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Nonperturbative Aspects of the Lattice Regularized Standard Electroweak Theory

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Nonperturbative aspects of the lattice regularized standard electroweak theory [1]

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Abstract

At present, due to conceptual as well as technical difficulties, a nonperturbative study of the electroweak sector of the Standard Model may only be done by reducing the field content to a small set, believed to be the relevant one for the phenomenon under study. In this article we analize, using analytical methods, some nonperturbative aspects of two interesting limiting cases of the Standard Model on the lattice: The SU(2) Higgs model and the sigma model with two fermion doublets transforming as mirror partners.

First, a linked cluster expansion for the infinite gauge coupling limit of the SU(2) Higgs model is developed. This method, when combined with strong gauge coupling expansions, is used to obtain the phase transition surface and the behaviour of scalar and vector masses in the lattice regularized theory. The method, in spite of the low order of truncation of the series applied, gives a reasonable agreement with Monte Carlo data for the phase transition surface and a qualitatively good picture of the behaviour of Higgs, glueball and gauge vector boson masses. Some limitations of the method are discussed, and an intuitive picture of the different behaviour for small and large bare self coupling λ is given.

A linked cluster expansion of an $SU(2)_L \otimes SU(2)_R$ symmetric model on the lattice is also developed. Scalar fields interact via Yukawa couplings with a doublet of fermions and their mirror partners. In the perturbative regime, species doubling is avoided by the introduction of a chiral invariant Wilson term. The mirror femion doublet, however, can not be kept at the cutoff level in the continuum limit and appears in the physical spectrum of the theory. We demonstrate that in the limit of infinite bare Yukawa couplings the spectrum contains an additional doublet of fermions and their mirror partners. The possibility of a change in the behaviour of the model at large bare Yukawa coupling is analysed and contrasted with the results of the linked cluster expansion. The nonperturbative behaviour of renormalized couplings and masses obtained from a linked cluster expansion up to the 8^{th} order in the fermion hopping parameter is presented and compared with the perturbative predictions.

Contents

Chapter 1

Introduction

At present, all available data in experimental high energy physics may be described by the Standard Model of the electroweak and strong interactions, a gauge theory based on the symmetry group $SU(3) \otimes SU(2)_L \otimes U(1)_Y$. The electroweak sector of the Weinberg Salam Standard Model[1] is a chiral gauge theory since only left handed fermions interact with the $SU(2)$ gauge field. The strong sector is a vector gauge theory because, in contradistinction to the electroweak sector, fermions of both chiralities interact in the same way with the $SU(3)$ gauge field.

In spite of its great success, the Standard Model can only be considered as an effective low energy theory, describing physical reality only until some possibly very high energy scale. The main reason is the existence of a fourth interaction that cannot be described by a renormalizable quantum field theory, namely gravitation. However, gravitational interactions are expected to be relevant only at energies as high as the Planck mass ($\simeq 10^{19} GeV$), and consequently may be neglected while analysing the experimental data. Even when the gravitational interactions were absent, there are strong indications that, for reasons that will be explained below, the energy cutoff introduced in the regularization procedure cannot be pushed to arbitrary high energy values without spoiling mathematical consistency in the Standard Model. This problem may be solved by embedding the theory of electroweak and strong interactions in an enlarged ultraviolet stable quantum field theory or in a more complicated structure describing also gravitation.

No explicit mass term for the gauge bosons and fermion fields is introduced in the Standard Model. Their masses are originated by the interactions with condensates of scalar fields (Higgs fields) which are included in the model for that purpose. Fermion and gauge boson masses are thus proportional to the renormalized Yukawa and gauge gouplings, which couple them to the Higgs field. The Standard Model needs a lot of free independent parameters in order to explain the observed physical spectrum and it gives no clue to fix them. There are several ways of reducing the number of free parameters. Most of them are based on enlarging, at high energies, the symmetry content of the theory. We will not discuss them in this article. Let us only mention that for any of these possibilities to work (compositeness, grand unification, supersymmetry, etc.) new physics has to be observed at relatively low energy scales, say $E \sim \mathcal{O}(1 TeV)$.

Since, until now, there is no experimental evidence of physics beyond the Standard Model, it is a natural question to ask whether the apparently large parametrization freedom is originated in an incomplete understanding of this theory. The answer to this question may only come from a formulation of quantum field theories which subsumes, but is not limited to perturbation theory. The lattice formulation gives an appropiate framework for this study

since it allows the treatment of the full path integral defining the quantum field theory.

Lattice gauge theories were first formulated in order to study nonperturbatively the SU(3) theory of strong interactions[2]. One of the greatest successes of the Wilson formulation of this theory was the proof that in the pure SU(3) gauge theory static sources are confined and thus asymptotic fermions transforming nontrivially under the color gauge group (quarks) are never seen in the spectrum [2,3]. Other non perturbative effects, such as the spontaneous breakdown of the global chiral symmetry and the (qualitative) meson and baryon spectrum have also been successfully computed on the lattice [4]. Although our understanding of the strong interactions is far from being complete, most of the conceptual questions of this theory have found an answer in the framework of the lattice formulation. However, if we want to understand the mechanism behind the origin of masses the inclusion of the electroweak interactions and Yukawa coulings in the model is necessary.

A technical remark is in order: When studying a lattice model with many independent parameters one has to localize the points in the bare parameter space where a continuum theory can be defined. A continuum theory can only be defined at the points where the ratio of any physical mass to the cutoff, which in the lattice formulation is given as the inverse of the lattice spacing a, vanishes. Using the language of statistical mechanics, the continuum theory is defined at a second order phase transition, where correlation lengths, measured in lattice units, acquire infinitely large values. Thus, a first step in any lattice study of a complicated model is to determine the phase diagram in the space of bare parameters.

Once the critical points are localized, one must determine the behaviour (flow) of the renormalized couplings under a change of the lattice spacing in the neighbourhood of the critical points. The critical point may belong to an ultraviolet or an infrared fixed point in this flow. If there is an ultraviolet fixed point in the space of bare parameters, the curves of constant renormalized couplings converge to it when $1/a \to \infty$. Consequently, fixing the value of the renormalized couplings at zero momentum, the continuum limit may always be defined. On the contrary, if there is an infrared fixed point in the space of renormalized parameters the curves of constant bare couplings converge to it when $1/a \to \infty$. In this case, the continuum limit can only be defined when the value of the renormalized couplings at zero momentum coincides with the fixed point value. If, for example there is only an infrared fixed point at zero renormalized couplings, the continuum limit will be trivial, that is noninteracting, independently of the value we give to the bare couplings. The O(N) symmetric ϕ^4 theory is an example of a theory where an infrared fixed point at zero renormalized couplings exists. QCD, instead, is an example of a theory with an ultraviolet fixed point at zero bare gauge coupling.

One may naively think that, since all masses vanish in lattice units the obtained continuum theory should be scale invariant. However, renormalization imposes the choice of an arbitrary energy scale in the theory. This scale spoils the naive dimensional analysis. One can study the behaviour of the renormalized couplings under variations of this energy scale. In the presence of an ultraviolet fixed point, like in QCD, the renormalized gauge coupling decreases as the energy is increased and, at very high energy scales (at small distances) the quarks behave as free fermions. This important property of QCD is called asymptotic freedom.

At present, the difficulties associated with introducing chiral fermions on the lattice haven't been overcome. Consequently, only the pure bosonic sector of the electroweak theory has been extensively studied in the literature[5]. In the SU(2) Higgs model it was demonstrated that in the limit of infinite bare self coupling, at small bare gauge coupling the theory has two phases. One phase is connected with the symmetric phase of the scalar. O(4) symmetric, ϕ^4 theory at vanishing gauge coupling, and with the pure gauge theory at infinite scalar masses. We will call it confinement phase. The other phase is connected with the broken phase of the ϕ^4 model and we will call it Higgs phase. An important difference of the lattice approach, compared to the usual perturbative analysis, is the possibility of calculating correlation functions without the implementation of a gauge fixing constraint. It was shown long ago that the vacuum expectation value of gauge noninvariant quantities, such as the Higgs doublet, must vanish if no gauge fixing is made [6]. Therefore, in this sense, there is no spontaneous symmetry breaking [1]. No local order parameter can be used for distinguishing the Higgs or screening phase from the confinement phase in this theory. In fact, it was demostrated that in SU(N) scalar gauge theories, with fields in the fundamental representation of the gauge group, both phases are analytically connected when the radial degree of freedom of the Higgs field is frozen [3,8]. This analytical connection is due to the fact that the phase transition line which starts at vanishing gauge coupling, has an end point for sufficiently low $\beta = 4/g^2$. This connection disappears if the radial degree of freedom is not frozen, for low values of the self coupling λ, where the phase transition line doesn't have an end point, and in fact extends to negative values of the gauge coupling [9,10].

The phase diagram of the full $SU(2) \otimes U(1)$ bosonic gauge theory was studied in ref.[11]. It was demonstrated that this theory has two disconnected phases, one at low $U(1)$ gauge coupling, which is analytically connected with the confinement-Higgs phase of the $SU(2)$ model, and one where the $U(1)$ interaction confines. In the former, which is the physical one, a massless gauge boson (photon) appears in the physical spectrum[12].

Since the abelian gauge coupling has an infrared fixed point at vanishing renormalized gauge coupling, it was usually assumed that it can be reliably treated perturbatively and was consequently neglected in most nonperturbative investigations. In the limit of vanishing gauge coupling the $SU(2)$ Higgs model reduces to the O(4) symmetric ϕ^4 theory with only two free parameters: the bare mass and self coupling of the scalar fields. The scalar self coupling has an infrared fixed point at vanishing renormalized coupling, similar to what happens with the $U(1)$ gauge coupling. Were this the only fixed point of the theory, the continuum theory of the ϕ^4 theory would be trivial. Due to the accumulated evidence coming from analytical[14,17] and numerical[13,15,16] works there is now little doubt that the continuum ϕ^4 theory is in fact noninteracting. If the cutoff is kept at finite, but very large values, characterizing the scale at which physics can no longer be described by the Standard Model, an upper bound on the scalar self coupling (and in the broken phase on the scalar Higgs mass) can be obtained in terms of the ratio of the renormalized Higgs mass to the cutoff.

The inclusion of a small bare SU(2) gauge coupling doesn't modify this picture: Since the gauge coupling g_2 is asymptotically free it is natural to study the behaviour of the theory along the lines of constant renormalized gauge coupling and constant bare scalar self coupling. In the perturbative regime one can show that along these lines both the bare gauge coupling and the renormalized scalar self coupling vanish in the continuum limit.

Apart from the phase transition line at vanishing bare gauge coupling, the only expected critical points where a continuum limit of the regularized SU(2) Higgs model may be defined are the end points of the phase transition surface in the interior of the phase diagram. Since these points are localized at strong gauge couplings, a numerical simulation in their neighbourhood is very difficult. A similar program as the one developed in the pure ϕ^4 theory in ref.[14] may be applied in order to study the behaviour of the theory in the neighbourhood of the endpoints. To do this, an analytical tool similar to the high temperature expansions

[1] However, this doesn't completely invalidate the semiclassical picture [7]

in the pure ϕ^4 theory is necessary. If a similar convergent expansion of the physical correlation functions at vanishing β is defined, we will be able to combine it with strong coupling expansions, having an analytical tool to inquire into the properties of the confinement phase of the SU(2) Higgs theory[18]. Since an alternative, and still consistent description of the electroweak interactions may be obtained if the system is in the symmetric phase[19] this analysis may also be of phenomenological interest. It is the purpose of a part of this article to develop such an analytical tool (see Chapter 3).

Further nonperturbative information may be obtained with the inclusion of fermions. However, if we want to put only left handed fermions in the fundamental representation of $SU(2)$ interacting with the gauge fields, a well known no-go theorem [20] tell us that under some reasonable assumptions, for example the invariance of the action under SU(2) transformations, right handed fermions in the same representation as the left handed ones appear. Thus, it seems that it is impossible to find a lattice formulation of the electroweak sector of the Standard Model if we don't include explicit symmetry breaking terms in the action of the theory (These terms should be irrelevant in the continuum in order to recover the continuum formulation). As it was mentioned before, in order to define the continuum limit one has to know the critical hypersurface in the space of bare parameters together with the phase structure in its vicinity. Unfortunately, there is not enough known about the phase structure of theories with Yukawa couplings. Certainly there are phases with mirror fermion partners (see ref.[21]) but the existence of a phase without mirror fermions is presently unknown, in spite of the recent progress in the perturbative lattice formulation of chiral gauge theories [22]. The phase structure of the chiral lattice model proposed by Karsten, Smit and Swift[25,26,24] is also not known. These questions can only be decided in future non perturbative studies. In this work the phase with mirror fermion pairs will be studied by the linked cluster expansion. The appearence of mirror fermion partners in the pure $SU(2)$ theory is not a very serious problem. Since $SU(2)$ has only real representations, an $SU(2)$ theory with 2N left-handed doublets may be rewritten as a theory with N left-handed and N right handed doublets[19]. Morover, an even number of interacting fermion doublets is necessary, otherwise nonperturbative anomalies would render the theory inconsistent[29].

If we want to study the properties of the electroweak theory with fermions it is natural to start by neglecting the gauge interactions. For simplicity, only one fermion doublet and its mirror partner should be first introduced in the spectrum. Since the theory is free of anomalies there will be no problem with the introduction of the $SU(2)$ gauge interactions. In the absence of hypercharge interactions this model may be considered as a realistic formulation of the electroweak sector of the Standard model, as we will discuss in Chapter 2. A nonperturbative study of this model at vanishing $SU(2)$ gauge coupling will be presented in Chapter 4 of this article.

Chapter 2

The Standard model and lattice regularization

2.1 Electroweak sector of the Standard model

As we have said in the introduction, the electroweak sector of the Standard model is a chiral gauge theory based on the gauge group $SU(2)_L \otimes U(1)_Y$. The fermion spectrum is replicated in families, each of them containing 15 Weyl fermions transforming nontrivially under the gauge group. The eight left handed fermions transform in the fundamental representation of the SU(2) gauge group. The seven right fermions, instead, transform as singlets under SU(2) gauge transformations. The appearence of an even number of interacting doublets assure the cancellation of nonperturbative anomalies within each family of the model. Perturbative anomalies are also cancelled since the conditions

$$\sum_i (q_Y^L)_i = 0, \qquad \sum_i \left\{ \left((q_Y^L)_i\right)^3 - \left((q_Y^R)_i\right)^3 \right\} = 0 \tag{2.1}$$

are verified when the sum of the hypercharge quantum numbers, $(q_Y)_i$, is done over all left (L) and right (R) fermionic members of a given family[31,30] (Remember that pure SU(2) is free from perturbative anomalies).

The Standard model was formulated assuming that for each fermion family a massless left handed fermion (neutrino) should appear. No relevant property of the model is modified if we add to the above spectrum an additional, neutral, right handed neutrino in order to give the neutrino a finite, but very small, mass. In the following we will always assume that this is the case. Since explicit fermion masses are forbidden by the gauge symmetry, fermion masses arise in the Higgs phase through the interaction with a condensate of scalar fields. Only one scalar field doublet is necessary to induce all fermion and gauge boson masses, and the minimal Standard model contains no other scalar fields.

There are four gauge vector bosons associated to the generators of the gauge group $SU(2) \otimes U(1)$. In the Higgs phase, the two electrically charged gauge bosons and one of the neutral gauge bosons acquire nonvanishing mass through their interactions with the Higgs condensate. The additional neutral gauge boson becomes massless and is identified with the photon[32]. In the continuum formulation the Lagrangian density of the model reads

$$\mathcal{L}_{SM} = \mathcal{L}_g + \mathcal{L}_\psi + \mathcal{L}_H + \mathcal{L}_{Yuk.}, \tag{2.2}$$

$$\mathcal{L}_g = -\frac{1}{4} Tr[G^{\mu\nu} G_{\mu\nu}] - \frac{1}{4} F^{\mu\nu} F_{\mu\nu},$$

$$G^{\mu\nu} = -\frac{i}{g_2}[\partial_\mu + i g_2 A_\mu, \partial_\nu + i g_2 A_\nu],$$

$$F^{\mu\nu} = -\frac{i}{g_Y}[\partial_\mu + i g_Y B_\mu, \partial_\nu + i g_Y B_\nu], \tag{2.3}$$

$$\mathcal{L}_\psi = \sum_j \bar{\psi}_j i \gamma^\mu \mathcal{D}_\mu \psi_j, \tag{2.4}$$

$$\mathcal{L}_{Yuk} = \left(\sum_{ij} G_{ij} \bar{\psi}^a_{L,i} \varphi_a \psi_{R,j} + \sum_{ij} G'_{ij} \epsilon_{ab} \bar{\psi}^a_{L,i} (\varphi_b)^* \psi_{R,j} \right) + h.c., \tag{2.5}$$

$$\mathcal{L}_H = (\mathcal{D}_\mu \varphi)^+ (\mathcal{D}_\mu \varphi) - m^2 (\varphi^+ \varphi) - \lambda (\varphi^+ \varphi)^2 \tag{2.6}$$

where $A_\mu = A^a_\mu T^a$, B_μ are the gauge bosons corresponding to $SU(2)$ and $U(1)_Y$ respectively, T^a are the generators of the group $SU(2)$, ψ and φ characterize the fermions and the scalar fields and the covariant derivative $\mathcal{D}_\mu = \partial_\mu + i g_2 A_\mu + i g_Y B_\mu q_Y$. $\sum_{ij}$ represents a summation over the different fermions $\psi_{R,j}$ (fermion doublets $\psi_{L,i}$) in the case of right (left) handed fermions and a summation over the isospin indices a, b is understood. In order to preserve the gauge invariance of the theory, the sum of the hypercharge quantum numbers q_Y of the scalar and fermion fields in each term of eq.(2.5) must vanish.

The scalar doublet φ may be put in a one to one correspondence with the matrix ϕ defined as

$$\phi = \begin{pmatrix} \varphi_2^* & \varphi_1 \\ -\varphi_1^* & \varphi_2 \end{pmatrix} \tag{2.7}$$

where $\varphi_{1,2}$ are the upper and lower isospin components of φ. It follows from eq.(2.7) that

$$det(\phi) = \varphi^+ \varphi = \frac{1}{2} Tr[\phi^+ \phi] \tag{2.8}$$

and the relation between ϕ and φ is given by

$$\varphi = \phi \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.9}$$

$\mathcal{L}_H$ may now be easily rewritten in terms of ϕ by using eq.(2.8) while $\mathcal{L}_{Yuk}$ takes the compact form

$$\mathcal{L}_{Yuk} = \bar{\psi}_L G_\psi \phi \psi_R + h.c. \tag{2.10}$$

where ψ is a fermion doublet, G_ψ is the matrix

$$G_\psi = \begin{pmatrix} G_1 & 0 \\ 0 & G_2 \end{pmatrix} \tag{2.11}$$

and we have assumed, for simplicity, that only a fermion doublet is present in the physical spectrum of the theory.

2.2 Higgs mechanism and gauge invariant operators

The scalar sector of the Standard Model is necessary in order to be able to give masses to the fermions and gauge bosons of the theory. An explicit gauge boson mass would not only break gauge invariance but would also spoil the renormalizability and unitarity of the Standard Model, which are necessary requirements in order to define a consistent quantum field theory. However, even in the presence of explicit mass terms for the fermions and gauge bosons gauge invariance may be recovered by using the well known Stuckelberg formalism [33]. The theory is rendered gauge invariant by the appearence of a scalar field which mimics the scalar field doublet of the Standard model but with fixed radial mode (see below). This model was studied with perturbative methods and it was realized that either renormalizability or unitarity are not achieved. However, nonperturbative effects can render this theory consistent. In fact, if the triviality of the scalar theory is preserved after the introduction of fermions and gauge fields, in the continuum limit the Stuckelberg theory will belong to the same class of universality as the Standard Model.

In the following we will neglect the $U(1)_Y$ interactions, although all what we will say may be easily extended to the case where abelian gauge interactions are included. In the usual perturbative approach, the origin of masses is explained through the 'breakdown' of gauge symmetry produced by a vacuum expectation value of the scalar fields. In the path integral formulation of gauge theories, however, local gauge invariance is never broken and, furthermore, all gauge noninvariant correlation functions vanish identically if no gauge fixing is made.

In the semiclassical approach to the Higgs mechanism, one usually fixes the unitary gauge, that is writing ϕ in terms of a radial ρ and an angular degree of freedom given by an $SU(2)$ matrix σ

$$\phi = \rho\sigma, \qquad \rho = det(\phi)^{1/2}, \tag{2.12}$$

one can always make a local gauge transformation in order to absorb σ in the definition of the left handed fermions and gauge fields. In the Lagrangian, the unitary gauge is equivalent to set the matrix $\sigma = I$. If, working in the unitary gauge, the scalar field acquires a vacuum expectation value, it will induce masses for both the fermions and the gauge bosons of the theory. In the literature, this phenomenon is usually described as a spontaneous breakdown of the gauge symmetry, although since we have completely fixed the gauge no breakdown of the local symmetry is really occuring.

In a complete nonperturbative approach to the scalar gauge theories, such as that provided by the lattice formulation, a gauge fixing is not required. All physical information may be obtained through the correlation function of gauge invariant physical operators, and as we have already said, the vacuum expectation value of the scalar doublet vanishes. The problem of defining the Higgs mechanism in a gauge invariant way was studied by several authors. They realized that in the Higgs phase, masses arise through the interaction with a gauge invariant Higgs condensate, and that what really characterizes the Higgs mechanism is not the breakdown of gauge symmetry but the existence of a nontrivial gauge orbit where the Higgs potential is minimized[7]. In the gauge invariant description the physical left handed fermions are described by the gauge invariant operator $\psi_L^{phys.}$ given by

$$\psi_L^{phys.} = \phi^+ \psi_L \tag{2.13}$$

The upper and lower components of $\psi_L^{phys.}$ may be identified with the physical electron and

neutrino, respectively. The physical Higgs field is described by

$$R_H = \varphi^+ \varphi \tag{2.14}$$

A gauge invariant expression may also be given for the gauge fields:

$$A_\mu^{phys.} = -\frac{i}{g}\varphi^+ \mathcal{D}_\mu \varphi \tag{2.15}$$

What is interesting of this physical operators is that they coincide with the unitary gauge expressions when only the gauge orbit minimizing the Higgs potential is considered[7], allowing the contact of this formulation with the semiclassical approach.

Fermion and boson masses arise in a way that resembles, but is not identical to, the semiclassical description[7]. For example the gauge invariant expression of the fermion mass matrix is

$$M_f = G_\psi \left\langle \varphi^+ \varphi \right\rangle^{1/2} \tag{2.16}$$

which, at tree level coincides with the semiclassical value. A similar expression may be found for the gauge boson masses.

2.3 Complementarity and the Abbott-Farhi model

As we have already said in the introduction, there are only two phases in the scalar $SU(2) \otimes U(1)$ Higgs model[11]. In the Coulomb phase of the abelian interactions the theory is in an unique Coulomb-Higgs-Confinement phase. The operatorial representation of the physical fields given in the last section does not depend on the assumption of working in the Higgs phase. They are also valid if we are working in the confinement phase of the theory. This property of the theory was called complementarity in the literature, and was interpreted by some authors as an evidence of the possibility of having a complicated 'hadron-like' spectrum not only in the confinement but also in the Higgs or screening phase of the theory[5].

The origin of masses, however, is very different if the continuum limit is taken in one or other phase. The $SU(2)$ gauge coupling has an ultraviolet fixed point at vanishing bare coupling. In the continuum theory, the running coupling constant $g_2^R \to 0$ at very high energies and it increases for low energies. In the Higgs phase the condensate fixes the characteristic energy scale of the model. The renormalized gauge coupling constant at this scale is small and may reliably be treated in perturbation theory. All masses are given in terms of the scalar condensate as explained in the last section. On the contrary, in the confinement phase the condensate is small and the renormalized gauge coupling becomes strong at low energies. All physical masses are expected to depend on the scale Λ_2 at which the renormalized gauge coupling $g_2^R \simeq 1$ [19].

In the confinement phase the spectrum is given by composite, physical states and consequently even in the case of a strong renormalized coupling between the elementary fields the effective coupling between the physical fermions and gauge bosons might be small. In ref. [19] it was shown that the Strongly coupled Standard Model, that is the theory obtained by taking the continuum limit in the confinement phase at bare gauge coupling $g_2 = 0$ can describe all available experimental data provided some nontrivial assumptions are fulfilled. For example, spontaneous chiral symmetry breaking may be induced by fermion fields, in analogy to what happens with the nonvanishing fermion condensate $\left\langle \bar{\psi}\psi \right\rangle$ in QCD. In the Strongly coupled Standard Model this phenomenon must be prevented by the presence of scalar fields since if this is not the case fermions will acquire large masses induced by the presence of the fermion condensate. There were a lot of recent studies trying to verify the validity of this assumption. If only the SU(2) interactions are gauged, spontaneous chiral symmetry breaking is induced by the presence of a nontrivial fermionic condensate in the confinement phase of the theory at strong gauge coupling, destroying the analytical connection between the two phases[34,35]. However, the non-existence of a phase with vanishing fermion condensate and confinement properties has not been established[36]. Morover, it is not clear what happens with the introduction of general Yukawa couplings[37] and the abelian interactions, that have been neglected in these studies (The chiral condensate usually studied in the literature breaks the abelian local symmetry). A nonperturbative study at weaker bare SU(2) gauge coupling and nonvanishing Yukawa couplings is necessary in order to check if fermion condensates spoil the strong coupling formulation of the Standard Model. Until such study is done, or experimental evidence doesn't rule it out, the Strongly coupled Standard Model should be kept in mind as an interesting open possibility.

2.4 Attempts to put the Standard model on the lattice and the sigma model

Since the Standard model is replicated in families, it is a natural simplification to start a nonperturbative study by considering only one family of leptons and quarks. Even in this case the spectrum contains 16 Weyl fermions (including the right handed neutrino) and it would be technically very difficult to make any nonperturbative study of the complete model. In order to make further simplifications one cannot throw away arbitrary fermions from the spectrum due to the requirement of anomaly cancellation. The best approximation that can be made without spoiling gauge invariance is to replace the three left handed quark doublets by only one left handed doublet with hypercharge three time that of the elementary quarks, namely $q_Y = -1/2$. The hypercharge of the right handed doublet must also be chosen as the opposite one of the right handed lepton hypercharges. The model constructed in this way may be studied perturbatively, and there is an easy way to put it on the lattice. As a further simplification one can set the Yukawa couplings that couple the components of a fermion doublet with the neutral scalar field at equal values. Due to the fact that SU(2) has only real representations this theory is equivalent to one where a fermion doublet and its mirror partner are present in the spectrum (see next section).

To write the Lagrangian density of this model it is sufficient to keep only one fermion doublet (ψ) since the formulation for two fermion doublets is obtained by analogy. In the limit of vanishing gauge couplings the Lagrangian density of this simplified model reads

$$\mathcal{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi + G_\psi(\bar{\psi}_L\phi\psi_R + h.c.) + \mathcal{L}_H \tag{2.17}$$

which is the Lagrangian density of the continuum sigma model.

The mirror fermion Lagrangian is identical to eq.(2.17) but changing the fermion field ψ by a mirror fermion field χ, G_ψ by G_χ and ϕ by ϕ^+. Once the gauge interactions are considered this model has several properties in common with the Standard model. However, the appearence of mirror fermions in the spectrum, which are not present in the minimal Standard model, makes the existence of fermion condensates like $\langle\bar{\psi}\chi\rangle$ possible. There is no analogous gauge invariant fermion operator in the Standard Model. Let us mention that in the absence of the $U(1)$ interactions and neglecting Yukawa couplings, a model of

fermions and mirror fermions is equivalent to a vector gauge model. For nonvanishing Yukawa couplings, the model can still be formally written in a vector like form. However, the physical eigenstates don't interact in a vector like form. In many cases we will have two fermion states with different masses, one (the light one) with gauge interactions predominantly (V-A) and the other with gauge interactions predominantly (V+A). If mirror fermions are sufficiently heavy, they have very small influence on the present phenomenology. A detailed analysis of this phenomenon and its possible relevance for the Standard Model has been done in ref.[39].

With somewhat different notation in the absence of $U(1)$ interactions and with equal Yukawa couplings (see below) the model of ref.[38], which we intend to analyse, is exactly equivalent to the one studied in ref.[37] at strong gauge coupling. The choice of the Yukawa couplings in ref.[37] is different from the one we will make here. We find it very important to explain the relation of these two formulations in some detail, since they are the best way we have to simulate the Standard Model on the lattice without excluding chiral fermions from the spectrum.

