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[1] Ph. D. Thesis

Abstract

In this thesis we study the fermionic charge induced by topologically nontrivial scalar and gauge field configurations through vacuum polarization effects. A powerful and simple technique to analyse these effects is the adiabatic method. In general, this technique accurately gives the induced fermion charge, which is related to the ground state fermion charge through the number of zero energy level crossings occuring during the adiabatic process. We consider here the 3+1 dimensional σ model, whose scalar sector supports solitons and is mathematically identical to the Higgs sector of the Standard model. In models with scalar fields, like the one under consideration, the adiabatic current becomes ill defined at the points where the scalar fields vanish. This implies an important limitation of the adiabatic technique since, in such cases, it no longer reliably gives the correct induced fermion number. We propose here a method to extend the adiabatic procedure, even in the presence of somewhere vanishing scalar fields. After an enlarged analysis adding gauge fields this method allows us to compute the charge of the sphaleron. From the proposed procedure it follows that the induced fermion charge value depends not only on the final configuration, but also on the number of times that the adiabatically evolving scalar fields vanish. To check our method, we verify, for different intermediate scalar configurations, that the induced fermionic charge value and the path dependent spectral flow contributions arrange themselves to give a path independent ground state fermion charge for the final scalar configuration. We consider, for example, configurations evolving from the normal vacuum to a final Skyrmion of width ρ_S. No zero energy level crossings exist for a fermion mass $m_f < 1.5/\rho_S$ while one level crossing occurs for $m_f > 1.5/\rho_S$, whenever the intermediate path implies no fermion flux at spatial infinity, i.e. the scalar configuration must vanish somewhere. Exactly the opposite conditions are obtained whenever the path allows fermion current flow at spatial infinity. We explain how, in both cases, the Skyrmion carries the fermion number of any fermion with a mass $m_f > 1.5/\rho_S$. We also extend our analysis to a soliton of winding number 2 and find similar results. Vacuum polarization effects within 2+1 dimensional QED are also analyzed. For large fermion masses, in the 3+1 dimensional σ model, the value of the ground state fermion number of a bosonic topological configuration is related in a consistent way to its associated spin and statistics evaluated within the bosonic effective theory. In 2+1 dimensional theories of fermions minimally coupled to gauge fields, ambiguities in the regularization procedure imply a non unique value of the coefficient μ of the Chern-Simons term which appears in the gauge field effective action. In such case, the consistency between the induced fermion charge and spin only exists for two values of μ, namely $\mu = \pm 1/2\pi$.

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Contents

Chapter 1

Introduction

The effects of a scalar field soliton on the Dirac sea of a fermion field have already been well studied in the literature [1]-[4]. It is known that the modification of the fermionic Dirac sea by its interaction with solitons influences many properties of the fermion-soliton quantum system. Nonzero and even non integer quantum numbers such as the fermion number or spin may be induced in such a context. The ground state fermion number induced by a soliton depends, of course, on the fermion content of the theory and on the fermion-soliton interactions. While it may be susceptible to the local behaviour of the soliton configuration too, in many models its dependence reduces to the asymptotical topological properties of the scalar background. In general the fractional contribution to the ground state fermion number associated with a bosonic configuration remains invariant under local variations of such background fields. The integer part, however, generally depends on this type of variations, and it can be accurately computed in terms of index theorems [5],[6].

To evaluate these vacuum polarization effects, in the weak coupling limit and while neglecting back action of the fermions on the scalar fields, a general formalism is to characterize the charge within the frame of the Dirac's hole theory. In the normal vacuum the scalar field gives the fermion a mass and the energy spectrum of the Dirac equation implies positive and negative energy continua separated by a gap from $E=-m$ to $E=m$. The no particle state or ground state is defined as the one where the negative energy continuum is filled and the positive energy continuum is empty. The normal Dirac sea is defined so that it carries zero charge. Any other charge is measured considering the normal ordering prescription involved within this definition of the normal vacuum ground state value. As the energy spectrum of the Dirac equation is altered there will be positive and negative energy continua and bound states may also appear. To obtain the ground state charge, one must fill all the levels with $E<0$ and evaluate the contribution to the charge relative to the normal vacuum ground state one. This straightforward procedure, even applicable in simple models with charge conjugation symmetry [1], turns to be very difficult in most of the cases.

The adiabatic method [7]-[9], developed as a way to evaluate diagrammatically the fermion number, is a more powerful and elegant technique that enables one to easily calculate the fermion charge in more complicated models. Basically, it consists in building up the final configuration starting from the normal vacuum and performing slow changes of the fields

in space and time. The induced fermion current expectation value can be calculated as an expansion in powers of derivatives of the background fields. The charge of the final state can be obtained from the lowest order nonvanishing term in this expansion. Although this method has the virtue of computational simplicity, an important issue is whether or not the state reached by the adiabatic construction is, indeed, the ground state of the final soliton. While building up the soliton, the energy spectrum of the Dirac Hamiltonian will be modified. For infinitely slow variations of the background, the states evolving from the initially filled Dirac sea remain filled and there is no substantial change. However, if, during the adiabatic process, a bound state emerges from the Dirac sea and crosses zero, the state reached in this way has one fermion more than the solitonic ground state. Hence, the adiabatic charge will be one unit greater than the ground state charge. It has been argued in the literature [8]-[11] that, whenever the Compton wavelength of the fermion is much smaller than the characteristic spatial scale of the background fields variations, the adiabatic result is expected to correctly predict the ground state charge. Otherwise zero energy level crossings may occur and the results gotten with this technique are reliable only for the fractional part of the ground state fermion number of the final configuration. It is worth remarking that, although the adiabatic procedure is usually appropiate for solitons with small spatial gradients, if this is not the case, one can always construct the soliton in two steps; first building up a smooth soliton with the desired asymptotical behaviour and then performing local deformations to determine its sharper structure. One can apply the adiabatic technique to the first step and obtain accurately the fractional part of the fermion number; the second step may, at most, affect the integer part of the charge while inducing zero energy level crossings.

The above issue concerning the spectral flow contributions is the relevant one when considering the fermion number of the low energy state of the system. However, even before this, one must consider if the adiabatic computation gives, in fact, a trustable result for the induced charge, in order to be sure that, after taking into account the spectral flow issue, the correct ground state charge will be obtained. The important question, when one tries to compute the charges induced by a background field via the adiabatic method, has to do with the overall validity of this method , or better said of the formula for the induced current [12],[13]. The expression for the induced current becomes ill defined when one studies scalar field configurations, as those associated with t'Hooft instanton [14] or more specifically the static sphaleron configuration [15]-[18], which go through zero at some point. The belief that the adiabatic method gives the induced charge in such cases can be very misleading, as will be illustrated below, since fermion charge seems to be induced due to fictitious anomalies generated by scalar fields. D'Hoker and Goldstone [19] investigated the fermion number current to leading order in the derivative expansion, including also gauge fields as background fields. They proved that, whenever no current flow at spatial infinity is allowed, only the gauge fields contribute to the induced fermion charge. However, this general result, at first sight, is at variance with that obtained explicitly with the adiabatic method [8], when fermion charge appears to be induced in the case of pure background scalar fields, even when there is no charge flux at spatial infinity. In the course of this work we will clarify the above apparent discrepancy. It is our aim to gain, thereby, a better understanding of the adiabatic method and its limitations when it is applied to obtain fermion number, associated with topologically non-trivial scalar backgrounds. To that purpose it will be useful to study the appearence of zero energy modes, analysing carefully its dependence on the relation between the fermion mass and the typical mass scale of the soliton, while building up adiabatically any of those peculiar scalar configurations which involve somewhere an ill-definition of the adiabatic current expression.

