DEUTSCHES ELEKTRONEN-SYNCHROTRON DESY

DESY 89-072
ITP-UH-4-89
June 1989



# Gauge Theory of Volume Preserving Diffeomorphisms

M. Kreuzer

*Institut für Theoretische Physik, Universität Hannover*

ISSN 0418-9833

NOTKESTRASSE 85 · 2 HAMBURG 52

ISSN 0418-9833

DESY 89-072
June 1989

ITP-UH 4/89

# Gauge theory of volume preserving diffeomorphisms

Maximilian KREUZER*

Institut für Theoretische Physik, Universität Hannover,
Appelstraße 2, D – 3000 Hannover 1, WEST GERMANY

## ABSTRACT

Recent interest in volume preserving coordinate transformations arose in the context of the cosmological constant problem and in the light-cone formulation of membrane theory. Covariant quantization of the corresponding gauge theory, "restricted gravity", leads to a finite chain of ghosts for ghosts. We show that negative norm states resulting from higher derivative gauge fixing decouple from the physical sector. The metric describes an additional scalar degree of freedom, the dilaton. IR problems in perturbation theory are solved by introducing a trivial BRS multiplet acting as a Lagrange multiplier. We obtain an IR finite class of gauges which is stable to all orders of perturbation theory and has simple Feynman rules.

* Supported by Deutsche Forschungsgemeinschaft



## 1. Introduction

Progress in physics often was connected with the understanding of the meaning of fundamental constants (with dimension). Possible outcomes of such a process are to put the constant equal to 1 or to turn it into a field [1,2]. The gravitational Lagrangian

$$\mathcal{L}_{inv}(g_{\mu\nu}) = \sqrt{-g}\frac{M_{Pl}^2}{16\pi}R + \sqrt{-g}\Lambda \tag{1.1}$$

contains Newton's constant *and* the cosmological constant $\Lambda$. In Brans-Dicke theory [2,3] Newton's constant is turned into a scalar filed. On the other hand $M_{Pl} \equiv 1$ may set the fundamental mass scale of nature. Then the incredibly tiny experimental bound on the dimensionless ratio

$$\frac{\Lambda}{M_{Pl}^4} < 10^{-120} \tag{1.2}$$

urges an explanation. This is the cosmological constant problem, which currently is considered the biggest puzzle of particle theory by many physicists [4,5].

In *eq.* (1) the cosmological constant $\Lambda$ resembles a Lagrange multiplier $\Lambda(x)$ for the volume element $\sqrt{-g}$. It is thus tempting to fix $\sqrt{-g}$ to a constant or to some scalar function of all other fields. These two possibilities are connected by a Weyl transformation. With the second choice we have a Lagrangian of the form

$$\mathcal{L}(g_{\mu\nu}, \phi^i, \Lambda) = \mathcal{L}_{inv.}(g_{\mu\nu}, \phi^i) - \Lambda\left(\sqrt{-g}e^{\sigma(\phi^i)} - 1\right), \tag{1.3}$$

where $\mathcal{L}_{inv}$ is invariant under general coordinate transformations [6]. Treating the Lagrange multiplier field $\Lambda(x)$ as a scalar general coordinate invariance is broken only by the last term $\Lambda$. The equations of motion constrain $\Lambda(x)$ to an arbitrary constant [6]. A dilaton-like scalar $\sigma(\phi^i)$ appeares in the theory. Its vacuum expectation value $\sigma_0 = \langle\sigma\rangle$ determines the relation between $M_{Pl}$ and Newton's constant, indicating a connection with Brans-Dicke theory. Further, Einstein's equations are recovered with $\Lambda e^{\sigma_0}$ playing the role of the cosmological term. This establishes the classical equivalence of our theory with general relativity. The cosmological term, however, which in Einstein's theory is a parameter of the Lagrangian, has been transformed into an arbitrary integration constant of the equations of motion. The cosmological constant problem has an different guise: "Why is this integration constant zero?" This has nontrivial implications in quantum cosmology, because the values of the cosmological constant in "different universes" are not fixed by a fundamental theory and thus need not coincide.

The type of theory (1.3), in the sequel termed "restricted gravity", has been suggested in the course of translating (broken) scale invariance to curved space [7]. Another interesting approach [8] emerged already some years ago from little group considerations [9] for massless spin-2 particles. It turned out that the minimal gauge group for gravitons is the group of "restricted" coordinate transformations, i.e. diffeomorphisms which preserve the volume element. $\sqrt{-g}$ may simply be fixed to a constant. Then the divergence in the path integral due to the conformal degree of freedom [10] is avoided. Further, since one has arbitrary funtions of $\sqrt{-g}$ at disposal, a polynomial Lagrangian for gravity is easily constructed [8].

Instead of eliminating the trace degree of freedom from the metric by constraining $\sqrt{-g}$ we investigate the most general Lagrangian invariant under restricted coordinate transformations. A kinetic term and a potential for the determinant of the metric arise and the metric comprises an additional scalar degree of freedom, the dilaton. In the parameter limit in which the kinetic term for the dilaton vanishes the determinant is no longer dynamical and we recover the former approach. For convenience we only consider "pure" restricted gravity. Demanding at most second derivatives in the equations of motion and performing a Weyl transformation the most general classical Lagrangian may then be cast into

$$\mathcal{L}_{rg} = 2\sqrt{-g}R + k(\sqrt{-g})\partial^\mu\sqrt{-g}\partial_\mu\sqrt{-g} - V(\sqrt{-g}) \tag{1.4}$$

(we use units such that $\hbar = c = M_{Pl}/\sqrt{32\pi} = 1$). The Lagrangian (1.4) is invariant under infinitesimal coordinate transformations $x' = x - \xi_r$ where the gauge parameter $\xi$ is constrained by $\partial_\mu\xi^\mu_r = 0$. Note that there exists also a larger subgroup of general coordinate transformations where restricted coordinate transformations are combined with a global



scale transformation $\xi^\mu = \xi^\mu_r + \epsilon x^\mu$, $\epsilon = const.$ (i.e. $\partial_\nu\partial_\mu\xi^\mu = 0$) [7]. The additional symmetry excludes a potential for the volume element $V(\sqrt{-g}) = V_0$ and constrains its kinetic term to

$$k(\sqrt{-g}) = \frac{k_0}{\sqrt{-g}}, \tag{1.5}$$

where $k_0$ and $V_0$ are constant. This is the generalization of global scale invariance to curved space mentioned above.

Volume (or area) preserving coordinate transformations also arise as a residual gauge symmetry in the light-cone quantization of (super-) p-branes [11]. And they are the gauge symmetry of string amplitudes in the "dual model gauge" [12]. We investigate the quantization of restricted gravity, which poses nontrivial problems due to constrained gauge parameters (respectively ghosts). An appropriate ghost system in which the constraint $\partial_\mu\xi^\mu = 0$ for the ghosts of general relativity is solved in terms of local fields has been suggested recently [13]. In that approach gauge invariance of antisymmetric tensor fields leads to a finite chain of ghosts for ghosts and covariant quantization is accomplished with higher derivatives in the gauge fixing term.

In the present paper we show that these higher derivatives do not spoil the unitarity of the S-matrix, because the associated negative norm states decouple from the physical sector. Even more, they are essential for incorporating the dilaton as a physical degree of freedom into the metric field $g_{\mu\nu}$. In section 2 we review the BRS [14] invariant action as constructed in *ref.* [13] and perform the quantization by turning Dirac brackets into (anti-) commutators. In section 3 the linearized theory, is investigated. The equations of motion are solved in momentum space using a complete set of polarization tensors. We thus obtain the representation of the BRS operator on asymptotic states and prove unitarity [15] at tree level. Perturbation theory is investigated in section 4. Due to the higher derivatives in the gauge fixing sector some of the propagators have a bad IR-behaviour like $1/k^4$. Thus Green's functions are IR-divergent in 4 dimensions and only S-matrix elements and gauge independent correlations should be independent of an IR-cutoff. This technical problem is avoided by introducing an additional BRS-multiplet acting as a Lagrange multiplier for the gauge condition. In this way an IR-finite gauge can be obtained, which is stable in perturbation theory and for which all "ghosts for ghosts" decouple. The resulting Feynman rules are similar to those of general relativity. Section 5 contains our summary.

## 2. Quantization

Our main tool for a consistent quantization is the BRS-symmetry, a global supersymmetry which after gauge fixing controls the physical properties of a theory comming from its classical gauge invariance. Most important among these properties is the unitarity of the S-matrix in an appropriate "physical" subspace of the Hilbert space of states. The action of the BRS-operator $s$ on the fields is determined by turning the gauge parameters of a bosonic (fermionic) gauge symmetry into ghost fields with fermionic (bosonic) statistics and demanding $s^2 = 0$.

