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Abstract

We present a refined asymptotic expression of the trace of the heat kernel $\Theta(t) = \mathrm{Tr} e^{t\Delta}$ for billiard systems in two unbounded regions $\Omega \subset \mathbb{R}^2$ with infinite and finite area respectively. Simon [8], [9] gave several proofs for the first system to have discrete spectrum and determined the leading divergence of the trace of the heat kernel. We combine standard techniques for the evaluation of Θ for bounded region billiards with results by Van den Berg [13] for "horn-shaped regions" using an optimized way of dividing the region into "narrow" and "wide" parts to determine the first three terms in the asymptotic expansion of Θ. This yields in addition an asymptotic expression for the spectral staircase $N(\lambda)$ with corresponding accuracy.

We denote points $(x, y) \in \Omega \subset \mathbb{R}^2$ by z. Let Δ be the Laplacian on Ω and $G_\Omega(t|zz')$ be Green's function (heat kernel) for $\Delta - \frac{\partial}{\partial t}$ with Dirichlet boundary condition on $\partial\Omega$, then

$$\Theta(t) = Tre^{t\Delta} = \int_0^\infty dN(\lambda)\, e^{-\lambda t} = \int_\Omega df\; G_\Omega(t|zz) \quad , t > 0$$

where the "staircase function" $N(\lambda)$ is defined by $N(\lambda) = \sum_{\lambda_i \le \lambda} 1$ with λ_i being the ith eigenvalue of the system (degenerate eigenvalues are counted according to their multiplicity).

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Then our method can be described as follows:
i) Integration of $G_\Omega(t|zz)$ over the region $\Omega(t) = \{z \in \Omega | \exists \text{circle } K : z \in K \subset \Omega; \text{diam}(K) = t^{\frac{1}{2}-\epsilon}\}$ ($\epsilon > 0$) with classical methods. The boundary condition influences (roughly speaking) points within a distance of $\sqrt{t}$ from $\partial\Omega$, therefore most of the points of the above mentioned set are not affected by the boundary. For them we can use the free heat kernel with small corrections ("Kac's principle of not feeling the boundary" [13], [6]).
ii) Integration over $\Omega - \Omega(t)$ with Van den Berg's formula [13], quoted in Theorem 4 below. The points of this domain lie between two nearly parallel parts of $\partial\Omega$ ("horn") with distance smaller than $t^{\frac{1}{2}-\epsilon}$. They are strongly affected by both of these parts. Van den Berg obtained an estimate for $G_\Omega(t|zz)$ which ignores the variation of the width of the horn. (Note that Stewartson and Waechter [11] used a similar approximation to evaluate Θ for regions of finite area with cusps.)
This second integration is precise up to order $t^{-\epsilon}$; in our first example there is a competing error term of order $t^{-(\frac{1}{2}-\epsilon)}$ (due to $\Omega(t)$) requiring the choice $\epsilon = \frac{1}{4}$; in our second example any choice $\epsilon > 0$ is allowed.
Our result is

Theorem 1: *Let Δ be the Dirichlet-Laplace-operator on $\Omega = \{(x,y) \in I\!R^2_+ | 0 \leq xy \leq 1\}$, then for $t \searrow 0$*

$$\Theta(t) = Tr e^{t\Delta} \sim -\frac{\log t}{4\pi t} + \frac{A}{4\pi t} - \frac{B}{8\sqrt{\pi t}} + O(t^{-1/4})$$

where

$$\begin{aligned} A &= 1 - 2\left(\sqrt{\pi} - \sum_{n=1}^{\infty}\left(\mathrm{E}_1(n^2) + \frac{\mathrm{erfc}(n\pi)}{n}\right)\right) = -2.0985... \\ B &= -4\,\frac{\pi^{3/2}}{\Gamma^2(1/4)} = -1.6944... \end{aligned}$$

For an early reference about this system see e.g. [12]. Note that Van den Berg's method [13], [14] suffices to determine A though he did not exploit this. The coefficient A could also be found by a careful manipulation with rectangles following the line of Simon's proof 3 in [8]. There is no obvious interpretation of the first two terms (as there is for the classical case [6],[11],[1]). B is (in a short but very informal phrasing) the difference between the length of the hyperbola and the length of two half-axes and the remaining factor is that one which is expected for perimeter corrections.