Apart from the analysis of induced quantum numbers in even space-time dimensions, it is also an interesting point to investigate this issue for odd space-time dimensional models. In an independent line of study [20], the possibility of constructing a gauge invariant mass term for the gauge bosons in 2+1 dimensions was demostrated. This term, related to the Chern-Simons secondary characteristic class, has been found to arise through radiative corrections in theories of fermions interacting with gauge fields [21]. Within 2+1 dimensional QED, the induced fermion current may be directly deduced in terms of the effective action obtained by eliminating the fermion fields by functional integration. In the low momentum approximation, the resulting Euler-Heisenberg effective action contains the Chern-Simons topological mass term. Due to the explicit form of the Chern-Simons term, the induced fermion number is given in terms of the net magnetic flux [22]. Consequently, if the effective bosonic theory supports topological excitations with a net magnetic flux, they will carry fermion number. At very large fermion masses, or equivalently, at low energy momentum compared with the fermion mass, the fundamental fermion is not observed and its dynamical effects may only arise through radiative corrections. The effective action, which is a compact way of calculating all the radiative corrections, should consequently describe all the dynamics of the theory. The important point is whether a topological bosonic configuration, inducing fermion number one at large fermion masses, may indeed be identified with a fermion. To assure the above one must verify that the spin and statistics of the topological configuration correspond to those of a fermion, when the bosonic theory is described by the effective action. In 2+1 dimensions, where ambiguities in the regularization procedure are known to exist [20], [23], only an appropiate choice of the regularization scheme allows the identification of the topological bosonic field with a fermion.

The plan of this thesis is as follows. In chapter 2 we first recall that the σ model, whose scalar sector in the linear limit is mathematically identical to the Higgs sector of the Weinberg-Salam theory, possesses solitons. Then, we present and expose in detail the adiabatic method. For physical applications where the true ground state charge is relevant, such as the fermion number of the Skyrmion, we supplement the adiabatic computation by an analysis of the spectral flow structure. In chapter 3 we start analysing the charge induced adiabatically by scalar field configurations, which interpolate between the ordinary vacuum and a final soliton, but which are somewhere vanishing, and we confront this results with those obtained for the induced charge after the analytical evaluation of the energy states. We deduce empirically a way to extend the adiabatic technique, even in the presence of scalar fields where the adiabatic current is somewhere ill-defined. We enlarge our considerations to background configurations where gauge fields are also present, in particular, to the sphaleron. In chapter 4 we evaluate numerically the energy states for the same kind of peculiar backgrounds considered in the previous section. We obtain explicitly the dependence of the zero energy modes on the intermediate path used to build up the final Skyrmion. Even though our work in chapter 4 concentrates on the Skyrmion, our results confirm the method proposed in chapter 3. In chapter 5 we first recall the correlation between the soliton fermion number and its associated spin in 3+1 dimensions. We introduce then, in 2+1 dimensions, the vacuum polarization

effects on the Dirac sea in the presence of a gauge background field configuration. We analyse the fermion number and spin induced by this background in the presence of an external charged particle. Considering the effective bosonic theory, we obtained the values of the coefficient of the Chern-Simons term for which there exists consistency between the induced quantum numbers.

Chapter 2

Ground State Charge, Induced Charge and Spectral Flow Contributions

In this chapter, we introduce all the fundamental concepts concerning the definition of the ground state charge of a system and its relation with the one obtained from the current density, derived through adiabatic techniques. We explain first, succinctly, the basis of the adiabatic method and its application to evaluate the fermion charge induced by a background scalar field configuration in the fermionic Dirac sea. More in detail, since in the following chapters it will be of special interest to compute the fermionic charge induced by solitonic configurations, in section 2.1 we present the scalar sector of a σ model and illustrate the existence of solitons in it. In section 2.2 we develop the adiabatic technique and find the adiabatic current expression in terms of the scalar background fields. In section 2.3 we analyse the limitations of the adiabatic method in the presence of spectral flow contributions. We obtain the exact relation between the ground state fermion number, the one given within the frame of the adiabatic method and the zero energy level crossings occuring during this process.

2.1 Solitons in the σ Model

Let us consider the scalar sector of a σ model:

$$\mathcal{L}_{\sigma} = \frac{1}{2}\partial_\mu\phi_a\partial^\mu\phi_a - \frac{\lambda}{4}(\phi^2 - v^2)^2 \tag{2.1}$$

with ϕ_a a quartet of scalar fields $(\phi_0, \vec{\phi})$. This model, as will be detailed below, possesses solitons which are topologically stable in the nonlinear model ($\lambda \to \infty$) and topologically metastable in the linear one (finite λ). The Higgs sector of the standard weak interactions is a linear σ model [25], thus it may support metastable solitons. However, the quartic coupling of the Higgs fields is a free parameter not yet constrained by experiment. Imagining the Higgs sector as an effective low energy theory, one can take an infinite limit for the coupling

constant and obtain a model with stable solitons [26],[27]. For solitons to exist, it is necessary to add to the above Lagrangian a stabilizing term. When a convenient choice of such term is made the solitons are referred as Skyrmions [28].

Topological solitons are associated with a space of finite energy field configurations divided into disconnected sectors, which are separated by an infinite energy barrier from each other. Such an infinite energy barrier may result from a finite energy density at spatial infinity or from an infinite energy density over a finite region in the space. The latter possibility is the one responsible for the existence of solitons in the σ model, since considering the limit $\lambda \to \infty$ any region of the space where $\phi^2 \neq v^2$ has infinite energy density. A finite energy field configuration is a map from the coordinate space into the space of fields configurations satisfying the constraint $\phi^2 = v^2$, which is S^3. Compactifying the coordinate space R^3 to S^3 by requiring that the fields approach a unique value at spatial infinity, each finite energy configuration may be characterized by its winding number, i.e. the number of times that the configuration $\phi(x)$ wraps the space of finite energy configurations S^3. Since $\pi_3(S^3) = Z$, this implies the existence of solitons of different winding number. The pertinent expression for the winding number is

$$n = \frac{1}{12\pi^2 v^4}\epsilon_{abcd}\epsilon^{ijk}\int d^3x\ \phi_a\partial_i\phi_b\partial_j\phi_c\partial_k\phi_d\,, \tag{2.2}$$

which is an integer. Taking λ large but finite, field configurations of different topological charges are no longer separated by an infinite energy barrier, but by a large one. The solitons in this case will not be totally stable, but metastable.

To illustrate the above it is useful to consider a 1+1 dimensions analogue, since there the scalar potential $V(\phi) = \lambda(\phi^2 - v^2)^2$ and the fields components ϕ_0, ϕ_1 may be visualize in three dimensional space as shown in Fig. 1. It is clear, that any finite energy field configuration must fulfill the condition $|\phi(\pm\infty)|^2 = v^2$. After compactification of the configuration space, $\phi(+\infty) = \phi(-\infty)$, thus both ends of any curve characterizing such field configuration must be represented by the same point on the ring $\phi^2 = v^2$. Obviously, considering the nonlinear constraint, the whole curve must lie on the ring and any of these curves can be characterized by a winding number, which in 1+1 dimensions is given by

$$n = \frac{1}{2\pi v^2}\epsilon_{ab}\int_{-\infty}^{\infty} dx\ \phi_a\partial_x\phi_b\,. \tag{2.3}$$

n is an integer and measures the degree of a map from the coordinate space into the configuration space, which is the ring S^1. Configurations with different winding numbers are disconnected, since any deviation from the ring in order to change the number of loops of a curve around the origin means going over the region of infinite energy. The above implies the existence of stable solitons in the non-linear limit. However, if one considers a large, but finite value for λ, a vacuum field configuration, which within this frame is represented by a point on the ring, may change its topology, winding once around the origin, by going over the potential barrier while vanishing somewhere (as shown in Fig. 1). Thus, metastable solitons are plausible in the large λ linear limit.