As we have already said, the model with mirror fermions may be written in a vector form. To do this let us define the new fermion doublets

$$\Psi^1 = \frac{(1+\gamma_5)}{2}\chi + \frac{(1-\gamma_5)}{2}\psi \tag{2.18}$$

$$\Psi^2 = \frac{(1-\gamma_5)}{2}\chi + \frac{(1+\gamma_5)}{2}\psi \tag{2.19}$$

Once $SU(2)$ is gauged in the way defined in ref.[38] Ψ_1 becomes an interacting fermion doublet, while Ψ_2 remains noninteracting. The gauge invariant mass terms $\bar{\chi}\psi + h.c.$ may be rewritten in this notation as

$$\mathcal{L}_m = \bar{\Psi}^1\Psi^1 + \bar{\Psi}^2\Psi^2 \tag{2.20}$$

and the term defining the Yukawa interactions may be rewritten as

$$\mathcal{L}_{Yuk} = G_\psi\left\{\left[\left(\bar{\Psi}^1\right)_{i,L}\varphi_i\left(\Psi^2\right)_{2,R} + \epsilon_{ij}\left(\bar{\Psi}^1\right)_{i,L}\varphi_j^*\left(\Psi^2\right)_{1,R}\right] + h.c.\right\} +$$
$$G_\chi\left\{\left[\left(\bar{\Psi}^1\right)_{i,R}\varphi_i\left(\Psi^2\right)_{2,L} + \epsilon_{ij}\left(\bar{\Psi}^1\right)_{i,R}\varphi_j^*\left(\Psi^2\right)_{1,L}\right] + h.c.\right\} \tag{2.21}$$

where a summation over the isospin indices i, j is understood. As we can see from here, this Lagrangian is invariant under the $SU(2)$ local symmetry

$$\Psi^1 \to g\Psi^1, \qquad \varphi \to g\varphi \tag{2.22}$$

which is the one gauged in ref.[38]. When mirror fermions interact with the U(1) hypercharge interaction, they also appear in a vector like form when written in terms of Ψ^1 and Ψ^2.

The interacting vector doublet considered in ref.[37] may be identified with Ψ^1, while the two SU(2) singlet fermions may be identified with the two isospin components of Ψ^2. The Yukawa terms considered there may be written as

$$\mathcal{L}_{Yuk} = h_1\left\{\left[\left(\bar{\Psi}^1\right)_{i,L}\varphi_i\left(\Psi^2\right)_{2,R} + \left(\bar{\Psi}^1\right)_{i,R}\varphi_i\left(\Psi^2\right)_{2,L}\right] + h.c.\right\} +$$
$$h_2\left\{\left[\epsilon_{ij}\left(\bar{\Psi}^1\right)_{i,L}\varphi_j^*\left(\Psi^2\right)_{1,R} + \epsilon_{ij}\left(\bar{\Psi}^1\right)_{i,R}\varphi_j^*\left(\Psi^2\right)_{1,L}\right] + h.c.\right\} \tag{2.23}$$

which is also invariant under the transformations defined in eq.(2.22). Obviously, if both Yukawa couplings are equal ($h_1 = h_2$ and $G_\psi = G_\chi$) the two Lagrangians coincide. The different values of the Yukawa couplings make the global symmetry content of both models different. Apart from that they are completely equivalent. What this is telling us is that the model of ref.[37] may be also considered as one with a fermion doublet and a mirror fermion doublet. The relevance of both models as approximations to the Standard Model will be discussed in the next section.

2.5 Mirror fermions and the Standard Model

H. Georgi[19] was the first who realized that, if only the $SU(2)$ gauge interactions are present, the Standard model may be written in a vector like form. This relation is possible because $SU(2)$ has only real representations, and because in the Standard Model there are an even number of left handed doublets. In fact, given a fermion doublet Ψ, the field Ψ' satisfying

$$\Psi' = i\tau^2 \Psi^c,$$

$$\Psi^c = i\gamma_2\gamma_0\bar{\Psi}^T, \tag{2.24}$$

where the superscript c denotes charge conjugation and τ^2 is a Pauli matrix, transforms as the mirror fermion of Ψ. If we apply this transformation to one half of the left handed fermion doublets and to one half of the right handed fermion doublets the theory takes the form of a vector gauge theory. For example, if we restrict our study to a pair of quarks without colour we can identify [37]

$$\Psi^1 = \begin{pmatrix} u_L \\ d_L \\ s^c_R \\ -c^c_R \end{pmatrix} \tag{2.25}$$

$$\Psi^2 = \begin{pmatrix} s^c_L \\ -c^c_L \\ u_R \\ d_R \end{pmatrix} \tag{2.26}$$

From here it is obvious that in the absence of an explicit mass term the Lagrangian proposed in ref. [38] gives equal masses to both members of one family (u and d) while the one proposed in ref. [37] gives equal masses to u and c, and to d and s. A better approximation to the Standard model is to choose the Yukawa couplings in such a way that all masses are different. However, that choice would enlarge the bare parameter space and would make an analytical and numerical study (even) more difficult.

In the presence of the $U(1)$ hypercharge interactions, the model studied here is no more a realistic approximation to the Standard model. For example if Ψ^1 transforms in a vector way, a relation of Ψ^1 with two left handed doublets in the form given in eq.(2.25) would only be possible if both left handed fermion doublets transform with opposite quantum numbers under $U(1)$. This model, however, can still be considered as a formulation of the Standard Model in the presence of mirror fermions in the spectrum which, contrary to the general belief, are not completely excluded by present experimental data [39].

Chapter 3

Linked cluster expansion in the SU(2) lattice Higgs model at strong gauge coupling

3.1 Introduction

The nonperturbative study of the scalar sector of the Standard Model is of crucial importance since this sector provides the mechanism that gives masses to the elementary particles of the theory. It is usually assumed that the system is in its broken phase and that the masses appear through the vacuum expectation value of the scalar fields. However, as we have discussed in Chapter 2, an alternative and still consistent picture may arise if the system is in the symmetric phase [19] . Since in any case the mechanism behind the origin of masses is of nonperturbative nature, its complete understanding may only come through an analysis going beyond the limits of perturbation theory, and thus the lattice formulation gives an appropiate framework for this study [5].

There have been many papers lately considering the $\beta \to \infty$ limit of the theory. The triviality of the φ^4 theory is used to put restrictions on the allowed values of the renormalized self coupling λ_R in terms of the high energy cutoff Λ, which in the lattice formulation is given by the inverse of the lattice spacing a [13,15,16,17]. Recently, a combination of high temperature expansions with renormalization group analysis has been used as an analytical tool to solve the system in the symmetric phase at vanishing gauge coupling and to put interesting bounds on the possible Higgs mass value [14].

In this Chapter we will concentrate on the opposite side of the phase transition diagram, namely the region where the gauge coupling is strong: $\beta \leq 1$. This is a very interesting region, since a nontrivial fixed point may exist at the expected critical point at the edge of the phase transition surface. The renormalization group analysis of ref.[40] is not conclusive concerning the behaviour of the system in this neighbourhood. Furthermore, most Monte Carlo analysis concentrate on intermediate values of the bare parameters, namely $\beta \sim 2$ and $\lambda \geq 0.1$. In the interesting region, where both β and the bare self coupling λ take small values the numerical simulation is difficult. Due to the limitations in the numerical analysis, little is known about the SU(N) Higgs theory at strong gauge coupling. A first analytical study for these systems has been made in ref.[28] where the large N theory at infinite gauge coupling was analysed.

In this Chapter we follow an analytical approach to study the strong coupling region of the Higgs theory. Our work is inspired by the recent successful analysis of the theory in the vanishing gauge coupling limit, where high temperature expansions are used to solve the system deep in the symmetric phase, reaching correlation length values as large as two lattice spacings, where the system is already inside the scaling region[14]. In analogy, the method should allow the solution of the system deep in the confinement phase, at small values of the hopping parameter k. We put special emphasis here in showing how the linked cluster expansion used in the infinite gauge coupling limit may be combined with strong coupling expansions to calculate analytically the correlation lengths of the isoscalar and isovector states in the desired coupling region. In the confinement phase important questions related to the behaviour of masses as one changes the different bare parameters and the exact location of the phase transition may be answered with an accuracy that depends on the order of truncation of the series we apply. There are, in principle, no restrictions on the self coupling λ. Although, as we will show, the order of the strong gauge coupling expansion necessary to get an accurate picture of the theory is higher for larger values of the self coupling. We are more interested in showing the capabilities and limitations of the method than in getting an accurate high order expansion. Thus, all the expansions will be given up to eighth order of the hopping parameter. We will show also, that the method may easily be extended to any U(N) gauge group with scalar fields in the fundamental representation. In section 3.2 we formulate the model on the lattice. In section 3.3 we elaborate on the general technique used. Its combination with strong gauge coupling expansions is explained in section 3.4. Section 3.5 contains an analysis of the results and some concluding remarks . Useful formulas for the calculations, and details about the derivation of the expansions are given in the appendices.

3.2 Formulation of the model

For the formulation of the model on the lattice, we use the parametrization first introduced in ref.[9], which has the advantage of being more suitable for the definition of the high temperature expansion. We will choose lattice units, so all length scales will be measured in terms of the lattice spacing. The SU(2) Higgs action reads

$$S = S_g(U) + \lambda \sum_x (\varphi_x^\dagger \varphi_x - 1)^2 + \sum_{x,y} \varphi_x^\dagger Q_{xy} \varphi_y \,, \tag{3.1}$$

where $S_g = -\frac{\beta}{2}\sum_P Tr(U_P)$, U_P is the ordered product of link variables $U_{x,\mu}$ around the plaquette P, $U_{x,\mu}$ is the link gauge variable defining the lattice covariant derivative

$$U_{x,\mu} = \exp(i g_2 A_{x,\mu}) \tag{3.2}$$

and

$$Q_{x,y} = \delta_{xy} - k(\delta_{x,y-\mu} U_{x,\mu} + \delta_{x,y+\mu} U^\dagger_{x-\mu,\mu}) \tag{3.3}$$

The parameter k is usually called hopping parameter, and is related to the inverse of the bare mass in the continuum formulation. More especifically, the relation of the above parametrization with the one used in the more traditional continuum formulation of the theory

$$S_c = \int dx((\mathcal{D}_\mu \varphi^c)^2 + m_c^2(\varphi^c)^2 + \lambda_c(\varphi^c)^4) \tag{3.4}$$

is given by

$$\varphi^c = \varphi\sqrt{k}, \quad \lambda_c = \frac{\lambda}{k^2}, \quad m_c^2 = \frac{(1 - 2\lambda - 8k)}{k} \,. \tag{3.5}$$

Observe that for $\lambda = 0$, and vanishing gauge coupling, the theory becomes noninteracting, and the point where the renormalized mass vanishes in lattice units is obviously at $k_c = 1/8$. At values of $k < k_c$ the mass becomes positive, and at $k > k_c$ becomes negative. Setting λ to be an infinitesimal, positive bare parameter, the theory will have two physical phases, one where the $O(4)$ symmetry of the pure scalar theory is not broken, that is the symmetric or confinement phase of the theory, and the other where the global symmetry is spontaneously broken, that is the broken or Higgs phase of the theory. In general, the symmetric phase is localized at values at values of k smaller than the critical value $k < k_c$, while the Higgs phase is localized at values of the hopping parameter $k > k_c$. As we have already discussed in Chapter 1, both phases are analytically connected at large values of the bare self coupling. In Fig.3.1 the location of the Higgs and confinement phase in the plane $k - \beta$ of the bare parameter space, for different values of the bare self coupling λ is shown.

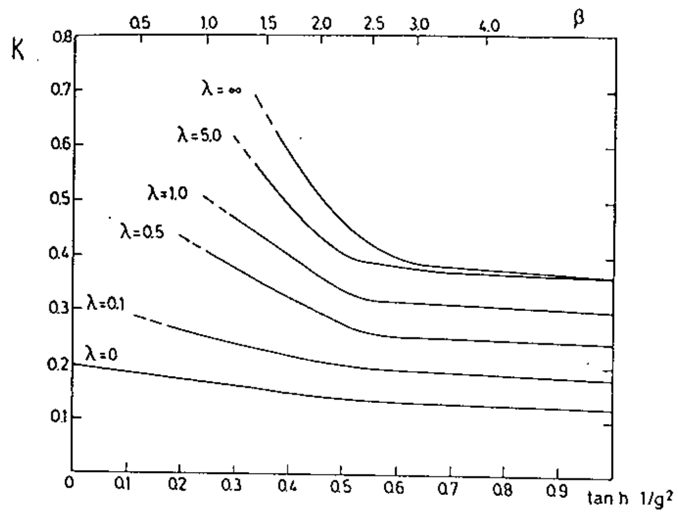


Figure 3.1 : Phase transition line in the $(\beta - k)$ plane for different values of the bare self coupling λ according to fig.3 in ref.[9]. At large values of λ the phase transition line ends without reaching $\beta = 0$.

The reader must observe (see Fig.3.1) that the symmetric phase is not always, as one naively would expect, at positive values of the bare mass m_c^2. In fact, what is important are not the bare but the renormalized parameters. For example, at sufficiently large values of λ, the bare mass m_c^2 becomes negative for all k. However, the symmetric phase doesn't disappear. Morover, for any λ, at $k=0$ the theory reduces to the pure $SU(2)$ gauge theory, where confinement occurs.

For future purposes, it is better to represent the Higgs doublet φ_x by its angular and radial degrees of freedom, $\varphi_x = \rho_x\alpha_x \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. The integration measure is then $\rho_x^3 d\rho_x d^3\alpha_x d^3U_{x,\mu}$, where d^3g denotes the SU(2) Haar measure. The action may be rewritten as

$$S = S_g(U) + \lambda\sum_x(\rho_x^2-1)^2 + \sum_x \rho_x^2 - k\sum_{x,\mu}\rho_x\rho_{x+\mu}Tr(\alpha_x^\dagger U_{x,\mu}\alpha_{x+\mu}) \tag{3.6}$$

where all isospinor like factors are absorbed in the definition of the traces. The angular part of the Higgs field may be integrated out for finite β. For this purpose it is useful to introduce a new gauge invariant link variable $V_{x,\mu} = \alpha_x^\dagger U_{x,\mu}\alpha_{x+\mu}$ [42]. Using the invariance of the Haar measure, eq.(A.1) the partition function may be rewritten in terms of the variables ρ_x and $V_{x,\mu}$, with the integration measure $\rho_x^3 d\rho_x d^3V_{x,\mu}$ and the action

$$S = S_g(V) + \lambda\sum_x(\rho_x^2-1)^2 + \sum_x \rho_x^2 - k\sum_{x,\mu}\rho_x\rho_{x+\mu}Tr(V_{x,\mu}), \tag{3.7}$$

where both $V_{x,\mu}$ and ρ_x are gauge invariant.

The integration of the angular variables of the Higgs field is equivalent to fix completely the gauge, by going to the unitary gauge (See Chapter 2). The unphysical, Goldstone fields disappear from the formulation. One may easily extend the definition of the physical gauge and Higgs fields of the continuum formulation to the lattice formulation. For example, the physical Higgs field acquires the same form as in the continuum

$$R_x = \varphi_x^+\varphi_x = \rho_x^2 \tag{3.8}$$

while the physical gauge fields are given by

$$\left(A_{x,\mu}^a\right)^{phys.} = Tr[\varphi_x^+ U_{x,\mu}\varphi_{x+\mu}T^a] = \rho_x\rho_{x+\mu}Tr[V_{x,\mu}T^a] \tag{3.9}$$

where $A_{x,\mu} = A_{x,\mu}^a T^a$. Due to the form of $U_{x,\mu}$, eq.(3.2), the expression given in eq.(3.9) reduces to the continuum expression when $a \to 0$.

3.3 β=0 boundary

3.3.1 Character expansion

For β=0, the link variable is random and may be integrated out exactly. In order to do this, it is convenient to rewrite

$$\int \mathcal{D}V \exp\left(k\sum_{x,\mu}\rho_x\rho_{x+\mu}Tr(V_{x,\mu})\right) = \prod_{x,\mu}\int \mathcal{D}V_{x,\mu}\exp\left\{\frac{2k\rho_x\rho_{x+\mu}}{2}\chi_f(V_{x,\mu})\right\}, \tag{3.10}$$

where $\chi_f(V_{x,\mu})$ denotes the character of $V_{x,\mu}$ in the fundamental representation of the group SU(2) and $\mathcal{D}V$ denotes the SU(2) Haar measure. Some basic properties of the character expansions are reviewed in Appendix A. The integrand in eq.(3.10), may be given in terms of a series in characters of $V_{x,\mu}$, with coefficients that are modified Bessel functions of argument $y_{x,\mu} = 2k\rho_x\rho_{x+\mu}$,

$$\exp\left(\frac{y_{x,\mu}\chi_f(V)}{2}\right) = \sum_j \frac{2(2j+1)I_{2j+1}(y_{x,\mu})\chi_j(V_{x,\mu})}{y_{x,\mu}}. \tag{3.11}$$

Once this expression is introduced in eq.(3.10), the only additional step for integrating out the gauge fields is an integration over the group variables of the characters of the group. The integral of all nontrivial characters vanishes identically, so this assures that only the term depending in the trivial character survives. Consequently, the result of the integration is

$$\prod_{x,\mu}\frac{2I_1(y_{x,\mu})}{y_{x,\mu}} = \prod_{x,\mu}\sum_{n=0}^{\infty}\frac{(k^2R_xR_{x+\mu})^n}{n!(n+1)!}. \tag{3.12}$$

This gives us an effective action for the physical Higgs field $\rho_x^2 = R_x$. The partition function in the infinite gauge coupling limit is then given by

$$Z = \int\prod_x(R_x\mathcal{D}R_x)\exp-\left\{\sum_x\lambda(R_x-1)^2 + R_x\right\}\prod_{x,\mu}\left\{\sum_{n=0}^{\infty}\frac{(k^2R_xR_{x+\mu})^n}{n!(n-1)!}\right\}. \tag{3.13}$$

Eq.(3.13) defines the effective action of the physical Higgs fields. In the $\beta=0$ limit of the theory only scalar excitations are present in the spectrum. The physical, composite Higgs field $R_x = \varphi_x^\dagger\varphi_x$ is like a meson in the fermion case[28], and the elementary scalar fields like the coloured quarks. In view of the results in the φ^4 theory, one hopes that the present theory may be solved by combining, at the expected second order phase transition point at the edge of the phase diagram, some convergent expansion of correlation functions with renormalization group methods. The linked cluster expansion explained in the next subsection is a first step in this direction[18].

3.3.2 Linked cluster expansion

The partition function of the SU(2) Higgs model at infinite gauge coupling is given by eq.(3.13). We intend now to define a linked cluster expansion that is suitable to evaluate the correlation functions of the Higgs field and to analyse other properties of the theory in this limit. Apart from the finite order of truncation of the expansions, no approximations are made in the calculations, and an accurate picture of the regularized theory may be obtained . The derivation of the method is given in Appendix B. At each order, the expansion coefficients are polynomial in the one point expectation values

$$< R^n > = \frac{\mathcal{I}_n}{\mathcal{I}_0} \qquad (n = 1, 2, ...) \tag{3.14}$$

$$\mathcal{I}_n(\lambda) = \int_0^\infty dR\ R^{n+1}\ \exp-\left(R + \lambda(R-1)^2\right) \tag{3.15}$$

More especifically, if we define the "one point" generating functional

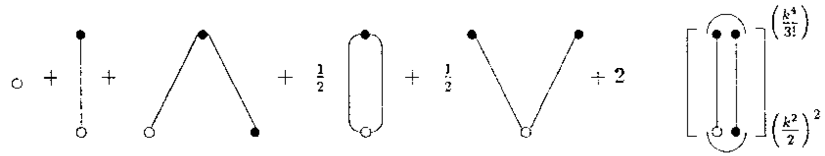
$$\mathcal{Z}(\lambda, h_x) = \int_0^\infty dR\, R \ \exp\left(h_x R - \lambda (R-1)^2\right) \tag{3.16}$$

the expansion coefficients are functions of the moments M_n

$$M_n = \langle R^n \rangle_{k=0}^c = \frac{\delta^n ln \mathcal{Z}(\lambda, h_x)}{\delta h_x^n}|_{h_x=-1} \tag{3.17}$$

In the limit $\lambda \to 0$ the moments reduce to a particularly simple form: $M_n = 2\,\Gamma(n)$. The correlation functions can be given in terms of graphical rules, which is in the same spirit as ref.[43]. For example, the vacuum expectation value of the Higgs field is represented, in low orders, by

$\langle R_x \rangle =$



where we draw edges as lines and external and interal vertices as open and filled circles respectively. The appearence of the last term which is absent in the rules of ref.[43] is due to the more complicated action we have and to the fact that k is kept as an expansion variable. This is analogous to what happens when trying to connect the linked cluster with the weak embedding expansions for the free energy in the φ^4 theory[43]. The factors present in front of the fourth and fifth terms are symmetry factors. The rules to calculate $< R_x >$ may be summarized as follows:

(a) Assign the label one to the external root vertex and a dummy label to each internal vertex.
(b)For each pair of vertices i and j write a factor $\frac{k^2}{2}$ when they are joined by one edge, a factor $\frac{(k^2)^2}{3!}$ when they are joined by two edges , and in general a factor $\frac{(k^2)^n}{(n+1)!}$ when they are joined by n edges.
(c)For each l-valent internal vertex i write a factor M_l.For each l-valent external vertex write a factor M_{1+l}.
(d)Sum each internal vertex label freely over the entire lattice.
(e) Divide by the symmetry factor of the 1-rooted graph.

This rules may be generalized for the calculation of any n point correlation function, by changing the number of external vertices. For example, the pair correlations for non coincident points may be calculated by the following rules:

(a) Assign the labels 1 and 2 to the external vertices and dummy labels to the internal vertices.
(b)For each pair of vertices i and j write a factor $\frac{k^2}{2}$ when they are joined by one edge, a factor $\frac{(k^2)^2}{3!}$ when they are joined by two edges , and in general a factor $\frac{(k^2)^n}{(n+1)!}$ when they are joined by n edges.
(c) For each l-valent internal vertex i write a factor $M_l(i)$. For each l valent external vertex

j=1,2 write a factor $M_{l+1}(j)$.
(d)Sum each internal vertex label freely over the entire lattice.
(e) Divide by the symmetry factor of the 2-rooted graph.

A special situation occurs when two or more of the n points coincide. Let us consider the simplest case, namely $< R^2 >^c$. In this case one has to sum a one rooted graph (changing the factor one by a factor two in rule c)) and a two rooted graph with both external vertices at the same point. This may be easily generalized for the case where there are j coincident points. Since the method at this point is the same as the one explained in ref.[43], we refer the reader to this article for a more detailed exposition.

The graphs typified by the last term in the expression of $\langle R_x \rangle$ above don't respect rules b) and e). This kind of graphs are always given by product of connected graphs that coincide in one or more points, signaled by the semicircles in the above example. For example, the dependence on k of this term may be obtained by substracting the value given above from the one given below the bracket. In the general case, both the factor in front of these graphs and their dependence on k may be computed in a systematic way, and is briefly explained in Appendix B. Since these graphs are dominated by low order moments M_n, they are more important for larger λ. The reason for this is the behaviour of the different M_n as a function of the self coupling. When large values of λ are considered, only M_1 and M_2 become relevant: M_2 goes to zero and $M_1 \to 1$ as $\lambda \to \infty$ (see Table 3.C.1). The graphs considered give important contribution to the correlation functions in this limit. On the contrary, for small values of λ, ($\lambda << 1$) they give only small contribution to the correlation functions. However, they are relevant in the interesting region where the end points are located. The presence of these graphs and the dependence of the k factor on the number of coincident edges make the calculation more difficult here compared to pure φ^4 theory.

3.3.3 Phase diagram and Higgs mass

Once the correlation functions of the Higgs field are computed, we can obtain the phase transition line and the Higgs mass in a a similar way as in the φ^4theory. We have computed expansions for the susceptibilities

$$\chi_2 = \sum_x < R(x)R(0) >^c \tag{3.18}$$

$$\mu_2 = \sum_x x^2 < R(x)R(0) >^c \tag{3.19}$$

in powers of k^2. From these expansions, we can get information about the location of the phase transition. For this purpose, one has to assume a given behaviour of the series at high orders of k. This is a difficult point, since even the order of the phase transition is unknown. We have assumed a second order phase transition and used some methods appropiate for this case.

We expect the radius of convergence of χ_2 to give us the value k_c^2, at which the phase transition takes place. In our calculation, for small enough λ, all the coefficients are positive.

Then, the value k_c^2 may be estimated, for large n, as

$$k_c^2 \sim \frac{a_n}{a_{n-1}} = r_n \tag{3.20}$$

where a_n is the n^{th} coefficient of the expansion. The ratios r_n, eq.(3.20), converge rapidly to a positive value . However, the order of truncation doesn't allow us to extract information directly from the last ratios computed and some extrapolation method is needed. An estimate of k_c can be given by [46]:

$$k_c^2 = \frac{(n' r_{n'} - n r_n)}{(n' - n)} \tag{3.21}$$

Here r_n is the approximate value defined in eq.(3.20). We have chosen the factor (n'-n)=2, because it allows us to obtain two different results which can be compared. Both estimates are usually very close to each other, for those values of λ where a phase transition clearly appears. The final value obtained by this procedure is also usually very close to the last evaluated ratio. We expect this value to give us a good approximation to the correct k_c.

The Higgs mass may be estimated in two ways. The first estimate assumes that, at low momentum p, the renormalized two point function behaves as

$$\Gamma_R^{(2,0)}(p,-p) = (m_R^2 + p^2 + high.ord.), \quad for \quad p \to 0. \tag{3.22}$$

As in the φ^4 theory, this gives a mass[44]

$$m_R = \left(8\frac{\chi_2}{\mu_2}\right)^{\frac{1}{2}}. \tag{3.23}$$

This definition, doesn't coincide with the mass obtained from the pole of the propagator

$$\Gamma^{(2,0)}(p,-p) = 0, \quad p = (\vec{0}, im), \tag{3.24}$$

but we expect both values to be close , and their difference to be computable in terms of some perturbative expansion around the second order phase transition point where the continuum limit can be defined. In the following we will keep eq.(3.23) as our definition of the Higgs mass. However, for comparision, we have also computed the mass obtained from the expected exponential decay behaviour of the two point Higgs correlation function at $\vec{p} = \vec{0}$, in a similar way as employed in computing the glueball masses[45] . First, we calculate the $\vec{p} = \vec{0}$ Fourier transformation of the correlation function at time τ

$$\sum_{\vec{x}} \left\langle R(\vec{0},0) \;\; R(\vec{x},\tau)\right\rangle = G(\vec{p} = 0, \tau) \tag{3.25}$$

Then, a second estimate of the Higgs mass can be given as

$$m_H = -\log\left(\frac{G(\vec{p} = \vec{0}, \tau = N)}{G(\vec{p} = \vec{0}, \tau = N-1)}\right) \tag{3.26}$$

The above quantity may be computed by the methods explained in the last subsection.

3.3.4 Mean field results

At infinite bare gauge coupling, where the effective action depends only on the physical, scalar variable R_x it is natural to make a mean field analysis of the theory, to gain a first insight on its nonperturbative behaviour. Two mean field approachs to this theory have been made in the literature. In ref.[28], a $1/N$ expansion was performed. Only the leading term in $1/N$ was considered and consequently the results can only be considered as reliable ones at large values of N. On the other hand, in ref.[27] the leading term of a mean field, $1/d$ expansion (d: space time dimension) was considered. In this case, the results should be reliable at very large values of d. Both expansions give similar results for the case of interest ($N = 2$, $d = 4$).

One interesting result obtained in ref.[27] is that the value of the critical hopping parameter at $\lambda = 0$ is $k_c = 1/8$ independently of the value given to the bare gauge coupling. The analysis of ref. [28] conduces,at $\beta = 0$, to the same result. Furthermore, a reasonable explanation of this result was given in ref. [27]. This value of the critical hopping parameter is, however, in contradiction with Monte Carlo data which shows a nontrivial variation of the critical hopping parameter for different values of β at $\lambda = 0$, and in particular a value $k_c \simeq 0.2$ at $\beta = 0$. We will compare these results with the results of the hopping parameter expansion in section 3.5.