Theorem 2: *Let Δ be the Dirichlet-Laplace-operator on $\Omega = \{(x,y) \in I\!R^2 | 0 \leq y \leq e^{-|x|}\}$, then for $t \searrow 0$*

$$\Theta(t) = Tr e^{t\Delta} \sim \frac{|\Omega|}{4\pi t} + \frac{2\log t}{8\sqrt{\pi t}} - \frac{B'}{8\sqrt{\pi t}} + O(t^{-\epsilon})$$

(for any ϵ with $0 < \epsilon < \frac{1}{2}$) where

$$\begin{aligned} B' &= -4\sum_{n=1}^{\infty}\left(\mathrm{E}_1(n^2\pi^2) + \frac{\mathrm{erfc}(n)}{n}\right) + \frac{4}{\sqrt{\pi}} + 2\left(\log 2 - 1 + \sqrt{2} - \log(1+\sqrt{2})\right) \\ &= 2.0701... \end{aligned}$$

The last billiard system was introduced in [13] and Van den Berg determined the two leading terms. Note that the first term is analogous to the standard situation, whereas the second and the third term lack any simple interpretation.
To gain direct information about the spectrum we need

Theorem 3: *Let* $0 < b < \lambda_1 \leq \lambda_2 \leq \ldots$ *and* $N(\lambda) = \sum_{\lambda_i \leq \lambda} 1$ *satisfy* $\int_b^\infty \lambda^{-r_0} |dN(\lambda)| < \infty$ *for some* $r_0 > 0$ *and let*

$$\int_b^\infty dN(\lambda) e^{-\lambda t} = \sum_{i=1}^{k} \left(t^{-r_i} (c_i + c_i' \log t) \right) + O(t^{-r_{k+1}})$$

over $0 < t \leq T$ *for some* $T \in I\!R_+$ *and* $r_{k+1} < r_k < \ldots < r_1$. *Then over* $\lambda > b$

$$N(\lambda) = \sum_{i=1}^{k} \frac{\lambda^{r_i}}{\Gamma(r_i + 1)} \left(-c_i' \log \lambda + \left(c_i + c_i' \left(\psi(r_i) + \frac{1}{r_i} \right) \right) \right) + \tilde{O}(\lambda^{r_{k+1}} \log \lambda)$$

where ψ *denotes the digamma function and* $\tilde{O}$ *refers to "log Gaussian error estimates": Let* F *be of bounded variation over every finite interval, where it is continuous and* $\int_0^\infty y^{-r_0} |dF(y)| < +\infty$ *for some* $r_0 \geq 0$; *let either* $f_r(y) = y^r \log y$ *or* $f_r(y) = y^r, r > 0$, *then*

$$F(y) = \tilde{O}(f_r(y)) \;\Leftrightarrow$$

$$\forall \rho > 0 \; \exists M_\rho \; : \quad \left| \int_b^\infty \exp\left(-\frac{1}{2} \rho^2 \left(\log \frac{v}{y} \right)^2 \right) dF(y) \right| \leq M_\rho f_r(v)$$

This theorem can easily be obtained by mimicking the proof of Theorem 7.13 in [4]. (Properties of $\tilde{O}$ are also quoted in [1], p. 30ff.).
Starting from Theorem 1 one finds

Corollary 1: *For the quantum billiard considered in Theorem 1 holds*

$$N(\lambda) \sim \frac{\lambda \log \lambda}{4\pi} + \frac{a}{4\pi}\lambda - \frac{b}{4\pi}\sqrt{\lambda} + \tilde{O}(\lambda^{1/4} \log \lambda)$$

for $\lambda \to \infty$ *where*

$$a = 2\left(\sum_{n=1}^{\infty} \left(\mathrm{E}_1(n^2) + \frac{\mathrm{erfc}(n\pi)}{n} \right) - \sqrt{\pi} \right) + \gamma = -2.5212\ldots$$

$$b = -4 \, \frac{\pi^{3/2}}{\Gamma^2(1/4)} = -1.6944\ldots$$

The staircase function for the quantum billiard in the region $\{0 \leq xy \leq k\} \cap I\!R_+^2$ can be found by replacing λ by $k\lambda$ in the r.h.s. of the corollary. E.g. for $k = 2$:

$$N_{k=2}(\lambda) = \frac{\lambda \log \lambda}{2\pi} + \frac{(a + \log 2)}{2\pi}\lambda - \frac{b\sqrt{2}}{4\pi}\sqrt{\lambda} + \tilde{O}(\lambda^{1/4} \log \lambda)$$

Berry [2] pointed out that the leading term in the last expression coincides with the first term of the asymptotic expansion for the number of the non-trivial zeroes of the Riemann zeta function with imaginary part less than λ :

$$N_{\mathrm{Riemann}}(\lambda) = \frac{\lambda \log \lambda}{2\pi} - \frac{(1 + \log(2\pi))}{2\pi}\lambda + \tilde{O}(\lambda^0)$$