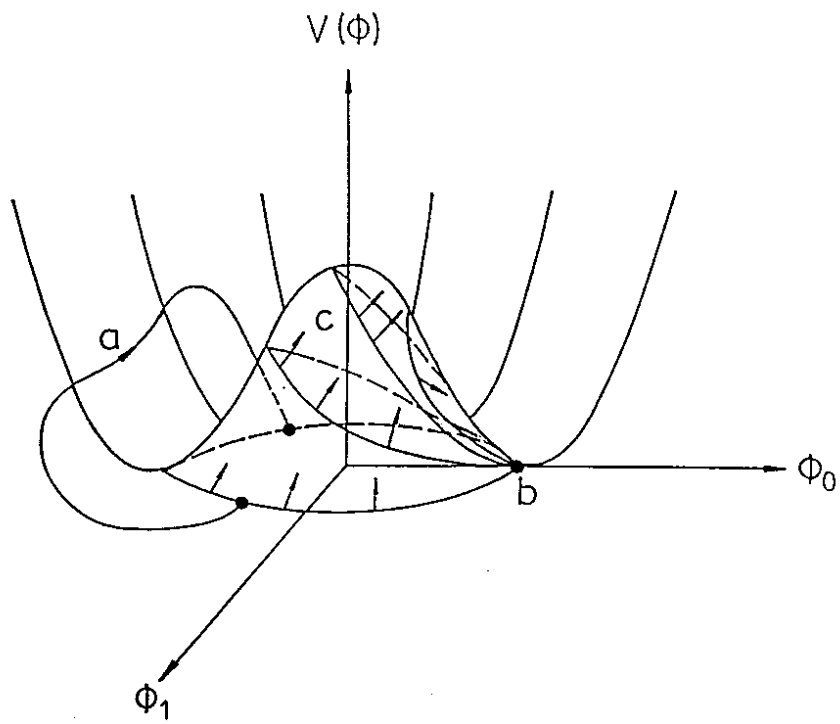


Figure 1: Potential energy density of a linear σ model as a function of the field configuration $\phi = (\phi_0, \phi_1)$. a) General finite energy field configuration (the two points on the ring $\phi^2 = v^2$ correspond to $\phi(-\infty)$ and $\phi(\infty)$). b) Vacuum field configuration (the whole configuration is represented by a point on the ring $\phi^2 = v^2$). c)Sequence of field configurations from the vacuum to a final soliton ($\phi(-\infty) = \phi(\infty)$ for any field configuration).

2.2 The Adiabatic Method

The adiabatic method was first developed by Goldstone and Wilczek [7] as a way to evaluate the fermionic induced charge in the background of a scalar soliton. They extended the procedure also for background configurations where gauge fields appear. This method provides, in general, a simple technique to compute the induced fermion number, giving reliable results for fields with small spatial gradients, and, quite generally, for the fractional part of any arbitary background field. The spirit of the method it to build up the final desired configuration starting from a convenient initial one, by slow changes of the fields in space and time. A convenient initial configuration means one whose associated fermion charge is exactly known. For simplicity, it is usually proposed to be the vacuum , so the comparison

fermion charge is zero. The expectation value of the induced fermion current can be evaluated as an expansion in powers of derivatives of the background fields. Keeping the lowest nonvanishing term in this expansion, neglecting any higher order derivatives, one obtains the induced charge of the final state within the adiabatic limit.

Consider the fermionic Lagrangian with a scalar quartet of background fields:

$$\mathcal{L}_F = i\,\bar{\psi}\partial_\mu\gamma^\mu\psi - \frac{g_y}{\sqrt{2}}\,\bar{\psi}\,(\phi_0 + i\gamma_5\vec{\phi}\cdot\vec{\sigma})\,\psi \tag{2.4}$$

This Lagrangian is SU(2) $\otimes$ SU(2) global invariant. This is easily seen by rewriting $\mathcal{L}_F$ in terms of the 2×2 matrix

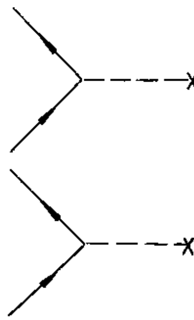
$$\Phi = \frac{1}{\sqrt{2}}\left(\phi_0 + i\vec{\phi}\cdot\vec{\sigma}\right) \tag{2.5}$$

as,

$$\mathcal{L}_F = i\,\bar{\psi}_L\partial_\mu\gamma^\mu\psi_L + i\,\bar{\psi}_R\partial_\mu\gamma^\mu\psi_R - g_y\left(\bar{\psi}_L\Phi\psi_R + \bar{\psi}_R\Phi^+\psi_L\right) \tag{2.6}$$

The Feymann rules derived from the Lagrangian, Eq. (2.4), are,

x ——▶—— y $\quad = \quad iS_F^0(x-y)$



$\phi_0(x) \quad = \quad \frac{-ig_y}{\sqrt{2}}\phi_0(x)$

$\phi_i(x) \quad = \quad \frac{g_y}{\sqrt{2}}\gamma_5\phi_i(x)\sigma_i$ (2.7)

with $S_F^0(x-y)$ being the propagator of the massless fermion.

Since the scalar fields are assumed to evolve very slowly, they may be written as a Taylor expansion around a point x, as

$$\phi_{0,i}(x_j) = \phi_{0,i}(x) + (x_j - x)^\nu\partial_\nu\phi_{0,i}(x) + \dots\,, \tag{2.8}$$

and the current will be given in terms of derivatives of these fields computed at the point x. In principle, the current at a point x is an infinite summation of Feymann diagrams, as illustrated in Fig. 2,

$$< j^\mu(x) > \; = -\sum_{j=0}^{\infty} \int d^4x_1 \dots d^4x_j \; Tr \left[\gamma^\mu \imath\, S_F^0(x-x_j)\gamma_5 \frac{g_y}{\sqrt{2}}\, \phi_{0,i}(x_j)\, \sigma_{0,i}\, \imath\, S_F^0(x_j - x_{j-1}) \dots \right.$$
$$\left. \dots \imath\, S_F^0(x_2 - x_1)\gamma_5 \frac{g_y}{\sqrt{2}}\, \phi_{0,i}(x_1)\, \sigma_{0,i}\, \imath\, S_F^0(x_1 - x) \right] , \qquad (2.9)$$

where $\phi_{0,i}(x_j)\,\sigma_{0,i}$ implies $\phi_0(x_j)\sigma_0$ or $\phi_i(x_j)\sigma_i$, with σ_0 defined as $\sigma_0 = -\imath\gamma_5 \mathrm{I}$. However, the adiabatic approximation implies that the current expansion converges rapidly enough, so that one may consider only the first nonvanishing term of the above expression. Due to the existence of an SU(2) global symmetry, one can always make a transformation $\Phi'(x) = \Phi(x)U(x)$ and choose $U(x) = \Phi^+(x_0)$, so that $\phi_i'(x_0) = 0$ and $\phi_0'(x_0) \equiv \sqrt{2}m/g_y$, and consider the current expansion around x_0. Thus, the zeroth order term in the Taylor expansion implies mass insertions.