In the mean field calculations there is a first order phase transition at the phase transition line at $\beta = 0$. At the endpoint a second order phase transition takes place. The scaling behaviour close to the critical point was analysed in ref.[28]. Mean field predicts that the theory behaves as a one component ϕ^4 theory in the scaling region and the continuum theory at the critical point is thus presumably noninteracting. However, all these properties, including the order of the phase transition at the phase transition line at $\beta = 0$, may change in the real $N = 2$, $d = 4$ theory.

3.4 Strong gauge coupling expansion

3.4.1 Higgs field effective action

The computation of the generating functional at $\beta = 0$ can be combined with a character expansion of the Wilson action to obtain information at low β. For $k \to 0$ the Higgs field decouples and we recover the pure gauge theory. The low energy excitations are given by glueballs, one of which has the quantum numbers of the vacuum and is supposed to be the lowest excitation. For $k \neq 0$ both the glueball and the Higgs boson acquire a finite mass, and a non-negligible mixing may appear since both have the same quantum numbers. For non-negligible k, a vector excitation, the $\vec{W}$ triplet, also appears and we are interested in obtaining the dominant behaviour for the $\vec{W}$ mass.

The Wilson action may be written in terms of a character expansion, similar to the expression of the kinetic part of the Higgs field action, eq.(3.11).

$$\prod_P \exp\left(\frac{\beta}{2} Tr(V_P)\right) = \prod_P \left(\sum_j \frac{2(2j+1) I_{2j+1}(\beta)}{\beta} \chi_j(V_P)\right) \tag{3.27}$$

Using eq.(3.27) and the character properties, all the usual strong coupling methods may be developed. In the pure gauge theory only closed surface plaquette configurations appear. A different result is obtained if Higgs loops are considered because open surface diagrams also contribute. The general strategy for obtaining physical information will be to integrate out first the gauge fields, which is easily done by using the character properties. One can show that once the gauge fields are integrated out all correlations of physical particles are given in series in β, with coefficients which are given in terms of correlation functions of the Higgs field at $\beta = 0$. Since we have an explicit expansion for correlations of the Higgs field in this limit, we can also consider this as an expansion in k, where the coefficients are polynomial functions of β. Using this fact, the phase transition surface and the Higgs mass can be obtained in the same way as explained in section 3.3.

The corrections up to β^2 to the partition function can be easily gotten. First we write explicitly the expansions

$$\mathcal{Z} = \int \left(\prod_x \mathcal{D}R_x R_x\right)\left(\prod_{x,\mu} \mathcal{D}V_{x,\mu}\right) \exp - \{S(k=0,\beta=0)\} \prod_P \prod_{x,\mu} \left\{\sum_j \frac{2(2j+1)I_{2j+1}(\beta)\chi_j(V_P)}{\beta}\right\}$$
$$\left\{\sum_i 2(2i+1)\frac{I_{2i+1}(y_{x,\mu})\chi_i(V_{x,\mu})}{y_{x,\mu}}\right\} \tag{3.28}$$

Then, for computing the first powers in β we expand the Wilson action keeping only the low order character terms. To integrate out the gauge field one has to compute some nontrivial character integrals (see Appendix A). Once this integration is done, one arrives at the following expression:

$$ln\mathcal{Z} = ln\mathcal{Z}(\beta=0) + \beta\sum_P \left\langle \prod_{links} \frac{I_2(y_{x,\mu})}{I_1(y_{x,\mu})}\right\rangle_P + \frac{\beta^2}{4}\sum_{P,P'}\left\langle \prod_{links}\frac{I_2(y_{x,\mu})}{I_1(y_{x,\mu})}\right\rangle_{P,P'}$$
$$+3\frac{\beta^2}{8}\sum_P\left\langle\prod_{links}\frac{I_3(y_{x,\mu})}{I_1(y_{x,\mu})}\right\rangle_P - \beta^2\sum_{P,P'}\left\langle\prod_{links}\frac{I_2(y_{x,\mu})}{I_1(y_{x,\mu})}\right\rangle_P\left\langle\prod_{links}\frac{I_2(y_{x,\mu})}{I_1(y_{x,\mu})}\right\rangle_{P'}$$
$$-\frac{\beta^2}{2}\sum_P\left(\left\langle\prod_{links}\frac{I_2(y_{x,\mu})}{I_1(y_{x,\mu})}\right\rangle_P\right)^2 + 3\frac{\beta^2}{4}\sum_{P,P'}\left\langle\frac{I_3(y_{x',\mu'})}{I_1(y_{x',\mu'})}\prod_{links}\frac{I_2(y_{x,\mu})}{I_1(y_{x,\mu})}\right\rangle_{P,P'} \tag{3.29}$$

where the brackets above always imply expectation values at $\beta = 0$. $\sum_P$ ($\sum_{P,P'}$) here means a summation over all different plaquettes P (pair of neighbour plaquettes P,P') and $\prod_{links} f(y_{x,\mu})$ means that for each link at the point x and with direction μ belonging to the contour of the plaquet P (pair of neighbour plaquettes P,P') we must write a factor $f(y_{x,\mu})$. The factor $\frac{I_3(y_{x',\mu'})}{I_1(y_{x',\mu'})}$ of the last term of eq.(3.29) must be understood as belonging to the link that joins both plaquettes. Once the expansions of the Bessel functions are done, each bracket is a disconnected correlation function of the Higgs field at $\beta = 0$. For example

$$\left\langle\prod_{links}\frac{I_2(y_{x,\mu})}{I_1(y_{x,\mu})}\right\rangle_P = \sum_{j,l,m,n=1}^{\infty} c_{jlmn}(k) < R_x^j R_{x+\mu}^l R_{x+\mu+\nu}^m R_{x+\nu}^n > \tag{3.30}$$

with coefficients $c_{jlmn}(k)$ to be determined order by order in k^2. Any disconnected correlation function

$$\langle R_x R_y \ldots R_z\rangle \tag{3.31}$$

is a sum of products of cumulant correlations in which each term corresponds to a partition of the arguments (x y...z) and every partition appears once and only once.

There is an equivalent and more compact way of calculating the disconnected correlation functions. For the computation of $< R_x^j R_{x+\mu}^l R_{x+\mu+\nu}^m R_{x+\nu}^n >$, we start putting a factor $c^j(x), c^l(x+\mu), c^m(x+\mu+\nu), c^n(x+\nu)$ at the vertices of the plaquette, where $c^j =< R^j >_{k=0}$. The factors c^j are the one point disconnected moments at $k = 0$, and can be given as a polynomial expression $c^j(h_x)$ of the connected moments $M_n(h_x)$ at $h_x = -1$. Once this is done, one can construct the linked cluster expansion of the disconnected correlation function, working as if one were calculting the more simpler correlation $< R_x R_{x+\mu} R_{x+\mu+\nu} R_{x+\nu} >$. In the latter case, one has to write a factor M_{1+l} whenever the plaquette vertex is l valent. In the general case, if one of the plaquette vertices is l valent, one has to replace c^j by

$$c_l^j = \frac{\delta^l c^j(h_x)}{\delta h_x^l}|_{h_x=-1}, \tag{3.32}$$

where $c^j(h_x)$ is the above defined polynomial function of c^j in terms of the moments $M_n(h_x)$. This compact form allows the computation of all the brackets in eq.(3.30), up to a given order, by replacing the factor $c_l^j(x)$ by the appropiate factor $c_l^n(x)$, whenever in the bracket the power of R_x changes from j to n.

We observe that, in the method explained in section 3.3, for $\beta \neq 0$ there is an additional complication one must face to obtain the phase transition location. This is due to the fact that the corrections in β appear at second order in k^2 . So the first evaluated ratio r_1 is the same even when the corrections in β are included. As a result, only one reliable estimate, using eq. (3.21), can be obtained by the previously explained method.

3.4.2 Glueball mass

For the glueball state we considered the symmetric combination of the three space like orientation single plaquette operators, which is often used in QCD glueball spectrum calculations. Let us observe, that this state has isospin $I_w = 0$ and spin parity $J^{PC} = 0^{++}$, which are the same as that of the Higgs field. For the computation of the n-plaquette disconnected correlation functions one has to calculate

$$\langle\chi_f(V_{P^1})\ldots\chi_f(V_{P^2})\rangle = \frac{\int \mathcal{D}V\mathcal{D}RR\ \ (\exp - S)\ \ \chi_f(V_{P^1})\ldots\chi_f(V_{P^2})}{\mathcal{Z}(\beta,k,\lambda)} \tag{3.33}$$

The denominator is the partition function, for which we have an explicit expansion, and the numerator may be calculated using the above discussed expansions, eqs.(3.27) and (3.11), and the character properties. All the computations are similar to the ones already done for the β dependent correction of the generating functional. For example, at first order in β, we have:

$$\left\langle\chi(V_P)^2\right\rangle^c = 1 - 4\frac{I_2(\beta)}{I_1(\beta)}\left\langle\prod_{links}\frac{I_2(y_{x,\mu})}{I_1(y_{x,\mu})}\right\rangle_P . \tag{3.34}$$

Keeping only the first eight orders in the hopping parameter, a correlation of two plaquettes connected by a third one is given by

$$\langle\chi_f(V_{P^1})\,\chi_f(V_{P^2})\rangle^c = \left(\frac{I_2(\beta)}{I_1(\beta)}\right)\left\langle\prod_{links}\frac{I_2(y_{x,\mu})}{I_1(y_{x,\mu})}\right\rangle_{P^1,P',P^2} \tag{3.35}$$

The brackets always indicate mean values at $\beta = 0$ and $\prod_{links}$ implies that we have to put a factor $\frac{I_2(y_{x,\mu})}{I_1(y_{x,\mu})}$ for each link belonging to the contour of the diagram formed by the three plaquettes P^1, P' and P^2

For the computation of the mass we consider[45]

$$\sum_{\vec{x}} \sum_{spat.dir.} \left\langle \chi_f(V_P)(\vec{0},0)\chi_f(V_{P'})(\vec{x},\tau)\right\rangle^c_{\tau=0,1} = G(\vec{p}=\vec{0},\tau=0,1) \tag{3.36}$$

where τ means the number of lattice spacings in the time direction. As a first approximation, one can take

$$m_g = -ln\left(\frac{G(\vec{p}=\vec{0},\tau=1)}{G(\vec{p}=\vec{0},\tau=0)}\right) \tag{3.37}$$

where the small time interval chosen is as the only possible one at eighth order in k. We have also evaluated the first pure gauge contribution, in order to take care of the behaviour for $k \to 0$ of the glueball mass, namely $m_g \sim -4ln(\frac{\beta}{4})$.

3.4.3 Gauge vector boson mass

The third component of the $\vec{W}$ vector operator is defined as

$$W_3 = \rho_x\rho_{x+\mu}\chi_{\frac{1}{2}}(\tau_3 V_{x,\mu}), \tag{3.38}$$

As we have already discussed in section 3.2 this is the analogue of the continuum operator $Tr(\tau_3\varphi^\dagger\mathcal{D}_\mu\varphi) = W^3_\mu(x)$, defined in Chapter 2. This state has weak isospin $I_w = 1$ and spin parity $J^{PC} = 1^{--}$. The relevant correlation functions of $W^3_{x,\mu}$ are easily calculated from the relations:

$$\left\langle W^3_{x,\mu}\right\rangle_\beta = 0 \tag{3.39}$$

If we connect two vector operators with a two dimensional array of n plaquettes we get:

$$\left\langle W^3_{x,\mu}W^3_{x+n\nu,\mu}\right\rangle^c_\beta = \left(\frac{I_2(\beta)}{I_1(\beta)}\right)^n \left\langle \rho_x\rho_{x+\mu}\rho_{x+n\nu}\rho_{x+n\nu+\mu}\prod_{links}\frac{I_2(y_{x,\mu})}{I_1(y_{x,\mu})}\prod_{links'}\left(1-\frac{I_3(y_{x',\mu'})}{I_1(y_{x',\mu'})}\right)\right\rangle \tag{3.40}$$

where $\prod_{links'}$ means that we have to write a factor $\left(1-\frac{I_3(y_{x',\mu'})}{I_1(y_{x',\mu'})}\right)$ for the links x,μ and $x+n\nu,\mu$ and for each link joining two plaquettes, while $\prod_{links}$ means that we have to write a factor $\frac{I_2(y_{x,\mu})}{I_1(y_{x,\mu})}$ for each other link belonging to the contour of the plaquettes. A useful relation here is the property of the modified Bessel function :

$$(I_1(z) - I_3(z)) = \frac{4}{z}I_2(z). \tag{3.41}$$

In the case of the vector state, we assumed exponential decay of the two point correlation functions and employed a similar computation to that used in the glueball case. Defining

$$\sum_x \left\langle W^3_\mu(\vec{0},0)\ \ W^3_\mu(\vec{x},\tau)\right\rangle^c_{\tau=1,2} = G(\vec{p}=\vec{0},\tau=1,2) \tag{3.42}$$

we get an estimate of the vector mass by using eq.(3.26), for N=2.

3.5 Results and discussion

3.5.1 Phase transition surface

The correlation function expansions were truncated in the eighth order in k and in the second order in β. In Appendix C we give the results of the susceptibility coefficients up to k^8 in terms of M_n, at $\beta = 0$. We have also included the values of the moments M_n, of the expansion coefficients, and of the estimated values of k_c for some representative values of λ. The obtained phase transition location is compared with Monte Carlo data in Fig.3.2.a. For $\beta < 1$ our results agree to a good approximation with Monte Carlo results, although the values we obtain by applying eq.(3.21) are higher than the values found in the Monte Carlo analysis. The values shown in Fig.3.2.a at $\beta = 0$ are an average of both estimates obtained using eq.(3.21). These tipically differ by less than 5% from each other (see Table 3.C.3). Our assumption of the existence of a phase transition relies on the positiveness of the coefficients and on the approximate constant values of the computed ratios. At higher λ, tipically $\lambda \sim 1$, the ratios begin to vary very fast at different orders and some coefficients become negative (see table 3.C.3). We interprete this behaviour as a signal of the end of the phase transition line. Obviously, at the order considered one gets only a range of possible (λ, k) values for the end point, which extends up to values of k as high as one for all the values of β considered.

For $\lambda \ll 1$, the position of the phase transition surface is surprisingly well predicted when one extrapolates to higher values of the gauge coupling. However, for larger λ the phase transition points obtained are systematically at higher values of k. The main reason for this behaviour is that the correlation functions of the Higgs field begin to be dominated by plaquette terms as the self coupling is increased, due to the fact that the $\beta = 0$ contribution carries higher order M_n factors. For this reason, we expect higher corrections in β to be more relevant for larger λ. In order to get an estimate about the inclusion of higher order terms in β, we consider the most relevant corrections of order β^3, generated by three joined plaquettes. For $\beta \leq 1$ we obtain a little correction to the estimated k_c, besides a shift of the order of 10% on the position of the end point at $\beta = 1$. However, when extrapolated to larger values of β, the coefficients become positive and the ratios become rapidly convergent even for values of $\lambda \gg 1$ and at $\beta \sim 2$ the phase transition end point disappears. In Fig. 3.2 we have also included values of β as large as 3, to show how the extrapolation behaves for small values of the self coupling. In Fig. 3.2.b, we show the dependence of the phase transition surface for different values of β, when the third order corrections are included. We see that not only the quantitative picture for low values of λ, but also the qualitative picture for higher values of λ begin to agree with the one found by Monte Carlo simulations in ref.[10].

Our results are in disagreement with the mean field prediction that at $\lambda = 0$ the critical hopping parameter must take the value $k_c = 1/8$, and, as we have already mentioned in this section, agree well with Monte Carlo results. It is still possible that the difference of our results with the mean field predictions is an artifact of the order of truncation of the series we applied and thus it will be reduced at higher orders. The same can be said about Monte Carlo results; they may change in larger lattices. It is also possible that the argument given in ref.[27] fails because the configurations with $V_{x\mu} = 1$ form a set of measure zero in the space of general configurations.

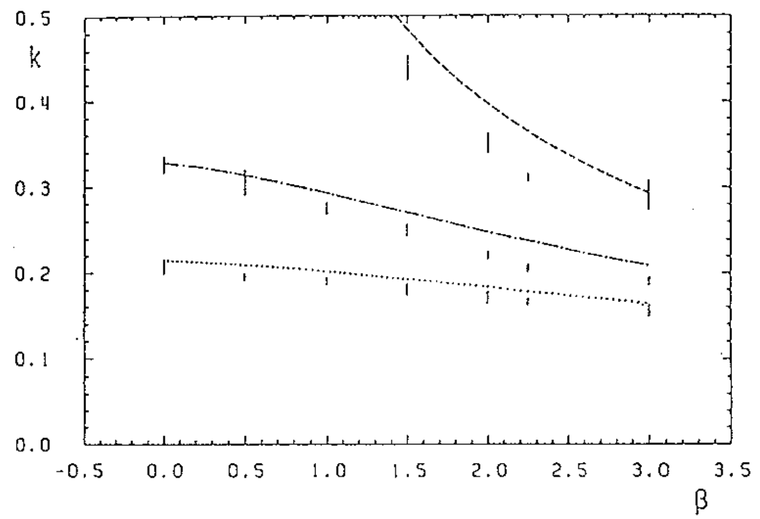


Figure 3.2.a : Dependence of the phase transition location on the inverse gauge coupling β, as estimated using eq.(3.21), for the bare self coupling $\lambda = 0.01$ (dotted line) ,0.1 (dashed-dotted line) and 1 (dashed line). Also shown in the figure is the Monte Carlo data for the same values of the bare couplings. The vertical segments give the errors in the Monte Carlo estimates.

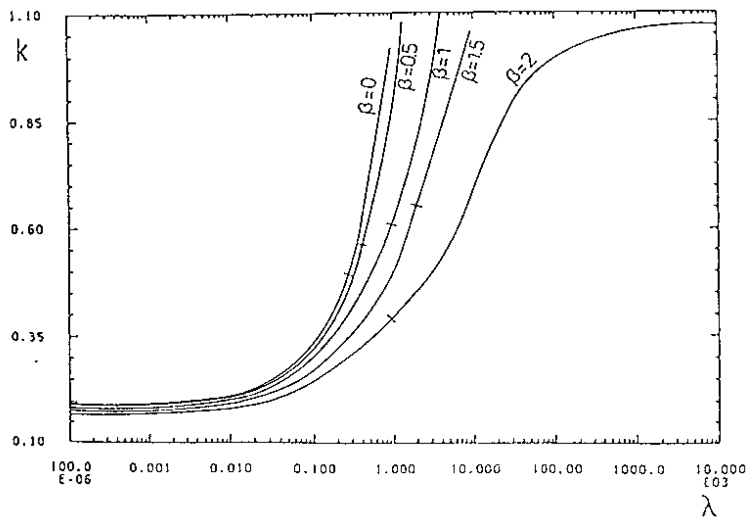


Figure 3.2.b : Dependence of the phase transition location on the bare self coupling λ. for the inverse gauge coupling β taking the values $\beta = 0$, 0.5 , 1 , 1.5 , 2. The endpoints are taken when the estimate of k_c, eq.(3.21), differs by 10% with the last evaluated ratio ($\sqrt{r_4}$). The vertical segment in each curve shows the point where this difference is already 5%.

3.5.2 Scalar and vector masses.

The masses obtained by the linked cluster expansions have a natural cutoff, namely the inverse size of the typical graph that contribute to the correlation functions. In view of the experience gained in the φ^4 theory. one expects, for example, that even when critical behaviour occurs masses of order $m_R \sim 0.5$ may only be obtained when the 10^{th} order of the series is computed (see table 2 of ref.[14]). Here we have two limitations to obtain small values for the masses. On one side, the phase transition is presumably of first order until the end points, and then the correlation length remains finite. On the other side, the order of truncation of the series is too low to expect low masses to appear. The obtained expansions are a sensible approximation up to some value k_f, lower than k_c, that depends on the order of truncation applied. The analysis of the behaviour of scalar and vector masses, that is strictly related to the picture obtained from our expansions, is made taking k_f coincident with the estimate of k_c , eq. (3.21). We don't expect our conclusions to depend strongly on a better estimate of k_f since in the neighbourhood of k_c the expansions have a smooth dependence on k and the mass values conform with the typical size of the graphs included.

In Fig.3.3 we show the dependence on k of the masses, for different values of β and λ. Let us begin with the discussion of the behaviour of the Higgs mass. The figure shows that this mass always decreases with the hopping parameter k. up to k_c. For $k \to 0$, the Higgs mass m_R diverges quadratically in k for any β . For greater values of β, when λ and k are fixed, the values of m_H become lower. a fact that can be understood since we are approaching the phase transition surface. For fixed k and β, instead, the Higgs mass values are larger for larger λ. Again. this behaviour may be understood since we are going away from the neighbourhood of the phase transition surface. The mass estimate m_H. Eq.(3.26), shows the same behaviour when the different bare parameters are varied, although it diverges logarithmically for $k \to 0$. We expect m_H and m_R to be close to each other in the neighborhood of a critical point. The figures show that m_H agrees well with m_R when k approaches k_c. This agreement is better near the end points than for small values of λ.

Information about the strength of the phase transition may be obtained by examining the values of the Higgs mass in the neighbourhood of k_c. For example, the curves show that at a fixed value of the gauge coupling, in the neighbourhood of the phase transition, m_R is smaller for larger values of λ. This is a signal of the weakening of the phase transition order, when higher values of λ are considered. At fixed values of the self coupling the figures show a weakening of the phase transition for lower values of β. when the values of the self coupling are near the predicted end points. We can now compare these results with the ones obtained from Monte Carlo data. At intermediate values of β. where the Monte Carlo analysis is done. the previously described dependence of the strength of the phase transition on λ is preserved. However. the dependence on β is reversed: the transition becomes weaker for larger values of β. although the variation is very soft. Our data is compatible with Monte Carlo. In fact, if the transition at the end point in the strong coupling regime is of second order. and at intermediate values of β is of first order [10], there must be a weakening of the phase transition transition with growing gauge coupling in the strong coupling region and this behaviour must be reversed at lower values of the gauge coupling.

As $k \to 0$ the glueball mass goes to the pure gauge value $m_g \sim -4ln(\frac{\beta}{4})$ and this behaviour dominates for low k. While reaching k_c. there is a more pronounced decreasing for higher λ. The greater decreasing is produced because open surface plaquette contribution become more relevant for larger λ. Plaquette contributions to the correlation functions are a source of mixing between the glueball and the Higgs states. So. for high values of λ. where most

Monte Carlo calculation are done, a substantial mixing between these two states seems most probable. This gives support to the qualitative picture described in ref.[41].

The vector excitation mass diverges logarithmically for $k \to 0$ and also for $\beta \to 0$. For finite values of the gauge coupling, the behaviour of the $\vec{W}_{x,\mu}$ and Higgs masses are similar whenever the various parameters are varied. Near the phase transition , for $\beta \leq 1$, the hierarchical relation $m_H < m_v < m_g$ is always verified. Since the same relation between the Higgs and $\vec{W}$ masses is also observed in Monte Carlo data at intermediate values of the gauge coupling, we expect the relation to be preserved in all the intermediate region and, perhaps, even in all the confinement phase of the theory. In the Higgs phase , instead, this relation is reversed [36].

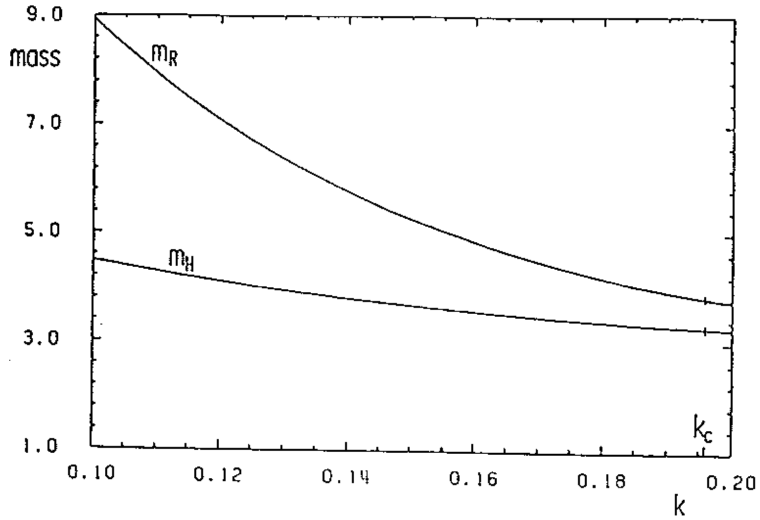


Figure 3.3 (a):Dependence of the Higgs (m_R,m_H), vector gauge boson (m_V) and glueball (m_g) masses on the hopping parameter k, for β=0 and $\lambda = 0.01$,

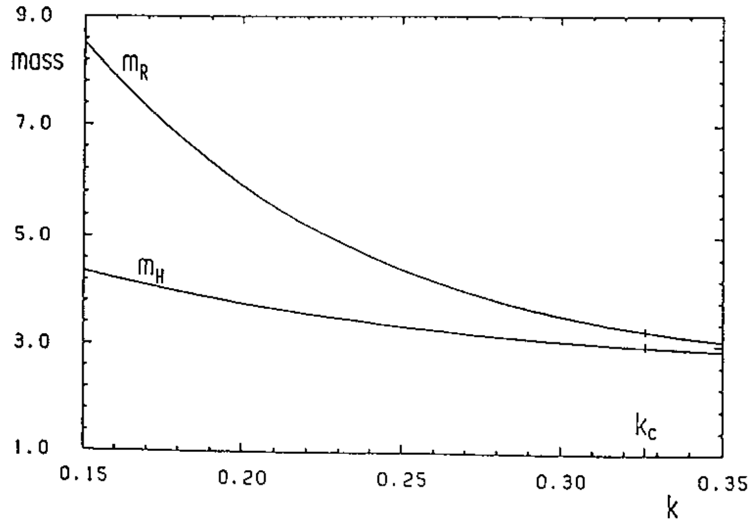


Figure 3.3 (b): The same as Fig. 3.3.(a) but for $\beta = 0$ and $\lambda = 0.1$,

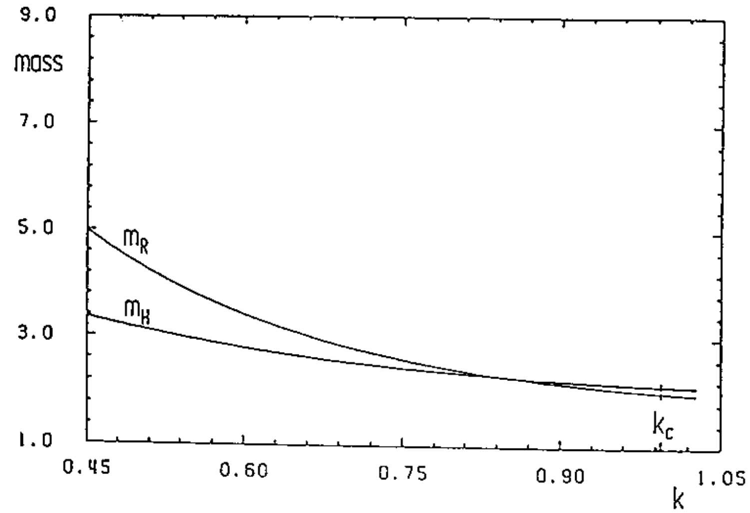


Figure 3.3 (c): The same as Fig. 3.3.(a) but for $\beta = 0$ and $\lambda = 1$,

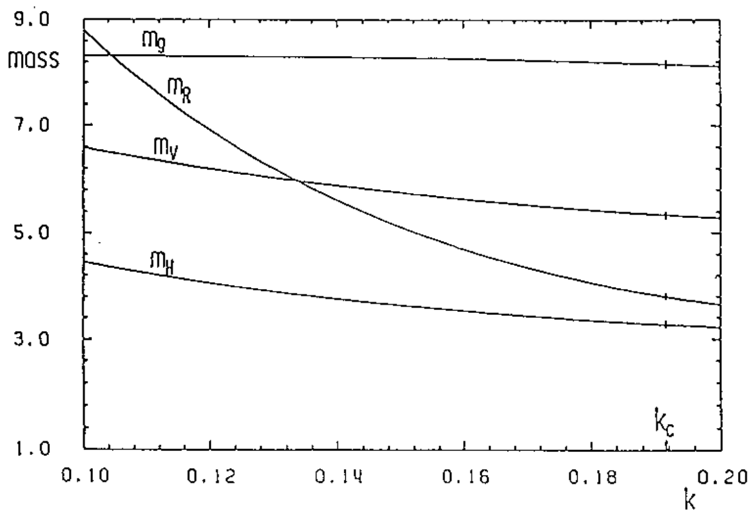


Figure 3.3 (d): The same as Fig. 3.3.(a) but for $\beta = 0.5$ and $\lambda = 0.01$,