The lack of time-reversal invariance which a hypothetical system whose eigenvalues are given by the imaginary parts of the Riemann zeroes is believed to be subject to [2] could be enforced upon a billiard system by introducing a magnetic field thus establishing an Aharonov-Bohm billiard (c.f. [3], [10]). Note, however, that the terms we determined are not affected by a magnetic field of the Aharonov-Bohm type, i.e. the above formulae also hold for Aharonov-Bohm quantum billiards.
Starting from Theorem 2 one finds

Corollary 2: *For the quantum billiard considered in Theorem 2 holds*

$$N(\lambda) = \frac{\lambda}{2\pi} - \frac{1}{2\pi}\sqrt{\lambda}\log\lambda - \frac{b'}{4\pi}\sqrt{\lambda} + \tilde{O}(\lambda^\epsilon)$$

for $\lambda \to \infty$ *where*

$$b' = B' - 2(2 - \gamma - 2\log 2) \quad .$$

(Here we have used Theorem 7.7 III in [4] which is also quoted in [1].)
To prove Theorem 1 we need the following definition:

Def.1: For t given with $0 < t \ll 1$ let the sets $\Omega(t), \Omega_i$ be as follows:

$$\begin{aligned}
\Omega(t) &= \{(x,y)|x < t^{-1/4} \wedge y < t^{-1/4}\} \cap \Omega \\
\Omega_1(t) &= \{(x,y)|\text{dist}(z,\{xy = 1\}) < \frac{1}{4}t^{1/4}\} \cap \Omega(t) \\
\Omega_3 &= \{(x,y)|x,y < 1/2\} \cap \Omega \\
\Omega_2(t) &= \{(x,y)|\text{dist}(z,\{xy = 0\}) < \frac{1}{4}t^{1/4}\} \cap \Omega(t) - \Omega_3
\end{aligned}$$

(We will sometimes omit the t-dependence of $\Omega_1(t), \Omega_2(t)$; the sets are illustrated in figure 1.)

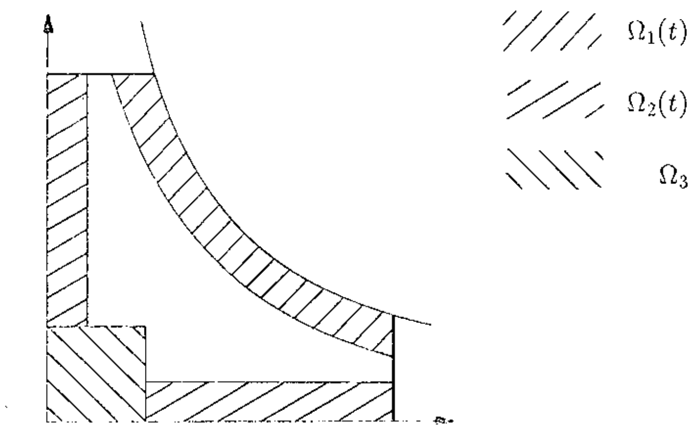

fig. 1

The integration over Ω required for the derivation of Theorem 1 shall be performed separately for $\Omega - \Omega(t), \Omega(t), \Omega_3$, the results being stated in Theorems 4, 5, 6 which together immediately prove Theorem 1. In Appendix 3 we will sketch the proof of Theorem 2.

Theorem 4: *Let $\Omega = \{(x,y) \in I\!R^2_+ | 0 \le xy \le 1\}$ and $\Omega(t)$ be as defined in Definition 1, then*

$$-\frac{1}{2\sqrt{\pi}} \le \lim_{t\to 0} t^{1/4} \left(\int_{\Omega-\Omega(t)} df\ G_\Omega(t|zz) - \left\{ \frac{-\frac{1}{2}\log t}{4\pi t} + \frac{A'}{4\pi t} + \frac{1}{t^{3/4}\sqrt{4\pi}} \right\} \right) \le \frac{1}{4\sqrt{\pi}}$$

where

$$A' = -2\left(\sqrt{\pi} - \sum_{n=1}^{\infty} \left(\mathrm{E}_1(n^2) + \frac{\mathrm{erfc}(n\pi)}{n} \right) \right) \quad .$$