A simplification of the calculations is achieved by using transformation properties of the current under global symmetries, for example, under charge conjugation. The fields ϕ_i are charge conjugation odd objects, while ϕ_0 is even under this transformation. Requiring the current to be a charge conjugation odd object and considering the properties of the traces of γ^μ, γ_5 and Pauli matrices, the lowest nonvashing term is the one with three γ_5 vertices carrying off momentum and infinite ϕ_0 vertices giving mass insertions. Since an infinite summation of mass insertions in the massless fermion propagator, S_F^0, turns it into a massive one, S_F, we finally have the corresponding diagram of a massive fermion loop, as shown in Fig. 3. If the scalar fields are massive enough and they do not propagate, no further graphs have to be considered. The current expression reads:

$$< j^\mu(x) > \; = \; - \int d^4y_1 d^4y_2 d^4y_3 \; Tr \left[\gamma^\mu \imath S_F(x-y_3)\gamma_5 \frac{g_y}{\sqrt{2}} \phi_k(y_3)\sigma_k \imath S_F(y_3 - y_2) \right.$$
$$\left. \gamma_5 \frac{g_y}{\sqrt{2}} \phi_j(y_2)\sigma_j \imath S_F(y_2 - y_1)\gamma_5 \frac{g_y}{\sqrt{2}} \phi_i(y_1)\sigma_i \imath S_F(y_1 - x) \right] \qquad (2.10)$$

In momentum space the above becomes,

$$< j^\mu(x) > \; = \; \frac{g_y^3}{\sqrt{2}^3} Tr[\sigma_i\sigma_j\sigma_k] \int \frac{d^4q_1 d^4q_2 d^4q_3}{(2\pi)^{12}} \phi_i(q_1)\phi_j(q_2)\phi_k(q_3) \exp[-\imath(q_1+q_2+q_3)x]$$
$$\int \frac{d^4p}{(2\pi)^4} Tr \left[\gamma^\mu \frac{1}{(p_\beta + q_{1\beta} + q_{2\beta} + q_{3\beta})\gamma^\beta - m} \gamma_5 \frac{1}{(p_\alpha + q_{1\alpha} + q_{2\alpha})\gamma^\alpha - m} \gamma_5 \right.$$
$$\left. \frac{1}{(p_\nu + q_{1\nu})\gamma^\nu - m} \gamma_5 \frac{1}{p_\delta \gamma^\delta - m} \right] \qquad (2.11)$$

Until now we have made the computations to all orders, but now we keep only the linear term in $q_1\, q_2\, q_3$ and get

$$< j^\mu(x) > \; = \; 2\imath\, \epsilon_{ijk} \frac{g_y^3}{\sqrt{2}^3} m \int \frac{d^4q_1 d^4q_2 d^4q_3}{(2\pi)^{12}} \exp[-\imath(q_1+q_2+q_3)x]$$
$$\phi_i(q_1)\phi_j(q_2)\phi_k(y_3) \frac{\epsilon^{\mu\nu\alpha\beta} q_{1\nu} q_{2\alpha} q_{3\beta}}{24\pi^2 m^4} \qquad (2.12)$$

Using the chiral invariance of the theory, the current expression in the adiabatic limit finally becomes [7],[9]:

$$< j^\mu(x) > = \frac{\epsilon_{dabc}\epsilon^{\mu\nu\alpha\beta} \phi_d \partial_\nu \phi_a \partial_\alpha \phi_b \partial_\beta \phi_c}{12\pi^2 |\phi|^4} \qquad (2.13)$$

The current, Eq. (2.13), is conserved, but it is singular at $|\phi| = 0$. Thus obviously, the adiabatic requirement $|\partial\phi| \ll g_y^2 |\phi|^2$ cannot be satisfied at the singularity. With $|\phi|$ never vanishing and $(\phi_0, \vec{\phi}) = (v, \vec{0})$ at spatial infinity, the charge formally constructed from this current measures the degree of a mapping from S^3 to S^3, that is to say - after compactification of R^3 to S^3 - from $\vec{r} \to (\phi^0(\vec{r}), \vec{\phi}(\vec{r}))/|\phi(\vec{r})|$, and, since $\pi_3(S^3) = Z$, it takes therefore integer values.

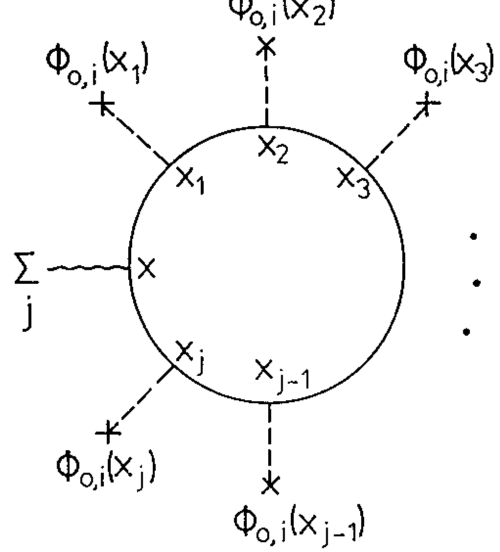


Figure 2: Feymann diagrams giving the fermion current at a point x. $\phi_{0,i}$ implies all the possibilities of having either ϕ_0 or ϕ_i with i = 1,2 or 3 for each point x_j. An infinite sum over j is understood.

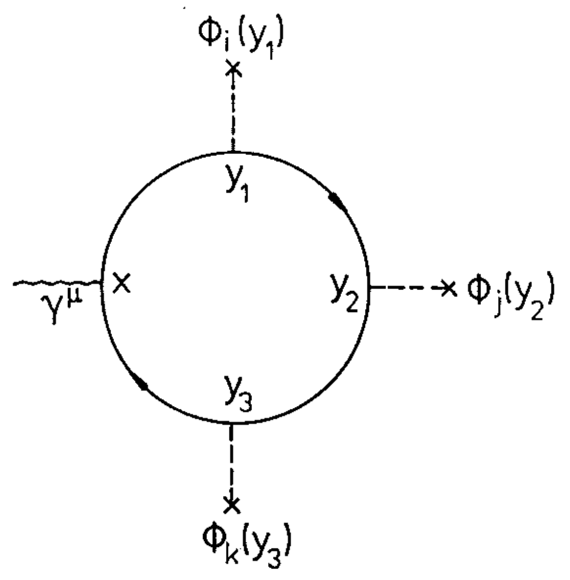


Figure 3: Fermion current given by the minimal contributing Feymann diagram with a massive fermion loop.

In terms of the 2 × 2 matrix Φ we can construct the SU(2) matrix

$$\hat{\Phi} = \sqrt{2}\frac{\Phi}{|\phi|}, \qquad (2.14)$$

so that $\hat{\Phi}^+\hat{\Phi} = 1$. This allows one to rewrite Eq.(2.13) as [29]:

$$j^\mu_{\hat{\Phi}}(x) = \frac{1}{24\pi^2}\epsilon^{\mu\nu\alpha\beta}Tr[\hat{\Phi}^+\partial_\nu\hat{\Phi}\hat{\Phi}^+\partial_\alpha\hat{\Phi}\hat{\Phi}^+\partial_\beta\hat{\Phi}]. \qquad (2.15)$$

The above equations (2.13), (2.15) show clearly that, in the adiabatic limit of the gradient expansion, the fermion number induced by a soliton background configuration can be identified with its topological charge. Therefore, the adiabatic charge depends only on the asymptotic behaviour of the fields, being insensitive to the exact form of the soliton. Considering higher orders in q_i, the current expression and particularly the charge density receive contributions from all orders in the expansion. However, the charge remains that given by the first nonvanishing term we have computed [9].