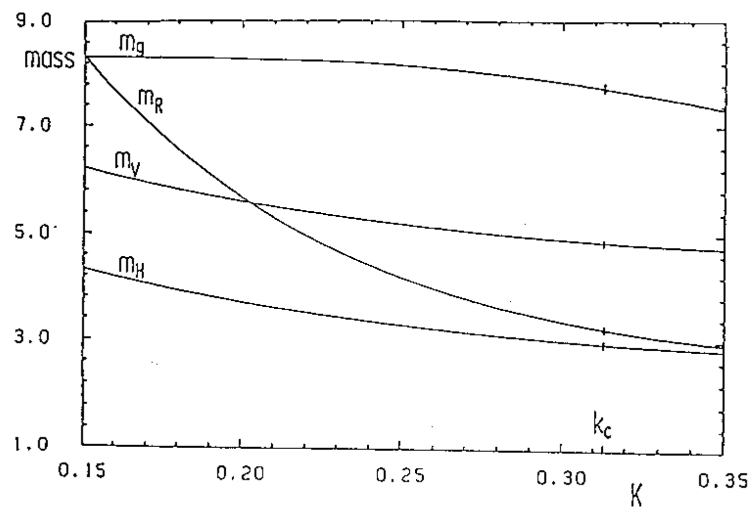


Figure 3.3 (e): The same as Fig. 3.3.(a) but for $\beta = 0.5$ and $\lambda = 0.1$,

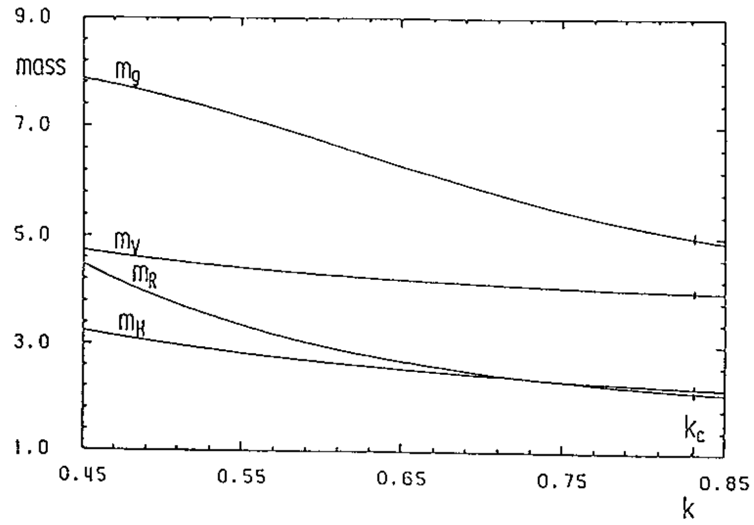


Figure 3.3 (f): The same as Fig. 3.3.(a) but for $\beta = 0.5$ and $\lambda = 1$,

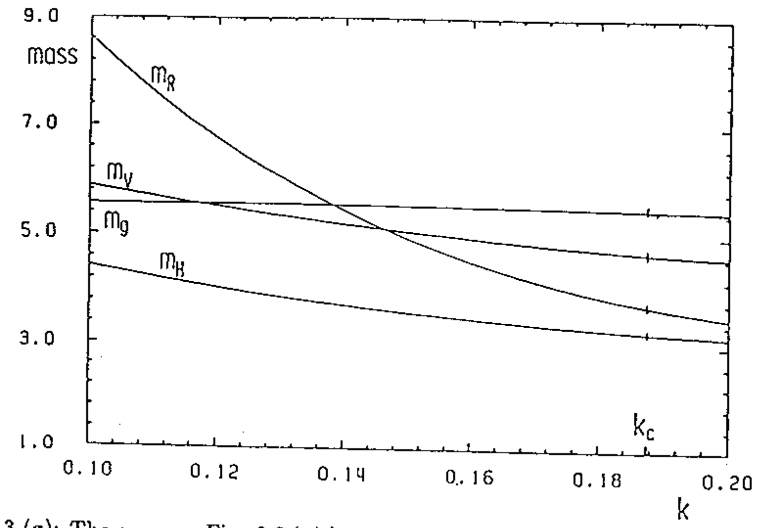


Figure 3.3 (g): The same as Fig. 3.3.(a) but for $\beta = 1$ and $\lambda = 0.01$,

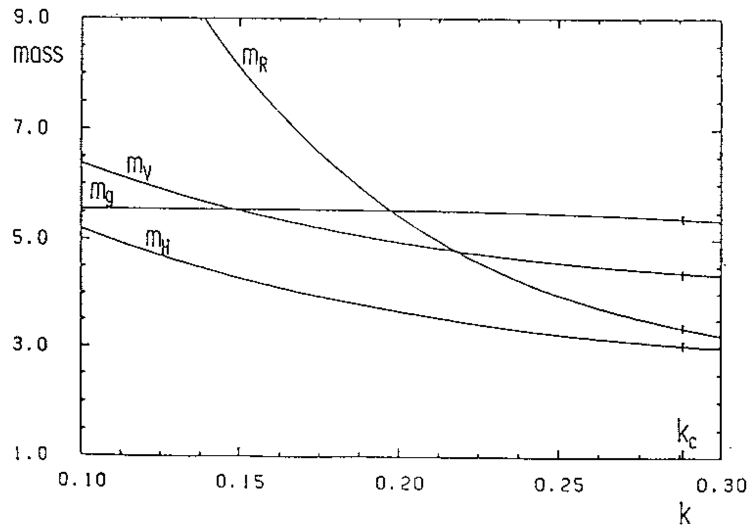


Figure 3.3 (h): The same as Fig. 3.3.(a) but for $\beta = 1$ and $\lambda = 0.1$, and

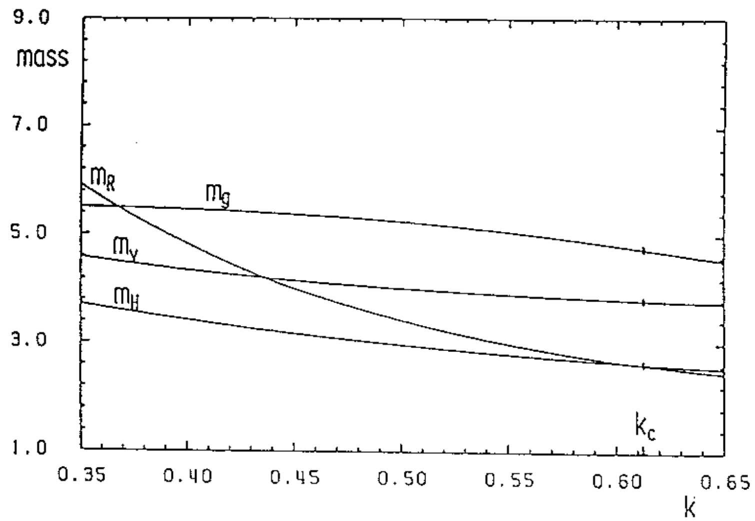


Figure 3.3 (i): The same as Fig. 3.3.(a) but for $\beta = 1$ and $\lambda = 1$.

3.5.3 Conclusions

The $\beta = 0$ linked cluster expansion, when combined with usual strong coupling expansion, is a useful technical tool to analyse the Higgs system in the strong coupling limit. The position of the phase transition surface and the behaviour of the masses can be obtained in the confinement phase. Due to the technical difficulties in obtaining higher order coefficients for larger β, this expansion cannot replace Monte Carlo data for intermediate $\beta \sim 2-3$. However, it is useful as a complementary analysis tool for smaller values of β and λ, where Monte Carlo calculations are difficult to perform . It also serves to get an analytical understanding of the behaviour of physical quantities, as a function of the different bare parameters inside the confinement phase. There are some interesting questions that may well have an answer with this and similar techniques. One of these is a more precise determination of the position of the end points, and the behaviour of masses and renormalized couplings in their neighbourhood. Another interesting question[47] is the change of the mixing of the glueball and Higgs states for different bare parameters. In this article we accomplished a first step in this direction obtaining expansions of the correlation functions up to eighth order in k and second order in β. A more accurate picture may be obtainable with a higher order expansion in k, but probably one needs similar orders as in the φ^4 theory. Furthermore, a similar program as the one developed in ref.[14] at vanishing gauge coupling may be carried out at strong gauge coupling, to study the behaviour of the renormalized self coupling in the neighbourhood of the critical point. In this case, the computation of further correlation functions besides those computed in the present work will be required. This will allow us to inquire into the nature of the resultant continuum theory at the end point of the phase transition diagram, namely if it is interacting or not.

The model that can be immediately tackled by similar methods is the $U(1)$ Higgs model. As explained in Appendix B, all the calculations done in this article can be applied with little variations to this case. An interesting application is to understand what happens with the location of the confinement phase when an external magnetic field is applied. Due to the limitations of mean field and Monte Carlo data in the study of this phenomenom[49], a link cluster expansion seems to be an interesting alternative technique.

Chapter 4

Nonperturbative study of a chiral $SU(2)_L \otimes SU(2)_R$ symmetric model on the lattice.

4.1 Introduction

In Chapter 3 some nonperturbative aspects of the bosonic sector of the Standard model were analysed. In order to make a nonperturbative study of the complete Standard Model one has to overcome the problems associated with the introduction of fermions on the lattice [24]. For example, if one rewrites the continuum action of a free fermion on the lattice by changing the derivatives to finite differences, the continuum theory describes instead of one, 16 degenerate fermion states, associated with the corners of the Brillouin zone in momentum space [25]. Wilson proposed to avoid this problem through the introduction of an additional term into the naive action[50], namely

$$S_W = r \sum_{x,\mu} \left(\bar{\psi}_x U_{x,\mu} \psi_{x+\mu} + \bar{\psi}_{x+\mu} U^+_{x,\mu} \psi_x \right) \tag{4.1}$$

where r is the Wilson parameter and the gauge variable $U_\mu = I$ in the free fermion case. Eq.(4.1) is essentially a mass term, and consequently it explicitly breaks the chiral invariance of the massless theory. After the introduction of the Wilson term all spurious fermions receive masses proportional to r, and only the state associated with the pole at $p = 0$ reaches the zero mass condition. Keeping r fixed the additional fermion states get masses proportional to the cutoff and decouple in the continuum limit.

The general features of the free fermion case are preserved in the interacting theory. In vector gauge theories with gauge fields interacting with massless fermion fields the well known problem of anomalies occurs: There is in general no way of regularizing the theory while keeping simultaneously the gauge and the global chiral symmetry[30,31]. Since both are good symmetries at the classical level, the quantized theory will have less symmetries than the classical one, i.e. an anomaly occurs. In particular, if gauge symmetry is preserved, chiral symmetry is broken. Let us mention that the gauge group SU(2) (but neither SU(3) nor SU(2)$\otimes$ U(1)) is free from perturbative anomalies (However, SU(2) is not free from nonperturbative anomalies as we will discuss below). The lattice formulation of quantum field theories is based on the Feynmann path integral formulation. In the framework of the path integral formulation, the breakdown of chiral symmetry is associated with the noninvariance of the fermionic measure under chiral transformations[51]. The analogous effect in the lattice formulation is absent, since the lattice fermionic measure is invariant under both gauge and global chiral symmetry transformations. Were the Wilson term not present, the lattice formulation would provide us a nonperturbative gauge and chiral invariant regularization of the theory in contradiction with a general theorem about the impossibility of this construction[30,31]. The Wilson term is necessary not only to achieve the desired spectrum but also to obtain the correct set of symmetries of the quantum theory. In fact, both aspects are closely related : The additional fermion states behave as mirror fermions of the original ones, that is they have the same quantum numbers but opposite chirality, and thus the cancellation of the chiral anomaly is assured[25]. The Wilson formulation conduces to the same expression for the consistent perturbative anomaly as, for example, the well known Pauli-Villars regularization [52]-[53].

So far, the formulation of vector gauge theories on the lattice seems to be well understood. The trouble of the Wilson formulation in the case of chiral gauge theories is that the Wilson term breaks the gauge symmetry explicitly. On the other hand, the suppression of the Wilson term would induce the appearence of unwanted fermion states in the spectrum. However, gauge invariance may be recovered in the lattice formulation even in the presence of the Wilson term: We can start with the free action and after that introduce gauge fields with general interactions with the left and right handed fermions. The introduction of the gauge variables in the action may be limited to the derivative terms because the Wilson and mass terms break the gauge invariance anyway (i.e. set $U_\mu = I$ in eq. (4.1)). One can show that scalar fields appear naturally in the partition function when one integrates over all possible gauge configurations [54]-[55], rendering the action gauge invariant [25]. This is a generalization of the well known Stuckelber method in the case of massive gauge theories. What is surprising in the case of fermionic theories is that the method is quite general and the gauge group may be an anomalous one. In this sense one can say that the anomalies associated with local symmetries are always cancelled in the quantum theory.

The interest in the gauge invariant formulation of anomalous guage theories has grown, since it was demonstrated that the chiral Schwinger model, a two dimensional anomalous theory of left fermions interacting with a U(1) gauge field, can be quantized in a consistent way when an appropiate regularization of the theory is chosen [56]-[57]. However, a careful analysis shows that the quantum theory which is obtained is not a theory of interacting left fermions and free right fermions, but a theory equivalent to the vector Schwinger model plus free massless fermions [59]. Even the nonabelian extension of this theory to a group U(N) or SU(N) is, for the choice of the regularization where a consistent theory is known to exist [58], equivalent to the vector theory plus free massless fermions [59]. We will not give here a proof of this statement. We refer the interested reader to refs.[58]-[59].

In four dimensions, the gauge invariant formulation of anomalous gauge theories behaves as a nonrenormalizable theory in the perturbative regime [60]-[61]. However, nonperturbative effects may render the theory consistent. More probably, due to the results of refs.[62]-[63], additional fermion states will appear in the spectrum cancelling the anomalies of the original fermions. It is well possible that the continuum limit of any given anomalous gauge theory belongs to the same class of universality as a theory in which additional fermion states are explicitly included in order to cancel the perturbative and nonperturbative anomalies.

Due to the results stated above one should first concentrate on the study of anomaly free theories. Even though the electroweak group $SU(2)_L \otimes U(1)_Y$ is not anomaly free, the fermion content of the Standard Model is such that anomalies are cancelled within each family of leptons and quarks, without the appearence of mirror fermions. Since each family in the

Standard Model includes 15 Weyl fermions, it is easier to begin the study with the most simple version of a non anomalous chiral gauge theory, namely an $SU(2)_L$ gauge theory with only two doublets of left handed fermions coupled to the gauge fields and two free right fermion doublets (One need an even number of doublets, since if this is not the case nonperturbative anomalies [29] will arise in the theory, rendering it inconsistent). One may ask if the proposal of Refs.[54]-[55] would lead to the desired chiral theory in this case. Recently there has been some progress toward the answer of this important question in the framework of perturbation theory [22]. The conclusion of this study is that lattice regularization can, in principle, also be used for the formulation of perturbation theory achieving the target theory in the continuum, if one adds a full set of gauge noninvariant counterterms in the action, besides the already included Wilson term. Even though the proposal of ref.[22] seems to work at the first orders of perturbation theory, a nonperturbative study of the phase structure of this beautiful but complicated formulation of chiral gauge theories is still lacking.

4.2 Theories with Yukawa couplings

In order to make a nonperturbative study of the Standard model on the lattice a first simplification may be achieved by using the Dashen and Neuberger approximation, by neglecting first the electroweak gauge interactions and assuming that they can be reliable described in the framework of perturbation theory. A further simplification may be achieved by considering the two components of the SU(2) weak isospin doublets to be degenerate in mass in the broken phase of the theory. If only one fermion doublet is kept in the spectrum the theory reduces to the well known sigma model.

Two different formulations of this globally chiral invariant model on the lattice have been studied. In ref.[62] a model with a Yukawa term and a Yukawa like Wilson term was analysed. When the radial degree of freedom of the Higgs field is fixed this model coincides with the limit of vanishing gauge coupling of the formulation of ref. [54]. Once the scalar field acquires a vacuum expectation value, the Yukawa like Wilson term behaves like an effective Wilson term in the action [26]. The basic problem of this formulation is that mirror fermions get masses directly proportional to the vacuum expectation value of the scalar fields, instead of getting masses proportional to the inverse lattice spacing. Thus mirror fermions may only be avoided by sending the renormalized Yukawa coupling to very large values. However, the theory has an infrared fixed point at vanishing renormalized Yukawa coupling and consequently, if no other fixed point exists, even sending the bare Yukawa coupling to infinity the renormalized Yukawa coupling stays at finite values in the scaling region of the theory and, in fact, it vanishes in the continuum limit. A careful analysis was made in ref.[62] and it was shown that mirror fermions can not be kept at the cutoff level in this formulation of the theory.

A second nonperturbative approach to the sigma model was studied in ref.[21]. There, the Wilson term was kept as a chiral symmetry breaking term. Nonsymmetric counterterms must be added to the action in order to find a symmetric phase in the large cutoff limit of perturbation theory. A nonperturbative study was done in the limit of infinite bare Yukawa coupling, a choice that seems to be justified due to the expected triviality of the continuum theory. The resulting spectrum is surprising: In addition to the original fermions a composite mirror fermion doublet appears dynamically in the spectrum. The fermion eigenstates correspond to two degenerate femion states with opposite parity. The conclusion of this study is there exists a phase of the lattice model where chiral symmetry is realized by the appearence of mirror fermions. The possibility of another phase with mirror asymmetric spectrum remains

open. Experimentally it is not yet clear whether nature is described by a phase with physical mirror fermions (at the 100 GeV mass scale [39]) or by an asymmetric phase without mirror fermions.

Thus, the simple formulations of the sigma model with Wilson femions on the lattice leads to the dynamical appearence of mirror fermions. A possible attitude towards this problem [21] is to formulate the model in such a way that mirror fermions are included explicitly in the action and to try to see under what conditions, if any, the mirror fermions can be removed from the physical spectrum in the large cutoff limit. This formulation has the advantage of allowing the introduction of a chiral invariant Wilson term, and the additional possibility of obtaining a local chiral $SU(2)_L \otimes SU(2)_R$ symmetry which can be gauged in the way it is required in the Standard Model [38]. Observe that only with the inclusion of this additional fermion state the Dashen-Neuberger approximation may be done, since if only one doublet is kept the gauge theory is rendered inconsistent by nonperturbative anomalies. Morover, due to the fact that SU(2) has only real representations, in absense of the $U(1)_Y$ interactions this lattice gauge theory may be regarded as an appropiate chiral invariant regularization of a chiral gauge model with two left handed doublets coupled to the gauge fields (See Chapter 2), that is the one we proposed to study in the last section. In the following we will follow this approach.

The lattice action of the model reads[38]

$$S = \sum_x \left\{ -\sum_\mu \frac{k}{2}\, Tr\left[\phi^+_{x+\mu}\phi_x\right] + \frac{\mu}{2}\, Tr\left[\phi^+_x\phi_x\right] + \frac{\lambda}{4}\left(Tr\left[\phi^+_x\phi_x\right]\right)^2 \right.$$
$$+\mu_{\psi\chi}\left(\bar{\chi}_x\psi_x + h.c.\right) - \sum_\mu \left[K_\psi(\bar{\psi}_{x+\mu}\gamma_\mu\psi_x) + K_\chi(\bar{\chi}_{x+\mu}\gamma_\mu\chi_x)\right.$$
$$\left.\left. +r\sqrt{K_\psi K_\chi}(\bar{\psi}_{x+\mu}\chi_x + \bar{\chi}_{x+\mu}\psi_x)\right] + G_\chi\left[\bar{\chi}_{L,x}\phi^+_x\chi_{R,x} + h.c.\right] + G_\psi\left[\bar{\psi}_{L,x}\phi_x\psi_{R,x} + h.c.\right]\right\} \quad (4.2)$$

Here $\sum_\mu$ runs over the eight neighbour directions: $\mu = \pm 1, \ldots, \pm 4$, r is the Wilson parameter and K_χ and K_ψ are the hopping parameters of χ and ψ respectively and the normalization of the fields has been left free. The transformation properties of the scalar and fermion fields under the global $SU(2)_L \otimes SU(2)_R$ transformations are

$$\psi'_{L,x} = U_L\psi_{L,x}, \quad \psi'_{R,x} = U_R\psi_{R,x}, \quad \bar{\psi}'_{L,x} = \bar{\psi}_{L,x}U_L^{-1}, \quad \bar{\psi}'_{R,x} = \bar{\psi}_{R,x}U_R^{-1},$$
$$\chi'_{L,x} = U_R\chi_{L,x}, \quad \chi'_{R,x} = U_L\chi_{R,x}, \quad \bar{\chi}'_{L,x} = \bar{\chi}_{L,x}U_R^{-1}, \quad \bar{\chi}'_{R,x} = \bar{\chi}_{R,x}U_L^{-1}, \quad \phi'_x = U_L\phi_x U_R^{-1}. \quad (4.3)$$

The scalar doublet field ϕ_x may be written in terms of real O(4) fields $\phi_{S,x}$ in the following way:

$$\phi = \phi_{0,x} + i\phi_{s,x}\tau_s \qquad s = 1,..3 \quad (4.4)$$

where τ_s are the hermitian Pauli matrices. In the broken phase the field $\phi_{0,x}$ acquires a non-zero vaccum expectation value giving chirally asymmetric masses to the fermions of the model, while the three scalar fields ϕ_s become massless Goldstone bosons. On the contrary,in the symmetric phase the four scalar fields remain massive and degenerate in mass.

The normalization freedom in the scalar sector may be used to set the mass parameter $\mu = 1 - 2\lambda$, a choice that is very useful for numerical simulations and for high temperature expansions, as we have seen in chapter 3. In the fermionic sector we have two normalization degrees of freedom. One of it can be used to choose the condition $K_\psi = K_\chi = K$. This choice is of course very useful for the linked cluster expansions. The Yukawa couplings G_ψ and G_χ

remain different. The other normalization degree of freedom may be used to set $\mu_{\psi\chi} = 1$. In the calculations below, however, we will keep an arbitrary $\mu_{\psi\chi}$. With the above choice of normalization the action reads:

$$S = \sum_x \Big\{ -\sum_\mu k\, \phi_{S,x+\mu}\phi_{S,x} + \phi_{S,x}\phi_{S,x} + \lambda\left(\phi_{S,x}\phi_{S,x} - 1\right)^2$$
$$+\mu_{\psi\chi}\left(\bar{\chi}_x\psi_x + h.c.\right) - \sum_\mu K\left[\bar{\psi}_{x+\mu}\gamma_\mu\psi_x + \bar{\chi}_{x+\mu}\gamma_\mu\chi_x\right.$$
$$\left.+r(\bar{\psi}_{x+\mu}\chi_x + \bar{\chi}_{x+\mu}\psi_x)\right] + G_\chi\, \bar{\chi}_x\Gamma_S^+\chi_x\, \phi_{S,x} + G_\psi\, \bar{\psi}_x\Gamma_S\psi_x\, \phi_{S,x}\Big\} \tag{4.5}$$

where $\Gamma_S = (I, i\gamma_5\tau_s)$ and a summation over the O(4) indices $S = 1,\ldots 4$ is understood. The Wilson parameter r has been left arbitrary, but since we expect it to be irrelevant in the scaling region of the theory, it can be fixed at some positive value. In the linked cluster expansion we will develop in sections 4.6-4.8 the value $r = 1$ will be used.

4.3 Renormalization group flow of the renormalized couplings

The definition of renormalized couplings and masses of this section will follow closely those of ref.[14] for the pure scalar field theory. We will define

$$M = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \qquad and \qquad \Psi = \begin{pmatrix} \psi \\ \chi \end{pmatrix}. \tag{4.6}$$

The renormalized fermion and scalar masses m_f and m_R will be defined assuming that at low momentum the scalar and fermion propagators have the behaviour

$$G_{S\,S'}(p) = \delta_{S,S'}\,\frac{Z_R}{m_R^2 + p^2} \qquad (p \to 0) \tag{4.7}$$

$$G_\Psi(p) = Z_\Psi\,(i\gamma p + m_f\, M)^{-1} \qquad (p \to 0) \tag{4.8}$$

These masses don't coincide with the physical masses obtained from the pole of the propagators in momentum space, but we expect both values to be close in the scaling region and their difference computable in a perturbative expansion in the neighbourhood of the multi-critical point. The renormalized self coupling and the renormalized Yukawa coupling will be defined in terms of the amputated 1-particle irreducible parts of the connected correlation functions in momentum space $\Gamma^{(n_f,n_B)}(p_1,\ldots,p_{n_f}, q_1,\ldots,q_{n_f})$ through

$$\Gamma^{(0,4)}(0,0,0,0)_{SS'S''S'''} = -\frac{1}{3}G_4(S,S',S'',S''')\, Z_R^{-2}\, g_R \tag{4.9}$$

$$\Gamma^{(2,1)}(0,0,0)_S = -\begin{pmatrix} G_\psi^R\Gamma_S & 0 \\ 0 & G_\chi^R\Gamma_S^+ \end{pmatrix} Z_\Psi^{-1} Z_R^{-\frac{1}{2}} \tag{4.10}$$

where $G_4(S,S',S'',S''')$ is the O(4) invariant factor defined in eq. (E.8) of Appendix E .

The renormalized vertex functions $\Gamma_R^{(n_f,n_B)}$ are defined from $\Gamma^{(n_f,n_B)}$ as follows[14]:

$$\Gamma_R^{(n_f,n_B)} = Z_R^{\frac{n}{2}} Z_\Psi^{n_f}\Gamma^{(n_f,n_B)} \tag{4.11}$$

From our definition of renormalized masses and couplings one find the following renormalization conditions

$$\Gamma_R^{(0,2)}(0,0)_{SS'} = \delta_{S,S'}\, m_R^2 \tag{4.12}$$

$$\Gamma_R^{(2,0)}(0,0) = m_f\, M \tag{4.13}$$

$$\Gamma_R^{(0,4)}(0,0,0,0)_{S,S',S'',S'''} = -\frac{g_R}{3}G_4(S,S',S'',S''') \tag{4.14}$$

$$\Gamma_R^{(2,1)}(0,0,0,0)_S = -\begin{pmatrix} G_\psi^R\,\Gamma_S & 0 \\ 0 & G_\chi^R\,\Gamma_S^+ \end{pmatrix} \tag{4.15}$$

Following ref.[14] we may now define the dimensionful quantities

$$\bar{\Gamma}_R^{(n_f,n_B)} = a^{n_B+\frac{3}{2}n_f-4}\Gamma_R^{(n_f,n_B)} \tag{4.16}$$

To obtain physical results from the renormalized vertex functions, $\bar{\Gamma}_R^{(n_f,n_B)}$ at low momentum ($p_i < m_R$) they should become independent on the lattice spacing, in the sense that[1]

$$|a\frac{\partial\bar{\Gamma}_R^{(n_f,n_B)}}{\partial a}| << |\bar{\Gamma}_R^{(n_f,n_B)}| \tag{4.17}$$

holds, when $m_R << 1$. A region of the phase diagram where eq. (4.17) is satisfied is called scaling region. This region is localized near the multicritical points in the bare parameter space and is there where the well known Callan-Symanzik equation gives nontrivial information about the behaviour of the theory. In particular, the behaviour of the renormalized couplings as a function of a physical mass, say m_R, is governed by the renormalization group equations

$$m_R\frac{\partial g_R}{\partial m_R} = \beta_\lambda \tag{4.18}$$

$$m_R\frac{\partial G_\psi^R}{\partial m_R} = \beta_{G_\psi} \tag{4.19}$$

$$m_R\frac{\partial G_\chi^R}{\partial m_R} = \beta_{G_\chi} \tag{4.20}$$

where β_λ, β_{G_ψ} and β_{G_χ} are independent of m_R up to scaling violations which vanish with some power of m_R when $a \to 0$.