Proof: Theorem 3 in [13] states for $f : [x_0, \infty) \to I\!R_+$ decreasing and $I\!R^2 \supset D \supset D' = \{(x,y) \in I\!R^2 | 0 \le y \le f(x),\ x \ge x_0\}$

$$-\frac{f(x_0)}{4\sqrt{\pi t}} \le \int_{D'} df\ G_D(t|zz) - \frac{1}{\sqrt{4\pi t}} \int_{x_0}^{\infty} dx\ \Theta(t,x) \le \frac{f(x_0)}{8\sqrt{\pi t}}$$

where $\Theta(t,x) = \sum_{n=1}^{\infty} \exp\left(-\frac{\pi^2 n^2 t}{f^2(x)}\right)$. We choose $x_0 = t^{-1/4}$ and $f(x) = \frac{1}{x}$; the integral can be carried out after use of the transformation formula of the Jacobi theta function:

$$\int_{t^{-1/4}}^{\infty} dx\ \sum_{n=1}^{\infty} \exp(-\pi^2 n^2 x^2 t) = \frac{1}{\sqrt{t}} \int_{t^{1/4}}^{\infty} dx\ \sum_{n=1}^{\infty} \exp(-\pi^2 n^2 x^2)$$

$$= \frac{1}{\sqrt{t}} \int_{t^{1/4}}^{1} dx\ \frac{1}{2} \left(\frac{1 + 2\sum_{n=1}^{\infty} \exp\left(-\frac{n^2}{x^2}\right)}{x\sqrt{\pi}} - 1 \right) + \frac{1}{\sqrt{t}} \int_{1}^{\infty} dx\ \sum_{n=1}^{\infty} \exp(-\pi^2 n^2 x^2) =$$

$$= -\frac{\frac{1}{4}\log t}{\sqrt{4\pi t}} - \frac{1 - t^{1/4}}{2\sqrt{t}} + \frac{1}{\sqrt{4\pi t}} \sum_{n=1}^{\infty} \left(\mathrm{E}_1(n^2) + \frac{\mathrm{erfc}(n\pi)}{n} \right) - \frac{1}{\sqrt{\pi t}} \int_0^{t^{1/4}} \frac{dx}{x} \sum_{n=1}^{\infty} \exp\left(-\frac{n^2}{x^2}\right)$$

where $E_1(x) = \int_x^\infty \frac{du}{u} e^{-u}$ denotes the exponential integral. For small x:

$$\sum_{n=1}^{\infty} \exp\left(-\frac{n^2}{x^2}\right) \le 2e^{-\frac{1}{x^2}} \le 2x^3 e^{-\frac{1}{2x^2}}$$

which gives the required bound for the last term. □

Theorem 5: *Let $H = \int_1^\infty \left(\sqrt{1 + \frac{1}{x^4}} - 1 \right) dx$, then*

$$0 \le \lim_{t\to 0} t^{1/4} \left(\int_{\Omega(t)-\Omega_3} df G_\Omega(t|zz) - \left\{ \frac{|\Omega(t) - \Omega_3|}{4\pi t} - \frac{2t^{-1/4} + (H - \frac{3}{2})}{4\sqrt{\pi t}} \right\} \right) \le \frac{1}{\pi}$$

Proof: Van den Berg [13] shows for regions with R-smooth boundary (for $z_0 \in \partial\Omega$ there is a circle K with radius R and $\partial K \cap \Omega = \{z_0\}$): Let $z \in \Omega$ and $\delta = \mathrm{dist}(z, \partial\Omega)$ then

$$G_\Omega(t|zz) \le \frac{1}{4\pi t} \left\{ 1 - e^{-\frac{\delta^2}{t}} + \frac{4\delta}{R} e^{-\frac{\delta^2}{t}} + 4\frac{t}{R^2} \right\} \quad .$$

This inequality will be used for $z \in \Omega_1 \cup \Omega_2$, whereas for $z \in \Omega(t) - (\Omega_1 \cup \Omega_2 \cup \Omega_3)$ we employ

$$G_\Omega(t|zz) \le \frac{1}{4\pi t} \quad .$$

We choose $R = 2$ and integrate (c.f. Lemma 2 in Appendix 2):

$$\begin{aligned} \int_{\Omega(t)-\Omega_3} df G_\Omega(t|zz) &\le \frac{|\Omega(t) - \Omega_3|}{4\pi t} - \frac{1}{4\pi t} \int_{\Omega_1 \cup \Omega_2} e^{-\frac{\delta^2}{t}} df + \frac{1}{2\pi t} \int_{\Omega_1 \cup \Omega_2} \delta e^{-\frac{\delta^2}{t}} df + \frac{|\Omega(t)|}{4\pi} \\ &\le \frac{|\Omega(t) - \Omega_3|}{4\pi t} - \frac{1}{4\sqrt{\pi t}} \left(2t^{-1/4} + (H - \frac{3}{2}) \right) + \frac{1}{2\pi} + \frac{1}{\pi t^{1/4}} + \frac{|\Omega(t)|}{4\pi} \end{aligned}$$