2.3 Spectral Flow Contributions

There are physical situations in which it is particularly interesting to establish whether or not the charge computed adiabatically is the ground state charge of the system. For example, in evaluating with the adiabatic method the charge induced by a final Skyrmion one obtains as a result its topological number. It is of particular interest to know, accurately, if its ground state charge is indeed determined by its topology. As was mentioned earlier, the absence or not of spectral flow contributions plays a decisive role in this point, since the adiabatic procedure does not take into account any issue associated with zero energy level crossings.

Consider the normal ordered fermion number operator given by [6],[30]

$$Q = \int dk\,\left(b_k^+b_k - d_k^+d_k\right) - \frac{1}{2}\eta_{[H]}\,. \qquad (2.16)$$

Here the operators b_k, d_k^+ and their Hermitian conjugates are the annihilation and creation operators of the fermion and antifermions states, respectively, which satisfy the standard anticommutation relations:

$$\left\{b_k, b_{k'}^+\right\} = \left\{d_k, d_{k'}^+\right\} = \delta(k-k') \qquad (2.17)$$

The c-number part of the charge operator is given in terms of the spectral asymmetry, $\eta_{[H]}$, which is essentially a ζ function regularization of the difference between the number of positive and negative eigenvalues of the Dirac Hamiltonian H. It is possible to construct a state which is annihilated by the operator part of (2.16), and this is defined as the ground state of the system. Then, the ground state charge is given by the c-number part of the fermion number operator,

$$Q_{GS} = -\frac{1}{2}\eta_{[H]}\,. \qquad (2.18)$$

All other states for a given background may be constructed from the ground state, by operating with creation and annihilation operators. Thus their fermion charge will differ by integers from the ground state charge. Observe that the fermion number operator in Eq. (2.16) is odd under charge conjugation, since this transformation interchanges fermions and antifermions $b_k \leftrightarrow d_k$.

The crucial point in applying the adiabatic evolution is that the normal order in the fermion number operator remains the same as in the comparison initial state. This means that, if during the process, an initially occupied state crosses the zero energy level and attains positive energy, it will be still quantized as a fermion of negative energy associated to the operator d_k. From the above, one understands that any discontinuous change in the fermion number operator is overlooked within the adiabatic procedure. Thus, in such cases, the adiabatic charge differs by integers from the final ground state charge.

We want now to consider the issue of nonvanishing spectral flow contributions in a more quantitative way, but first we analyse the case with no zero energy level crossings explicitly. From the above it is quite clear that, in the absence of spectral flow contributions, at any time t during the adiabatic evolution, the adiabatic charge coincides with the ground state charge of the system,

$$Q_{ad.}(t) = Q_{GS}(t) = -\frac{1}{2}\eta^c_{[H_t]} , \tag{2.19}$$

where the supraindex c in $\eta_{[H]}$ is only to remind one that the spectral asymmetry is, in this case, a continuous function of t. In the spirit of the adiabatic method, let us evaluate the charge at a final time, $t \to \infty$, by computing the change of the charge as the background evolves adiabatically from the initial configuration at time $t \to -\infty$ to the final one at $t \to \infty$.

$$\begin{aligned} Q_{GS}(t \to \infty) - Q_{GS}(t \to -\infty) &= -\frac{1}{2}\eta^c_{[H]} + \frac{1}{2}\eta^c_{[H_0]} \\ &= \int_{-\infty}^{\infty} dt\, \partial_t Q_{ad.}(t) \\ &= \int_{-\infty}^{\infty} dt\, \partial_t \int d^3x\, j^0(x,t) \\ &= \int d^4x \left(\partial_\mu j^\mu - \vec{\nabla}\cdot\vec{j}\right) , \end{aligned} \tag{2.20}$$

where we have defined the interpolating family of Dirac Hamiltonians H_t such that

$$\begin{aligned} lim_{t\to\infty} H_t &= H \\ lim_{t\to-\infty} H_t &= H_0 . \end{aligned} \tag{2.21}$$

As said before, one can, for simplicity, choose the comparison initial background as the normal vacuum, then

$$Q_{GS}(t \to -\infty) = -\frac{1}{2}\eta^c_{[H_0]} = 0 \tag{2.22}$$

and we have for the final ground state fermion number:

$$Q_{GS}(t \to \infty) = -\frac{1}{2}\eta^c_{[H]} = -\int_{-\infty}^{\infty} dt \oint d\vec{S}^i \cdot \vec{j}_i + \int d^4x\, \partial^\mu j_\mu , \tag{2.23}$$

where the index i in the surface integral implies a summation over all the possible spatial surfaces through which a flux of fermions into the system may appear.

Assuming that a non-trivial spectral flow contribution appears during the adiabatic process, then the spectral asymmetry is no more a continuous function of t, but one with discontinuity ± 2 for each energy eigenvalue that crosses from negative values to positive values or vice versa . Defining n_+ (n_-) as the number of energy states which cross in the positive (negative) direction of the energy axis at times $t_i < t$, we may then write the spectral asymmetry, in this case, as a sum of its continuous and discontinuous parts,

$$\eta_{[H_t]} = \eta^c_{[H_t]} + 2\,(n_+ - n_-) . \tag{2.24}$$

The fermion number operator must also have discontinuous changes, since any state quantized as a fermion (antifermion) before crossing the zero energy level must be quantized as its antiparticle after the zero cross. Of course, during the intermediate intervals, when no eigenvalue crossing ocurrs, the relations obtained in the absence of spectral flow are still valid. For the particular times t_i, for which a zero mode exists, the above description is only correct as a limiting one. In fact, to compute the fermion number at exactly those times when zero modes are present, one has to make a special consideration, as it will be shown below.

Still considering a trivial comparison configuration, the final ground state fermion number in the case of nonvanishing spectral flow contributions becomes:

$$Q_{GS}(t \to \infty) = -\frac{1}{2}\left(\eta^c_{[H]} + 2\,(n_+ - n_-)\right) . \tag{2.25}$$

Since the continuous part of the spectral asymmetry is unchanged by the spectral flow contribution, it is still given by the adiabatic evaluation as in Eq. (2.19). We can use Eq. (2.23) and rewrite the ground state fermion number finally as,

$$Q_{GS}(t \to \infty) = -\int_{-\infty}^{\infty} dt \oint d\vec{S}^i \cdot \vec{j}_i + \int d^4x\, \partial^\mu j_\mu - (n_+ - n_-) \tag{2.26}$$

The above formula is the version, in terms of the adiabatic technique, of the generalization to open space of the Atiyah-Patodi-Singer index theorem [31] derived by Niemi and Semenoff [5] to compute the fermion number in the presence of a general arbitrary Dirac Hamiltonian.