If the renormalized couplings g_R, G_ψ^R and G_χ^R are small the different β functions may be computed in perturbation theory. The calculation to one loop gives the result [38]

$$\beta_{G_\psi} = \frac{1}{16\pi^2}4G_\psi^R\left[\left(G_\psi^R\right)^2 + \left(G_\chi^R\right)^2\right] \tag{4.21}$$

$$\beta_{G_\chi} = \frac{1}{16\pi^2}4G_\chi^R\left[\left(G_\psi^R\right)^2 + \left(G_\chi^R\right)^2\right] \tag{4.22}$$

$$\beta_\lambda = \frac{1}{16\pi^2}\left[4g_R^2 + 16g_R\left(G_\psi^R\right)^2 + 16g_R\left(G_\chi^R\right)^2 - 96\left(G_\psi^R\right)^4 - 96\left(G_\chi^R\right)^4\right] \tag{4.23}$$

[1] For a more detailed discussion see ref.[14]

The β function of the Yukawa couplings is positive, meaning that once the Yukawa couplings are in the perturbative range, they go monotonically to lower values as $m_R \to 0$. Thus, in the continuum limit the renormalized Yukawa couplings vanish. The β function of the renormalized self coupling β_λ is either positive or negative depending on the value of the ratio of the self coupling to the renormalized Yukawa couplings. Due to the asymptotic behaviour of the Yukawa couplings, β_λ will become positive for sufficiently small lattice spacings, and the renormalized scalar self coupling will also go to zero as $a \to 0$ (we are assuming that $g_R \geq 0$). Thus, the perturbative behaviour of the theory suggests that it is probably trivial, that is it consists of noninteracting fermion and scalar fields in the continuum limit. However, the perturbative behaviour is not a proof of triviality. Triviality holds if the continuum theory obtained for all values of the bare parameters belongs to the same class of universality as the one described by perturbation theory, an information that may only be obtained using nonperturbative methods. In the pure scalar theory, for example, a linked cluster expansion of the correltaion functions was done in ref.[14]. The result was that, whenever the system enters the scaling region, the renormalized couplings are small making the perturbative treatment reliable. Morover, the behaviour of the physical parameters described by perturbation theory extrapolates continuosly the one obtained from the high temperature expansions. Thus, there is now little doubt that the pure scalar theory is trivial. A similar nonperturbative study is unavoidable in the present theory in order to understand the phase structure and investigate the critical behaviour near the continuum limit.

4.3.1 Solution of the renormalization group equations

To solve the renormalization group equations to one loop it is convenient to choose as independent physical parameters one Yukawa coupling, say G_ψ, the ratio of both Yukawa couplings and the ratio of the scalar self coupling to the square of one of the two Yukawa couplings, namely

$$m_R \frac{\partial r_y}{\partial m_R} = 0 \tag{4.24}$$

$$m_R \frac{\partial G_\psi^R}{\partial m_R} = \beta_3 \left(G_\psi^R\right)^3 \tag{4.25}$$

$$m_R \frac{\partial r_\lambda}{\partial m_R} = \left(G_\psi^R\right)^2 \left(\beta_2 r_\lambda^2 + \beta_1 r_\lambda + \beta_0\right) \tag{4.26}$$

where

$$r_y = \frac{G_\psi^R}{G_\lambda^R}, \qquad r_\lambda = \frac{g_R}{\left(G_\psi^R\right)^2}, \tag{4.27}$$

and $\hat{\beta}_i = 16\pi^2 \beta_i$ is given by

$$\hat{\beta}_3 = 4\left(1 + r_y^2\right), \tag{4.28}$$

$$\hat{\beta}_2 = 4r_\lambda^2, \qquad \hat{\beta}_1 = 8r_\lambda\left(1 + r_y^2\right), \qquad and \qquad \hat{\beta}_0 = -96\left(1 + r_y^2\right). \tag{4.29}$$

Eq.(4.24) tell us that the ratio of both Yukawa couplings is preserved while approaching the continuum limit. Eq.(4.25) may be easily solved giving

$$m_R = C_y \exp\left(-\frac{1}{2\beta_3 \left(G_\psi^R\right)^2}\right) \tag{4.30}$$

or, inverting this relation

$$\left(G_\psi^R\right)^2 = \left(2\beta_3(C' - \tau)\right)^{-1} \tag{4.31}$$

where C_y and C' are integration constants and $\tau = \ln m_R$. This means that, while approaching the continuum limit ($m_R \to 0$) the Yukawa couplings goes to zero as $|\ln(m_R)|^{-\beta_3}$.

The last equation may be simplified by inserting the solution of eq.(4.25) in it and making an appropiate change of variables:$\tau \to \tau' = 2\beta_3(C' - \tau)$. We get

$$-2\beta_3 \frac{\partial r_\lambda}{\partial \ln \tau'} = \left(\beta_2 r_\lambda^2 + \beta_1 r_\lambda + \beta_0\right) \tag{4.32}$$

Observe that $\tau' \to +\infty$ as $m_R \to 0$. The renormalization group flow of r_λ has two nontrivial fixed points which may be found by solving the equation

$$\beta_2 \left(r_\lambda^2\right)_{f.p.} + \beta_1 \left(r_\lambda\right)_{f.p.} + \beta_0 = 0 \tag{4.33}$$

The two values of $(r_\lambda)_{f.p.}$ are then given by

$$r_{1,2} = -\left(1 + r_y^2\right) \pm \sqrt{r_y^4 + 26r_y^2 + 25} \tag{4.34}$$

Once we know the location of the fixed points we can get the qualitative behaviour of r_λ without computing the exact solution of the renormalization group equation. The expected renormalization flow is shown in Fig.4.1. If we start with a given initial ratio $r_\lambda > r_1$, the renormalization flow will drive it to values closer to r_1 for smaller lattice spacings. The ratio r_λ reachs asymptotically the fixed point r_1 when $m_R \to 0$. The same happens for an initial value $r_1 \geq r_\lambda > r_2$, namely the renormalization flow drives r_λ always to r_1 in the continuum limit. Thus, the 'triviality' of the theory is verified in these situations since both renormalized coupling constants vanish in the continuum limit.

On the other hand, for an initial value $r_\lambda < r_2$, the renormalization flow drives the ratio to negative infinite values in the continuum limit. This case corresponds to the situation where while reaching the multicritical point the renormalized self coupling is negative. Since β_λ is always positive at sufficiently small Yukawa couplings, the renormalization flow will carry the self coupling to even lower values. Obviously, at the moment that the renormalized coupling becomes strong the perturbative analysis looses sense. A negative self coupling in the quasi continuum limit breaks the stability of the scalar potential, and thus this is an undesired situation.

Observe that even starting with a positive value of the self coupling at very small m_R, the renormalization flow may drive it to negative values at greater lattice spacings. This corresponds to the case where $r_2 < r_\lambda < r_1$, that is to say for sufficiently strong Yukawa couplings. What this is telling us, and is important for the high temperature expansions, is that for sufficiently large renormalized Yukawa couplings, the values of the scalar self coupling obtained from the high temperature expansions may go through negative values when the hopping parameters are not sufficiently close to their critical values (see Fig.4.1). In the next sections we will discuss this point in more detail.

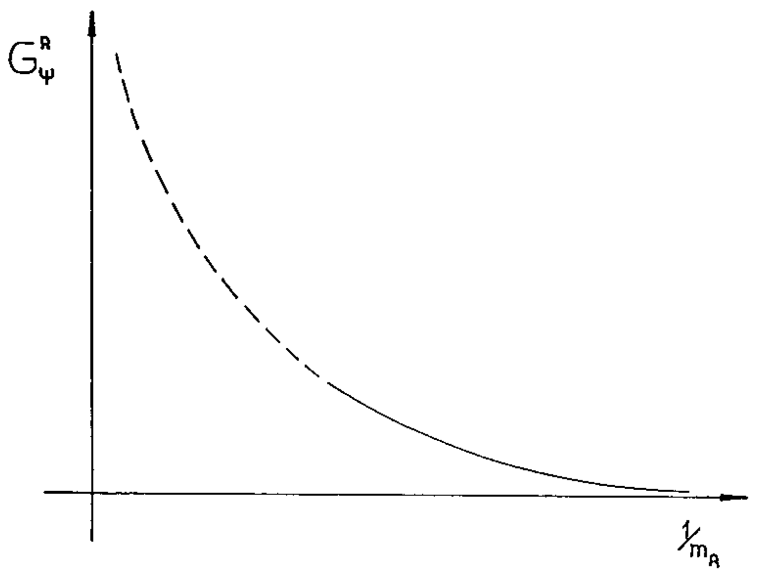


Fig. 4.1 (a): Expected nonperturbative (dashed line) and perturbative (continuous line) flow, of the renormalized Yukawa couplings,

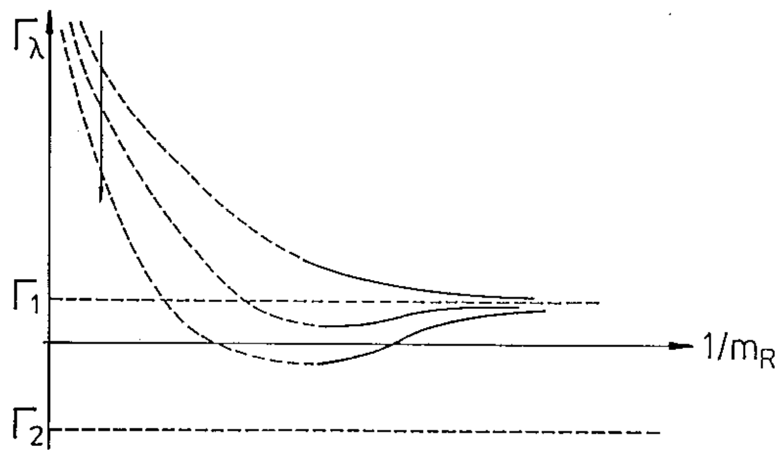


Figure 4.1 (b): The same as Fig. 4.1 (a) but for the ratio r_λ. The arrow shows the direction of increasing renormalized Yukawa couplings.

4.4 Linked cluster expansion

4.4.1 Basic formulas

To develop the linked cluster expansion we need to calculate the general one point expectation value

$$\left\langle \phi_{S_1} \cdots \phi_{S_k} \psi_{a_1} \bar{\psi}_{b_1} \cdots \psi_{a_l} \bar{\psi}_{b_l} \chi_{c_1} \bar{\chi}_{d_1} \cdots \chi_{c_j} \bar{\chi}_{d_j} \right\rangle^c_{k=0;K=0} \tag{4.35}$$

where the superscript c means that we have to compute the connected one point correlation function. For this purpose, let us define the fermion dependent part of the partition function, $Z_{\psi\chi}$, by

$$Z_{\psi\chi} = \int d\bar{\Psi} d\Psi \exp - \left\{ \bar{\Psi} O \Psi - \bar{N} \Psi - \bar{\Psi} N \right\} \tag{4.36}$$

where

$$O = \begin{pmatrix} G_\psi \Gamma_S \phi_S & \mu_{\psi\chi} \\ \mu_{\psi\chi} & G_\chi \Gamma_S^\mp \varphi_S \end{pmatrix} \tag{4.37}$$

and, as usual, a current term $\bar{N}\Psi + h.c.$ has been added. Apart from a term independent of the currents and the scalar fields, the result of the Grassmann integration is

$$Z_{\psi\chi} = (detO) \ \exp\left\{ \bar{N} O^{-1} N \right\} \tag{4.38}$$

The expression of O^{-1} may be easily obtained from eq.(4.37) and the algebra of gamma and Pauli matrices:

$$O^{-1} = \frac{1}{[G_\chi G_\psi(\phi_S\phi_S) - \mu^2_{\psi\chi}]} \begin{pmatrix} G_\chi \Gamma_S^+ \phi_S & -\mu_{\psi\chi} \\ -\mu_{\psi\chi} & G_\psi \Gamma_S \varphi_S \end{pmatrix} \tag{4.39}$$

Comparing the matrix form of O and of O^{-1} we get a relation between $detO$ and $det(O^{-1}) = (detO)^{-1}$, namely

$$det(O^{-1}) = \frac{detO}{[G_\chi G_\psi(\varphi_S\varphi_S) - \mu^2_{\chi\psi}]^{16}}, \tag{4.40}$$

from where we get the expression of detO

$$detO = [G_\chi G_\psi(\phi_S\phi_S) - \mu^2_{\psi\chi}]^8. \tag{4.41}$$

The expectation value, eq.(4.35), can now be computed by introducing the result of the Grassmann integral in the full partition function of the theory. For doing this, it is convenient to express O^{-1} as a sum of two matrix terms, $O^{-1} = O_S^{-1}\varphi_S + \tilde{\mu} M$ where the 16x16 matrix $O_S^{-1}\phi_S$ is given by the two diagonal 8x8 blocks of the matrix O^{-1},

$$O_S^{-1} = \frac{1}{[G_\psi G_\chi(\phi_S\phi_S) - \mu^2_{\psi\chi}]} \begin{pmatrix} G_\chi \Gamma_S^+ & 0 \\ 0 & G_\psi \Gamma_S \end{pmatrix}, \tag{4.42}$$

and

$$\tilde{\mu} = \frac{-\mu_{\psi\chi}}{[G_\psi G_\chi(\phi_S\phi_S) - \mu^2_{\psi\chi}]} \tag{4.43}$$

Thus, at vanishing hopping parameters the partition function of the theory may be written as

$$Z(N, J) = \int d\phi_S \ \exp\left\{ \tilde{\mu} \bar{N} M N \right\} \ detO \ \exp - \left\{ \phi_S\phi_S + \lambda(\varphi_S\varphi_S - 1)^2 - (J_S + \bar{N} O_S^{-1} N)\phi_S \right\} \tag{4.44}$$

where J_S are the currents associated with the scalar fields. We will assume that the continuum theory belongs to the same class of universality independently of the value we give to the bare scalar self coupling. In other words, the quasi continuum theories, in the scaling region near the critical points where all physical masses are small with respect to the inverse lattice spacing a^{-1}, are equivalent for arbitrary values of the bare self coupling. We can consequently give the bare self coupling an infinite value in order to study the nonperturbative behaviour of the theory. Moreover, based on the experience gained in the pure scalar theory, the value infinite for the bare self coupling should be prefered since is in this plane of the bare parameter space where the upper bound of the renormalized self coupling is obtained [14]. To take the limit $\lambda \to \infty$ means that the unrenormalized scalar fields will satisfy the relation $\phi_S \phi_S = 1$ (However, the renormalized fields don't satisfy this relation). In the limit of infinite scalar self coupling the scalar dependent part of the partition function at vanishing hopping parameters reads

$$Z(N,J)_\phi = \int d\phi_S \ \exp\{(J_S + \bar{N} O_S^{-1} N)\phi_S\} \ \delta(\phi_S \phi_S - 1) \tag{4.45}$$

The integral may be done by using the character properties of the group SU(2) explained in Appendix C. Remembering that $\phi = \phi_S T_S$, $(T_S = (I, i\tau_s))$ we can define $j_S = (J_S + \bar{N} O_S^{-1} N)$ and the matrix $j = j_S T_S$. The matrices ϕ and j may be rewritten in terms of their radial and angular degrees of freedom

$$j = |j| \ \tilde{j}, \qquad \phi = \rho \ \tilde{\phi} \tag{4.46}$$

where

$$|j| = \sqrt{j_S j_S}, \qquad \rho = \sqrt{\phi_S \phi_S} \tag{4.47}$$

and $\tilde{j}$ and $\tilde{\phi}$ are matrices that belong to the fundamental representation of the group SU(2). Consequently,

$$Z(N,J)_\phi = \int d\tilde{\phi} \ \rho^3 d\rho \ \exp\left\{ \frac{|j| \ \rho \ Tr[\tilde{\phi} \ \tilde{j}]}{2} \right\} \ \delta(\rho^2 - 1) \tag{4.48}$$

The integral over ρ may be now easily done. The integral over the angular degrees of freedom may be also done by using eqs.(A.4)-(A.10) . It follows that

$$Z(N,J)_\phi = \frac{2 \, I_1(|j|)}{|j|} \tag{4.49}$$

where $I_1(|j|)$ is a modified Bessel function. When $Z(N,J)_\phi$ is replaced in eq.(4.44) we obtain the final expression of $Z(N,J)$ at infinite bare self coupling:

$$Z(N,J) = \exp\left\{\bar{\mu} \bar{N} M N\right\} \ \left(\sum_{n=0}^{\infty} \frac{[(J_S + \bar{N} O_S^{-1} N)(J_S + \bar{N} O_S^{-1} N)]^n}{n! \ (n+1)! \ 4^n} \right) \tag{4.50}$$

This expression is ill defined at the values of the bare parameter where $detO = 0$, namely $G_\chi \ G_\psi = \mu_{\chi\psi}^2$. In the following we will assume that this condition does not occur. However, by developing the linked cluster expansion we will be able to discuss the behaviour of the system in this particular limit.

Since we now have a formal expression for $Z(N,J)$ in terms of the currents N and J, we can easily derive from here the general one point connected correlation functions given in eq.(4.35). The details will be given in Appendix E . The general result is

$$\left\langle \phi_{S_1} ... \phi_{S_k} \ \Psi_{a_1} \bar{\Psi}_{b_1} ... \Psi_{a_l} \bar{\Psi}_{b_l} \right\rangle^c_{k=0;K=0} = \bar{\mu} \ M_{a_l b_l} \ \delta_{l,1} \delta_{k,0} +$$

$$\frac{C_{k+l} \sum_{(k+l)\{S_i,S_j\}} \prod_{(S_i,S_j)} \delta_{S_i,S_j} \sum_{\pi(l)} \sigma_\pi \ O^{-1}_{S_{k+1},a_1 b_{\pi(1)}} ... O^{-1}_{S_{k+l},a_l b_{\pi(l)}}}{(k+l-1)!! \ \ 2^{k+l}} \tag{4.51}$$

Here $\sum_{(k+l)\{S_i,S_j\}}$ is a summation over all different pairings of the O(4) indices $S_1, S_2, .., S_{k+l}$ and $\sum_{\pi(l)}$ means summation over the permutations$\{\pi(1), \pi(2), ..., \pi(l)\}$ of $\{1, 2, ...l\}$ with parity σ_π. It follows from here that the sum of the number of scalars (k) and the number of fermion pairs (l) in the vertices is always even, with the only exception of the chiral invariant $\langle \bar{\chi}\psi \rangle$ (remember the expression of Ψ, eq.(4.6)). The coefficients C_{k+l} entering in eq.(4.51) are given by

$$C_{2N} = \sum_{m,n_1,n_2,...=1}^{\infty} \delta_{N,n_1+2n_2+3n_3+...} \ \delta_{m,n_1+n_2+n_3+...} \frac{(2N)! \ (-1)^{m-1}(m-1)!}{n_1! n_2! n_3! ... (1!2!)^{n_1} \ (2!3!)^{n_2} (3!4!)^{n_3} ...} \tag{4.52}$$

The expression of the connected moments, eq.(4.51), is very similar to the one obtained by Montvay [21] in his study of the σ model with Wilson fermions at infinite Yukawa coupling. As in that case, we see that chiral symmetry will never be broken in the vertices. Morover, since in the present formulation the Wilson term preserves chiral symmetry, it will be an exact symmetry of the theory in the range of validity of the linked cluster expansion. In particular,

$$\langle \phi_S \rangle \ = \ \left\langle \bar{\Psi} O_S^{-1} \Psi \right\rangle = 0 \tag{4.53}$$

The chiral invariant fermion bilinear form $\bar{\Psi} M \Psi$, instead, will get a nonzero vacuum expectation value.

4.4.2 General features of the expansion.

The derivation of the linked cluster expansion for theories that include fermions may be done in the same lines as the one done for theories with only scalar fields in ref. [43]. However, let us mention some special features of the linked cluster expansions at $r = 1$

a) The definition of the topologically equivalent graphs must be done taking into account the orientation of the fermion lines. Two graphs which only differ in the orientation of the fermion lines must be considered nonequivalent. This has important implications over the symmetry factors of the graphs. For example, the diagrams of Fig.4.2 (a),(b) have symmetry factor S(G)=2,1 respectively.

b) For each pair of vertices i and $i + \hat{\mu}$ one must write a factor $K_{\pm\mu} = K(\pm\gamma_\mu + M)$ if they are joined by one edge with direction $\pm\hat{\mu}$. Since both the link factor K_μ as the one point expectation value (see eq.(4.51)) are given by matrix expressions, one has to write all factors in the order given by the fermion number flow. A general rule to make it correctly is given below.

Fig.4.2 (a)-(b): Example of two non-equivalent graphs which only differ in the orientation of the fermion lines.

c) It may be seen from the derivation of the expansion of the partition function, in the way of ref.[37], that if we write all one point expectation values in the order given in eq.(4.51) a minus sign appears each time a fermion loop is closed. This is a consequence of the Grassmann algebra of the fermion variables. The general recipe to make no confusion with the signs is to follow the direction of the fermion number flow, writing the matrix factors O_S^{-1} and M which appear at the vertices with positive sign, setting at the end one minus sign if and only if an odd number of fermion loops appear in the contribution.

What is peculiar of these expansions in comparison with perturbation theory is that the same diagram can give contributions where fermion loops appear and contributions where fermion loops are absent. An example are the two contributions of the diagram of Fig.4.3 which is given in eq.(4.90). The change in sign and the trace are due to the presence of one fermion loop. Obviously, as more complicated graphs are considered the amount of contributions of the same graph increases. This is another important difference with the calculations with scalars, which obviouly makes the computations rather more difficult in this case.

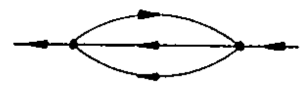


Fig.4.3: First correction to the irreducible part of the fermion propagator at $G_\psi \neq G_\chi$.

d) Since the link factor K_μ carries a γ_μ matrix the contribution of a graph depends on its embedding on the lattice. For example two different embeddings of a same graph (with a graph we simbolizes all the graphs belonging to the same class of equivalence) are given in Fig.4.4 (a),(b). In view of the following relations satisfied by the K_μ matrices (see Appendix D)

$$K_\mu K_{-\mu} = 0, \qquad [K_\mu, M] = 0 \tag{4.54}$$

it is clear that the contribution of Fig.4.4 (a) vanishes, while the one of Fig.4.4 (b) doesn't. Thus, in contradistinction to what happens in the pure scalar theory, there is no way of multiplying the contribution of one embedding by some embedding factor to obtain the total contribution of a graph.

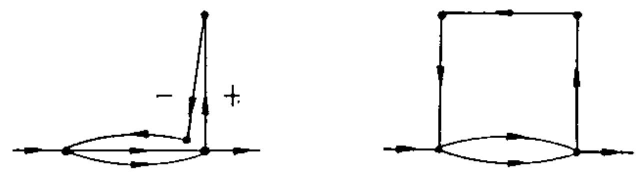


Fig.4.4 (a)-(b): Two different embeddings on the lattice of the graph of Fig.2 (a). In Fig.4.4 (a) the links + and - coincide, while in Fig.4.4 (b) the fermion loop links are the sides of one lattice's plaquette.

4.4.3 Renormalized couplings and masses

In order to calculate the masses, couplings and renormalization constants defined in section 4.3 it is convenient to define the following susceptibilities:

$$\sum_x \langle \phi_S(x)\phi_{S'}(0)\rangle^c = \lambda_2\, \delta_{S,S'} \tag{4.55}$$

$$\sum_x \left\langle \Psi(x)\bar{\Psi}(0)\right\rangle^c = \lambda_2^f = \bar{\lambda}_2^f M \tag{4.56}$$

$$\sum_x x^2 \langle \phi_S(x)\phi_{S'}(0)\rangle^c = \mu_2\, \delta_{S,S'} \tag{4.57}$$

$$\sum_{i,x} x_i\gamma_i \left\langle \Psi(x)\bar{\Psi}(0)\right\rangle^c = \lambda_1^f \tag{4.58}$$

$$\sum_{x,y,z} \langle \phi_S(x)\phi_{S'}(y)\phi_{S''}(z)\phi_{S'''}(0)\rangle^c = \frac{1}{3}G_4(S,S',S'',S''')\lambda_4 \tag{4.59}$$

$$\sum_{x,y} \left\langle \phi_S(x)\Psi(y)\bar{\Psi}(0)\right\rangle = \lambda_3^f = \bar{\lambda}_3^f O_S^{-1} \tag{4.60}$$

The relation between the susceptibilities and the renormalized masses and couplings is given by

$$m_R^2 = 2d\frac{\lambda_2}{\mu_2} \tag{4.61}$$

$$m_f = d\frac{\bar{\lambda}_2^f}{\lambda_1^f} \tag{4.62}$$

$$Z_R = 2d\frac{(\lambda_2)^2}{\mu_2} \tag{4.63}$$

$$Z_\Psi = d\frac{\left(\bar{\lambda}_2^f\right)^2}{\lambda_1^f} \tag{4.64}$$

$$g_R = -(2d)^2\frac{\lambda_4}{(\mu_2)^2} \tag{4.65}$$

$$G_\psi^R = -\sqrt{2}\, d\frac{\bar{\lambda}_3\, b}{\lambda_1^f\, \sqrt{\mu_2}} \tag{4.66}$$

$$G_\chi^R = -\sqrt{2}\, d\frac{\bar{\lambda}_3\, c}{\lambda_1^f\, \sqrt{\mu_2}} \tag{4.67}$$

where the coefficients b and c are defined in eq.(D.1) of Appendix D and d is the number of dimensions of the euclidean lattice we are considering. Thus, the linked cluster expansion of the quantities of physical interest is easily derived from the one of the susceptibilities.

By making the expansions of the susceptibilities a first simplification is achieved while computing only the one particle irreducible graphs [64]-[43]. A connected graph G is called one particle irreducible if it cannot be broken into two disconnected pieces by removing a single (fermionic or bosonic) internal line. This definition includes the case where the graph consists of only a single vertex and no internal line. Following the notation of ref.[64] we

will define $\lambda_2^{1pi}, \left(\bar{\lambda}_2^f\right)^{1pi}, \mu_2^{1pi}, \left(\lambda_1^f\right)^{1pi}, \lambda_4^{1pi}$ and $\bar{\lambda}_3^{1pi}$ as the irreducible part of the corresponding susceptibilities. One may show that

$$\lambda_2 = \lambda_2^{1pi}\left(1 - 4dk\lambda_2^{1pi}\right)^{-1} \tag{4.68}$$

$$\bar{\lambda}_2^f = \left(\bar{\lambda}_2^f\right)^{1pi}\left(1 - 2dK\left(\bar{\lambda}_2^f\right)^{1pi}\right)^{-1} \tag{4.69}$$

$$\mu_2 = \left[\mu_2^{1pi} - 4dk\left(\lambda_2^{1pi}\right)^2\right]\left(1 - 4dk\lambda_2^{1pi}\right)^{-2} \tag{4.70}$$

$$\lambda_1^f = \left(\lambda_1^{1pi} + 2dK\left[\left(\bar{\lambda}_2^f\right)^{1pi}\right]^2\right)\left(1 - 2dK\left(\bar{\lambda}_2^f\right)^{1pi}\right)^{-2} \tag{4.71}$$

$$\lambda_4 = \lambda_4^{1pi}\left(1 - 4dk\lambda_2^{1pi}\right)^{-4} \tag{4.72}$$

$$\bar{\lambda}_3 = \bar{\lambda}_3^{1pi}\left(1 - 4dk\lambda_2^{1pi}\right)\left(1 - 2dK\left(\lambda_2^f\right)^{1pi}\right)^{-2} \tag{4.73}$$

From here we can easily get the expression of the renormalized physical quantities. Thus, it is sufficient to calculate the one particle irreducible graphs in order to get the desired physical information.

A further simplification of the expansions is achieved by replacing the one point connected moments defined in eq.(4.51) by the so called renormalized moments[64]-[43]. To understand what is it meant with renormalized moments suppose that there are no external lines but l internal fermionic and k scalar internal lines at one vertex V. We can consider a new graph V_R in which the initial internal lines act as external lines at one vertex, and there is no other external line. The contribution of V_R at zero order in the hopping parameters is simply the original one point connected moment. The idea is to generate higher order contributions to the original graph G by replacing the one point connected moment in the vertex V by the contribution of V_R which is called renormalized moment $m_{k,l}$. To be sure of not counting twice the same contribution one has only to consider the one vertex irreducible graphs in the high temperature expansions. A graph G is said to have this property, if each connected piece of G which remains after removal of an arbitrary vertex together with all the lines ending there has a nonzero number of external lines. In this way, the number of graphs which one has to compute is considerably reduced. We refer the reader to refs. [43]-[64] for a more detailed discussion.

4.5 Physical spectrum and mirror fermions

We will now procede to demonstrate a very important property of the model in the region of convergence of the linked cluster expansion: In the quasi continuum theory the fermion spectrum contains fermions and mirror fermions. There is no possibility of keeping the mirror fermions at the cut-off level while having zero mass fermions (in lattice units), independently of the value we give to the Yukawa couplings. In fact, the impossibility of removing the mirror fermions in this theory may be considered as a special case of a general no-go theorem [20]. However, we find it very instructive to present a short demonstration of this dynamical restriction in the special case under consideration.