In general relativity the transformation of tensors under coordinate transformations is given by the Lie-derivative

$$\delta_\xi T^{\rho\ldots}_{\alpha\ldots} = \mathcal{L}_\xi T^{\rho\ldots}_{\alpha\ldots} = \xi^\mu \partial_\mu T^{\rho\ldots}_{\alpha\ldots} + \partial_\alpha \xi^\mu T^{\rho\ldots}_{\mu\ldots} + \ldots - \partial_\mu \xi^\rho T^{\mu\ldots}_{\alpha\ldots} - \ldots \tag{2.1}$$

and in particular for the determinant of the metric tensor $\delta_\xi \sqrt{-g} = \partial_\mu(\xi^\mu \sqrt{-g})$. Demanding that the volume element $\sqrt{-g}$ transforms like a scalar

$$\delta_\xi \phi = \xi^\mu \partial_\mu \phi \tag{2.2}$$

we have to fulfil the constraint

$$\partial_\mu \xi^\mu = 0. \tag{2.3}$$

In order to have off shell $s$-invariance of the gauge fixed action we solve the constraint (2.3) of the gauge parameter $\xi^\mu$ in terms of local unconstrained fields. Turning the gauge parameters into ghosts we let

$$\xi^\mu \to c^\mu \equiv \partial_\nu A^{\nu\mu}, \qquad s g_{\mu\nu} = \delta_c\, g_{\mu\nu}, \tag{2.4}$$



and assign ghost number 1 and fermionic statistics to the ghost fields $A^{\nu\mu}$. Physical fields $g_{\mu\nu}, \phi, \ldots$ have vanishing ghost number.

For the construction of an $s$-invariant action which preserves the ghost number we need an additional field $\bar{A}^{\nu\mu}$ with ghost number $-1$. This field can be used to define the additional (anti-)BRS [16] transformation $\bar{s}$

$$\bar{s}T = \delta_{\bar{c}}\, T, \quad \bar{c}^\mu = \partial_\nu \bar{A}^{\nu\mu} \tag{2.5}$$

Due to the gauge invariance

$$A^{\nu\mu} \to A^{\nu\mu} + \partial_\lambda A^{\lambda\nu\mu} \tag{2.6}$$

of *eq.* (2.4) the algebra $s^2 = 0$ does not close on $g_{\mu\nu}$ and $A^{\nu\mu}$. Additional ghosts with higher ghost number have to be introduced. The enlarged algebra

$$s^2 = \bar{s}^2 = \{s, \bar{s}\} = 0 \tag{2.7}$$

proves useful in order to derive the complete multiplets and may also be used to restrict the form of the action by demanding $s$- and $\bar{s}$-invariance. $s$ and $\bar{s}$ commute with partial derivatives $[s, \partial_\mu] = [\bar{s}, \partial_\mu] = 0$ and act as real anti-derivatives on functions of the fields

$$s(AB) = (sA)B + (-)^{|A|} A\, sB, \qquad \bar{s}(AB) = (\bar{s}A)B + (-)^{|A|} A\, \bar{s}B, \tag{2.8}$$

$$(sA)^* = (-)^{|A|} sA^*, \qquad (\bar{s}A)^* = (-)^{|A|} \bar{s}A^*. \tag{2.9}$$

For integer spin fields $A$ the grading $|A|$ coincides with the ghost number.

From the algebra (2.4) – (2.8) the complete ghost multiplet has been obtained in *ref.* [13]. All ghosts are completely antisymmetric tensors $(A^I_G)^{\mu_1\ldots\mu_I}$ of rank $I$, $I \geq 2$, and ghost number $G$ which ranges from $1 - I$ to $I - 1$. (We identify $A^{\nu\mu}$ with $(A^2_1)^{\nu\mu}$ and $\bar{A}^{\nu\mu}$ with $(A^2_{-1})^{\nu\mu}$.) The rank $I$ and the ghost number $G$ with

$$2 \leq I, \qquad 1 - I \leq G \leq I - 1 \tag{2.10}$$

uniquely characterize the ghost fields. They are commuting (anticommuting) if $G$ is even (odd) and are real or purely imaginary

$$(A^I_G)^* = (A^I_G)\,(-)^{\frac{I(I-1)}{2}+G(I-1)}. \tag{2.11}$$

In 4 space-time dimensions $I \leq 4$ and one has altogether 45 ghosts of which 24 anticommute and 21 commute. They are necessary to compensate for the 3 gauge degrees of freedom of restricted coordinate transformations (2.1-3). It is convenient to denote the ghost fields $A^I_G$ by $B^I_G$ if $G+I$ is even, because these fields will become auxiliary (i.e. have algebraic equations of motion).

The complete action of $s$ and $\bar{s}$ on the ghosts contains nonlinear terms which depend on $c^\mu = \partial_\nu(A^2_1)^{\nu\mu}$ and $\bar{c}^\mu = \partial_\nu(A^2_{-1})^{\nu\mu}$ and appear only in the combinations

$$(S^{N+\bar{N}}_{N-\bar{N}})^{\mu_1\ldots\mu_{N+\bar{N}}} \equiv \frac{1}{N!\bar{N}!}\sum_\pi \mathrm{sign}(\pi) c^{\mu_{\pi(1)}}\ldots c^{\mu_{\pi(N)}}\bar{c}^{\mu_{\pi(N+1)}}\ldots\bar{c}^{\mu_{\pi(N+\bar{N})}} \tag{2.12}$$

with antisymmetrized Lorentz-indices. The sum runs over all permutations $\pi$ of $N+\bar{N}$ indices and $G = N-\bar{N}$ is the ghost number. Once the relation

$$s(S^I_{G-1})^{\mu_1\ldots\mu_I} + \bar{s}(S^I_{G+1})^{\mu_1\ldots\mu_I} = \partial_\nu(S^{I+1}_G)^{\nu\mu_1\ldots\mu_I}. \tag{2.13}$$

is established, it is easy to verify that the algebra (2.7) closes on the fields $A^G_I$ if the action of $s$ and $\bar{s}$ is given by

$$sA^I_{I-1} = -\partial A^{I+1}_I + S^I_I, \tag{2.14a}$$

$$sA^I_G = B^I_{G+1}, \qquad I+G \text{ odd}, G \leq I-3, \tag{2.14b}$$

$$sB^I_G = 0, \qquad I+G \text{ even}, \tag{2.14c}$$

$$\bar{s}A^I_G = -B^I_{G-1} - \partial A^{I+1}_{G-1} + S^I_{G-1}, \qquad I+G \text{ odd}, \tag{2.15a}$$

$$\bar{s}B^I_G = \partial B^{I+1}_{G-1} - sS^I_{G-2}, \qquad I+G \text{ even}, \tag{2.15b}$$

$\partial A^I_G$ is a shorthand for $\partial_{\mu_1}(A^I_G)^{\mu_1\ldots\mu_I}$.



The most general $s$- and $\bar{s}$-invariant local action with ghost number 0 is given by the integral over a scalar Lagrangian $L_{inv}$ composed out of the metric (we consider pure gravity), and the gauge fixing and ghost part which always is of the form $s\bar{X} = \bar{s}X$ [17],

$$\mathcal{L} = \mathcal{L}_{inv}(g) + s\bar{X}. \tag{2.16}$$

The classical invariant Lagrangian $\mathcal{L}_{inv}$ is required to contain two derivatives at most and therefore consists of Einstein's action and a kinetic energy and a potential for $\sqrt{-g}$ [7].

$$\mathcal{L}_{inv} = 2\sqrt{-g}R + k(\sqrt{-g})\,g^{\mu\nu}\partial_\mu\sqrt{-g}\partial_\nu\sqrt{-g} - V(\sqrt{-g}). \tag{2.17}$$

(A Weyl transformation is used to bring the curvature term into this form.) The last two terms are responsible for the fact that for $k > 0$ the metric contains a massive physical particle, the dilaton, in addition to the two helicity states of the graviton. As far as the linearized theory is concerned the gauge fixing and ghost Lagrangian enters the action only via $s\bar{s}Y$, where $Y$ is a Lorentz-scalar with vanishing ghost number. $Y$ can be suitably chosen such that the propagators of all fields exist (up to IR-problems; see chapter 4). We require $Y$ to conserve parity and to contain no derivatives in order to avoid higher derivatives and to keep the ghosts $B^I_G$ ($I+G$ even) auxiliary. If in addition $Y$ consists only of the lowest powers of the fields it is given by

$$Y = -\alpha\eta^{\mu\nu}g_{\mu\nu} + \sum_{\substack{0\leq G\leq I-1\\ I+G\ odd}} \alpha^I_G(A^I_{-G}|A^I_G), \qquad \bar{X} = \bar{s}Y \tag{2.18}$$

with real or imaginary parameters $\alpha$, $\alpha^G_I$ according to

$$\alpha^* = -\alpha, \qquad (\alpha^I_G)^* = (-)^{G+1}\alpha^I_G, \tag{2.19}$$

$\eta_{\mu\nu}$ is the Lorentz metric $diag(1,-1,-1,-1)$ and the scalar product ( | ) is introduced as a shorthand for

$$(A^I_{-G}|A^I_G) \equiv \frac{1}{I!}(A^I_{-G})^{\mu_1\ldots\mu_I}(A^I_G)^{\nu_1\ldots\nu_I}\eta_{\mu_1\nu_1}\ldots\eta_{\mu_I\nu_I}. \tag{2.20}$$

Because of the reality properties (2.9), (2.11) and (2.19) the action is real and the S-matrix is (pseudo-) unitary.