The adiabatic charge is a dynamic value induced through polarization effects in the Dirac sea during the evolving process of the background configuration. It is associate only with the asymptotic properties of the current and its contribution to the ground state charge must then be invariant under local deformations of the background fields. From now on we call the adiabatic charge, in general, the induced charge, $Q_{ind.}$. As we will show extensively in the next chapters, this identification, $Q_{ad.} = Q_{ind.}$, is correct in all the cases where the adiabatic current has a good behaviour at any point during the adiabatic process. However it may happen that an apparently harmless ill definition in the current expression at one point destroys the power of the adiabatic method to determine the correct induced charge value. In these cases we will have the adiabatic result, $Q_{ad.}$, due to the wrong extrapolation of the adiabatic procedure, and, of course, the true induced charge, $Q_{ind.}$. We write then, in general, from Eqs. (2.19), (2.25),

$$Q_{GS} = Q_{ind.} - (n_+ - n_-) , \tag{2.27}$$

a relation to be used in the following chapters.

In all the above computations, we have assumed that the Hamiltonian $H \equiv lim_{t\to\infty} H_t$ does not have any zero modes. However, suppose we want to evaluate the ground state charge at a time t_i in which $H_{t=t_i} \equiv H_i$ has zero modes. This implies that the ground state charge is degenerate and an expression for it may be deduced as follows. We consider initially the Dirac sea with the trivial vacuum as background configuration. The positive and negative continua are related to one another by charge conjugation, thus, the spectral asymmetry vanishes,

$$\eta_{[H_0]} = N_{0_{E+}} - N_{0_{E-}} = 0 \qquad (2.28)$$

with $N_{0_{E\pm}}$ being, schematically, the number of energy states in the trivial background, with positive and negative energy, respectively. The total number of energy states of this spectrum can be written as

$$N_0 = N_{0_{E+}} + N_{0_{E-}} = 2N_{0_{E-}} \qquad (2.29)$$

At any time t_i, we can always write the spectral asymmetry as

$$\eta_{[H_i]} = N_{i_{E+}} - N_{i_{E-}} \qquad (2.30)$$

and the number of energy states will be given by

$$N_i = N_{i_{E+}} + N_{i_{E-}} + dimKer(H_i)\,, \qquad (2.31)$$

with $dimKer(H_i)$ being the number of zero modes of the operator H_i; that is, the number of zero energy modes, $N_{i_{E=0}}$. One can always write $N_{i_{E=0}}$ as a sum of the number of occupied zero energy modes plus the number of the empty ones. Then,

$$dimKer(H_i) = N^{occ.}_{i_{E=0}} + N^{emp.}_{i_{E=0}} \qquad (2.32)$$

Since the total number of states in either energy spectrum is the same - the two sets of states are complete - this leads to:

$$N_0 \;=\; N_i \qquad (2.33)$$

Using Eqs. (2.29), (2.31) and (2.32), the above relation of completness may be written as,

$$2N_{0_{E-}} \;=\; N_{i_{E-}} + N_{i_{E+}} + N^{occ.}_{i_{E=0}} + N^{emp.}_{i_{E=0}} \qquad (2.34)$$

or in terms of the final spectral asymmetry,

$$2N_{0_{E-}} \;=\; 2N_{i_{E-}} + \eta_{[H_i]} + N^{occ.}_{i_{E=0}} + N^{emp.}_{i_{E=0}} \qquad (2.35)$$

We want to determine a expression for the ground state charge in terms of manageable quantities. We already know that, if no zero energy mode is present, the ground state charge is given, by definition, by filling all the negative energy states and then evaluating this result relative to the normal vacuum ground state charge. Thus,

$$Q_{GS}(t) = N_{t_{E-}} - N_{0_{E-}}\,, \qquad (2.36)$$

otherwise,

$$Q_{GS}(t_i) = N_{i_{E-}} - N_{0_{E-}} + N^{occ.}_{i_{E=0}} \qquad (2.37)$$

From the above equations (2.35), (2.37), we finally have

$$Q_{GS}(t_i) \;=\; -\frac{1}{2}\eta_{[H_i]} - \frac{1}{2}\left(N^{emp.}_{i_{E=0}} - N^{occ.}_{i_{E=0}}\right) \qquad (2.38)$$

or

$$Q_{GS}(t_i) \;=\; -\frac{1}{2}\left(\eta_{[H_i]} + dimKer(H_i) - 2N^{occ.}_{i_{E=0}}\right)\,. \qquad (2.39)$$

In the case where all the zero energy modes are empty, one can observe that the existence of zero energy modes means to replace $\eta_{[H_i]} \rightarrow \eta_{[H_i]} + dimKer(H_i)$. For practical reasons Eq.(2.38) is the one we are going to use below.

Chapter 3

Adiabatic Charge from Scalar Fields : Physical Interpretation

In this chapter we consider several bosonic fields as background configurations for the fermion fields and we discuss the vacuum polarization effects. In section 2.1 we use the adiabatic method to analyse the fermionic charge induced by a general scalar field, which interpolates between the ordinary vacuum and a final soliton, but which is somewhere vanishing. In section 2.2 we evaluate the adiabatic fermion charge induced by the scalar field associated with t'Hooft instanton. After making a careful study of the energy level crossings and observing the existence of a symmetry in the energy states, we will be able to obtain the correct fermion charge induced by such field and to confront this result with the one obtained employing naively the adiabatic method. The same technique will then be used, in section 2.3, in the case in which also gauge fields are present. We will show that, proceeding in this way the result of the induced fermion charge is in agreement with the general statement in Ref. [19]. Finally, in section 2.4, we particularize our analysis to the sphaleron solution due to its relevance in connecting topologically distinct vacuum states near the weak phase transition temperature. We restrict our work to the limit where θ_W vanishes and the U(1) field decouples and we show, within this limit, that the sphaleron fermionic charge is 1/2.

3.1 Ill-definition of the Adiabatic Current

Treating the scalar fields as background fields for the fermions, we write first, as in the Weinberg Salam model, an SU(2) ⊗ U(1) invariant Yukawa interaction, considering an SU(2) doublet fermion $\psi_L = \begin{pmatrix} \psi_L^{(1)} \\ \psi_L^{(2)} \end{pmatrix}$ and two SU(2) singlets $\psi_R^{(1)}$, $\psi_R^{(2)}$,

$$\mathcal{L}_F = i\bar{\psi}_L\partial^\mu\gamma_\mu\psi_L + i\bar{\psi}_R^{(1)}\partial^\mu\gamma_\mu\psi_R^{(1)} + i\bar{\psi}_R^{(2)}\partial^\mu\gamma_\mu\psi_R^{(2)} - \left(g_{y_1}\bar{\psi}_L^\alpha\varphi^\alpha\psi_R^{(2)} + g_{y_2}\bar{\psi}_L^\alpha\varphi^{*\beta}\epsilon_{\alpha\beta}\psi_R^{(1)} - h.c.\right) \quad (3.1)$$

with φ a complex doublet, which may be written in terms of the quartet of scalar fields of the σ model as:

$$\varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \phi_2 + i\phi_1 \\ \phi_0 - i\phi_3 \end{pmatrix}. \quad (3.2)$$

In the above Lagrangian we have considered different Yukawa couplings, g_{y_1}, g_{y_2}, so that we have no chiral symmetry. However, by considering a unique coupling constant g_y and writing the φ doublet in terms of the matrix Φ defined in Eq. (2.5), or equivalently, of the scalar quartet ϕ_0, ϕ_i,

$$\Phi = \frac{1}{\sqrt{2}}(\phi_0 + i\vec{\phi}\cdot\vec{\sigma}) = \begin{pmatrix} \varphi_2^* & \varphi_1 \\ -\varphi_1^* & \varphi_2 \end{pmatrix}, \quad (3.3)$$

so that $\hat{\Phi} = \Phi/|\varphi|$, we recover the SU(2) ⊗ SU(2) invariant Lagrangian of Eq. (2.6), or equivalently, of Eq. (2.4). Using the adiabatic method, one finds, from the calculations done in the above section, the following expression for the current induced by the background fields

$$< j^\mu(x) > = \frac{\epsilon_{dabc}\epsilon^{\mu\alpha\beta\gamma}\phi_d\partial_\alpha\phi_a\partial_\beta\phi_b\partial_\gamma\phi_c}{12\,\pi^2\,|\phi|^4}. \quad (3.4)$$