To obtain a lower bound we integrate the following inequalities:

Lemma 1:

$$(i)\quad z \in \Omega_1(t) \cup \Omega_2(t) \Longrightarrow G_\Omega(t|zz) \geq \frac{1-e^{-\frac{\delta^2}{t}}}{4\pi t} - \frac{1}{\pi t}\exp\left(-\frac{1}{32\sqrt{t}}\right)$$

$$(ii)\quad z \in \Omega(t) - (\Omega_1(t) \cup \Omega_2(t) \cup \Omega_3(t)) \Longrightarrow G_\Omega(t|zz) \geq \frac{1}{4\pi t} - \frac{1}{\pi t}\exp\left(-\frac{1}{32\sqrt{t}}\right)$$

(The proof will be given below.)
We find

$$\begin{aligned}\int_{\Omega(t)-\Omega_3} df G_\Omega(t|zz) &\geq \frac{|\Omega(t)-\Omega_3|}{4\pi t} - \frac{1}{4\pi t}\int_{\Omega_1(t)\cup\Omega_2(t)} e^{-\frac{\delta^2}{t}} df - \frac{|\Omega(t)|}{4\pi}\exp\left(-\frac{1}{32\sqrt{t}}\right) \\ &\geq \frac{|\Omega(t)-\Omega_3|}{4\pi t} - \frac{1}{4\sqrt{\pi t}}\left(2t^{-1/4} + (H-\frac{3}{2})\right) - \frac{1}{4\pi} - \frac{|\Omega(t)|}{4\pi}\exp\left(-\frac{1}{32\sqrt{t}}\right)\end{aligned}$$

□

Proof of Lemma 1: (i) A trivial geometric consideration shows that for $z_0 \in \partial\Omega(t) \cap \partial\Omega$ there is a square $Q \subset \Omega$, with one side centred at z_0, tangential to $\partial\Omega$ whose sides have length $l = \frac{1}{2}t^{1/4}$. Therefore

$$G_\Omega(t|zz) \geq G_Q(t|zz) = \frac{1}{4\pi t}\left\{\sum_{n=-\infty}^{+\infty}\left(e^{-\frac{n^2l^2}{t}} - e^{-\frac{(\delta+nl)^2}{t}}\right)\right\}\left\{\sum_{n=-\infty}^{+\infty}\left(e^{-\frac{n^2l^2}{t}} - e^{-\frac{(n+1/2)^2l^2}{t}}\right)\right\} \quad .$$

The first series is $\geq 1 - \exp(-\frac{\delta^2}{t}) - \exp(-\frac{l^2}{4t})$ (because of $\delta < l/2$). Depending on the sign of this bound one uses that the second series is ≤ 1 or $\geq 1 - 2\exp(-\frac{l^2}{4t})$.
(ii) In this case there is a square $Q' \subset \Omega$ with center at z and length $l/\sqrt{2}$ and

$$\begin{aligned}G_\Omega(t|zz) \geq G_{Q'}(t|zz) &= \frac{1}{4\pi t}\left\{\sum_{n=-\infty}^{+\infty}\left(e^{-\frac{n^2l^2}{t}} - e^{-\frac{(n+1/2)^2l^2}{t}}\right)\right\}^2 \\ &\geq \frac{1}{4\pi t}\left(1 - 2\exp\left(-\frac{l^2}{8t}\right)\right)^2 \quad .\end{aligned}$$

This completes the proof. □

Theorem 6:

$$0 \leq \int_{\Omega_3} df\ G_\Omega(t|zz) - \left(\frac{|\Omega_3|}{4\pi t} - \frac{1}{8\sqrt{\pi t}}\right) \leq \frac{1}{\pi} \quad .$$

Proof: Let $G_Q(t|zz')$ be Green's function for the unit square and $G_1(t|zz')$ Green's function in the case of Dirichlet boundary conditions on $\{xy = 0\}$. Then

$$\int_{\Omega_3} df\ G_Q(t|zz) = \left\{\frac{1}{2}\sum_{n=1}^{\infty} e^{-\pi^2n^2t}\right\}^2 \geq \frac{1}{4}\left(\frac{1}{\sqrt{4\pi t}} - \frac{1}{2}\right)^2$$

(for $0 < t < 1/\pi$) by virtue of the Jacobi theta function transformation formula.
Furthermore

$$\int_{\Omega_3} df\ G_1(t|zz) = \frac{1}{4\pi t}\left(\int_0^{1/2} dx\left(1 - e^{-\frac{x^2}{t}}\right)\right)^2 \leq \frac{|\Omega_3|}{4\pi t} - \frac{1}{8\sqrt{\pi t}} + \frac{1}{\pi}$$

for $0 < t < \frac{1}{10}$ (because of erfc$x \leq \frac{2}{x\sqrt{\pi}}$). □
It is evident that a corner can at most affect the constant term in the asymptotic expansion of Θ ([11], quoted also in [1]). Therefore the integration over Ω_3 could (with little loss of rigour) have been taken into account by increasing $L(p(\Omega_1))$ (c.f. Appendix 2) by 1.