The proof is as follows :

a) For each vertex where no scalar particle ϕ_S (or no fermion bilinear $\bar{\Psi}O_S\Psi$ with the same

quantum numbers) is emited there is a mirror flip, that is to say when a ψ fermion is annihilated, a χ fermion is created.
b)For each vertex where a scalar particle, or in general an odd number of scalar particles (or fermion bilinears) are emited, there is no mirror flip
c) In the links, a factor (rM) is associated with a mirror flip, while a factor γ_μ ($\hat{\mu}$: direction of the link) produces no mirror flip

In short, we have that the number of mirror flips is

$$N_{flips} = \sum_{v_i}(1 - n_{B_i}) + N_r = N_v - N_B + N_r \tag{4.74}$$

where $\sum_{v_i}$ is a summation over all vertices, n_{B_i} is the number of scalars (or fermion pairs with the same quantum numbers) in a vertex and N_r is the number of links with a factor (rM) associated with them. For a general propagation of the fermions, we have that

$$N_v = N_r + N_{\gamma_\mu} + 1 \tag{4.75}$$

where N_{γ_μ} is the number of links with a factor γ_μ associated with them. On the other side, due to the chiral symmetry a scalar particle that is emited in one vertex must be absorbed in other vertex, and consequently N_B is an even number. Inserting these relations in eq.(4.74) we get

$$N_{flips} - N_{\gamma_\mu} = odd \tag{4.76}$$

This means that if we want to have no mirror flips in a general propagation of the fermion fields, the propagator will have an odd number of γ_μ matrices. On the contrary, to have a mirror flip the propagator will have an even number of γ_μ matrices. Thus, $G(0,x)$ may be written as

$$G(0,x) = \begin{pmatrix} \sum_{\{n_i\}}\left(F_i(r,G_\chi,G_\psi,\mu_{\psi\chi})\prod_{j=1}^{i}\gamma_{\mu_{n_j}}\right) & \sum_{\{n_k\}}\left(H_k(r,G_\chi,G_\psi,\mu_{\psi\chi})\prod_{l=1}^{k}\gamma_{\mu_{n_l}}\right) \\ \sum_{\{n_i\}}\left(B_m(r,G_\chi,G_\psi,\mu_{\psi\chi})\prod_{p=1}^{m}\gamma_{\mu_{n_p}}\right) & \sum_{\{n_i\}}\left(K_q(r,G_\chi,G_\psi,\mu_{\psi\chi})\prod_{s=1}^{q}\gamma_{\mu_{n_s}}\right) \end{pmatrix} \tag{4.77}$$

where $\sum_{\{n_i\}}$ is a sum over all different contributions (F_i) each one carrying a factor of i gamma matrices with direction μ_{n_j}, and i, q and k, m are odd and even integer numbers respectively. Observe that, due to chiral symmetry no Pauli matrix factor will appear in the different contributions.

Apart from a renormalization constant that is irrelevant for the following analysis, the inverse renormalized fermion mass matrix is equal to the zero momentum fermion propagator (see Eq.(4.8)). The zero momentum fermion propagator may be given in term of the two point fermionic Green function $G(0,x)$ as

$$G(p=0) = \sum_x G(0,x) \tag{4.78}$$

By calculating $G(p=0)$ we can add to each contribution to $G(0,x)$ the contribution to $G(0,-x)$ in which the fermion propagates through the links in the same order but with opposite direction with respect to the ones of $G(0,x)$. Obviously, the 8x8 diagonal block matrices in $G(p=0)$ vanish by applying this procedure. Thus, the inverse mass matrix consists of only the 8x8 antidiagonal block matrices.

It is now very easy to show that both block matrices are equal and proportional to the 8x8 identity matrix. The considerations we have to make are very similar to the ones presented

above. For example if we sum to $G(0,x)$ the contribution that results from changing the orientation of all links in one direction $\hat{\mu}$ but preserving all other properties, it is clear that the nonpaired γ_μ factors will vanish in $G(p=0)$. Consequently, the antidiagonal 8x8 matrices are proportional to the identity. To prove that both matrices are equal, one can procede in the following way: If there are no nonpaired element, a product of an even number of gamma matrices is invariant under mirror reflection. With this we mean that

$$\gamma_\alpha\gamma_\delta\cdots\gamma_\mu = \gamma_\mu\cdots\gamma_\delta\gamma_\alpha \tag{4.79}$$

This property is also satisfied by a product of an even number of Pauli matrices with no nonpaired element. The contributions to the fermion propagator in which a mirror flip occurs may be represented graphically as shown in Fig.4.5 (a). We can define a mirror graph of this one as that obtained by putting all links in the same direction but in opposite order while changing $\psi \to \chi$ in the vertices at 0 and x (see Fig.4.5 (b)). Due to the invariance under mirror reflection of gamma and Pauli matrices, it is easy to see that both graphs represent equal contributions to the fermion-mirror fermion and mirror fermion-fermion propagation between 0 and x. Since for each graph its mirror graph exists and is unique, both 8x8 antidiagonal matrices are equal. Consequently, the form of the inverse fermion mass matrix will be

$$\mathcal{M}_f^{-1} = (m_f(r, G_\psi, G_\chi, \mu_{\psi\chi}))^{-1}\, M \tag{4.80}$$

This shows that in the unbroken phase near $k = K = 0$ chiral symmetry is realized by a parity doublet pair of degenerate fermion doublets. The eigenvectors of $\mathcal{M}_f$ are indeed

$$\eta_{1,2} = \frac{(\psi \pm \chi)}{\sqrt{2}} \tag{4.81}$$

which have opposite parity. One may ask what happens with the fermion spectrum in some particular limits, for example when we take the bare Yukawa couplings to infinite values. We will analyse this question in the following sections.

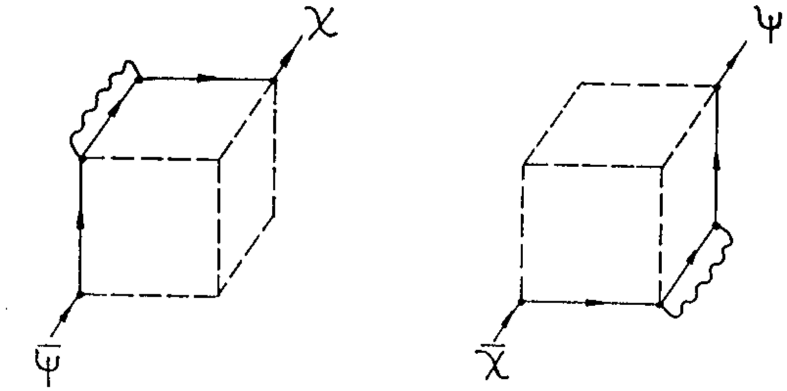


Fig. 4.5 (a)-(b): Fig.4.5 (a) gives an example of a graph contributing to the propagation of the fermions with a mirror flip. The twisted line represents the interchange of a scalar boson. Fig 4.5 (b) is the mirror graph of the one given in Fig.5 (a), according to the definition we give in the text.

4.6 Random walk approximation to the linked cluster expansion

The linked cluster expansion is useful to obtain nonperturbative information of the theory, in a region where correlation lengths are usually smaller than or of the order of two lattice spacings. The aim of these calculations is to apply the Lüscher and Weisz procedure [14], that is to use the results of the high temperature expansions as initial data for the integration of the renormalization group equations obtained in perturbation theory. For this method to work two nontrivial conditions should be satisfied. The first one is that at correlation lengths as small as the ones obtained in the linked cluster expansions the theory is already in the scaling region where lattice artifacts can be neglected. The second one is that the obtained renormalized couplings are in a region where perturbation theory can give reliable results. Due to the infrared fix point of the Yukawa and scalar couplings in this theory the hope is that these two conditions are fulfilled as in the pure scalar theory. Of course, one is always assuming that the theory arising from the high temperature expansion belongs to the same class of universality as the one analysed in perturbation theory. For this to be satisfied, while approaching the continuum limit at the multicritical points where all relevant masses vanish in lattice units, all spurious states should be preserved at the cutoff level.

In the following only the linked custer expansion at vanishing scalar hopping parameter will be analysed in detail. Scalar fields will appear as composite fermion antifermion states at $k = 0$. As we will shown in section 4.8, multicritical points where the continuum limit may be defined, are reached in this limit. If the obtained theories belong to the same class of universality as the desired one is another question to which we will address in this and the following sections.

Before developing the exact linked cluster expansion of the theory it is convenient to get a first insight on the nonperturbative behaviour of fermion and boson masses under some approximation. The random walk approximation was first developed in QCD by Kawamoto [65], in order to study the behaviour of physical masses at strong gauge coupling. In the present context it means to restrict the irreducible part of the scalar and fermion propagators to a given set of graphs, and summing the complete series as these were the only contributions present in the theory.

To illustrate the method let us calculate first the free fermion propagator, setting both Yukawa couplings to zero. The recursion relation reads

$$G(0,x) = \tilde{\mu}\, M\, \delta_{x,0} + \tilde{\mu}\, M\, K\, (\gamma_\mu + r\, M) G(0, x - \hat{\mu}) \tag{4.82}$$

The expression of the free fermion propagator in momentum space is then

$$G(k) = \left(1 - \tilde{\mu}\, M\, K \sum_{\hat{\mu}} \exp(-ik_\mu)(\gamma_\mu + r\, M)\right)^{-1} \tilde{\mu}\, M \tag{4.83}$$

As can be seen from Eq.(4.8) the fermion mass matrix may be obtained by inverting $G(k)$

$$G(k)^{-1} = \begin{pmatrix} i\, 2\, K \gamma \bar{k} & -K\, r(8 - \hat{k}^2) + \mu_{\psi\chi} \\ -K\, r(8 - \hat{k}^2) + \mu_{\psi\chi} & i\, 2\, K \gamma \bar{k} \end{pmatrix} \tag{4.84}$$

where, using the notation employed in ref.[21]

$$\hat{k}_\mu = 2 \sin \frac{k_\mu}{2}, \qquad \bar{k}_\mu = \sin k_\mu \tag{4.85}$$

Comparing this expression with eq.(4.8) we see that the renormalization constant is $Z_\Psi = \frac{1}{2K}$. The renormalized fermion mass matrix may be obtained from here by setting $k_\mu = 0$. The condition for a zero mass pole is then

$$K_c = \frac{\mu_{\psi\chi}}{8\, r} \tag{4.86}$$

It is easy to see from the above equations that the states in the corner of the Brillouin zone, with at least one of the components of the four vector k_μ satisfying $k_\mu = \pi$ get masses

$$m = \frac{1}{2K} |\mu_{\psi\chi} - r\, K\, (8 - 4\, n_\pi)| \tag{4.87}$$

where n_π are the number of nonvanishing components of k_μ. Consequently, the Wilson mechanism is working in this theory and the spurious states decouple in the continuum limit.

The interesting question is if this property is preserved once we include the interactions. The first correction to the irreducible fermion propagator is given in Fig.4.3. We will call the contribution of this graph F_μ and $K_\mu = K\, (\gamma_\mu + r\, M)$ the factor appearing in the links. The result for the fermion propagator in momentum space may be obtained from a recurtion relation similar to the one given previously in the free fermion case, namely

$$G(k) = (1 - (\tilde{\mu} M + F) K_0)^{-1} (\tilde{\mu} M + F)\,. \tag{4.88}$$

where

$$K_0 = \sum_{\hat{\mu}} \exp(-ik_\mu) K_\mu, \qquad F = \sum_{\hat{\mu}} \exp(-ik_\mu) F_\mu \tag{4.89}$$

The contribution F_μ of the diagram of Fig.4.3 is given by

$$F_\mu = \frac{1}{16} \left(\sum_{s,s'} \left\{ O_S^{-1} K_\mu O_{S'}^{-1} K_{-\mu} O_S^{-1} K_\mu O_{S'}^{-1} - Tr[O_S^{-1} K_\mu O_{S'}^{-1} K_{-\mu}] O_S^{-1} K_\mu O_S^{-1} \right\} \right) \tag{4.90}$$

where a factor 2 that appears in the calculation has been cancelled with the symmetry factor of the diagram. This expression may be worked out using the relations given in Appendix D. From now on we will fix $r = 1$. The final result is

$$F_\mu = \frac{5K^3 (G_\chi - G_\psi)^2}{2[G_\chi G_\psi - \mu_{\psi\chi}^2]^4} \begin{pmatrix} G_\chi^2 \gamma_\mu & G_\psi G_\chi \\ G_\psi G_\chi & G_\psi^2 \gamma_\mu \end{pmatrix} \tag{4.91}$$

and consequently the fermion propagator at $\bar{k} = 0$ reads

$$G(\bar{k} = 0) = \frac{\left(\tilde{\mu} + \frac{5K^3 (G_\chi - G_\psi)^2 G_\psi G_\chi (8 - \hat{k}^2)}{2[G_\chi G_\psi - \mu_{\psi\chi}^2]^4}\right)}{\left(1 - \left(\tilde{\mu}(8 - \hat{k}^2) + \frac{5K^3 (G_\chi - G_\psi)^2 G_\psi G_\chi (8 - \hat{k}^2)^2}{2[G_\chi G_\psi - \mu_{\psi\chi}^2]^4}\right) K\right)} M \tag{4.92}$$

The critical hopping parameter K_c where the fermion propagator has a zero mass pole satisfies the following relation

$$1 = \tilde{\mu}(8 - \hat{k}^2) K + \frac{5K^4 (G_\chi - G_\psi)^2 G_\psi G_\chi (8 - \hat{k}^2)^2}{2[G_\chi G_\psi - \mu_{\psi\chi}^2]^4} \tag{4.93}$$

The critical hopping parameter K_c is then positive when $\bar{\mu} > 0$, goes to zero as $\bar{\mu} \to \infty$ and varies continuously to negative values when $\bar{\mu} < 0$. From eq.(4.93) one can see that, unless $\bar{\mu} = 0$, at the critical hopping parameter K_c, where the state at $k_\mu = 0$ acquires zero mass, the states at the corner of the Brillouin zone remain massive. Furthermore, one may easily see from the above expression that at values of $|K| < |K_c|$ no of this states reach zero mass. The Wilson mechanism seems to work even when the interactions are present and the spurious states are decoupled from the spectrum.

Let us discuss now the particular case $\mu_{\psi\chi} = 0$. Due to the renormalization freedom, this case is completely equivalent to the one where both Yukawa couplings $G_\psi, G_\chi \to \infty$. The critical hopping parameter is determined by the equation

$$K_c^4 = \frac{2\, G_\chi^3\, G_\psi^3}{5\,(G_\chi - G_\psi)^2\ (8 - \hat{k}^2)^2} \tag{4.94}$$

This expression shows that in this limit we have a duplication of the spectrum because the state with $k_\mu = (\pi, \pi, \pi, \pi)$ will acquire zero mass together with the state at $k_\mu = 0$. The spectrum in the continuum will consist of four fermion doublets instead of two. Thus, the behaviour of the theory in this limit seems to be different from the one expected.

One may ask if the fermion doubling of the spectrum at $\mu_{\psi\chi} = 0$ is a condition which arises only in the context of this approximation. The answer is no, and the exact chiral symmetry of the theory is responsible for this situation. The explanation is the following: At $\mu_{\psi\chi} = 0$ the fermion can only propagate by emiting in each vertex an odd number of scalars or fermion antifermion pairs . Because of the preservation of chiral symmetry the sum of the number of scalar bosons emited and absorbed in the different vertices N_B must be even. Thus, the total number of vertices is even, too. This implies that independently of the number of mirror flips the fermion can't propagate to a point separated an even number of lattice spacings. A relation between the fermion propagator $G(k)$ and $G(k_\pi) = G(k + (\pi, \pi, \pi, \pi))$ arises from here, namely

$$G(k_\pi) = \sum_n \exp(-ik_\pi\, n) G(0, n) = \sum_n \exp(-ikn) G(n) \exp(-i\pi \sum n_i) = -G(k) \tag{4.95}$$

since $\sum n_i$, that is the length of the propagation of the fermion measured in terms of lattice spacings, is an odd number. The fermion spectrum is doubled; besides a pole at $k_1 = k_2 = k_3 = 0$, $k_4 = iaE$ there will be always a pole at $k_1 = k_2 = k_3 = \pi$, $k_4 = \pi + iaE$.

In the scaling region and at small renormalized couplings the theory may be studied with perturbative methods. In the perturbative approach the spurious states may be kept at the cutoff level and the spectrum consists of two fermion doublets as in the free case. Thus, the theory at infinite bare Yukawa couplings $G_\chi, G_\psi \to \infty$ does not belong to the same class of universality as the one described in perturbation theory. This may mean that either nonperturbative effects spoil the Lüscher and Weisz procedure in this formulation of the theory, or there is a change of behaviour of the theory at strong Yukawa couplings. A procedure to answer this question is to develop the linked cluster expansion in the way of ref.[14] looking for the existence of multicritical points. If the multicritical line is preserved for large bare Yukawa couplings one must check if while entering the scaling region the renormalized couplings are already in the perturbative regime. If this is the case, the β functions should be compared with those ones obtained in perturbation theory. The appearence of extra fermion states will be signalized by a change in the flow of the renormalized couplings. We will partially follow this procedure in section 4.8

The scalar propagator may also be studied in a convenient approximation. We expect, in general, a common K_c for the four O(4) components ϕ_S of the scalar fields, due to the exact chiral symmetry of the theory. This is an important difference with the case of the sigma model with Wilson fermions where, due to the breakdown of the chiral symmetry produced by the Wilson term, this condition is not easily reached. The results of the random walk approximation of the scalar field propagator are in complete agreement with these expectations. However, they are not very instructive and we will not present them here. Instead, we prefer directly to present the results of the full linked cluster expansion in the theory, where no approximation is made. This will be done in section 4.8.

4.7 Evaluation of the high temperature expansion

At zero scalar hopping parameter k, the propagation of scalars is only possible through fermion antifermion pairs with the same O(4) quantum numbers. Thus, the size of a typical diagram contributing to the scalar two point correlation function at a given power of the hopping parameter K will be of about one half the size of a typical diagram contributing to the fermion two point correlation function at the same power of K. This does not mean that the multicritical point can not be reached in this limit, but that the obtained femion mass will be usually smaller than the scalar one, as far as one doesn't make an expansion up to very high orders. While in the pure scalar theory it was sufficient to develop the expansions up to the 10^{th} order in the scalar hopping parameter k [14], one expects that here an expansion at least up to the 16^{th} order will be necessary in order to include the boundary of the scaling region in the range of validity of the high temperature expansions. Such a high order expansion is in principle possible but very difficult, unless one has a powerful 'graph generation' program as the one developed for the pure scalar theory in ref.[64]. On the other hand, an expansion up to the 8^{th} order in K cannot allow us to reach the scaling region but can give us the first answers to questions such as the existense of the multicritical points at $k = 0$ and the behaviour of the renormalized couplings in the theory. Furthermore, the algebra of the Γ_S and K_μ matrices is simplified if both Yukawa couplings are fixed at equal values (see Appendix D), that is to fix the bare ratio $r_y = 1$. The hope is that this election doesn't modify the qualitative behaviour of the theory. In the following we will only analyse this special case, although the asymmetric case deserves further study since only in this case, if any , mirror fermion masses may be kept at large values in the large cutoff limit.

In the next section we will present the results of an expansion up to the 8^{th} order in K of the renormalized masses, Yukawa couplings and scalar self coupling. Since these results are obtained from the expansion of the six susceptibilities defined in the text, the critical fermion hopping parameter, namely the value of K at which the divergence of the susceptibilities χ_2 and χ_2^f is located, may be approximately computed by looking at the behaviour of the ratio of two successive coefficients as has been done in Chapter 3. Since fermion loops only appear at the 4^{th} order in the fermion two point correlation functions, only the last three ratios may give us a good approach to the correct K_c. Our assumption of the existence of a phase transition relies, as in chapter 3, on the approximately constant value of the last three ratios. To get an estimate of K_c we have decided to make an average of these three values. The average differs usually in less than a 10% with the last evaluated ratio. For similar reasons, we decided to take the last ratio as the best approximation to K_c which can be gotten from the expansion of the scalar susceptibility. We will then define

$$\bar{r}_f = \frac{r_f^6 + r_f^7 + r_f^8}{3}, \qquad \bar{r}_B = \sqrt{r_B^4} \tag{4.96}$$

where

$$r_f^i = \frac{(\chi_2^f)^i}{(\chi_2^f)^{i-1}}, \qquad r_B^i = \frac{\chi_2^{2i}}{\chi_2^{2i-2}} \tag{4.97}$$

and $(\chi_2^f)^i$, χ_2^j are the coefficients of the i^{th} and j^{th} order in the expansion in K of the fermion and scalar susceptibilities respectively. In general, from χ_2 and χ_2^f we obtain different estimated values of K_c ($\hat{r}_f$ and $\hat{r}_B$) that can be compared to obtain information about the position of the multicritical points in the space of bare parameters.

4.8 Results and discussion

4.8.1 Phase transition line and multicritical points

In the calculations presented in this section we fix the fermion normalization by choosing $\mu_{\psi\chi} = 1$. We will define G_Ψ as the common value of the Yukawa couplings. In table 4.1 we give the values of $\bar{r}_f$ and $\bar{r}_B$ for some representative values of the Yukawa couplings. We see that up to values of the bare Yukawa coupling $G_\Psi \simeq 3$ the estimates of K_c gotten form the fermion and scalar susceptibilities coincide within an error of the order of a 10% of the values of the averaged ratios $\bar{r}_f$ and $\bar{r}_B$. This is a good indication of the existense of multicritical points at $k = 0$. Thus, it is possible to take the continuum limit at these points and hopefully, the theory obtained will belong to the same class of universality as the one described by perturbation theory. The values of K_c are positive for $G_\Psi < 1$ and negative for $G_\Psi > 1$, in agreement with our results in the random walk approximation.

At $G_\Psi \simeq 3$ the ratio r_B^4 begin to vary rapidly and for larger values, $G_\Psi > 3.5$ it takes negative values. This may be an indication that the multicritical point is lost, and would mean a different behaviour of the theory in the scaling region at large Yukawa couplings. Only a higher order expansion will allow us to decide whether this is an artifact of the order of truncation of the series we applied or if this effect is preserved at higher orders. However, if the loss of the multicritical point, possibly characterized by a tricritical point, is in fact present we would have the answer to one of the questions we asked in the present article: In the symmetric phase, at small Yukawa couplings the critical line consists of multicritical points where both scalar and fermion masses vanish in the continuum limit. At large Yukawa couplings, instead, the scalars don't reach the zero mass condition and decouple in the continuum limit. Thus, the desired continuum theory will only be obtained in the former case and is there where the nonperturbative results should be compared with the perturbative predictions.

One may ask if these results are preserved at finite hopping parameter of the scalar fields. A first order correction in the scalar hopping parameter k shows that the convergence is improved in this case, and that the bare Yukawa coupling at which the loss of the multicritical point is observed moves to higher values.

G_Ψ	$\hat{r}_f$	$\hat{r}_B$
0.01	0.125	0.132
0.10	0.124	0.131
0.50	0.0932	0.0947
1.01	-0.250 10^{-2}	-0.235 10^{-2}
1.50	-0.144	-0.132
2.00	-0.289	-0.280
2.50	-0.442	-0.456
3.00	-0.626	-0.738
3.50	-0.854	-2.46
3.60	-0.904	-

Table 4.1: The two estimated values of K_c, according to eq.(4.96) as a function of the bare Yukawa coupling G_Ψ.

4.8.2 Masses, Couplings and renormalization constants

As in the SU(2) Higgs model analysed in Chapter 3 the masses and couplings obtained from the low order expansions we have done are only reliable up to some value of K, K_u, lower than K_c. It is very difficult to make an estimate of K_u with the series we have at our dispossal. Since the qualitative behaviour of the renormalized masses and coupling constants depends only weakly on the exact value of K_u (see Figs.4.6-4.11), we have decided to take $K_u = 0.9K_c$ as a first approximation to the correct value. Even though at $k = 0$ and $G_\Psi = 0$ the scalar particle can't propagate, i.e. its mass become infinite, the presence of the multicritical point shows that this situation is modified as far as we give to G_Ψ a non vanishing value. At very small bare Yukawa couplings fermions propagate almost free and scalars only propagate through large fermion antifermion loops. In the hopping parameter expansion the scalar mass get a contribution proportional to $(1/G_\Psi)^2$ at the lowest order in K, which dominates as far as high orders are not considered. Thus, for small Yukawa couplings the scalar masses are very large and consequently a high order expansion will be necessary to reach the scaling region where all masses should be smaller than, say $1/2a$. Fig.4.6 show the behaviour of scalar and fermion masses for different values of the bare Yukawa couplings. At values of the bare Yukawa coupling $G_\Psi \simeq 0.5$ the scalar mass is already of a few lattice spacings when we extrapolate our expansions up to values of $K = 0.9K_c$. Both masses decrease monotonically from infinite values at $K = 0$ up to some small positive value at $K = 0.9K_c$. The scalar mass is always larger than the fermion one, and at strong Yukawa couplings ($G_\Psi \simeq 2$) its value becomes of the order of twice the fermion mass, in agreement with our previous expectations.

The behaviour of the Yukawa couplings for different bare couplings is shown in Fig.4.7. The result obtained is in agreement with the predictions of the renormalization group equations, in the sense that it decreases monotonically from infinite values at $K = 0$ up to small positive values at $K = 0.9K_c$. Near the critical point the values of the renormalized Yukawa coupling slowly increase when the value of the bare Yukawa coupling is increased: At $K = 0.9K_c$ it varies from $G_\Psi^R \simeq 4$ at bare coupling $G_\Psi \simeq 0.01$ until $G_\Psi^R \simeq 16$ at bare coupling $G_\Psi \simeq 3$.

The ratio of the self coupling and the square of the Yukawa coupling r_y shows a behaviour consistent with the perturbative predictions, too. As can be seen from Fig.4.8 it decreases from infinite values at $K = 0$ until values that are close to the perturbative fixed points. At large bare Yukawa couplings, $G_\Psi > 1$, the ratio r_y decreases rapidly until values of the hopping parameter $K \simeq 0.5K_c$ and after that it evolves very slowly keeping almost a constant small value $r_{as.}$, which always satisfies $|r_{as.}| < r_1$ (r_1 is the value of the perturbative fixed point, Eq.(4.34)). This may be an indication that the ratio stop decreasing at some value of K to begin to increase while approaching the scaling region. That for sufficiently strong Yukawa couplings($G_\Psi > 2$) the value of $r_{as.}$ can take negative values, but with very small $|r_{as.}|$, as is shown in Fig.4.8 (c), is something that should not surprise in view of the perturbative predictions.

The behaviour of the renormalized self coupling is in agreement with the above expectations. For the values of the Yukawa couplings for which it is positive defined, it becomes monotonically smaller while approaching K_c. For strong values of the Yukawa couplings, its flow goes through negative values, but at values of $K \simeq 0.8K_c$ the renormalized self coupling begin to increase, flowing towards positive values (see Fig.4.9).

In Fig.4.10-11 we have also plotted the behaviour of both renormalization constants Z_Ψ and Z_R for different bare Yukawa couplings. We can see that both $|Z_\Psi|$ and $|Z_R|$ decrease while approaching the critical hopping parameter. An interesting physical parameter is the 'normalized' fermion renormalization constant defined as

$$Z'_\Psi = 2KZ_\Psi. \tag{4.98}$$

$Z'_\Psi = 1$ in the free fermion case. What is interesting is that for all values we have computed Z'_Ψ is close to one, whenever the hopping parameter is in the neighbourhood of K_c. Values of Z'_Ψ for some representative values of G_Ψ at $K = 0.9K_c$ are given in Table 4.2.

At $G_\Psi \to 1$ the critical hopping parameter $K_c \to 0$. The renormalization constant $Z_\Psi \to \pm\infty$ depending on whether the limit is taken from values greater or lower than one. All renormalized physical couplings and masses remain finite in the limit. What is interesting is that the limiting value of the physical parameters at $K = 0.9K_c$ is independent of whether the limit is being taken from values of G_Ψ greater or lower than one. In conclusion, there is no evidence of a change in the behaviour of the system while crossing $G_\Psi = 1$.

G_Ψ	Z_Ψ	Z'_Ψ
0.01	4.44	1.00
0.10	4.50	1.00
0.50	5.95	0.998
1.01	-2.14 10^2	0.964
1.50	-3.51	0.906
2.00	-1.68	0.876
2.50	-1.06	0.846

Table 4.2: Values of the renormalization constant Z_Ψ and of the normalized renormalization constant Z'_Ψ for different values of the bare Yukawa coupling G_Ψ.