It is shown in [9] that, if one considers different values for the fermions masses, $g_{y_1} \neq g_{y_2}$, the following factor appears in the above expression,

$$F = \frac{1}{2}\int_0^1 du\,\left\{1 + \frac{g_{y_1}^2 g_{y_2}^2}{\left[\left(g_{y_1}^2 - g_{y_2}^2\right)u + g_{y_2}^2\right]^2}\right\} \quad (3.5)$$

It is easy to see that, for any non zero value of the coupling constants g_{y_i}, this integral is one; leaving unchanged the adiabatic current result from Eq. (3.4). In the special case that g_{y_1} or $g_{y_2} = 0$, the integral gives 1/2, and would imply an unexpected result for the adiabatic charge. This result might correspond to an interesting case physically, since neutrinos may be massless. However, some arguments are given in Ref. [9] against the validity of this limit. We are not interested to discuss here the presumably illegitimacy of the $g_{y_i} = 0$ limit. So, we restrict our analysis to the case of a unique Yukawa coupling, which may be trivially extended to the case of different, non zero mass values, since no extra factor for the current must be considered.

As we already said, in a solitonic background, the charge associated with the adiabatic current is the topological charge of the scalar configuration. If $|\phi|$ vanishes at some value of r, as may happen in the linear model, then the current in Eq. (3.4) is ill defined at that point and considering this somewhere vanishing field as a static configuration of a time evolving background, such time dependent background may change its topological charge there.

We are going now to compute the charge induced by a soliton configuration evaluating the flow of current as the scalar fields slowly evolve from the vacuum to a final soliton [8],[9]. This final configuration may be, for example, the Skyrmion Ansatz with winding number one

$$(\phi_0, \vec{\phi})_{Sk.} = v(f_1(r), f_2(r)\frac{\vec{r}}{r}) \quad (3.6)$$

where f_1 and f_2 are illustrated in Fig. 4.

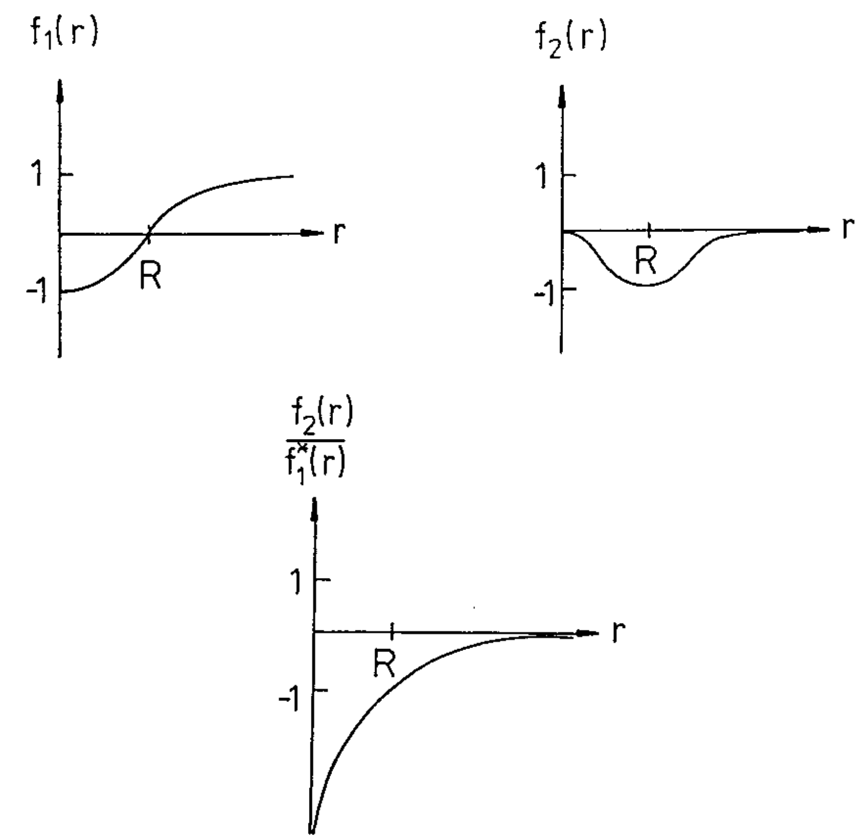


Figure 4: The Skyrmion Ansatz functions $f_1(r)$, $f_2(r)$, and $f_2(r)/f_1^*(r)$, with $f_1^* = 1 + f_1$ (to be used below).

Consider the fields given by:

$$(\phi_0, \vec{\phi}) = v\left\{[1 - h(t)(1 - f_1(r))]\ ,\ h(t) f_2(r)\frac{\vec{r}}{r}\right\}\ , \tag{3.7}$$

where h(t) is a function which varies slowly and monotonically from 0 to 1. During this process, the background configuration is at the vacuum value at spatial infinity. The change in the charge may be written as:

$$Q_{ad.sk.} = \Delta Q = \int_{-\infty}^{\infty} dt \partial_t \int d^3x\, j^0(r,t). \tag{3.8}$$

In evaluating this expression one must, of course, be careful at the point where $|\phi| = 0$, because j_0 is ill defined there. This calculation has been already done by Mackenzie and Wilczek [8]. In order to apply the adiabatic method they have excluded from the configuration space a sphere surrounding the origin. A flux appears through this outward surface giving

the charge changing from zero to one. Explicitly, invoking current conservation, they write Eq. (3.8), as in the last chapter, in terms of a surface integral, where the relevant surface is a small sphere at the origin:

$$Q_{ad.sk.} = \int d^4x(\partial_\mu j^\mu - \vec{\nabla}\cdot\vec{j}) = \int_{-\infty}^{\infty} dt \oint_{S0} d\vec{S}\cdot\vec{j} \tag{3.9}$$

With the fields expression from Eq. (3.7) the current is

$$j_k = -\frac{h^2 f_2^3\, \dot{h}\, r_k}{2\pi^2 r^3 \left\{[1 - h(1 - f_1)]^2 + h^2 f_2^2\right\}^2} \equiv j(r)\frac{r_k}{r} \tag{3.10}$$

For the surface S_0 of infinitesimal radius $r = \epsilon$, with $f_1 = -1 + \epsilon^2$ and $f_2 = -\epsilon$, the integral from Eq. (3.9) turns to be,

$$\begin{aligned} Q_{ad.sk.} &= \int_{-\infty}^{\infty} dt\, 4\pi\epsilon^2 j(\epsilon) \\ &= \frac{2}{\pi}\int_{-\infty}^{\infty} dt \frac{\dot{h}\, h^2 \epsilon^3}{\left\{[1 - h(2 - \epsilon^2)]^2 + h^2\epsilon^2\right\}^2} \\ &= \frac{2}{\pi}\int_0^1 dh \frac{h^2\epsilon^3}{\left\{[1 - h(2 - \epsilon^2)]^2 + h^2\epsilon^2\right\}^2} \end{aligned} \tag{3.11}$$

and after some work it gives $Q_{ad.sk.} = 1$. Thus, in Ref. [8], they obtain the value one for the adiabatic charge induced while reaching the final soliton configuration. Even though the exclusion of an infinitesimal circle centered at the origin, based on the sufficiently mild divergence of the charge density, is mathematically correct, it has, as we will show below, unexpected consequences for the interpretation of the above result.