Appendix 1: The Hyperbola Integral

Denote the arclength between (1,1) and $(x, \frac{1}{x})$ on the hyperbola $y = \frac{1}{x}$ by $L((x, \frac{1}{x}))$, then (for $x \geq 1$)

$$\begin{aligned} L((x, \frac{1}{x})) &= \int_1^x \sqrt{1 + \frac{1}{u^4}} du \\ &= x - 1 + \int_1^x \left(\sqrt{1 + \frac{1}{u^4}} - 1\right) du \\ &\leq x - 1 + \int_1^\infty \left(\sqrt{1 + \frac{1}{u^4}} - 1\right) du < x \end{aligned}$$

(because $H = \int_1^\infty \left(\sqrt{1 + \frac{1}{x^4}} - 1\right) dx < 1$, see below). Furthermore

$$\begin{aligned} L((x, \frac{1}{x})) &= x - 1 + H - \int_x^\infty \left(\sqrt{1 + \frac{1}{u^4}} - 1\right) du \\ &\geq x - 1 + H - \int_x^\infty \frac{du}{2u^4} = x - 1 + H - \frac{1}{6x^3} \quad . \end{aligned}$$

The quantity H can be determined by partial integration:

$$\begin{aligned} H &= \int_1^\infty \left(\sqrt{1 + \frac{1}{x^4}} - 1\right) dx \\ &= 1 - \sqrt{2} + 2 \int_1^\infty \frac{dx}{x^2\sqrt{1 + x^4}} \\ &= 1 - \sqrt{2} + \mathrm{K}(\frac{1}{\sqrt{2}}) - 2\mathrm{E}(\frac{1}{\sqrt{2}}) + 2\sqrt{\frac{1}{2}} \end{aligned}$$

(E, K : complete elliptic normal integrals of the first and second kind, c.f. formula 3.166,2 in [5]). Legendre's relation between K, E, K', E' yields for $k = \frac{1}{\sqrt{2}}$:

$$\mathrm{E}(\frac{1}{\sqrt{2}}) = \frac{\frac{\pi}{2} + \mathrm{K}^2(\frac{1}{\sqrt{2}})}{\mathrm{K}(\frac{1}{\sqrt{2}})} \quad .$$

With $\mathrm{K}(\frac{1}{\sqrt{2}}) = \frac{1}{4\sqrt{\pi}}\Gamma^2(1/4)$ ([7], p. 359) we finally find:

$$H = 1 - 2\,\frac{\pi^{3/2}}{\Gamma^2(1/4)} = 0.1527...$$

Appendix 2: Boundary Corrections

For merely technical reasons we need two additional sets and for their definition a projection onto the hyperbola:

Def.2: For

$$z \in \{(x,y)|\text{dist}(z,\{xy=1\}) < \frac{1}{4}t^{1/4}\} \cap \Omega$$

there is a unique $z_0 \in \{xy = 1\}$ with $|z - z_0| = \text{dist}(z, \{xy = 1\})$. We define the projection p onto the hyperbola by

$$z_0 = p(z)$$

Def.3:

$$\begin{aligned} \Omega_1'(t) &= p^{-1}(\partial\Omega_1(t) \cap \{xy=1\}) \\ \Omega_1''(t) &= p^{-1}(p(\Omega_1(t)) \end{aligned}$$

(The sets are illustrated in figure 2.)

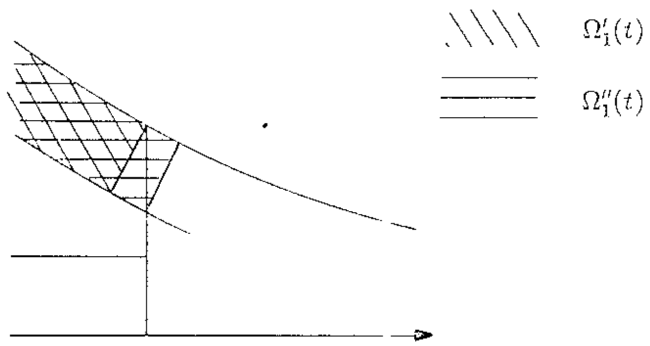

fig. 2

The integration over Ω_1 is performed in new coordinates $s(z) := L(p(z))$ (arclength between (1,1) and the projection of z onto the hyperbola) and $\delta(z) := \text{dist}(z, \{xy\} = 1)$. For a connected $M \subset \{xy = 1\}$ let $L(M)$ be the length of M on the hyperbola, then