We perform the canonical quantization in the linearized theory, which is sufficient to calculate the norms of asymptotic states and the propagators. Thus we expand the Lagrangian (2.16) – (2.18) to second order in

$$h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu} \tag{2.21}$$

and the ghosts $A^I_G$, $B^I_G$. (In our units $M_{Pl} = \sqrt{32\pi}$). Ordered according to ghost number one has

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3 + vertices, \tag{2.22}$$

$$\begin{aligned}\mathcal{L}_0 = \tfrac{1}{2}&\left(\partial^\mu h^{\rho\sigma}\partial_\mu h_{\rho\sigma} - \partial^\mu h^\lambda_\lambda \partial_\mu h^\rho_\rho - 2\partial_\rho h^{\rho\mu}(\partial^\lambda h_{\lambda\mu} - \partial_\mu h^\lambda_\lambda)\right)\\ &+ \tfrac{\kappa}{4}\left(\partial_\mu h^\rho_\rho \partial^\mu h^\lambda_\lambda - M_D^2 (h^\lambda_\lambda)^2\right) + 2\alpha\partial_\mu\partial^\lambda h_{\lambda\nu}(B^2_0)^{\mu\nu}\\ &+ \alpha^2_1(B^2_0|B^2_0 + \partial A^3_0) + 2\alpha^3_0(B^4_0|dA^3_0) + \alpha^4_1(B^4_0|B^4_0),\end{aligned} \tag{2.23}$$

$$\begin{aligned}\mathcal{L}_1 = -2\alpha\partial^\mu&(\partial A^2_{-1})^\nu\partial_\mu(\partial A^2_1)_\nu + \alpha^2_1(dA^2_{-1}|B^3_1) + \alpha^2_1(B^3_{-1}|dA^2_1)\\ &+ (2\alpha^3_0 - \alpha^3_2)(B^3_{-1}|B^3_1) - \alpha^3_2(B^3_{-1}|\partial A^4_1) + 2\alpha^3_0(\partial A^4_{-1}|B^3_1),\end{aligned} \tag{2.24}$$

$$\mathcal{L}_2 = \alpha^2_1(\partial A^3_{-2}|\partial A^3_2) + \alpha^3_2(B^4_{-2}|dA^3_2) + \alpha^3_2(dA^3_{-2}|B^4_2) + (\alpha^4_3 - \alpha^4_1)(B^4_{-2}|B^4_2), \tag{2.25}$$

$$\mathcal{L}_3 = -\alpha^3_2(\partial A^4_{-3}|\partial A^4_3). \tag{2.26}$$

$d$ denotes the exterior derivative , i. e.

$$(B^{I+1}|dA^I) = \frac{1}{I!}(B^{I+1})^{\mu_1\cdots\mu_{I+1}}\partial_{\mu_1}(A^I)_{\mu_2\cdots\mu_{I+1}}, \tag{2.27}$$

and $\partial$ denotes the divergence $(\partial A)^{\mu_2\cdots\mu_I} = \partial_{\mu_1}(A^I)^{\mu_1\cdots\mu_I}$. With these notations the formula for partial integration in scalar products (2.20) of forms is

$$(A^I|dB^{I-1}) = -(\partial A^I|B^{I-1}) + d(\ldots) \tag{2.28}$$

We have anticipated that

$$M_D^2 = \tfrac{1}{2\kappa}\left(\tfrac{\partial}{\partial\sqrt{-g}}\right)^2 V\Big|_{\sqrt{-g}=1}, \qquad \kappa \equiv k(\sqrt{-g})\Big|_{\sqrt{-g}=1}. \tag{2.29}$$



gives the mass $M_D$ of the dilaton $h^\lambda_\lambda$. Further we have chosen $V'(1) = 0$ without loss of generality to make the expansion (2.21) consistent. The choice of the determinant of the background metric in the expansion (2.21) corresponds to the choice of the cosmological term in the Einstein equations of motion.

To simplify our notation we omit in the following indices of partial derivatives if they are contracted with the subsequent index and the indices of the trace of $h_{\mu\nu}$

$$\partial h_\nu \equiv \partial^\mu h_{\mu\nu}, \quad h \equiv h^\lambda_\lambda. \tag{2.30}$$

The same rule is understood for the spatial part of partial derivatives $\nabla = (0, \partial_i)$ (indices $i, j, \ldots$ run over spacial components only). For antisymmetric tensors we need a generalization of the Kronecker tensor $\delta$

$$\delta^{\nu_1\ldots\nu_n}_{\mu_1\ldots\mu_n} = \det(\delta^{\nu_b}_{\mu_a}) \quad a, b = 1, \ldots, n. \tag{2.31}$$

We also identify $B \equiv B^2_0$. Note that the gauge fixing term (2.23)

$$2\alpha B^{\mu\nu}\partial_\mu\partial h_\nu, \qquad B^{\mu\nu} \equiv (B^2_0)^{\mu\nu} \tag{2.32}$$

and the kinetic term (2.24) for the ghosts $A^2_{\pm 1}$ contain second derivatives.

A Hamiltonian formulation of higher derivative theories with the Lagrangian

$$L(q, \dot{q}, \ddot{q}, \ldots, q^{(N)}) \tag{2.33}$$

requires additional canonical coordinates $q^n$, $1 \le n \le N$

$$q^1 = q, \quad q^2 = \dot{q}, \ldots \quad q^N = q^{(N-1)}. \tag{2.34}$$

With the recursively defined canonical momenta

$$p_N = \frac{\partial L}{\partial q^{(N)}}, \quad p_n = \frac{\partial L}{\partial q^{(n)}} - \dot{p}_{n+1} \quad n < N \tag{2.35}$$

and the Hamiltonian

$$H(q^n, p_n) = \sum_{n=1}^{N} \dot{q}^n p_n - L(q, \ldots, q^{(N)}) \tag{2.36}$$

we have the correct Hamiltonian equations of motion [18]

$$\dot{p}_n = -\frac{\partial H}{\partial q^n}, \quad \dot{q}^n = (-)^{|q|}\frac{\partial H}{\partial p_n}. \tag{2.37}$$

If – returning to the notation of relativistic field theory – a Lagrangian depending on second derivatives of the fields has a continuous symmetry

$$s\mathcal{L}(\phi, \partial_\mu\phi, \partial_\mu\partial_\nu\phi) = \partial_\mu \Lambda^\mu, \tag{2.38}$$

the conserved Noether current is given by

$$j^\mu = (s\phi)\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} - \partial_\nu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_\nu\phi)}\right) + (s\partial_\nu\phi)\frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_\nu\phi)} - \Lambda^\mu \tag{2.39}$$

(which reduces to the better known current if $\frac{\partial\mathcal{L}}{\partial(\partial\partial\phi)} = 0$).

In our case 2nd derivatives of the metric field $h_{\mu\nu}$ can be avoided by partial integration of the gauge fixing term in (2.23)

$$2\alpha B^{\mu\nu}\partial_\mu\partial h_\nu = -2\lambda\partial B^\nu\partial h_\nu + \partial(\ldots). \tag{2.40}$$

Then 3 components of the field $B$ become dynamical. (Note that $B$ still is auxiliary in the sense that it has an algebraic equation of motion. It could be eliminated, but then we would need 3 additional canonical coordinates $h_i \equiv \dot{h}_{0i}$ and 3 corresponding canonical momenta). For the ghosts $A^2_{\pm 1}$ we have to introduce $2\cdot 3 = 6$ additional pairs of phase space variables.

According to the decomposition (2.23) – (2.26) of the Lagrangian quantization can be performed for fields with a definite modulus of the ghost number. We denote the canonical momenta by

$$\pi_{\mu\nu} = \frac{\partial\mathcal{L}}{\partial\dot{h}^{\mu\nu}}, \quad (\pi^I_G)^{\mu_1\ldots\mu_I} = \frac{\partial\mathcal{L}}{\partial(\dot{A}^I_G)_{\mu_1\ldots\mu_I}} \tag{2.41}$$



For ghost number 0 we have the constraints

$$(\pi^2_0)^{ij} = 0, \quad (\pi^3_0)^{0ij} = \alpha_1^2 B^{ij}, \quad (\pi^3_0)^{ijk} = 2\alpha_0^3(B^4_0)^{0ijk}, \quad \pi^4_0 = 0. \tag{2.42}$$