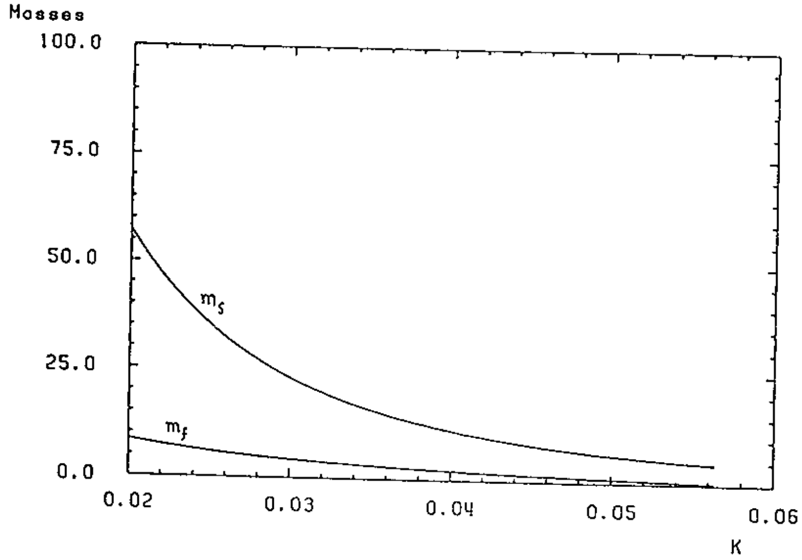


Fig.4.6 (a)

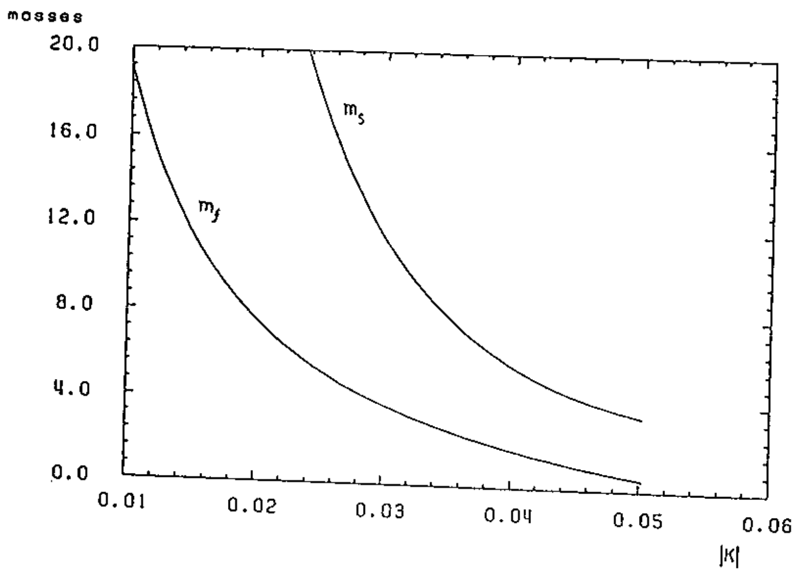


Fig.4.6 (b)

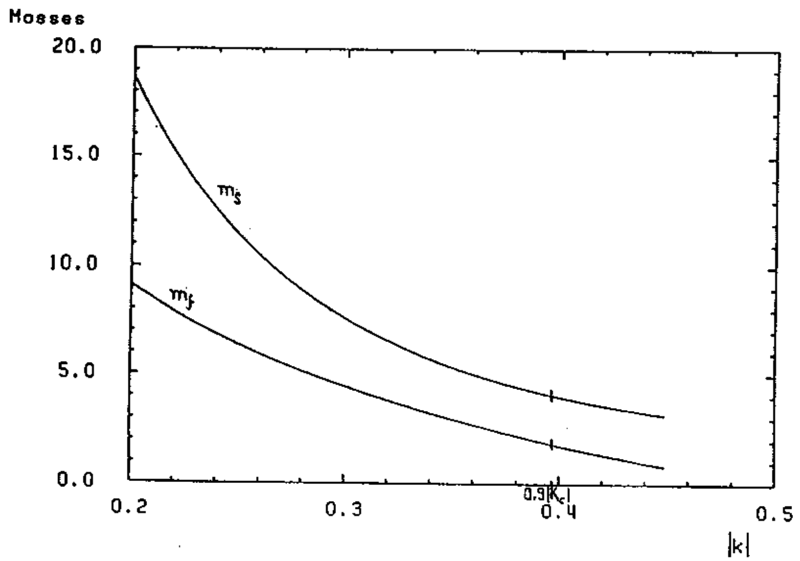


Fig.4.6 (c)

Figure 4.6: Dependence of the scalar (m_S), and fermion (m_f) masses on the hopping parameter K, for a) $G_\Psi = 0.7$, b) $G_\Psi = 1.2$ and c) $G_\Psi = 2.5$. All expansions are extrapolated up to $K_u = 0.9K_c$, where K_c is the estimate $\bar{r}_f$ of the critical hopping parameter.

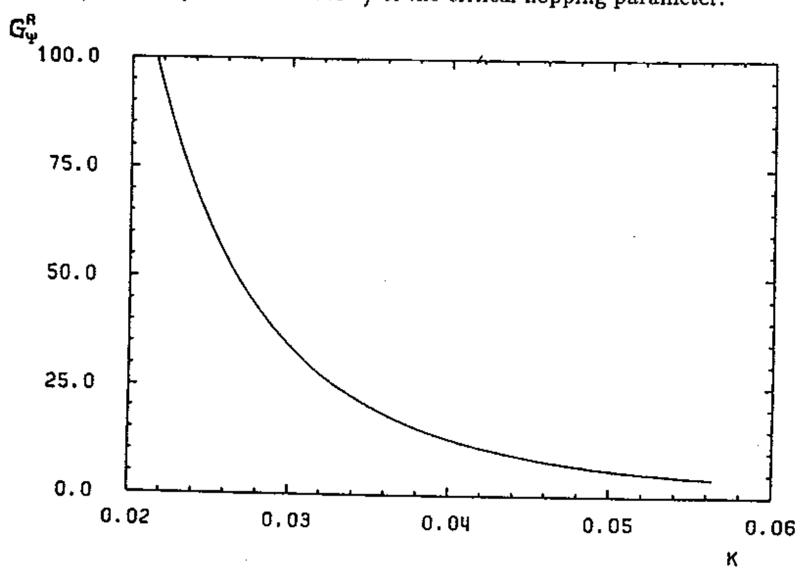


Fig.4.7 (a)

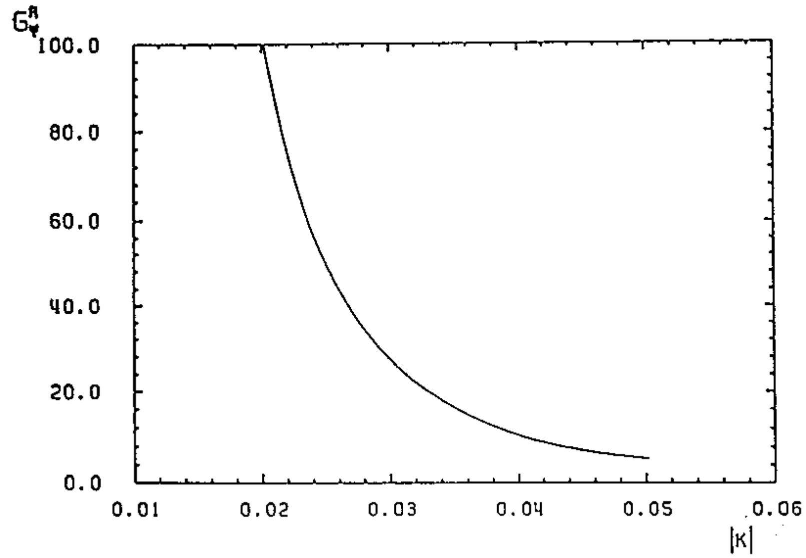


Fig.4.7 (b)

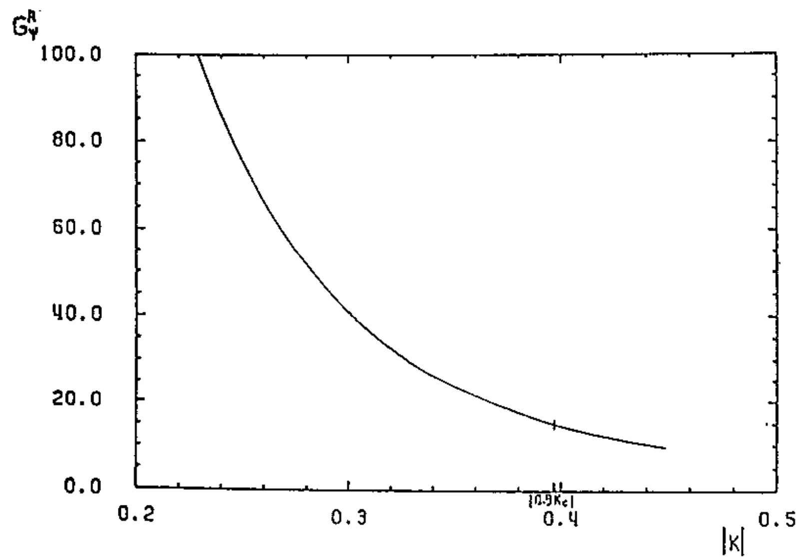


Fig.4.7 (c)

Figure 4.7: The same as Fig.4.6 but for the renormalized Yukawa coupling G_Ψ^R.

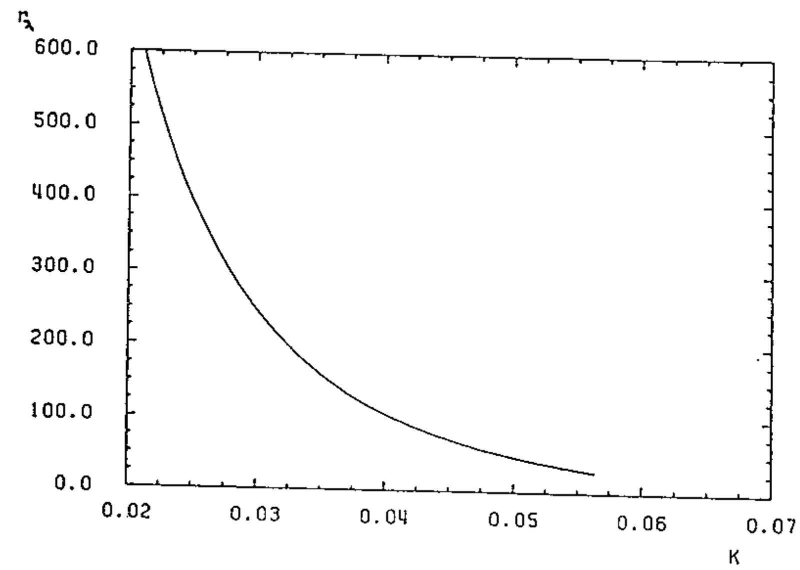


Fig.4.8 (a)

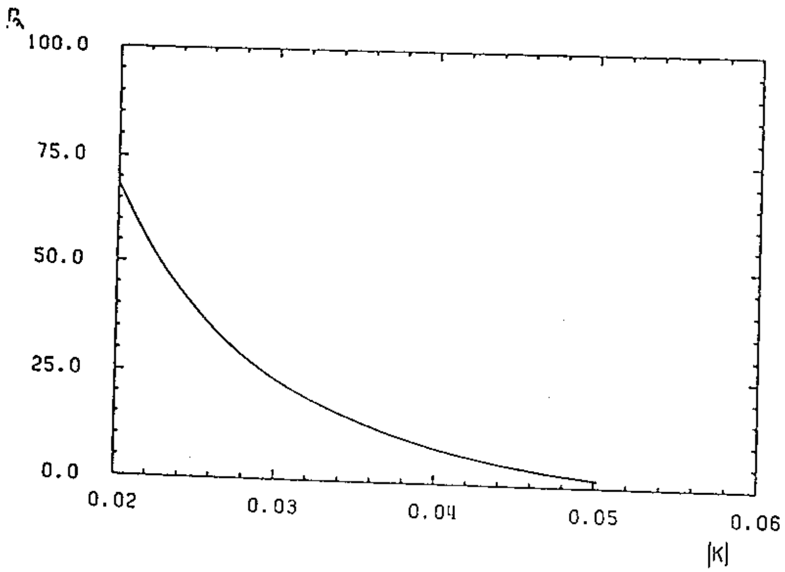


Fig.4.8 (b)

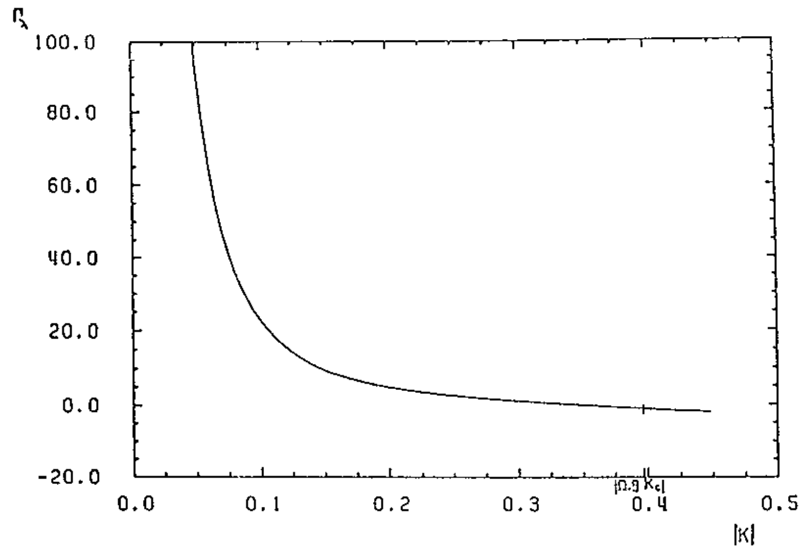


Fig.4.8 (c)

Figure 4.8: The same as Fig.4.6 but for the ratio r_λ of the renormalized self coupling to the square of the renormalized Yukawa couplings.

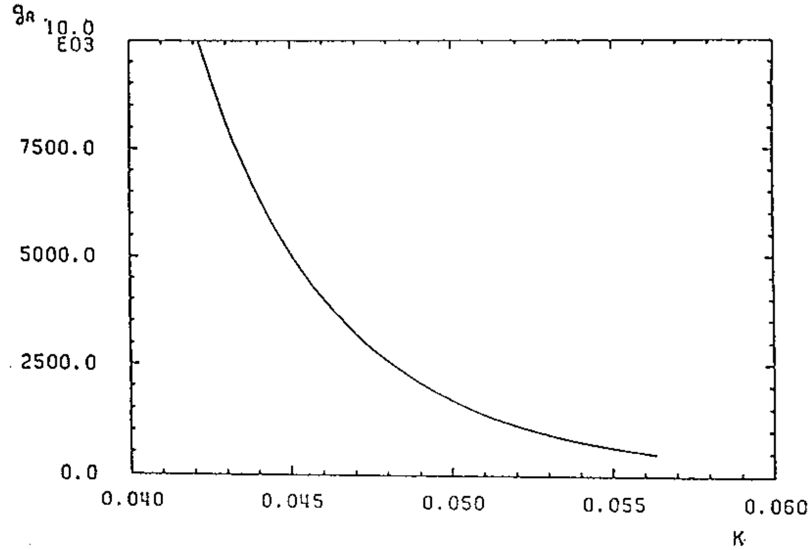


Fig.4.9 (a)

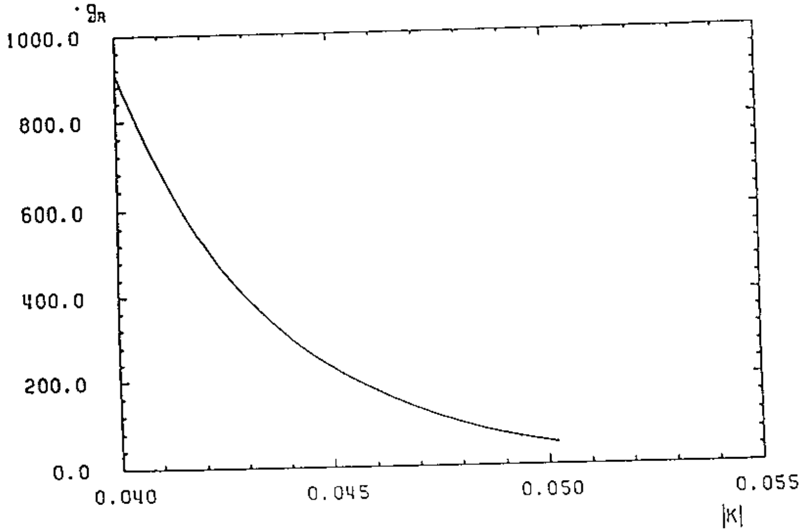


Fig.4.9 (b)

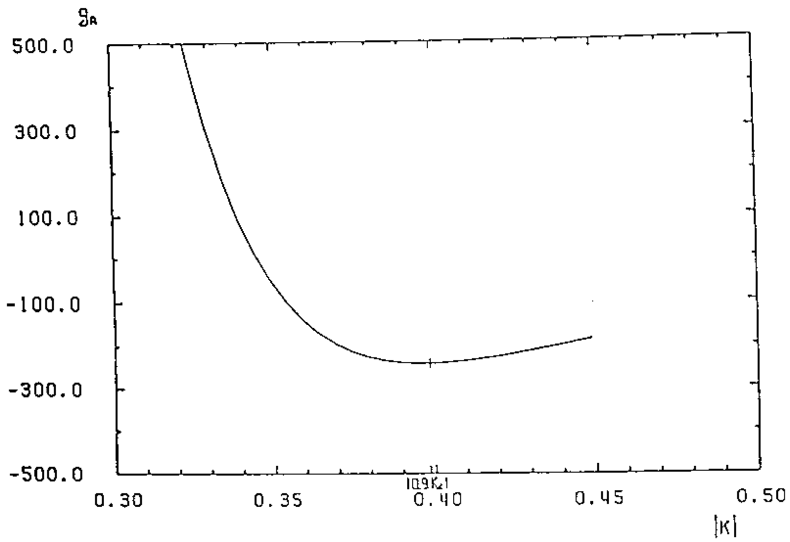


Fig.4.9 (c)

Figure 4.9: The same as Fig.4.6 but for the renormalized Self coupling g_R.

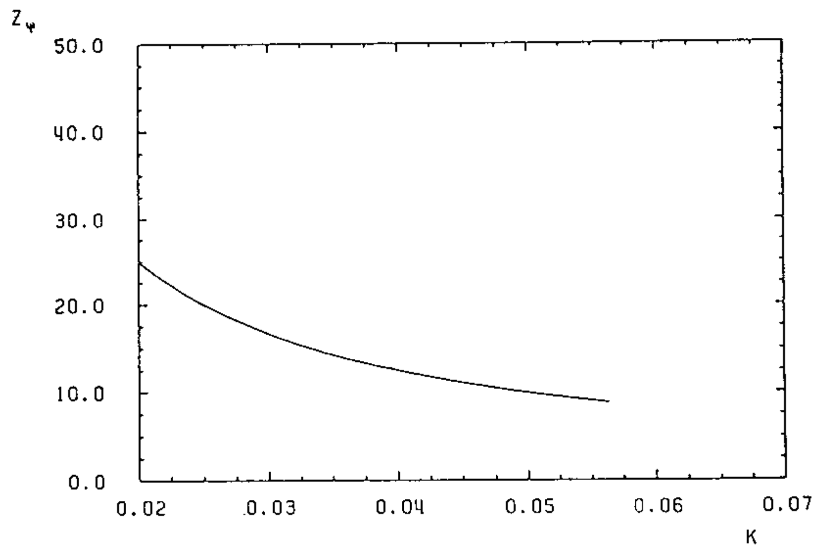


Fig.4.10 (a)

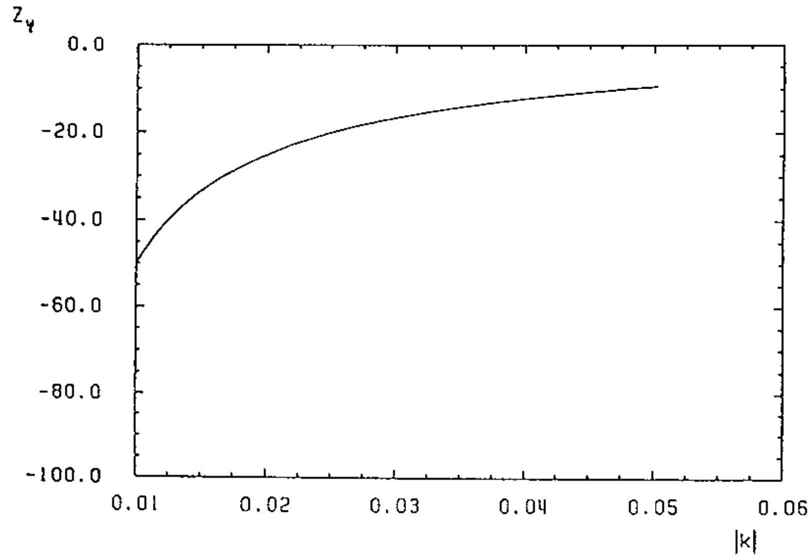


Fig.4.10 (b)

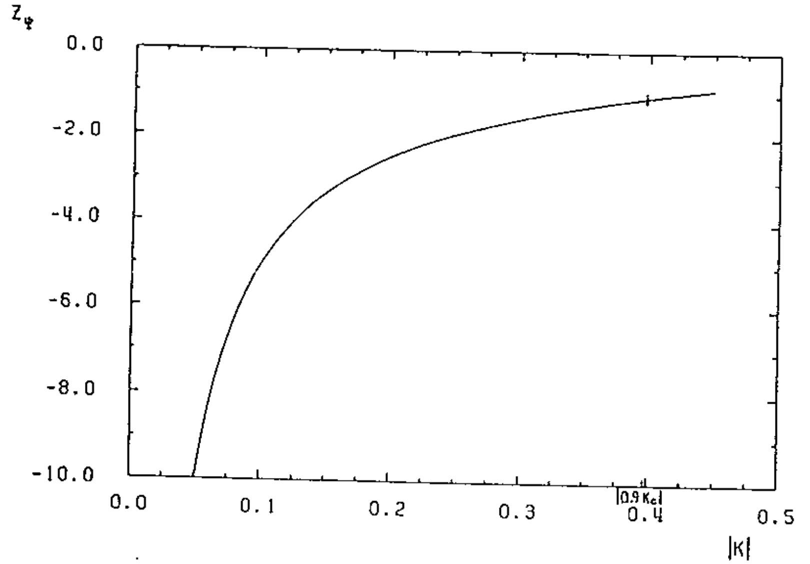


Fig.4.10 (c)

Figure 4.10: The same as Fig.4.6 but for the fermionic renormalization constant Z_Ψ.

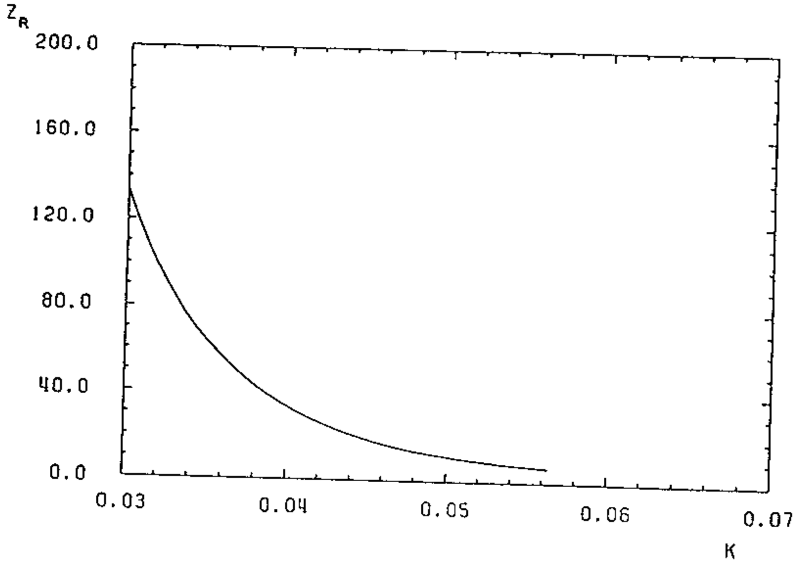


Fig.4.11 (a)

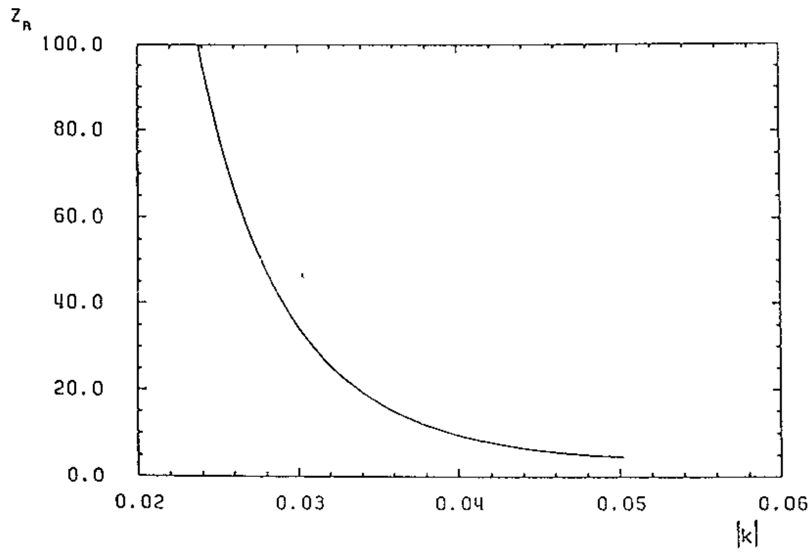


Fig.4.11 (b)

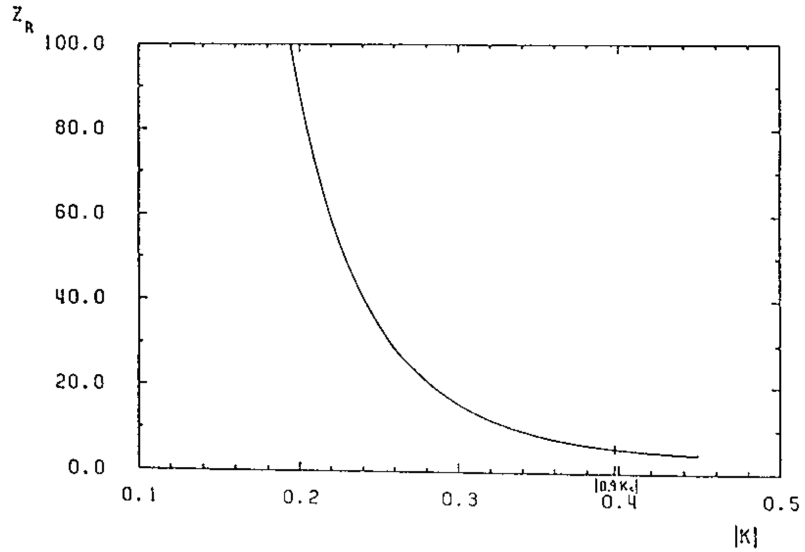


Fig.4.11 (c)

Figure 4.11: The same as Fig.4.6 but for the bosonic renormalization constant Z_R.

4.9 Conclusions

The perturbative analysis of the chiral $SU(2)_L \otimes SU(2)_R$ symmetric model indicates that the continuum limit of this theory is probably trivial, i.e. a noninteracting theory. If one considers this formulation as a first approach to the Standard model in the Dashen-Neuberger approximation [66] of small gauge couplings, this would mean that the interacting Standard model can only be considered as an effective theory valid until some finite scale of energy at which new physics should appear.

To test the triviality of the theory we need nonperturbative methods, which allow us determine the phase structure and to localize and approach the critical points at which the continuum limit may be defined, independently of the value of the bare Yukawa and scalar self couplings. In this Chapter a linked cluster expansion was performed to obtain the desired nonperturbative information. We obtained that if both bare Yukawa couplings are infinite the fermionic spectrum in the continuum limit consists of two degenerate fermion doublets with their mirror fermion partners, in contrast with the perturbative result where only one fermion doublet and its mirror partner appear.

