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Comment on "Boundary Conditions from Path Integrals"

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We want to clarify some points in a recent Letter by Jaroszewicz[1]. In our opinion this author missed some important facts and recent results on the subject.

1) The classical Lagrangian in Eq.(2) of Ref.1 is not the correct one to be used in the corresponding path integral. Following our general theory as described in Refs.2-4 it rather must be replaced by the *effective Langrangian*:

$$L_{eff} = \frac{m}{2}(\dot{r}^2 + r^2\dot{\phi}^2) - \frac{\lambda - \frac{1}{4}}{2mr^2}, \tag{1}$$

where the path integral is defined in the "product form" [5] (real time):

$$\begin{aligned} K(\tau; r, r_0, \phi, \phi_0) &= \lim_{N\to\infty} \left(\frac{m}{2\pi i\epsilon}\right)^N \prod_{j=1}^{N-1} \int_0^\infty r^{(j)} dr^{(j)} \int_{-\infty}^\infty d\phi^{(j)} \\ &\times \exp\left(i \sum_{j=1}^N \left\{ \frac{m}{2\epsilon}\left[(r^{(j)} - r^{(j-1)})^2 + r^{(j)}r^{(j-1)}(\phi^{(j)} - \phi^{(j-1)})^2 \right] - \epsilon \frac{\lambda - \frac{1}{4}}{2mr^{(j)}r^{(j-1)}} \right\} \right) \\ &= \int_{-\infty}^\infty \frac{dk}{2\pi} e^{ik(\phi-\phi_0)} \lim_{N\to\infty} \left(\frac{m}{2\pi i\epsilon}\right)^{\frac{N}{2}} \prod_{j=1}^{N-1} \int_0^\infty dr^{(j)} \prod_{j=1}^N \mu_\nu[r^{(j)}] e^{\frac{im}{2\epsilon}(r^{(j)} - r^{(j-1)})^2}. \end{aligned} \tag{2}$$

In the last step standard path-integration methods where applied (Fourier transformation in ϕ). The variable ϕ has to be interpreted as lying on the multisheeted plane, see Eq.(3) of Ref.1. ν is given by $\nu = \sqrt{\lambda + k^2}$, and the functional measure $\mu_\nu[r]$ reads as given in Refs.2 and 9:

$$\mu_\nu[r^{(j)}] = \sqrt{\frac{2\pi m}{i\epsilon} r^{(j)}r^{(j-1)}} \exp\left(-\frac{m}{i\epsilon} r^{(j)}r^{(j-1)}\right) I_\nu\left(\frac{m}{i\epsilon} r^{(j)}r^{(j-1)}\right). \tag{3}$$

The additional "quantum potential" $\Delta V = -\frac{1}{8mr^2}$ is of crucial importance in order that the path integral in polar coordinates (r, ϕ) is well defined. The necessity of such quantum corrections was already observed by DeWitt[6]; for the special case discussed here it was first calculated by Arthurs[7].

2) The space-time transformation yielding Eq.(10) of Ref.1 is not correct. The author's remark that higher order terms in the transformation do not contribute is wrong. In contrary: Again an additional quantum potential $V_{qu} = \frac{1}{8mr^2}$ appears if the prescription of Ref.2 is applied; in fact, the resulting path integral which is nothing but the path integral for Liouville quantum mechanics was completely solved in Ref.3. However, the quantum potential coming from the space-time transformation cancels exactly the quantum potential ΔV of the effective Lagrangian, so that Eq.(10) of Ref.1 is correct by chance. Furthermore, the author did not really calculate (and did not claim to do so) any path integral so that his discussion concerning path integrals seems at least incomplete to us. In Ref.3 we computed the path integral for Liouville quantum mechanics by performing a space-time transformation yielding a radial path integral with generalized angular momentum. The latter could easily be solved using well-known path integral identities[8,9].

3) The author's claim that path integrals contain more information than the corresponding differential equation is rather misleading. The path integral as the kernel of a unitary operator corresponds to a self-adjoint Hamiltonian and implies a specific Schrödinger equation which is defined on some domain D in some Hilbert space

$\mathcal{H}$. The problem of self-adjointness is, of course, one of the hardest tasks in Hilbert space theory, see e.g. Ref.10 and 11. However, the correct path integral is not determined entirely by the action but requires in addition a careful definition of the path-integration measure which in turn depends on the correct boundary conditions. Of course, for particular examples it might be easier to analyse these boundary conditions from the path integral prescription than from the operator formalism.

4) Following our prescription given in Refs.2-4 we can immediately write down the path integral solution of the path integral for the original problem of Ref.1 yielding (we restrict ourselves to the case $D = 2$):

$$\begin{aligned} K(\tau; r, r_0, \phi, \phi_0) &= \frac{m}{i\tau} \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ik(\phi-\phi_0)} \exp\left[\frac{im}{2\tau}(r^2 + r_0^2)\right] I_\nu\left(\frac{m}{i\tau} r r_0\right) \\ &= \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ik(\phi-\phi_0)} \int_0^\infty p\, dp \exp\left(-i\frac{p^2}{2m}\tau\right) J_\nu(pr) J_\nu(pr_0), \end{aligned} \tag{4}$$

where an integral relation for Bessel functions[12] ($\mathrm{Re}(\nu) > -1$) has been used. As noted by Jaroszewicz the sign ambiguity in the order of the Bessel functions J_ν must be clarified for values of ν lying in the intervall $0 < |\nu| < 1$. According to Eq.(8) of Ref.1, an additional term containing Bessel functions J_ν with negative order implies an additional contribution $\sin\pi\nu K_\nu$ instead of only I_ν in Eq.(4) above. Because a term like this is absent in the first line of Eq.(4), one immediately concludes $\nu > 0$ and the sign is fixed. Therefore the path integral formalism gives the correct solution of the original potential problem in a unique way.

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