The other momenta are related to time derivatives by the nonsingular system of equations

$$\begin{aligned}
\pi_{00} &= \tfrac{\kappa}{2}\dot{h} + \nabla h_0 + 2\alpha\nabla B_0, & \dot{h}^\lambda_\lambda &= \tfrac{2}{\kappa}(\pi_{00} - 2\alpha\nabla B_0 - \nabla h_0),\\
\pi_{0i} &= \tfrac{1}{2}\nabla_i h + \nabla h_i - \alpha\partial B_i, & \dot{h}_{0i} &= \nabla h_i - \tfrac{1}{2\alpha}(\pi^2_0)_{0i},\\
\pi_{ij} &= \dot{h}_{ij} - \delta_{ij}\left((\tfrac{\kappa}{2} - 1)\dot{h} + \partial h_0\right), & \dot{h}_{ij} &= \pi_{ij} + \delta_{ij}\left(\tfrac{1}{2}\pi - \pi_{00} + \nabla h_0 + \alpha\nabla B_0\right),\\
(\pi^2_0)^{0i} &= -2\alpha\partial h^i, & \dot{B}_{0i} &= \nabla B_i - \tfrac{1}{\alpha}\pi_{0i} + \tfrac{1}{\alpha}(\tfrac{1}{2}\nabla_i h + \nabla h_i).
\end{aligned} \tag{2.43}$$

The Dirac brackets [19] coincide with the Poisson brackets for $h_{\mu\nu}$, $\pi^{\mu\nu}$, $B_{0i}$, $(\pi^2_0)^{0i}$, $A^3_0$ and $\pi^3_0$ and follow from the constraints (2.42) for $B_{ij}$, $(\pi^2_0)^{ij}$, $B^4_0$ and $\pi^4_0$. Thus we have the nonvanishing equal time commutation relations

$$\begin{aligned}
[h_{\mu\nu}(t,\mathbf{x}), \pi_{\alpha\beta}(t,\mathbf{y})] &= i\tfrac{1}{2}(\eta_{\mu\alpha}\eta_{\nu\beta} - \eta_{\mu\beta}\eta_{\nu\alpha})\delta^3(\mathbf{x}-\mathbf{y}),\\
[B_{0i}(t,\mathbf{x}), (\pi^2_0)^{0j}(t,\mathbf{y})] &= i\delta^j_i\delta^3(\mathbf{x}-\mathbf{y}),\\
[(A^3_0)_{\alpha\beta\gamma}(t,\mathbf{x}), (\pi^3_0)^{\mu\nu\rho}(t,\mathbf{y})] &= i\delta^{\mu\nu\rho}_{\alpha\beta\gamma}\delta^3(\mathbf{x}-\mathbf{y}).
\end{aligned} \tag{2.44}$$

All constraints are second class confirming that the gauge fixing is complete. In toto we have $10+3+4 = 17$ dynamical degrees of freedom. Due to the 2nd derivatives in the gauge fixing term 3 more initial conditions have to be specified for $h_{\mu\nu}$ on a Cauchy surface so that the metric describes 13 degrees of freedom.

Note that for $\kappa \to 0$ (but $\kappa M_D^2 = m^2 = const.$) there is the additional constraint $\pi_{00} = 2\alpha\nabla B_0 + \nabla h_0$. In that case for $m^2 > 0$ the equations of motion yield a secondary constraint which is second class and these 2 constraints eliminate 1 degree of freedom from the metric field $h_{\mu\nu}$ (the "dilaton" is no longer dynamical). In the patological case $m^2 = \kappa = 0$ the linearized equations of motion are degenerate (there is a gauge freedom $\delta h_{\mu\nu} = \partial_\mu\partial_\nu\Lambda$).

For ghost number $\pm 1$ we have to deal with 2nd derivatives. We introduce the "coordinates" $(\hat{A}_{\pm 1})_i = (\dot{A}^2_{\pm 1})_{0i}$ and the corresponding momenta $(\hat{\pi}_{\pm 1})^i$. The Lagrangian (2.24) implies the relations (see (2.35))

$$(\hat\pi_{\pm1})^i = \mp 2\alpha\partial(\dot{A}^2_{\pm1})^i, \quad (\pi^2_{\pm1})^{0i} + (\dot{\hat\pi}_{\pm1})^i = \mp 2\alpha(\nabla^i\partial(\dot{A}^2_{\pm1})^0 - \nabla^2\partial(A^2_{\pm1})^i) \tag{2.45}$$

and the constraints

$$(\pi^2_{\pm1})^{ij} + \nabla^{[i}(\hat\pi_{\pm1})^{j]} = \pm\alpha_1^2(B^3_{\pm1})^{0ij}, \quad \pi^3_{\pm1} = 0, \quad (\pi^4_{\pm1})^{0ijk} = -\alpha^3_{\mp2}(B^3_{\pm1})^{ijk}, \tag{2.46}$$

where $\alpha^3_{-2} \equiv 2\alpha^3_0$. Again the Dirac brackets coincide with the Poisson brackets for the unconstrained momenta and the corresponding canonical coordinates. Thus the nonvanishing (anti)commutation relations are

$$\begin{aligned} \{(\pi^2_{\pm1})^{\mu\nu}(t,\mathbf{x}),(A^2_{\mp1})_{\alpha\beta}(t,\mathbf{y})\} &= -i\delta^{\mu\nu}_{\alpha\beta}\delta^3(\mathbf{x}-\mathbf{y}), \\ \{(\hat\pi_{\pm1})^i(t,\mathbf{x}),(\hat{A}_{\mp1})_j(t,\mathbf{y})\} &= -i\delta^i_j\delta^3(\mathbf{x}-\mathbf{y}), \\ \{(\pi^4_{\pm1})^{\mu\nu\rho\sigma}(t,\mathbf{x}),(A^2_{\mp1})_{\alpha\beta\gamma\delta}(t,\mathbf{y})\} &= -i\delta^{\mu\nu\rho\sigma}_{\alpha\beta\gamma\delta}\delta^3(\mathbf{x}-\mathbf{y}) \end{aligned} \tag{2.47}$$

and follow from the constraints for the remaining degrees of freedom.

For higher ghost number the calculations are simple. We have the equations

$$(\pi^3_{\pm2})^{0ij} = \alpha_1^2(\partial A^3_{\pm2})^{ij}, \quad (\pi^4_{\pm3})^{0ijk} = \pm\alpha_2^3(\partial A^4_{\pm3})^{ijk} \tag{2.48}$$

and the constraints

$$(\pi^3_{\pm2})^{0ij} = \alpha_2^3(B^4_{\pm2})^{0ijk}, \quad \pi^4_{\pm3} = 0. \tag{2.49}$$

Again all constraints are second class and are valied in the strong sense when Poisson brackets are replaced by Dirac brackets. The nonvanishing canonical commutation relations are

$$\begin{aligned} [(\pi^3_{\pm2})^{\mu\nu\rho}(t,\mathbf{x}),(A^3_{\mp2})_{\alpha\beta\gamma}(t,\mathbf{y})] &= -i\delta^{\mu\nu\rho}_{\alpha\beta\gamma}\delta^3(\mathbf{x}-\mathbf{y}), \\ \{(\pi^4_{\pm3})^{\mu\nu\rho\sigma}(t,\mathbf{x}),(A^4_{\mp3})_{\alpha\beta\gamma\delta}(t,\mathbf{y})\} &= -i\delta^{\mu\nu\rho\sigma}_{\alpha\beta\gamma\delta}\delta^3(\mathbf{x}-\mathbf{y}). \end{aligned} \tag{2.50}$$

The complete number of degrees of freedom is 47. However, the ghosts are introduced in order to compensate for the unphysical degrees of freedom which propagate after gauge fixing. The effective number of degrees of freedom is obtained by counting negative all fields which have the wrong spin-statistics connection, i.e. all ghosts with odd ghost number. Including +3 from the second derivative terms of the metric and −6 from the second derivative terms in the sector with ghost number ±1 we have an effective number of 25 − 22 = 3 pairs of phase space variables.



## 3. Mode decomposition and unitarity

In this section we reveal the particle content of our theory. Grouping the one-particle states into BRS-representations we prove tree-level unitarity according to the quartet mechanism of Kugo and Ojima [15]. (Perturbation theory will be the subject of section 4.)

We first solve the linearized equations of motion by Fourier expansion. For the metric field $h_{\mu\nu}$ and the auxiliary field $B$ we have the coupled system

$$\begin{aligned} -(\Box h_{\mu\nu} - \partial_\mu\partial h_\nu - \partial_\nu\partial h_\mu + \partial_\mu\partial_\nu h - \eta_{\mu\nu}(\Box h - \partial\partial h)) \\ -\frac{\kappa}{2}\eta_{\mu\nu}(\Box + M_D^2)h + \alpha(\partial_\mu\partial B_\nu + \partial_\nu\partial B_\mu) = 0 \\ B = -\frac{\alpha}{\alpha_1^2}d\partial h - \frac{1}{2}\partial(A_0^3) \quad \Rightarrow \quad \alpha_1^2\partial B_\mu + \alpha(\Box\partial h_\mu - \partial_\mu\partial\partial h) = 0. \end{aligned} \tag{3.1}$$

Taking the trace and the divergence of the first equation one derives

$$\Box^4(\Box + M_D^2)h_{\mu\nu} = 0 \quad \text{if } M_D^2 > 0, \qquad \Box^4 h_{\mu\nu} = 0 \quad \text{if } M_D^2 = 0. \tag{3.2}$$

(The case $M_D^2 = 0$ is related to global scale invariance.) If $k^2 = M_D^2 \neq 0$ the polarization of the state has to be proportional to $k_\mu k_\nu$. So there is only one massive state. This implies the ansatz

$$\begin{aligned} h_{\mu\nu}(x) = \int d^4k\,\theta(k_0)e^{ikx}\Big\{ & \delta(k^2 - M_D^2)\tfrac{k_\mu k_\nu}{M_D^2}h_D^\dagger + \delta(k^2)h^{0\dagger}_{\mu\nu} + \delta'(k^2)h^{1\dagger}_{\mu\nu} \\ & + \tfrac{\delta''(k^2)}{2}h^{2\dagger}_{\mu\nu} + \tfrac{\delta'''(k^2)}{3!}h^{3\dagger}_{\mu\nu}\Big\} + h.c. \end{aligned} \tag{3.3}$$

To simplify our notation we have suppressed the momentum dependence of the creation operators $h^\dagger = h^\dagger(\mathbf{k})$.