The important question is in which sense this is an indication that nonperturbative effects are spoiling the Lüscher and Weisz procedure in this formulation of the model. To answer this question we developed a linked cluster expansion of the physical parameters at infinite bare scalar self coupling and vanishing scalar hopping parameter. The expansions were done up to the 8^{th} order in the hopping parameter K. The obtained results indicate the possible existence of a tricritical point in the phase transition line at a large value G^c_Ψ of the bare Yukawa coupling. At values of the bare Yukawa coupling $G_\Psi < G^c_\Psi$ the phase transition line is given by multicritical points where both the fermion and the scalar masses vanish in lattice units. At $G_\Psi > G^c_\Psi$, the partial results indicate that the scalar critical point is lost and thus scalar particles decouple in the continuum limit. If we assume that this behaviour is not a product of the order of truncation of the series we applied, the perturbative results should only be contrasted with the results of the linked cluster expansion at $G_\Psi < G^c_\Psi$. Although the scaling region may not be reached at the 8^{th} order in K, all the obtained results show consistency with the expectations obtained from perturbation theory. The continuum limit of the theory at the multicritical points is most probably noninteracting and the Lüscher and Weisz procedure must work as in the pure scalar theory.

Appendix A

Character expansion properties

The Haar measure of the group SU(2) satisfies[45]

$$d^3U = d^3U^\dagger = d^3(UV) \tag{A.1}$$

$$\int d^3U = 1 \tag{A.2}$$

where the second property is a normalization condition. The irreducible characters form an orthonormal basis under integration over the Haar measure of the group:

$$\int d^3U \chi_r(U)\chi_s^*(U) = \delta_{rs} \tag{A.3}$$

$$\int d^3U \chi_r(U)\chi_s(U^\dagger V) = \delta_{rs} d_r^{-1}\chi_r(V) \tag{A.4}$$

where d_r is the dimension of the representation. For SU(2), a matrix U is parametrized as

$$U = \cos\left(\frac{\theta}{2}\right) + i\vec{\sigma}\hat{n}\sin\left(\frac{\theta}{2}\right) \qquad (0 \le \theta < 4\pi) \tag{A.5}$$

where σ_i are the 2x2 Pauli matrices. In terms of θ and $\hat{n}$, the Haar measure is given by

$$d^3U = \sin^2\left(\frac{\theta}{2}\right)\frac{d\theta}{2\pi}\frac{d^2\hat{n}}{4\pi} \tag{A.6}$$

and the characters read

$$\chi_j(U) = \frac{\sin\left((j+\frac{1}{2})\theta\right)}{\sin(\frac{\theta}{2})} \qquad j = 0, \frac{1}{2}, 1, \ldots \tag{A.7}$$

Any class function f(U), that satisfies

$$f(U) = f\left(VUV^\dagger\right) \tag{A.8}$$

may be decomposed in components as

$$f(U) = \sum_r \chi_r(U) f_r \tag{A.9}$$

In particular , the pure gauge action may be decomposed as

$$\exp\left(\frac{\beta\chi_{\frac{1}{2}}(U)}{2}\right) = \exp\left(\beta\cos\left(\frac{\theta}{2}\right)\right) = \sum_j \frac{2(2j+1)I_{2j+1}(\beta)\chi_j(U)}{\beta} \tag{A.10}$$

where $I_n(\beta)$ are the modified Bessel functions of argument β. The fact that SU(2) has only real representations assures the equality

$$\chi_j(U) = \chi_j\left(U^\dagger\right) \tag{A.11}$$

a property that may be easily obtained from eq.(A.7) by changing $\theta \to -\theta$. Another useful relation for the expansions we consider is

$$\chi_1(G) = \left(\chi_{\frac{1}{2}}(G)\right)^2 - 1 \tag{A.12}$$

Finally, we list some nontrivial integrals we used in the expansions:

$$I_1 = \int \mathcal{D}U\ \left(\chi_{\frac{1}{2}}(UV)\right)^3\ \chi_{\frac{1}{2}}(U) = \ \chi_{\frac{1}{2}}(V) \tag{A.13}$$

$$I_2 = \int \mathcal{D}U\ \chi_{\frac{1}{2}}(U)\ \ \chi_{\frac{1}{2}}(UV)\,\chi_1(UV) = \ \frac{\chi_{\frac{1}{2}}(V)}{2} \tag{A.14}$$

$$I_3 = \int \mathcal{D}U\ \chi_1(U)\,\chi_1(UV)\ \left(\chi_{\frac{1}{2}}(UV)\right)^2 = \ \frac{2\chi_1(V)}{3} \tag{A.15}$$

$$I_4 = \int \mathcal{D}U\ \chi_{\frac{1}{2}}(UV)\,\chi_{\frac{1}{2}}(UA)\,\chi_1(U) = \ \frac{1}{6}\left(\chi_{\frac{1}{2}}\left(AV^\dagger\right) + 2\chi_{\frac{1}{2}}(AV)\right) \tag{A.16}$$

$$I_5 = \int \mathcal{D}U\ \chi_{\frac{1}{2}}(i\tau_3U)\,\chi_{\frac{1}{2}}(U)\,\chi_{\frac{1}{2}}\left(i\tau_3\left(U^\dagger\right)^2\right) = \frac{2}{3} \tag{A.17}$$

$$I_6 = \int \mathcal{D}U\ \chi_1(i\tau_3U)\ (\chi_1(U))^2 = -\frac{1}{3} \tag{A.18}$$

Observe that eqs.(A.13) and (A.14) are not independent, but related by eq.(A.12), while eq.(A.18) may be deduced by using eqs.(A.12),(A.15) and the invariance of the Haar measure eq.(A.1).

Appendix B

Linked cluster expansion at $\beta = 0$

According to eq.(3.13) the partition function in the infinite gauge coupling limit is ($h_x = -1$)

$$\mathcal{Z} = \int \prod_x (R_x \mathcal{D}R_x) \exp -(\sum_x(\lambda(R_x-1)^2 - h_x R_x) \prod_{x,\mu} \left\{ \sum_{n=0}^{\infty} \frac{(k^2 R_x R_{x+\mu})^n}{n!(n+1)!} \right\} \tag{B.1}$$

This may be rewritten as

$$\mathcal{Z}(h_x,\lambda,k) = \mathcal{Z}(h_x,\lambda,k=0)\sum_n \frac{1}{2^n n!} \sum_{1,2,3,..} v(1,2,..,2n) < R(1)...R(2n) >_{k=0} \tag{B.2}$$

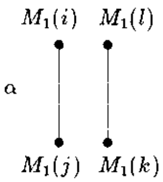
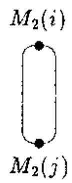
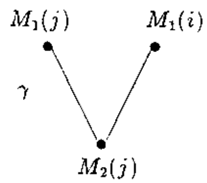
where

$$v(1,2,...,2n) = \prod_\alpha \left(\frac{n_\alpha!(k^2)^{n_\alpha}}{n_\alpha!(n_\alpha+1)!}\right) \tag{B.3}$$

when in the term $\{n_\alpha\}$ pairs of points coincide. The summation is for all points over the lattice. The factor $\left(\frac{1}{2}\right)^n$ appears because we make no distinction between the elements of the pairs. The additional factor $\left(\frac{\prod_\alpha n_\alpha!}{n!}\right)^{-1}$ is the number of different ways we can change the indices while keeping $\{n_\alpha\}$ coincident pairs. The whole factor

$$\sum_{1,2,...,2n} v(1,2,...,2n) < R(1)...R(2n) >= \mathcal{J} \tag{B.4}$$

may be given by a graphical expansion [43]. For example for n=2,

$\mathcal{J} = \alpha$  $+\ \beta$  $+\ \gamma$ 

The factors M_n that appear above are given by eq.(3.17), each free index i, j, k, l must be summed over the entire lattice and a factor $\prod_\alpha \left(\frac{(k^2)^{n_\alpha}}{(n_\alpha+1)!}\right)$ must be written for each time $\{n_\alpha\}$ edges coincide. The coefficients α, β, γ are given by

$$2^n n! \prod_\beta \frac{1}{g_\beta^{n_\beta} n_\beta!} \tag{B.5}$$

where n is the order in k^2 of the graph and n_β is the number of repetitions of a connected graph w_β of symmetry g_β. [43] If only coincident edges belonging to the same connected graph w_α are considered, this allows an easy computation of $\mathcal{Z}$

$$\mathcal{Z} = \mathcal{Z}(h_x, \lambda, k=0) \sum_{n_\beta} \prod_\beta \left(\frac{w_\beta}{g_\beta}\right)^{n_\beta} \frac{1}{n_\beta!} \tag{B.6}$$

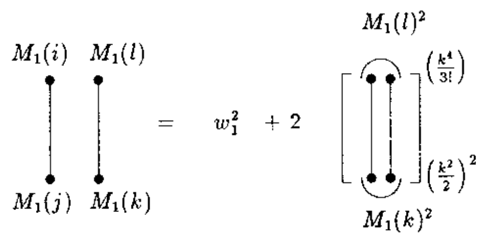
so that

$$ln\mathcal{Z} = ln\mathcal{Z}(h_x, \lambda, k=0) + \sum_\beta \left(\frac{w_\beta}{g_\beta}\right) \tag{B.7}$$

For obtaining an n-point correlation function one has only to perform h derivations

$$\frac{\delta^n (ln\mathcal{Z})}{\delta h_x ... \delta h_y}|_{h_x=-1} = < R_x ... R_y >^c \tag{B.8}$$

From this equation and eq.(B.7) we obtained the rules given in section 3.3. There is still a problem one has to face. It originates in the coincident edges that appear when two different connected graphs coincide in one or more links. Such a case is not included in the derivation of eq.(B.7). For example , the simplest case is



(B.9)

The last expresion above means that we have considered a factor $\left(\frac{k^2}{2}\right)^2$, while the correct factor was $\frac{(k^2)^2}{3!}$. The factor 2 is the number of times the last graph appears in the product. In fact, for any graph of this type, the factor that appears in front of it is only a symmetry reduction factor

$$\frac{(\prod_\alpha g_\alpha^{n_\alpha})}{\prod_i g_i'^{n_i}} \tag{B.10}$$

where $\prod_\alpha$ means the product of the symmetry factors of the initial connected graphs w_α that are repeated n_α times and g_i' are the symmetry factors of the final connected graphs w_i. For

example, in Eq.(B.9) the factor is $\frac{(2)^2}{2}$. With this prescription, one can calculate, in a tedious but straightforward way, the corrections to eq.(B.6) that appear due to coincident connected graphs. For simplicity, we will simbolized them as corr. in the following. Then, returning to eq.(B.6), we have

$$\mathcal{Z} = \mathcal{Z}(h_x, \lambda, k=0) \left(\sum_{n_\beta} \left(\prod_\beta \left(\frac{w_\beta}{g_\beta}\right)^{n_\beta} \frac{1}{n_\beta!} \right) + corr. \right) \tag{B.11}$$

and

$$ln\mathcal{Z} = ln\mathcal{Z}(h_x, \lambda, k=0) + \sum_\beta \left(\frac{w_\beta}{g_\beta}\right) + ln\left(1 + corr. \exp(-\sum_\beta \frac{w_\beta}{g_\beta})\right) \tag{B.12}$$

Obviously, this apparently complicated expresion is highly simplified due to the fact that disconnected graphs disappear in the final result.

The method may be easily generalized to any SU(N) group, by observing that for a different SU(N) (or U(N)) theory, with fields in the fundamental representation of the gauge group, the generating functional at $\beta = 0$ is given by[28]

$$\mathcal{Z} = \int \prod_x (R_x^{N-1} \mathcal{D}R_x) \exp -(\sum_x (\lambda (R_x - 1)^2 - h_x R_x) \prod_{x,\mu} \left\{ \sum_{n=0}^{\infty} \frac{(k^2 R_x R_{x+\mu})^n}{n!(n+N-1)!} \right\} \tag{B.13}$$

Then, the only change with respect to the SU(2) theory, appart from a redefinition of the integration measure, is the change of the factor (n+1)! by the factor (n+N-1)!. If we return to the derivation of the link cluster expansion, the only changes we have to make so that it applies to a general U(N) group, is to redefine the moments M_n by the appropiate change of measure and replace the factor $\frac{k^2}{(n+1)!}$ by a factor $\frac{k^2}{(n+N-1)!}$ in rule c) of section 3.3.

Appendix C

Susceptibility coefficients at the infinite gauge coupling boundary

The expansion coefficients of the susceptibility χ_2 are polynomial functions of the moments M_n defined in section 3.3. The first coefficients in k^2, at $\beta = 0$,are

$$SUS_0 = M_2 \tag{C.1}$$

$$SUS_1 = 4(M_3M_1 + M_2^2) \tag{C.2}$$

$$SUS_2 = \frac{1}{3}(2M_4M_2 + 138M_3M_2M_1 + 23M_4M_1^2 - 2M_3^2 + 46M_2^3 - 2M_3M_1^3 - 6M_2^2M_1^2) \tag{C.3}$$

$$SUS_3 = \frac{1}{18}(M_5M_3 + 2340M_3^2M_1^2 + 169M_1^3M_5 + 135M_4M_3M_1 + 45M_5M_2M_1 + 135M_3^2M_2 + 6279M_3M_1M_2^2 + 2067M_4M_2M_1^2 + 3M_1^5M_3 - 540M_2M_3M_1^3 - 45M_4M_1^4 + M_4^2 + 90M_4M_2^2 + 1053M_2^4 + 15M_2^2M_1^4 - 540M_2^3M_1^2) \tag{C.4}$$

$$SUS_4 = \frac{1}{1080}(3M_5^2 + 19794M_5M_1^2M_3 + 29718M_5M_1M_2^2 + 666M_5M_1M_4 + 1320M_5M_2M_3 - 336M_1^6M_2^2 + 31104M_1^5M_2M_3 + 49908M_1^4M_2^3 - 96444M_1^4M_2M_4 - 73752M_1^4M_3^2 - 564360M_1^3M_2^2M_3 + 370818M_1^3M_3M_4 - 280152M_1^2M_2^4 + 1157850M_1^2M_2^2M_4 + 1727682M_1^2M_2M_3^2 + 14946M_1^2M_4^2 + 2168688M_1M_2^3M_3 + 132696M_1M_2M_3M_4 - 30750M_1M_3^3 - 194868M_2^5 + 43266M_2^3M_4 + 74562M_2^2M_3^2 + 882M_2M_4^2 - 660M_3^2M_4 - 3M_4M_6 - 4848M_5M_1^5 + 1728M_1^6M_4 - 48M_1^7M_3 + 4848M_1^2M_2M_6 + 222M_1M_3M_6 - 219M_2^2M_6 + 177666M_5M_1^3M_2 + 8928M_1^4M_6) \tag{C.5}$$

where SUS_n are the n-th order expansion coefficients of χ_2.

In table 3.C.1 and 3.C.2 we give the results of the moments M_n and the susceptibility coefficients for different values of λ ,at $\beta = 0$. We also show the values of the ratios r_n, eq.(3.20), for the same values of λ, in Table 3.C.3. The first three values of λ are representative of a situation where a phase transition seems to take place. The value 1 of the self coupling is an example where, although all the computed coefficients are positive, the ratios are not rapidly convergent. The existence of a phase transition is not clear, and even if it takes place, the estimated k_c may be far from the real value. The last λ value is a value where a phase transition surely doesn't take place. The coefficients vary rapidly and their positiveness is lost.

λ	M_1	M_2	M_3	M_4	M_5	M_6
0.001	1.992	1.972	3.883	11.40	44.37	214.6
0.01	1.928	1.766	3.078	7.662	24.16	90.12
0.1	1.607	0.983	0.948	1.054	1.024	1.335
1.0	1.133	0.281	0.069	$0.54\ 10^{-3}$	-0.014	-0.004
10.0	1.003	0.047	$0.29\ 10^{-3}$	$-0.45\ 10^{-4}$	$0.912\ 10^{-5}$	$-0.206\ 10^{-5}$

Table 3.C.1 Values of the one point moments M_n, defined in the text, for different values of the bare self coupling λ.

λ	SUS_0	SUS_1	SUS_2	SUS_3	SUS_4
0.001	1.997	$0.47\ 10^2$	$0.12\ 10^4$	$0.29\ 10^5$	$0.72\ 10^6$
0.01	1.766	$0.36\ 10^2$	$0.76\ 10^3$	$0.16\ 10^5$	$0.35\ 10^6$
0.1	0.982	$0.99\ 10^1$	$0.98\ 10^2$	$0.94\ 10^3$	$0.90\ 10^4$
1.0	0.281	0.631	$0.11\ 10^1$	$0.13\ 10^1$	$0.16\ 10^1$
10.0	$0.47\ 10^{-1}$	$0.10\ 10^{-1}$	$-0.28\ 10^{-2}$	$-0.12\ 10^{-2}$	$0.31\ 10^{-2}$

Table 3.C.2. Values of the susceptibility coefficients in k^2, as a function of the bare self coupling λ at $\beta = 0$.

λ	r_1	r_2	r_3	r_4	rm_1	rm_2
0.001	0.0417	0.0401	0.0392	0.0391	0.1974	0.1966
0.01	0.0487	0.0475	0.0468	0.0469	0.2151	0.2141
0.1	0.0986	0.102	0.104	0.104	0.3277	0.3261
1.0	0.445	0.574	0.831	0.804	1.016 (?)	1.012 (?)
10.0	4.68	-3.62	2.34	-0.38	-	-

Table 3.C.3 Values of the ratios r_n, defined in the text, together with the two estimated values $(\sqrt{rm_i})$ of k_c, in accordance to eq.(3.21), for different values of the bare self coupling λ at $\beta = 0$.

Appendix D

Summary of matrix formulas

To develop the linked cluster expansion at $r = 1$ it is convenient to list a set of very useful formulas that make use of the algebra of Pauli and gamma matrices in euclidean space. We will use the notation

$$O_S^{-1} = \begin{pmatrix} c\Gamma_S^+ & 0 \\ 0 & b\Gamma_S \end{pmatrix}, \qquad O_S = \frac{1}{c\,b} \begin{pmatrix} b\Gamma_S & 0 \\ 0 & c\Gamma_S^+ \end{pmatrix} \tag{D.1}$$

where O_S is then the inverse of O_S^{-1} defined in the text. We will call $K_{\pm\mu} = \pm\gamma_\mu + M$ and we will define the matrix

$$B = \frac{1}{c\,b} \begin{pmatrix} 0 & c^2 \\ b^2 & 0 \end{pmatrix} \tag{D.2}$$

The following properties are verified

$$B^2 = M^2 = \gamma_\mu\,\gamma_\mu = I \tag{D.3}$$

$$[B, \gamma_\mu] = [M, \gamma_\mu] = 0, \tag{D.4}$$

$$M\;B = B^T\;M \tag{D.5}$$

$$O_S^{-1}\;K_\mu\;O_S^{-1} = (B\;M\;\gamma_\mu\; + M)\;cb \tag{D.6}$$

$$(O_S^{-1})^+ K_\mu O_S = \gamma_\mu + B \tag{D.7}$$

$$M\;O_S\; = O_S^+\;B \tag{D.8}$$

$$O_S\;M = B^T\;O_S^+ \tag{D.9}$$

$$O_S^{-1}\;\left(O_S^{-1}\right)^+ \;=\; BM\;cb \tag{D.10}$$

$$M\;O_S^{-1} M = cb\,O_S \tag{D.11}$$

$$K_\mu\;K_\mu\; = 2\;M\;K_\mu, \qquad K_\mu K_{-\mu} = 0 \tag{D.12}$$

where no sum over the O(4) indices is understood. From the above properties some interesting relations between the K_μ matrices appear,

$$K_\beta K_\alpha K_{-\beta} \;=\; \begin{cases} 2\;M\;\gamma_\alpha K_{-\beta} & \text{if } \hat\alpha \neq \pm\hat\beta \\ 0 & \text{if } \hat\alpha = \pm\hat\beta \end{cases} \tag{D.13}$$

$$K_\mu K_\alpha K_\mu \;=\; 2\;K_\mu\left(1 + \tilde\delta_{\alpha\mu}\right), \quad \tilde\delta_{\alpha\beta} = \begin{cases} 1 & \text{if } \hat\alpha = \hat\beta \\ -1 & \text{if } \hat\alpha = -\hat\beta \\ 0 & \text{if } \hat\alpha \neq \pm\hat\beta \end{cases} \tag{D.14}$$

while the ordered product of K_μ matrices in a plaquette satisfies

$$K_\nu K_\mu K_{-\nu} K_{-\mu} = -2 K_\nu K_{-\mu} \tag{D.15}$$

The matrices O_S^{-1} and O_S satisfy the following modified anticommutation relation,

$$O_S^{-1}\,O_{S'} \;+\; O_{S'}^{-1}\,O_S \;=\; 2\delta_{S,S'} \tag{D.16}$$

From here the following relations arise

$$\sum_{S'} O_{S'}^{-1} O_S O_{S'}^{-1} \;=\; -2\,O_S^{-1}, \qquad \sum_{S'} O_{S'} O_S^{-1} O_{S'} \;=\; -2\,O_S \tag{D.17}$$

$$\sum_{S,S'} O_S O_{S'}^{-1} O_S O_{S'}^{-1} \;=\; -8 \tag{D.18}$$

$$\sum_{S,S',S''} O_S O_{S'}^{-1} O_{S''} O_S^{-1} O_{S'} O_{S''}^{-1} \;=\; 16 \tag{D.19}$$

$$\sum_{S,S',S'',S'''} O_S O_{S'}^{-1} O_{S''} O_{S'''}^{-1} O_S O_{S'}^{-1} O_{S''} O_{S'''}^{-1} \;=\; -128 \tag{D.20}$$

Finally, let us note that in the case of equal Yukawa couplings some of the relations above are simplified. In particular,

$$B = M \tag{D.21}$$

$$O_S^{-1} K_\mu O_S^{-1} \;=\; K_\mu c^2, \quad \left(O_S^{-1}\right)^+ K_\mu O_S \;=\; K_\mu \tag{D.22}$$

Consequently,

$$\left[O_S^{-1}\;,\;K_\mu M\right] \;=\; 0 \tag{D.23}$$

Appendix E

One point connected moments

The expression of the partition function $Z(N,J)$ was obtained in eq. eq.(4.50). The one point connected moments may be calculated by computing $\ln Z(N,J)$ and taking functional derivatives of it with respect to $N, \bar{N}$ and J_S. Remembering that

$$\ln(1+y) \;=\; \sum_{n=1}^{\infty}(-1)^{n-1}\,\frac{y^n}{n} \qquad \text{(E.1)}$$

and

$$(b_1+b_2+\ldots+b_N)^m = \sum_{n_1,\ldots,n_N} \delta_{m,n_1+n_2+\ldots+n_N}\frac{m!b_1^{n_1}\,b_2^{n_2}\ldots b_N^{n_N}}{n_1!n_2!\ldots n_N!} \qquad \text{(E.2)}$$

it is straightforward to obtain

$$\ln Z(N,J) \;=\; \bar{\mu}\,\bar{N}MN \;+\; \sum_{n=1}^{\infty}\frac{b_n}{4^n}\left[(J_S+\bar{N}O_S^{-1}N)(J_S+\bar{N}O_S^{-1}N)\right]^n \qquad \text{(E.3)}$$

where the coefficients b_n are given by

$$b_n = \sum_{m,n_1,n_2,\ldots=1}^{\infty} \delta_{n,n_1+2n_2+3n_3+\ldots}\,\delta_{m,n_1+n_2+n_3+\ldots}\frac{(-1)^{m-1}(m-1)!}{n_1!n_2!n_3!\ldots(1!2!)^{n_1}\,(2!3!)^{n_2}(3!4!)^{n_3}\ldots} \qquad \text{(E.4)}$$

Comparing this expression with eq.(4.52) we see that the relation between b_n and C_{2n} defined in the text is given by

$$b_n \;=\; \frac{C_{2n}}{(2n)!} \qquad \text{(E.5)}$$

Let us begin computing the one point connected moments of only scalar fields:

$$\langle\phi_{S_1}\phi_{S_2}\ldots\phi_{S_n}\rangle \;=\; \frac{\delta^k \ln Z(N,J)}{\delta J_{S_1}\ldots\delta J_{S_n}}\;|_{J=N=\bar{N}=0} \qquad \text{(E.6)}$$

From the expression of $\ln Z(N,J)$, eq.(E.3) , it is clear that n must be an even number. On the other hand, since at the end we put all currents $J_S=0$ it is clear that all O(4) indices must be paired and that only the term proportional to b_k with $2k=n$ will survive. The result must also be invariant under permutation of two indices S_i and S_j. Then, it is obvious that it should be proportional to the O(n) symmetry factor $G_{2k}(a_1,\ldots,a_{2k})$ which is defined as [64] :

$$G_0 \;=\; 1 \qquad \text{(E.7)}$$

$$G_{2k}(a_1,\ldots,a_{2k}) \;=\; \sum_{i=2}^{2k}\delta_{a_1a_i}G_{2k-2}(a_2,\ldots,a_{i-1},a_{i+1},\ldots,a_{2k}) \qquad \text{(E.8)}$$

The normalization is such that

$$G_{2k}(1,1,1,\ldots1) \;=\; (2k-1)!! \qquad \text{(E.9)}$$

The coefficient that appears in front of the whole expression may be easily obtained by considering the case where all scalars indices are equal. When one makes the j^{th} derivative of eq. (E.3) with respect to a particular $J_{S'}$ one picks up a factor $(2j)!$. Looking at the relation given in eq.(E.5) and the normalization of the O(n) symmetry factors, it is easy to get the final result:

$$\langle\phi_{S_1}\ldots\phi_{S_{2k}}\rangle \;=\; \frac{C_{2k}}{4^k(2k-1)!!}G_{2k}(S_1,\ldots,S_{2k}) \qquad \text{(E.10)}$$

which coincides with eq.(4.51) when we put no fermion fields.

Now we may include the fermion fields.We want to compute

$$\left\langle\phi_{S_1}\ldots\phi_{S_k}\Psi_{a_1}\bar{\Psi}_{b_1}\ldots\Psi_{a_l}\bar{\Psi}_{b_l}\right\rangle = \frac{\delta\ln Z(N,J)}{\delta J_{S_1}\ldots\delta J_{S_k}\delta N_{b_1}\delta\bar{N}_{a_1}\ldots\delta N_{b_l}\delta\bar{N}_{a_l}}\;|_{J_S=N=\bar{N}=0} \qquad \text{(E.11)}$$

The only term independent of the O(4) indices is easily calculated

$$\left\langle\Psi_{a_l}\bar{\Psi}_{b_l}\right\rangle \;=\; \bar{\mu}\,M_{a_l,b_l} \qquad \text{(E.12)}$$

The symmetry of eq.(E.3) under interchange of $J_S \to \bar{N}O_S^{-1}N$ makes the calculations very easy. The final result is obtained while noting that

a)We need the number of pairs of fermions plus the number of scalars (k+1) to be an even number. If not, there will be always a factor J_S or $\bar{N}O_S^{-1}N$ in the final expression and the connected moments would vanish. Obviously only the term proportional to C_{k+l} will contribute.

b) The pairing of the O(4) indices may be realized between two scalars, between two fermion pairs or between a scalar and a fermion pair. For each fermion pair a factor O_S^{-1} appears.

c) Due to the Grassmann algebra Ψ and $\bar{\Psi}$ a permutation of two of two indices b_i, b_j must give a change of sign in the final expression. The same must happen with a permutation of two indices a_i, a_j.

d) Due to the symmetry between J_S and $\bar{N}O_S^{-1}N$ all other numerical factors are the same as in the pure scalar case.

From the above rules we get the expression of the connected moments, namely

$$\left\langle\phi_{S_1}\ldots\phi_{S_k}\;\Psi_{a_1}\bar{\Psi}_{b_1}\ldots\Psi_{a_l}\bar{\Psi}_{b_l}\right\rangle^c_{k=0;K=0} = \bar{\mu}\,M_{a_lb_l}\,\delta_{l,1}\delta_{k,0}+$$

$$\frac{C_{k+l}G_{k+l}(S_1,S_2,\ldots,S_{k+l})\sum_{\pi(l)}\sigma_\pi\,O^{-1}_{S_{k+1},a_1b_{\pi(1)}}\ldots O^{-1}_{S_{k+l},a_lb_{\pi(l)}}}{(k+l-1)!!\;\;2^{k+l}} \qquad \text{(E.13)}$$

where the only difference with eq.(4.51) is that there we wrote the symmetry factor G_{k+l} by its explicit expression. The sign of the parity σ_π is defined as +1 when the indices a_ib_i are in the same order as the ones of the Ψ fields in the left hand side of eq.(E.13)

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