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Generalized Periodic-Orbit Sum Rules for Strongly Chaotic Systems [1]

by

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Abstract

The periodic-orbit theory of Gutzwiller is the only known semiclassical quantization scheme that can be applied to non-integrable systems. We present a generalization of this theory that leads to absolutely convergent periodic-orbit sum rules, using as an example strongly chaotic billiard systems.

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A semiclassical technique that is often used in studies of energy spectra of non-integrable quantum systems is the periodic-orbit theory of Gutzwiller.[1,2] It is derived from Feynman's path integral and culminates in an expansion of the density of states as a sum over all periodic orbits of the corresponding classical system. One fundamental problem of this theory is the fact that the sum over periodic orbits is at best conditionally convergent. In this Letter we show that for a large class of non-integrable systems generalized periodic-orbit sum rules can be written down that are absolutely convergent. As an explicit example we discuss the Gaussian level density, i. e. a smeared level density using a Gaussian smearing.

We consider a quantum system with two degrees of freedom, whose corresponding classical system has only isolated unstable hyperbolic periodic orbits. Then the density of states of the system with Hamiltonian operator $\hat{\mathrm{H}}$ can semiclassically ($\hbar \to 0$) be represented as

$$\begin{aligned} \sum_n \delta(E-E_n) &= -\frac{1}{\pi}\lim_{\varepsilon\to 0} \mathbf{Im}\,\mathrm{Tr}\left(\frac{1}{E+i\varepsilon-\hat{H}}\right) \\ &= \langle d(E)\rangle + \frac{1}{\pi\hbar}\mathbf{Re}\sum_\gamma\sum_{k=1}^\infty T_\gamma \frac{\exp\{ik(\frac{S_\gamma}{\hbar}-\nu_\gamma\frac{\pi}{2})\}}{\sqrt{|2-\mathrm{Tr}M_\gamma^k|}} \quad . \end{aligned} \tag{1}$$

Here $< d(E) >$ is the mean level density corresponding to the Thomas-Fermi approximation. γ labels all primitive (oriented) periodic orbits and k is the number of their multiple traversals. $S_\gamma = \oint_\gamma p\mathrm{d}q$ is the classical action around γ for the given energy E, and $T_\gamma = \frac{\partial S_\gamma}{\partial E}$ is the period of γ. The phases ν_γ depend on the focussing of trajectories close to the periodic orbit γ, and M_γ is a 2×2 matrix which describes the stability properties of γ. In case of unstable hyperbolic orbits, M_γ has eigenvalues $\lambda_{1,2} = e^{\pm u_\gamma}$, where $u_\gamma > 0$ is the stability exponent.[4] This means

$$\sqrt{|2-\mathrm{Tr}M_\gamma^k|} = 2\sinh\left(\frac{ku_\gamma}{2}\right) \quad . \tag{2}$$

In general the double sum on the right-hand side of Eq. (1) is not absolutely convergent. The usual way to ensure convergence is to introduce complex energies $E \to E + i\frac{\Gamma}{2}$. This corresponds to a substitution of the δ-functions by Breit-Wigner curves

$$\delta(E-E_n) \implies -\frac{1}{\pi}\mathbf{Im}\frac{1}{E+i\frac{\Gamma}{2}-E_n} = \frac{1}{\pi}\frac{\frac{\Gamma}{2}}{(E-E_n)^2+(\frac{\Gamma}{2})^2} \quad , \tag{3}$$

i. e. a smoothing of the level density. In general it turns out, however, that the smoothing parameter Γ has to be a function of the energy E in order to ensure absolute convergence.