In order to solve the equations of motion (3.1) with the ansatz (3.3) we expand the creation operators in a basis of polarization tensors $e^{(\lambda)}_{\mu\nu}$

$$h^{n\dagger}_{\mu\nu} = \sum_{(\lambda)} h^{n\dagger}_{(\lambda)}e^{(\lambda)}_{\mu\nu}, \quad n = 0,\ldots,3. \tag{3.4}$$

For massless particles we construct the polarizations in terms of the 4 (frame dependent) linearly independent vectors

$$k_\mu, \quad \bar{k}_\mu = \frac{(k_0,-\mathbf{k})_\mu}{k_0^2 + \mathbf{k}^2}, \quad n^1_\mu, \quad n^2_\mu, \qquad kn^i = \bar{k}n^i = n^1n^2 = 0, \quad (n^1)^2 = (n^2)^2 = -1. \tag{3.5}$$

$n^t$, $t=1,2$ are orthogonal to $k$ and $\bar{k}$. For lightlike $k$ the vector $\bar{k}$ is also lightlike and we have $k\bar{k}=1$. A convenient basis for the graviton polarizations is

$$\begin{aligned} e^l_{\mu\nu} &= k_\mu k_\nu, \quad e^{\bar{l}}_{\mu\nu} = k_\mu \bar{k}_\nu + k_\nu \bar{k}_\mu, \quad e^{\hat{l}}_{\mu\nu} = \bar{k}_\mu \bar{k}_\nu, \quad e^n_{\mu\nu} = n^1_\mu n^2_\nu + n^2_\mu n^1_\nu, \\ e^t_{\mu\nu} &= k_\mu n^t_\nu + k_\nu n^t_\mu, \quad e^{\bar{t}}_{\mu\nu} = \bar{k}_\mu n^{\bar{t}}_\nu + \bar{k}_\nu n^{\bar{t}}_\mu, \quad e^{\pm}_{\mu\nu} = n^1_\mu n^1_\nu \pm n^2_\mu n^2_\nu. \end{aligned} \tag{3.6}$$

The polarizations are labeled by $(\lambda) = l, \bar{l}, \hat{l}, t, \bar{t}, n, +, -$, where the indices $t$ and $\bar{t}$ assume 2 values and summation over contracted indices is understood (Lorentz symmetry is broken by the momentum $k$ and by the choice of the time axis, so we can keep an $O(2)$-symmetry manifest). This basis for symmetric tensors is related to $\eta_{\mu\nu}$ by

$$\eta_{\mu\nu} = e^{\bar{l}}_{\mu\nu} - e^{+}_{\mu\nu} - \frac{k^2 e^{\hat{l}}_{\mu\nu} + \bar{k}^2 e^l_{\mu\nu} - k^2\bar{k}^2 e^{\bar{l}}_{\mu\nu}}{1 - k^2\bar{k}^2}. \tag{3.7}$$

$e^-$ and $e^n$ are transverse and traceless and span the two helicity states of the graviton.

Inserting the ansatz (3.3) – (3.7) into (3.1) a tedious calculation shows that there are 13 independent operators. In the general case $M_D \neq 0$ these are

$$h_D, \quad h^0_l, \quad h^0_{\bar{l}}, \quad h^0_t, \quad h^0_{\bar{t}}, \quad h^0_\pm, \quad h^0_n, \quad h^1_l, \quad h^1_t. \tag{3.8}$$

The remaining operators vanish except $h^1_{\bar{l}}$, $h^1_+$, $h^2_l$, $h^2_{\bar{l}}$, $h^2_t$ and $h^3_l$, which are given by

$$\begin{aligned} h^1_{\bar{l}} &= -\tfrac{\alpha_1^2}{(\alpha)^2} h^0_+, \quad h^1_+ = -\tfrac{\kappa}{2} M_D^2 (2h^0_{\bar{l}} - 2h^0_+ - h^1_l), \\ h^2_l &= 2h^1_{\bar{l}} - 2h^1_+, \quad h^2_{\bar{l}} = -\tfrac{\alpha_1^2}{(\alpha)^2} h^1_+, \quad h^2_t = -\tfrac{\alpha_1^2}{(\alpha)^2} h^0_{\bar{t}}, \quad h^3_l = 2h^2_{\bar{l}}. \end{aligned} \tag{3.9}$$

For $M_D = 0$ all particles are massless. Thus $h_D$ does not occur in the ansatz (3.3). This degree of freedom is replaced by $h^1_+$ which is no longer constrained by equation (3.9b) and equation (3.9c) has to be replaced by

$$h^2_l = 2h^1_{\bar{l}} + (\tfrac{2}{\kappa} - 2)h^1_+, \qquad M_D = 0. \tag{3.10}$$

For the antisymmetric tensor fields we also need a basis of polarization tensors. We choose

$$\epsilon^l_{\mu\nu} = k_\mu \bar{k}_\nu - k_\nu \bar{k}_\mu, \quad \epsilon^t_{\mu\nu} = k_\mu n^t_\nu - k_\nu n^t_\mu, \quad \epsilon^{\bar{t}}_{\mu\nu} = \bar{k}_\mu n^t_\nu - \bar{k}_\nu n^t_\mu, \quad \epsilon^n_{\mu\nu} = n^1_\mu n^2_\nu - n^1_\nu n^2_\mu, \tag{3.11a}$$



$$v^l_{\mu\nu\rho} = 3! k_{[\mu} n^1_\nu n^2_{\rho]}, \quad v^t_{\mu\nu\rho} = 3! k_{[\mu} n^1_\nu \bar{k}_{\rho]}, \quad v^{\bar{l}}_{\mu\nu\rho} = 3! n^1_{[\mu} n^2_\nu \bar{k}_{\rho]}, \tag{3.11b}$$

$$s_{\mu\nu\rho\sigma} = 4! k_{[\mu} n^1_\nu n^2_\rho \bar{k}_{\sigma]} \tag{3.11c}$$

so that we have simple algebraic relations like $d\partial e^{\bar{l}} = -k^2 \epsilon^l$, $d\partial e^t = -k^2 \epsilon^t$ and $d\partial e^{\bar{t}} = -\epsilon^t$.

Now we can solve the remaining equations of motion. For ghost number 0 we have

$$B = \tfrac{\alpha}{\alpha_1^2} d\partial h - \tfrac{1}{2}\partial A^3_0, \qquad B^4_0 = -\tfrac{\alpha_0^3}{\alpha_1^4} dA^3_0. \tag{3.12}$$

These equations imply $\Box^2 A^3_0$ and from an ansatz with terms proportional to $\delta(k^2)$ and $\delta'(k^2)$ we calculate the general solution

$$A^3_0(x) = \int d^4k\, \theta(k_0) e^{ikx} \left\{ \delta(k^2) \sum_{(\lambda)=l,t,\bar{l}} (a^3_0)^\dagger_{(\lambda)} v^{(\lambda)} + (1 - \tfrac{4(\alpha_0^3)^2}{\alpha_1^2\alpha_1^4}) \delta'(k^2) (a^3_0)^\dagger_{\bar{l}} \right\} - h.c. \tag{3.13}$$

$$\begin{aligned} B(x) = \int d^4k\, \theta(k_0) e^{ikx} \Big\{ & \delta(k^2) \left( \tfrac{1}{\alpha} h^{0\dagger}_+ \epsilon^l + \left( \tfrac{\alpha}{\alpha_1^2}(h^{0\dagger}_{\bar{t}} - h^{1\dagger}_t) - \tfrac{i}{2}(a^3_0)^\dagger_t \right) \epsilon^t - 2i \tfrac{(\alpha_0^3)^2}{\alpha_1^2\alpha_1^4} (a^3_0)^\dagger_{\bar{l}} \epsilon^n \right) \\ & + \delta'(k^2) \left( \tfrac{1}{\alpha} h^{1\dagger}_+ \epsilon^l + \tfrac{1}{\alpha} h^{0\dagger}_{\bar{t}} \epsilon^t \right) \Big\} - h.c. \end{aligned} \tag{3.14}$$

$$B^4_0(x) = -i \tfrac{\alpha_0^3}{\alpha_1^4} \int d^4k\, \theta(k_0) e^{ikx} \left\{ \delta(k^2) (a^3_0)^\dagger_{\bar{l}} s \right\} + h.c. \tag{3.15}$$

In addition to the 13 operators contained in the metric there are the 4 independent operators $(a^3_0)_{(\lambda)}$, $(\lambda) = l, t, n$ in the ghost number 0 sector.