$$\begin{aligned} \int_C \epsilon^{-\frac{\delta^2}{t}}\, df &= \int_{p(C)} ds \int_0^{\frac{1}{4}t^{1/4}} d\delta\ (1+\kappa\delta)\epsilon^{-\frac{\delta^2}{t}} \\ &= L(p(C))\frac{\sqrt{\pi t}}{2}\left(1 - \text{erfc}\left(\frac{1}{4t^{1/4}}\right)\right) + \frac{t}{2}\left(1 - \epsilon^{-\frac{1}{16\sqrt{t}}}\right)\int_{p(C)} \kappa ds \end{aligned}$$

for $C = \Omega_1'(t), \Omega_1''(t)$ respectively (κ: curvature of the hyperbola).
With $\Omega_1' \subset \Omega_1 \subset \Omega_1''$:

$$\frac{\sqrt{\pi t}}{2}L(p(\Omega_1'))\left(1 - \text{erfc}\left(\frac{1}{4t^{1/4}}\right)\right) \leq \int_{\Omega_1(t)} \epsilon^{-\frac{\delta^2}{t}}\, df \leq \frac{\sqrt{\pi t}}{2}L(p(\Omega_1'')) + \frac{\pi t}{4}$$

Since

$$\sup_{x\in\Omega_1''(t)} x \leq t^{-1/4} + \frac{t^{3/4}}{4}$$

we have in combination with the bounds in Appendix 1:

$$\begin{aligned} 2(t^{-1/4} - 1 + H - \frac{t^{3/4}}{6}) &\leq L(p(\Omega_1')) \\ L(p(\Omega_1'')) &\leq 2(t^{-1/4} - 1 + H) \end{aligned}$$

finding

$$\int_{\Omega_1(t)} e^{-\frac{\delta^2}{t}}\, df \leq \sqrt{\pi t}(t^{-1/4} - 1 + H) + \frac{\pi t}{4}$$

and

$$\begin{aligned} \int_{\Omega_1(t)} e^{-\frac{\delta^2}{t}}\, df &\geq \sqrt{\pi t}(t^{-1/4} - 1 + H)\left(1 - \operatorname{erfc}\left(\frac{1}{4t^{1/4}}\right)\right) \\ &\geq \sqrt{\pi t}(t^{-1/4} - 1 + H) - t \end{aligned}$$

for $t \to 0$.
The integration over Ω_2 is obtained by inserting the correct length and setting $\kappa = 0$:

$$\sqrt{\pi t}\ (t^{-1/4} - \frac{1}{2}) - t \leq \int_{\Omega_2(t)} e^{\frac{\delta^2}{t}}\, df \leq \sqrt{\pi t}(t^{-1/4} - \frac{1}{2}) \quad .$$

For the integral over $\delta \exp\left(-\frac{\delta^2}{t}\right)$ one finds

$$\begin{aligned} \int_{\Omega_1''} \delta e^{-\frac{\delta^2}{t}}\, df &= \int_{p(\Omega_1'')} ds \int_0^{\frac{1}{4}t^{1/4}} d\delta\ (1 + \kappa\delta)\delta e^{-\frac{\delta^2}{t}} \\ &= \frac{t}{2} L(p(\Omega_1''))\left(1 - e^{-\frac{1}{16\sqrt{t}}}\right) + t^{\frac{3}{2}} \int_{p(\Omega_1'')} \kappa ds \left\{\frac{\sqrt{\pi}}{4}\left(1 - \operatorname{erfc}\left(-\frac{1}{16\sqrt{t}}\right)\right) - \frac{t^{-\frac{1}{4}}}{8} e^{-\frac{1}{16\sqrt{t}}}\right\} \\ &\leq \frac{t}{2} L(p(\Omega_1'')) + \frac{(\pi t)^{3/2}}{8} \leq 2t^{3/4} \end{aligned}$$

for $t \to 0$. The results can be summarized in

Lemma 2: *There is a $t_0 > 0$ with*

$$\frac{1}{4\sqrt{\pi t}}\left(2t^{-1/4} + H - \frac{3}{2}\right) - \frac{1}{2\pi} \leq \frac{1}{4\pi t}\int_{\Omega_1\cup\Omega_2} e^{-\frac{\delta^2}{t}}\, df \leq \frac{1}{4\sqrt{\pi t}}\left(2t^{-1/4} + H - \frac{3}{2}\right) + \frac{1}{16}$$

and

$$\frac{1}{4\pi t}\int_{\Omega_1\cup\Omega_2} \delta e^{-\frac{\delta^2}{t}}\, df \leq \frac{1}{2\pi t^{1/4}}$$

for any $0 < t \leq t_0$.