To keep the discussion as simple as possible, from now on we confine ourselves to plane billiard systems with elastic reflections at the boundary. In this case $S_\gamma = pl_\gamma$ and $T_\gamma = \frac{m}{p}l_\gamma$, where l_γ is the length of the orbit γ and $p = \sqrt{2mE}$ is the momentum of the 'bouncing ball' with mass m. The phases ν_γ are twice the number of reflections from the boundary. The stability exponents u_γ depend only on geometrical quantities and are energy-independent. Then the introduction of a positive imaginary part of p corresponds to an exponential damping of the contribution of orbits with large length l_γ. In order to estimate under what conditions the periodic-orbit sum converges absolutely we consider

$$\begin{aligned} \sum_\gamma\sum_{k=1}^\infty\left|\mathbf{Re}\left[\frac{ml_\gamma}{p}\frac{\exp\{ik(\frac{p}{\hbar}l_\gamma-\nu_\gamma\frac{\pi}{2})\}}{\sqrt{|2-\mathrm{Tr}M_\gamma^k|}}\right]\right| &< \sum_\gamma\sum_{k=1}^\infty\frac{2ml_\gamma}{|p|}\frac{\exp\{-\frac{\mathbf{Im}p}{\hbar}kl_\gamma\}}{\exp\{\frac{ku_\gamma}{2}\}-\exp\{-\frac{ku_\gamma}{2}\}} \\ &< \frac{\mathrm{const}}{|p|}\sum_\gamma\sum_{k=1}^\infty l_\gamma\exp\{-k(\frac{\mathbf{Im}p}{\hbar}l_\gamma+\frac{u_\gamma}{2})\} \ , \end{aligned} \tag{4}$$

where in the last step we assumed that $u_\gamma \geq u_{min} > 0$ for all γ. The last double sum converges if the sum over terms with $k = 1$ converges. Assuming that the average number $N(l)$ of periodic orbits with length l_γ smaller than l increases asymptotically like an exponential[5]

$$N(l) \sim \frac{e^{\tau l}}{\tau l} \; , \; l \to \infty \; , \tag{5}$$

where $\tau > 0$ denotes the topological entropy, and further that the stability exponents satisfy asymptotically

$$\exp\{-\frac{u_\gamma}{2}\} = O(\exp\{-\frac{\bar{u} l_\gamma}{2}\}) \; , \; l_\gamma \to \infty \; , \; \bar{u} \geq 0 \; , \tag{6}$$

the sum is convergent if the integral

$$\int_{l_0}^{\infty} \mathrm{d}l \, \exp\{\tau l\} \exp\{-\frac{\mathbf{Im}p}{\hbar} l - \frac{\bar{u}}{2} l\} \tag{7}$$

converges. This leads to the condition

$$\frac{\mathbf{Im}p}{\hbar} > \tau - \frac{\bar{u}}{2} =: \frac{\sigma}{\hbar} \tag{8}$$

for the imaginary part of p.[6] Eq. (8) is a condition on the minimum width Γ_{min} of the Breit-Wigner curves (3) of the smoothed level density, $\Gamma > \Gamma_{min} := 4\sigma\mathbf{Re}p$. For billiards with finite area A the mean level spacing is asymptotically constant. When Γ_{min} gets larger than the mean level spacing it is impossible to resolve different peaks. Therefore it would be favourable to have a different kind of smoothing which leads to convergent sums but with no condition on the minimum width of the peaks.

An example of a strongly chaotic system for which an absolutely convergent periodic-orbit theory exists is the Hadamard-Gutzwiller model,[7,8,9] a quantum mechanical system which describes the motion of a particle on a surface of constant negative curvature. For this model the periodic-orbit sum rules are even exact, i.e. they are valid not only in the semiclassical limit $\hbar \to 0$, but hold for all values of $\hbar$, since they can be rigorously derived from the Selberg trace formula,[10] a celebrated theorem in the mathematics of compact Riemann surfaces. Into these sum rules there enters a general 'smearing function' $h(p)$ which ensures absolute convergence.

In analogy to the Selberg trace formula we can derive a similar trace formula for our billiard systems, that we call *generalized periodic-orbit sum rule* and which is given by

$$\sum_n h(p_n) = \int_0^{\infty} \mathrm{d}p \, \bar{d}(p) h(p) + \frac{1}{\hbar} \sum_\gamma \sum_{k=1}^{\infty} l_\gamma \chi_\gamma^k \frac{g\left(\frac{k l_\gamma}{\hbar}\right)}{2 \sinh \frac{k u_\gamma}{2}} \; , \tag{9}$$

where $\bar{d}(p) := \frac{p}{m} < d(E) >$, $p_n = \sqrt{2mE_n}$, $\chi_\gamma = \exp\{-i\nu_\gamma \frac{\pi}{2}\} \in \{1, -1\}$ and

$$g(x) = \frac{1}{\pi} \int_0^{\infty} \mathrm{d}p \, h(p) \cos(px) \; . \tag{10}$$

