 1$. If $\vec{E}\cdot\vec{\beta}$ and $\vec{\Omega}\cdot\vec{a}$ appear in the opposite order to the one above , the sign should be reversed.

### References for Appendices


- *Berestetskii,Lifshitz and Pitaevskii*
  **V.B.Berestetskii,E.M.Lifschitz and L.P.Pitaevskii**,*Quantum Electrodynamics*,Landau and Lifschitz Course of Theoretical Physics Vol.4 2nd Edition,Pergamon Press

- *SMP*
  The **SMP**,"Symbolic Manipulation Program", was used in the version 1.5.0 which is maintained on the R02 Vax at DESY.

- *Gutbrod*
  **F.Gutbrod**,*Z.Phys.*,**168**,177(1962). The table of integrals is on page 194.