For ghost number $\pm 1$ the equations of motion are

$$\begin{aligned} 2\alpha \Box\, d\partial A^2_{\pm 1} + \alpha_1^2 \partial B^3_{\pm 1} &= 0, \quad dB^3_{\pm 1} = 0, \\ (2\alpha_0^3 - \alpha_2^3) B^3_{\pm 1} - \alpha^3_{\pm 2} A^4_{\pm 1} + \alpha_1^2 dA^2_{\pm 1} &= 0, \end{aligned} \tag{3.16}$$

where $\alpha^3_{-2} \equiv -2\alpha_0^3$. They imply $\Box^3 A^2_1 = \Box^3 A^2_{-1} = 0$ and thus

$$\begin{aligned} A^2_{\pm 1}(x) = \int d^4k\, \theta(k_0) e^{ikx} \Big\{ & \delta(k^2) \sum_{(\lambda)=l,t,n,\bar{t}} (a^2_{\pm 1})^\dagger_{(\lambda)} \epsilon^{(\lambda)} + \delta'(k^2) \left( (a^2_{\pm 1})^{1\dagger}_l \epsilon^l + (a^2_{\pm 1})^{1\dagger}_t \epsilon^t \right) \\ & \pm \tfrac{\delta''(k^2)}{2} \tfrac{(\alpha_1^2)^2}{2\alpha(2\alpha_0^3 - \alpha_2^3)} (a^2_{\pm 1})^{0\dagger}_{\bar{t}} \epsilon^t \Big\} + h.c., \end{aligned} \tag{3.17}$$

$$B^3_{\pm 1}(x) = \int d^4k\, \theta(k_0) e^{ikx} \frac{-i\alpha_1^2}{2\alpha_0^3 - \alpha_2^3} \delta(k^2) \left( ((a^2_{\pm 1})^{0\dagger}_n + i \frac{\alpha^3_{\pm 2}}{\alpha_1^2} (a^4_{\pm 1})^\dagger_n) v^l - (a^2_{\pm 1})^{0\dagger}_{\bar{l}} ) v^{\bar{l}} \right) - h.c., \tag{3.18}$$

$$A^4_{\pm 1}(x) = \int d^4k\, \theta(k_0) e^{ikx} \left\{ i\delta(k^2) (a^4_{\pm 1})^\dagger s \right\} - h.c. \tag{3.19}$$

For ghost number $G = \pm 2$ we have

$$\alpha_1^2 d\partial A^3_G + \alpha_2^3 \partial B^4_G = 0 \quad \wedge \quad \alpha^4 B^4_G + \alpha_2^3 d A^3_G = 0 \quad \Rightarrow \quad \Box^2 A^3_G = 0, \tag{3.20}$$

$$A^3_G(x) = \int d^4k\, \theta(k_0) e^{ikx} \left\{ \delta(k^2) \sum_{(\lambda) = l, t, \bar{l}} (a^3_G)^\dagger_{(\lambda)} v^{(\lambda)} + \left(1 + \frac{(\alpha_2^3)^2}{\alpha_1^2 \alpha^4}\right) \delta'(k^2) (a^3_G)^\dagger_{\bar{l}} \right\} - h.c., \tag{3.21}$$

$$B^4_G(x) = -i \frac{\alpha_2^3}{\alpha^4} \int d^4k\, \theta(k_0) e^{ikx} \left\{ \delta(k^2) (a^3_G)^\dagger_{\bar{l}} s \right\} + h.c., \tag{3.22}$$

where $\alpha^4 \equiv \alpha_3^4 - \alpha_1^4$. And finally for $G = \pm 3$

$$A^4_G(x) = \int d^4k\, \theta(k_0) e^{ikx} \left\{ i\delta(k^2) (a^4_G)^\dagger s \right\} - h.c. \tag{3.23}$$

The number of independent operators is in agreement with the counting of phase space variables in section 2.

As the Lagrangian is $s$- and $\bar{s}$-invariant we also must have an on-shell representation of the algebra $s^2 = \bar{s}^2 = \{s, \bar{s}\} = 0$. A nilpotent operator can have only two types of irreducible representations: singlets and doublets. Physical states are characterized by the condition [15]

$$s|phys\rangle = 0. \tag{3.24}$$

In order to prove unitarity of the S-matrix (which is pseudo-unitary in the complete Hilbert space) we have to show that the norm is nonnegative in the physical subspace (3.24). According to the quartet mechanism of Kugo and Ojima [15] it is sufficient to check that all "unphysical" creation operators group into doublets under the BRS transformation and that the norm is positive for the remaining physical states. This means that there must be no BRS-singlets with nonvanishing ghost number and that the commutation relations of the BRS-singlets with ghost number 0 must have the correct signs.



The anti-BRS symmetry is useful since we can have larger multiplets. In fact all $s$-doublets in our theory group into 11 representations of the form $a, sa, \bar{s}a, s\bar{s}a$,

| $a$ | $sa$ | $\bar{s}a$ | $s\bar{s}a$ |
|---|---|---|---|
| $h^0_l$ | $2(a^2_1)^0_l$ | $2(a^2_{-1})^0_l$ | $\frac{2}{\alpha} h^0_+$ |
| $h^0_{\bar{l}}$ | $(a^2_1)^1_l$ | $(a^2_{-1})^1_l$ | $\frac{1}{\alpha} h^1_+$ |
| $h^0_t$ | $(a^2_1)^1_t - (a^2_1)^0_{\bar{t}}$ | $(a^2_{-1})^1_t - (a^2_{-1})^0_{\bar{t}}$ | $\frac{1}{\alpha} h^0_{\bar{t}}$ |
| $(a^3_0)_l$ | $-\frac{i\alpha_1^2}{2\alpha_0^3 - \alpha_2^3} (a^2_1)^0_n + \frac{\alpha_2^3}{2\alpha_0^3 - \alpha_2^3} a^4_1$ | $\frac{i\alpha_1^2}{2\alpha_0^3 - \alpha_2^3} (a^2_{-1})^0_n + \frac{\alpha_2^3}{2\alpha_0^3 - \alpha_2^3} a^4_{-1}$ | $\frac{\alpha_0^3}{\alpha_1^4} (a^3_0)_{\bar{l}}$ |
| $(a^2_1)^0_t$ | $-i(a^3_2)_t$ | $-\frac{\alpha}{\alpha_1^2}(h^0_{\bar{t}} - h^1_t) - \frac{i}{2}(a^3_0)_t$ | $\frac{\alpha_1^2}{2\alpha_0^3 - \alpha_2^3} (a^2_1)^0_{\bar{t}}$ |
| $(a^2_{-1})^0_t$ | $\frac{\alpha}{\alpha_1^2}(h^0_{\bar{t}} - h^1_t) - \frac{i}{2}(a^3_o)_t$ | $-i(a^3_{-2})_t$ | $\frac{\alpha_1^2}{2\alpha_0^3 - \alpha_2^3} (a^2_{-1})^0_{\bar{t}}$ |
| $(a^3_2)^0_l$ | $-(a^4_3)$ | $\frac{i\alpha_1^2}{2\alpha_0^3 - \alpha_2^3} (a^2_1)^0_n - \frac{2\alpha_0^3}{2\alpha_0^3 - \alpha_2^3} a^4_1$ | $\frac{\alpha_2^3}{\alpha_3^4 - \alpha_1^4} (a^3_2)_{\bar{l}}$ |
| $(a^3_{-2})^0_l$ | $-\frac{i\alpha_1^2}{2\alpha_0^3 - \alpha_2^3} (a^2_{-1})^0_n - \frac{2\alpha_0^3}{2\alpha_0^3 - \alpha_2^3} a^4_{-1}$ | $-(a^4_{-3})$ | $-\frac{\alpha_2^3}{\alpha_3^4 - \alpha_1^4} (a^3_{-2})_{\bar{l}}$ |

(3.25)

The remaining 3 creation operators are $h^{0\dagger}_-$ and $h^{0\dagger}_n$, generating the 2 transverse graviton states and $h^\dagger_D$, generating the scalar dilaton state. They commute with $s$ and $\bar{s}$ and generate the physical Hilbert space modulo null-states. The corresponding nonvanishing commutation relations are

$$[h^0_n(\mathbf{k}), h^{0\dagger}_n(\mathbf{k}')] = [h^0_-(\mathbf{k}), h^{0\dagger}_-(\mathbf{k}')] = \frac{\sqrt{\mathbf{k}^2}}{8\pi^3} \delta^3(\mathbf{k} - \mathbf{k}') \tag{3.26}$$

$$[h_D(\mathbf{k}), h^\dagger_D(\mathbf{k}')] = \frac{\sqrt{\mathbf{k}^2 + M_D^2}}{2\pi^3 \kappa} \delta^3(\mathbf{k} - \mathbf{k}') \tag{3.27}$$