A rigorous derivation of Eq. (9) that uses only convergent sums leads to the following three conditions for the otherwise *arbitrary* function $h(p)$:

- $h(p)$ is an even function of p
- $h(p)$ is analytic in the strip $|\mathbf{Im}p| \leq \sigma + \varepsilon$ for some $\varepsilon > 0$, where σ is defined in Eq. (8)
- $|h(p)| \leq a|p|^{-2-\delta}$ for some $\delta > 0, a > 0, |p| \to \infty$.

Under these conditions all series and the integral in Eq. (9) converge absolutely.

The 'trace formula' (9) is the main result of this Letter. For any choice of an admissible 'smearing function' $h(p)$ one obtains a nicely convergent periodic-orbit sum rule which establishes a striking and 'apparently paradoxical' *duality relation* between the quantal energy spectrum $\{E_n\}$ and the length spectrum $\{l_\gamma\}$ of the classical periodic orbits. These sum rules provide a substitute, appropriate for quantum systems with hard chaos, for the Bohr-Sommerfeld-Einstein (WKB) quantization rules.

In Ref. 9 we have presented first numerical results based on the generalized periodic-orbit sum rule (9) for the Hadamard-Gutzwiller model. For this model the trace formula (9) is exact with[11] $\bar{d}(p) = \frac{A}{2\pi} p \tanh(\pi p)$, A = area of the compact Riemann surface, $u_\gamma = l_\gamma$, $\nu_\gamma = 0$, $\tau = 1$, i.e. $\chi_\gamma = 1$, $\bar{u} = 1$ and $\sigma = \frac{1}{2}$. It was found in Ref. 9 that a very useful smearing function is

$$h_{Gauss}(p') = \exp\{-\frac{(p-p')^2}{\varepsilon^2}\} + \exp\{-\frac{(p+p')^2}{\varepsilon^2}\} \ . \qquad (11)$$

For this choice we obtain from Eq. (9) the *Gaussian level density*

$$\sum_n [\exp\{-\frac{(p-p_n)^2}{\varepsilon^2}\} + \exp\{-\frac{(p+p_n)^2}{\varepsilon^2}\}]$$

$$= \int_0^\infty \mathrm{d}p' \, \bar{d}(p') h_{Gauss}(p') + \frac{\varepsilon}{2\hbar\sqrt{\pi}} \sum_\gamma \sum_{k=1}^\infty l_\gamma \frac{\chi_\gamma^k \cos(\frac{pkl_\gamma}{\hbar})}{\sinh \frac{ku_\gamma}{2}} \exp\{-\frac{\varepsilon^2}{4\hbar^2}(kl_\gamma)^2\} \ , \qquad (12)$$

which is absolutely convergent for any $\varepsilon > 0$ because of the Gaussian damping of orbits with large length l_γ. For small ε the left-hand side of Eq. (12) represents a series of delta like functions of width $\triangle p \sim \sqrt{2}\varepsilon$ having peaks exactly at the level positions $p = p_n$. If ε is made smaller, more and more terms in the sum over classical orbits on the right-hand side of (12) contribute with faster oscillations until eventually they sum up to give peaks at the quantal energies. Each periodic orbit γ contributes an oscillation to the smeared spectral density, which has a 'wavelength' $\triangle p \sim 2\pi\hbar/l_\gamma$, which implies that a resolution of order $\triangle p \sim \sqrt{2}\varepsilon$ requires a summation over the length spectrum up to lengths of order $l_\gamma \sim \sqrt{2}\pi\hbar/\varepsilon$. In Ref. 9 we have demonstrated numerically for the Hadamard-Gutzwiller model that the Gaussian level density (12) can indeed be used to determine the energy eigenvalues.

In summary, we have presented in this Letter the generalized periodic-orbit sum rule (9) which is absolutely convergent and thus avoids the divergencies of the original periodic-orbit theory.

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