Appendix 3: Proof of Theorem 2

Since the proof follows exactly the lines of Theorem 1 we only give the main steps:
The calculations can be confined to the region $x > 0$. In analogy to Theorem 4 one has to choose $f(x) = e^{-x}$ and $x_0 = \frac{1}{2}(\epsilon - 1)\log t$ and evaluates

$$\int_{x_0}^{\infty} dx \sum_{n=1}^{\infty} e^{-n^2\pi^2 t e^{2x}} =$$

$$= \frac{1}{\sqrt{4\pi}}\left(t^{-\frac{\epsilon}{2}} - 1\right) + \frac{\epsilon}{4}\log t + \frac{1}{2}\left(\sum_{n=1}^{\infty} \mathrm{E}_1(n^2\pi^2) + \frac{\mathrm{erfc}(n)}{n}\right) - \frac{1}{\sqrt{\pi}}\int_0^{t^{\frac{\epsilon}{2}}} \frac{dy}{y}\sum_{n=1}^{\infty} e^{-\frac{n^2}{y^2}} \quad .$$

Let $\Omega_1 = \{z \in \Omega | x > 0,\ \mathrm{dist}(z, \{ye^x = 1\}) < \frac{1}{4}t^{\frac{1}{2}(1-\epsilon)}\}$, p be a map

$$p : \{(x,y) \in \Omega | \mathrm{dist}(z, \{ye^x = 1\}) < t^{\frac{1}{2}(1-\epsilon)}\} \longrightarrow \{ye^x = 1\}$$

and

$$\begin{aligned} \Omega_1' &= p^{-1}(\partial\Omega_1 - \{ye^x = 1\} - \{x = 0\} - \{x = x_0\}) \\ \Omega_1'' &= p^{-1}(\partial\Omega_1 - \{ye^x = 1\} - \{x = 0\}) \end{aligned}$$

(The sets are illustrated in figure 3)

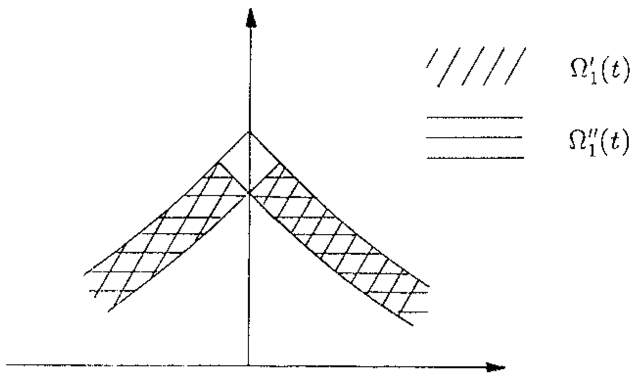

fig. 3

In the bounds of Appendix 2 we have to use

$$\begin{aligned} L(p(\Omega_1'')) &\leq -\frac{1-\epsilon}{2}\log t + \log 2 - 1 + \sqrt{2} - \log(1 + \sqrt{2}) + \frac{1}{4}t^{\frac{1-\epsilon}{2}}(1 + t^{\frac{1-\epsilon}{2}}) \\ L(p(\Omega_1')) &\geq -\frac{1-\epsilon}{2}\log t + \log 2 - 1 + \sqrt{2} - \log(1 + \sqrt{2}) - t^{\frac{1-\epsilon}{2}} \end{aligned}$$

where we have used the (elementary) arclength

$$\int \sqrt{1 + e^{-2x}}dx = \sqrt{1 + e^{-2x}} + \log(e^x + \sqrt{1 + e^{2x}})$$

The upper bound corresponds to the hyperbola billiard; a suitable lower bound is

$$\int_\Omega df G_\Omega(t|zz) \geq \frac{|(\Omega(t) - \Omega_1(t)) \cup (\Omega_1'' \cap \Omega(t))|}{4\pi t} - \frac{1}{4\pi t} \int_{\Omega_1'' \cup \Omega_2} e^{-\frac{\delta^2}{t}} df - \frac{|\Omega(t)|}{\pi t} e^{-\frac{1}{32t^\epsilon}}$$

The complication of the first term of the lower bound is a result of the edge at (0,1); in its neighbourhood the bound of Lemma 1 does not hold, therefore it has to be excluded from the integration.

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