Thus the norms of all physical states are positive if $\kappa > 0$, i.e. if the coefficient $k(\sqrt{-g})$ of the kinetic term in the Lagrangian (1.4) is positive at the expansion point $g_{\mu\nu} = \eta_{\mu\nu}$. (3.27) is obtained from the canonical commutation relations (2.44) by observing that the trace of $h_{\mu\nu}$ is given by

$$h(x) = \int d^4k\, \theta(k_0) e^{ikx} \left\{ \delta(k^2 - M_D^2) h^\dagger_D(\mathbf{k}) - \frac{2\alpha}{\kappa M_D^2} \delta(k^2) s\bar{s} h^{0\dagger}_{\bar{l}} \right\} + h.c. \tag{3.28}$$

and that $s\bar{s}h^0_{\bar{l}}$ cannot contribute to the equal time commutators of $h$ with $s$-invariant operators. If the dilaton mass $M_D$ vanishes all particle states propagate on the light cone

$k^2 = 0$. Then $h_+^{1\dagger}$ becomes an independent operator and $\tilde{h}_D^\dagger \equiv 2h_l^{0\dagger} - h_l^{1\dagger}$ becomes $s$-invariant, replacing the creation operator $h_D^\dagger$ for the massive dilaton. Again the nontrivial contribution to the trace comes from the dilaton degree of freedom $\tilde{h}_D$,

$$h(x) = \int d^4k\, \theta(k_0) e^{ikx} \left\{ \delta(k^2)[\tilde{h}_D^\dagger + \alpha s\bar{s}h_l^{0\dagger}] - \tfrac{2\alpha}{\kappa}\delta'(k^2) s\bar{s}h_l^{0\dagger} \right\} + h.c. \tag{3.29}$$

$\tilde{h}_D$ is normalized according to (3.27) with $M_D = 0$.

## 4. Perturbation theory

Having established unitarity at the tree level we now turn to perturbation theory. Although the higher derivative gauge fixing term (2.32) did not generate negative norm states in the physical sector of our theory it leads to severe problems in perturbation theory since it causes IR divergences in the loop expansion. The calculation of the Feynman rules is straightforward. We only give the expressions for the propagators since these reveal the essential features of our theory. Note that in the Lagrangian (2.16) - (2.18) the metric only couples to the first generation of ghosts so that only these ghosts enter 1-loop amplitudes of physical particles.

For the metric field and for the first generation of ghosts we have

$$\begin{aligned} i\langle T h_{\alpha\beta}(x) h_{\mu\nu}(y)\rangle = &\left( \frac{1}{2\Box}(\eta_{\alpha\mu}\eta_{\beta\nu} + \eta_{\alpha\nu}\eta_{\beta\mu} - \eta_{\alpha\beta}\eta_{\mu\nu}) + \frac{1}{\Box^2}(\partial_\alpha\partial_\beta\eta_{\mu\nu} + \partial_\mu\partial_\nu\eta_{\alpha\beta}) \right. \\ &- \frac{1}{2\Box^2}(\partial_\alpha\partial_\mu\eta_{\beta\nu} + \partial_\alpha\partial_\nu\eta_{\beta\mu} + \partial_\beta\partial_\mu\eta_{\alpha\nu} + \partial_\beta\partial_\nu\eta_{\alpha\mu}) - \frac{2}{\Box^3}\partial_\alpha\partial_\beta\partial_\mu\partial_\nu \\ &+ \frac{\alpha_1^2}{2(\alpha)^2}\left( \frac{1}{\Box^3}(\partial_\alpha\partial_\mu\eta_{\beta\nu} + \partial_\alpha\partial_\nu\eta_{\beta\mu} + \partial_\beta\partial_\mu\eta_{\alpha\nu} + \partial_\beta\partial_\nu\eta_{\alpha\mu}) - \frac{4}{\Box^4}\partial_\alpha\partial_\beta\partial_\mu\partial_\nu \right) \\ &\left. + \frac{2}{\kappa}\frac{\partial_\alpha\partial_\beta\partial_\mu\partial_\nu}{\Box^2(\Box + M_D^2)} \right)\delta^4(x-y), \end{aligned} \tag{4.1}$$

$$\begin{aligned} i\langle T A^{\alpha\beta}(x)\bar{A}^{\mu\nu}(y)\rangle = &\left( \left( \frac{1}{2\alpha\Box^3} - \frac{\alpha_2^3 - 2\alpha_0^3}{(\alpha_1^2)^2\Box^2} \right)(\partial^\alpha\partial^\mu\eta^{\beta\nu} - \partial^\alpha\partial^\nu\eta^{\beta\mu} - \partial^\beta\partial^\mu\eta^{\alpha\nu} + \partial^\beta\partial^\nu\eta^{\alpha\mu}) \right. \\ &\left. + \frac{\alpha_2^3 - 2\alpha_0^3}{(\alpha_1^2)^2\Box}(\eta^{\alpha\mu}\eta^{\beta\nu} - \eta^{\alpha\nu}\eta^{\beta\mu}) \right)\delta^4(x-y). \end{aligned} \tag{4.2}$$



IR-divergent contributions render these expressions formal [20]. Thus loop corrections to Green's functions can be defined only by introducing an IR cutoff, which entails broken gauge invariance and coupling of negative norm states. (Needless to say one also needs a regularization for the UV-divergences and renormalizability cannot be expected).

The remaining propagators are IR-finite

$$i\langle T A_0^3(x) A_0^3(y)\rangle = -\left( \frac{2}{\alpha_1^2}\frac{d\partial}{\Box^2} + \frac{\alpha_1^4}{2(\alpha_0^3)^2}\frac{\partial d}{\Box^2} \right)\delta(3)\delta^4(x-y), \tag{4.3}$$

$$i\langle T A_1^4(x) A_{-1}^4(y)\rangle = \frac{2\alpha_0^3 - \alpha_2^3}{2\alpha_0^3\alpha_2^3}\frac{\delta(4)}{\Box}\delta^4(x-y), \tag{4.4}$$

$$i\langle T A_2^3(x) A_{-2}^3(y)\rangle = \left( \frac{1}{\alpha_1^2}\frac{d\partial}{\Box^2} + \frac{\alpha_1^4 - \alpha_3^4}{(\alpha_2^3)^2}\frac{\partial d}{\Box^2} \right)\delta(3)\delta^4(x-y), \tag{4.5}$$

$$i\langle T A_3^4(x) A_{-3}^4(y)\rangle = -\frac{\delta(4)}{\alpha_2^3\Box}\delta^4(x-y), \tag{4.6}$$

where we have used an index free notation with the divergence $\partial$ and the exterior derivative $d$ acting according to their definition on $p$-forms and $\delta(p)$ is the generalized Kronecker tensor (2.31) which is antisymmetric in its $p$ upper and in its $p$ lower indices. We do not display the mixed propagators with the auxiliary fields $B_I^G$ because these fields can be eliminated by their algebraic equations of motion.

As the IR-divergent contributions occur in $\alpha$-dependent parts of the graviton propagator *and* of the (first generation) ghost propagator there is a hope that these divergences cancel in loop corrections to physical amplitudes. However, there are hard technical problems in IR-divergent theories with additional massive poles in some propagators [21] and it would be preferable to have a gauge which is manifestly IR-finite. Parameter limits eliminating the IR-divergent part of the graviton propagator would be $\alpha_1^2 \to 0$ or $\alpha \to \infty$. In the first case the ghost propagator (4.2) diverges. Also the second limit is not viable because there are vertices proportional to $\alpha$ and these yield – in spite of some cancellations – contributions to loop corrections which are proportional to powers of $\alpha$.

We thus propose to enforce a homogeneous gauge $d\partial h = 0$ by introducing a trivial BRS-representation of "Lagrange multipliers"

$$X^{\mu\nu}, \quad Y^{\mu\nu} = sX^{\mu\nu}, \quad \bar{Y}^{\mu\nu} = \bar{s}X^{\mu\nu}, \quad Z^{\mu\nu} = s\bar{s}X^{\mu\nu} \tag{4.7}$$

and the gauge fixing term

$$\begin{aligned}\mathcal{L}_{fix} &= \tfrac{1}{2}s\bar{s}(dX|dX) + s\bar{s}(\partial X^\mu \partial h_\mu)\\ &= (dX|dZ) + (dY|d\bar{Y}) + \partial Z^\mu \partial h_\mu - \partial\bar{Y}^\mu \Box \partial A_\mu + \partial Y^\mu \Box \partial \bar{A}_\mu + \partial X^\mu \Box \partial B_\mu + \mathcal{L}_{int},\end{aligned} \tag{4.8}$$

replacing the term $s\bar{s}(-\alpha\eta^{\mu\nu}g_{\mu\nu})$ in (2.18). Since $s$ and $\bar{s}$ act linearly on the multiplet (4.7) all vertices in (4.8) arise from the term $s\bar{s}\partial X^\mu \partial h_\mu$ and describe derivative couplings of the fields $X^{\mu\nu}$, $Y^{\mu\nu}$ and $\bar{Y}^{\mu\nu}$. Calculating the propagators from the total bilinear Lagrangian, i.e. the sum of (4.8), (2.23) and (2.24) with $\alpha = 0$, we find that the only IR-divergent propagators are

$$i\langle TY(x)\bar{A}(y)\rangle = i\langle TA(x)\bar{Y}(y)\rangle = -\frac{d\partial}{\Box^3}\delta(2)\delta^4(x-y), \tag{4.9}$$

$$i\langle TX(x)X(y)\rangle = 2\alpha_1^2 \frac{d\partial}{\Box^5}\delta(2)\delta^4(x-y). \tag{4.10}$$

The IR-behaviour of the mixed propagators (4.9) is harmless because $Y$ and $\bar{Y}$ only have derivative couplings. Due to the same argument there are 4 momenta from the vertices associated with each internal $X$-line in a Feynman graph. But as the $X$-propagator scales like $1/k^8$ this still is not sufficient to have IR-convergence in 4 dimensions.

A simple way out of this dilemma is to give up $\bar{s}$-invariance. If we keep only the linearized action of $\bar{s}$ in (4.8)

$$\tilde{\mathcal{L}}_{fix} = s(d\bar{Y}|dX) + s(\partial\bar{Y}^\mu \partial h_\mu) + s(\partial X^\mu \Box \partial \bar{A}_\mu) \tag{4.11}$$

the linearized theory and the propagators are not modified. The only interaction term in (4.11) is

$$\partial^\nu \partial \bar{Y}^\mu (\partial A^\rho \partial_\rho h_{\mu\nu} + \partial_\mu \partial A^\rho h_{\rho\nu} + \partial_\nu \partial A^\rho h_{\rho\mu}). \tag{4.12}$$



In this gauge we have manifestly IR-finite Feynman rules: The propagators of the metric $h_{\mu\nu}$ and of the ghosts $A_I^G$ are given by (4.1) – (4.6), where all $\alpha$-dependent terms have to be dropped. The IR-scaling behaviour of the mixed propagators (4.9) is balanced by two derivatives on $\bar{Y}$ in the vertex (4.12). All other propagators in this sector are irrelevant, because there are no corresponding vertices to generate loop contributions to vertex functions. As no counterterms for $X$, $Y$ and $Z$ can be generated these properties hold to all orders of perturbation theory. If the UV-regularization respects BRS-invariance not even terms in the effective action which are proportional to $\bar{Y}$ can get renormalized.

It is then obvious also to keep only the linearized action of $\bar{s}$ in (2.18). $s$ acts linearly on all fields except $A_{I-1}^I$ and $h_{\mu\nu}$. Thus, with $\bar{s}$ linearized, $s\bar{s}\alpha_G^I(A_{-G}^I|A_G^I)$ in (2.16) generates only vertices connecting powers of $c = \partial A$ with $A_{-2}^3$ and with $A_{-3}^4$. These vertices cannot contribute in loop corrections to vertex functions due to ghost number conservation. Again these properties hold to all orders of perturbation theory. Effectively all ghosts except $A^{\mu\nu}$ and $\bar{Y}^{\mu\nu}$ decouple and the only vertex for ghosts is (4.12).

Summarizing these results we start from the $s$-invariant Lagrangian

$$\mathcal{L}_{inv}(g) + s\left(\sum \alpha_G^I \bar{s}_0(A_{-G}^I|A_G^I) + (d\bar{Y}|dX) + \partial\bar{Y}\partial h + \partial X \Box \partial \bar{A}\right), \tag{4.13}$$

where

$$\bar{s}_0 A_G^I = -B_{G-1}^I - \partial A_{G-1}^{I+1}, \qquad \bar{s}_0 B_G^I = \partial B_{G-1}^{I+1} \tag{4.14}$$

and $s$ is given by (2.4), (2.14) and $sX = Y$, $s\bar{Y} = Z$, $s^2 = 0$. The resulting vertices

$$\alpha_1^2(\partial A_{-2}^3|(\partial A)^2) - \alpha_2^3(\partial A_{-3}^4|(\partial A)^3) \tag{4.15}$$

can be absorbed by shifting the fields $A_2^3$ and $A_3^4$ (the functional determinant of this change of variables in the path integral is 1). After eliminating all auxiliary fields by their equations of motion the path integrals over the fields $A_G^4$, $A_G^3$, $\bar{A}$, $Y$ and $X$ are gaussian. Thus, introducing sources only for the fields $h_{\mu\nu}$, $\bar{Y}^{\mu\nu}$ and $A^{\mu\nu}$, all connected Green's functions of these fields coincide with those calculated from the ($s$-variant) Lagrangian

$$\mathcal{L}_{inv}(g) + s(\partial\bar{Y}\partial h) + (d\bar{Y}|dA) + \tfrac{1}{2}(dZ|dZ). \tag{4.16}$$

$Z = s\bar{Y}$ acts as a Lagrange multiplier for the gauge condition and (4.12) is the only vertex in the gauge fixing sector. The resulting Feynman rules are similar to those of general relativity with the ghosts $\bar{c}$ and $c$ replaced by $\partial\bar{Y}$ and $\partial A$. In the physical sector we have additional vertices for $\sqrt{-g}$ and a $\kappa$-dependent term in the graviton propagator (4.1) describing the dilaton. (The $\alpha$-dependent term in (4.1) has to be omitted.)

## 5. Conclusion

We have established that volume preserving diffeomorphisms are a perfectly suited gauge group to describe a unitary theory of gravitation. The classical theory is equivalent to general relativity, but certain aspects of the quantum theory are different. The cosmological constant of general relativity occurs as an integration constant of the equations of motion. The divergence of the path integral due to the conformal degree of freedom is avoided.

Crucial for perturbation theory are IR-divergences stemming from the higher derivative gauge fixing. We have solved this problem by introducing additional fields. In this way a class of gauges was found which is manifestly IR-finite to all orders of perturbation theory and for which loop calculations are no more complicated than for general relativity apart from additional vertices for the determinant of the metric and an additional pole in the graviton propagator describing the dilaton.

I am indebted to C. Becchi for an instructive discussion concerning IR-divergences. I would like to thank W. Buchmüller for interesting discussions and N. Dragon for enjoyable collaboration.



## References

[1] R. D. Peccei and H. R. Quinn, Phys. Rev. Lett. **38** (1977) 1440;
Phys. Rev. **D16** (1977) 1791

[2] C. Brans and R. H. Dicke, Phys. Rev. **124** (1961) 925

[3] S. Weinberg, Gravitation and Cosmology (John Wiley and Sons, New York, 1972)

[4] A. Zee, *in* High-Energy Physics: Proceedings of the 20th Annual Orbis Scientiae, *eds.* B. Kursunoglu, S. L. Mintz and A. Perlmutter (Plenum Press, New York 1985)

[5] S. Weinberg, Rev. Mod. Phys. **61** (1989) 1

[6] W. Buchmüller and N. Dragon, Phys. Lett. **207B** (1987) 292

[7] W. Buchmüller and N. Dragon, Phys. Lett. **195B** (1988) 417;
Phys. Lett. B (to appear); Nucl. Phys. B (to appear)
M. Hennaux and C. Teitelboim, Austin preprint (1988)
W. G. Unruh, NSF-ITP 88-169 (1989) preprint

[8] J. J. van der Bij, H. van Dam and Y. J. Ng, Physica **110A** (1982) 307

[9] E. P. Wigner, Ann. Math. **40** (1939) 149

[10] G. W. Gibbons, S. W. Hawking and J. Perry, Nucl. Phys. **B138** (1978) 141

[11] E. Bergshoeff, E. Sezgin, Y. Tanii and P. K. Townsend, IC/88/186 preprint
J. Hoppe, in *Constraint's theory and relativistic dynamics* (Florence 1986, proceedings)
B. de Wit, J. Hoppe and H. Nicolai, Nucl. Phys. **B305** (1988) 545

[12] J. Scherk and J. H. Schwarz, Nucl. Phys. **B81** (1974) 118

[13] N. Dragon and M. Kreuzer, Z. Phys. **41C** (1988) 485

[14] C. Becchi, A. Rouet and R. Stora, Ann. Phys. **98** (1976) 287;

[15] T. Kugo and I. Ojima, Suppl. Prog. Theor. Phys. **66** (1979) 1;
Nucl. Phys. **B144** (1978) 234

[16] L. Baulieu, Phys. Rep. **129** (1985) 1

[17] F. Brandt, Diplomarbeit (Univ Hannover, 1989) unpublished

[18] E. T. Whittaker, Analytische Dynamik (Grundlehren der Mathematik XVII, Springer, Berlin, 1924)

[19] P. A. M. Dirac, Lectures on Quantum Mechanics (Yeshiva University Press, New York, 1964)

[20] N. Nakanishi, Prog. Theor. Phys. **51** (1974) 952

[21] P. Breitenlohner and D. Maison, Comm. Math. Phys. **52** (1977) 55