The adiabatic charge value at any fixed time t may be also calculated by integrating the charge density of the corresponding static background. From Eqs. (3.4), (3.7), j_0 is initially given by

$$j^0(r,t) = \frac{1}{2\pi^2 r^2}\frac{h^2 f_2^2\{[1 - h(1 - f_1)]\, h f_2' - h^2 f_2 f_1'\}}{\left\{[1 - h(1 - f_1)]^2 + h^2 f_2^2\right\}^2} \tag{3.12}$$

First of all we are going to examine the charge density at a fixed time t so that $h(t) = \frac{1}{2} + \alpha$, with infinitesimal α and at infinitesimal $r = \epsilon$, setting, once more, f_1 and f_2 as given by their asymptotic behaviour. The above leads to

$$\epsilon^2 j^0 = \frac{\epsilon^2}{2\pi^2}\frac{(\alpha/4 + \epsilon^2/16)}{(4\alpha^2 + \epsilon^2/4)^2} \tag{3.13}$$

Eq.(3.13) implies that $\epsilon^2 j^0$ behaves as ϵ^2/α^3 for $\epsilon \ll |\alpha|$. On the other hand, if we set $h = \frac{1}{2}$ we get the finite value $\epsilon^2 j^0 = 1/2\pi^2$. We evaluate now the charge induced in the background of the scalar configuration, Eq. (3.7), at $h(t = 0) = \frac{1}{2}$. The scalar quartet reads,

$$\left(\phi_0, \vec{\phi}\right) = v\left(\frac{(1 + f_1)}{2}\ ,\ \frac{f_2}{2}\frac{\vec{r}}{r}\right)\ ; \tag{3.14}$$

and the charge density from Eq. (3.12) becomes,

$$j_0(r,0) = \frac{1}{2\pi^2 r^2} \frac{f_2^2\left[(1+f_1)\, f_2' - f_2 f_1'\right]}{\left[(1+f_1)^2 + f_2^2\right]^2} \, . \tag{3.15}$$

We write then the adiabatic charge as

$$\begin{aligned} Q_{ad.}(t=0) &= \int d^3x\, j^0(r,0) \\ &= \frac{2}{\pi}\int_0^\infty dr\, \frac{f_2^2\left(f_1^* f_2' - f_2 f_1^{*'}\right)}{\left(f_1^{*^2} + f_2^2\right)^2} \, , \end{aligned} \tag{3.16}$$

with $1 + f_1 = f_1^*$. We simplified the above integral by replacing $f_1^* f_2' - f_2 f_1^{*'} = f_1^{*^2}\,(f_2/f_1^*)'$. This equivalence is not valid at r = 0 where f_1^* vanishes (see Fig. 4), however since $r^2 j_0$ has a finite value there, we can still use the above simplification by considering the integral

$$Q_{ad.}(t=0) = \frac{2}{\pi}\int_{0+\xi}^\infty dr\, \frac{f_2^2 f_1^{*^2}\,(f_2/f_1^*)'}{\left(f_1^{*^2} + f_2^2\right)^2} \, , \tag{3.17}$$

with infinitesimal ξ and then taking the limit $\xi \to 0$. Defining $f_2/f_1^* \equiv y$ we obtain

$$Q_{ad.}(t=0) \;=\; \frac{2}{\pi}\int_{-\infty}^0 dy\, \frac{y^2}{1+y^2} = \frac{1}{2} \, . \tag{3.18}$$

Before proceeding further, it is worthwhile to observe that the scalar configuration we have just considered in Eq. (3.14), with the general expressions for the radial functions, may be taken as the scalar field of the sphaleron solution evaluated in an appropiate gauge. Postponing additional considerations to the end of this chapter, we write the sphaleron scalar configuration explicitly as,

$$\begin{aligned} \phi_{0_{sph.}} &= v\, h(gvr)\, \sin\frac{\theta(r)}{2} \\ \phi_{i_{sph.}} &= -v\, h(gvr)\, \cos\frac{\theta(r)}{2}\, \frac{r_i}{r} \end{aligned} \tag{3.19}$$

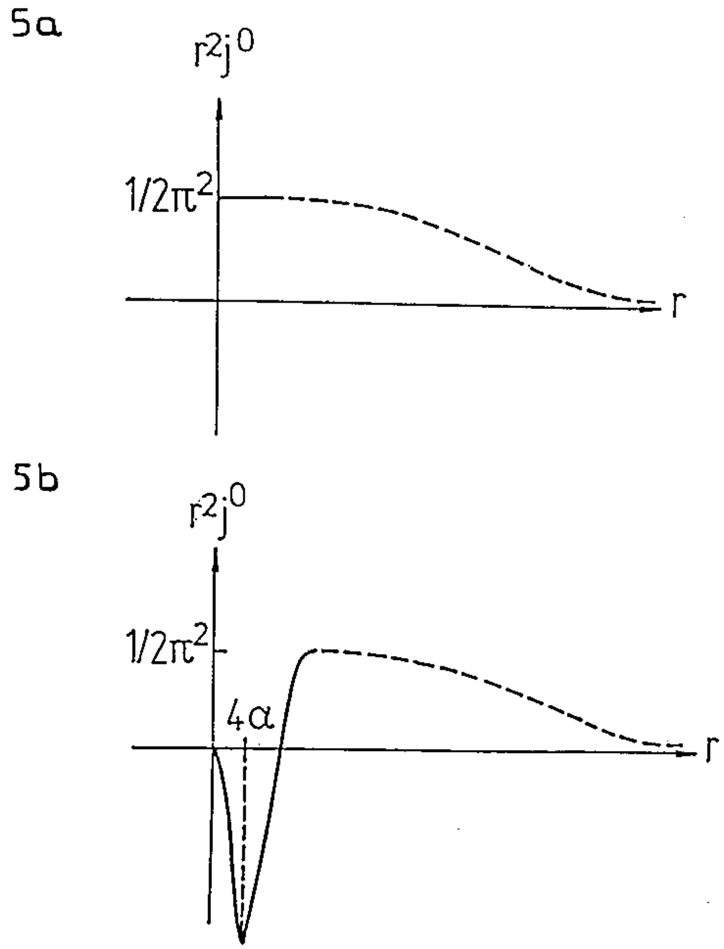
with $h(gvr) \to 0$ as $r \to 0$ and $h(gvr) \to 1$ as $r \to \infty$, and $\theta(r) \to [0, \pi]$ as r runs from 0 to ∞. Thus, it is plausible to identify

$$\begin{aligned} h(gvr)\sin\frac{\theta(r)}{2} &\equiv \frac{1+f_1(r)}{2} \\ -h(gvr)\cos\frac{\theta(r)}{2} &\equiv \frac{f_2(r)}{2} \, . \end{aligned} \tag{3.20}$$

This means that the charge value $Q_{ad.}(t=0) = 1/2$ would give, if physically correct, a 1/2 contribution to the sphaleron induced fermion number coming from the scalar fields. This

extra contribution would change the one half fermion number associated to the sphaleron, in the literature [18], to an integral value!. We are going to return to this item below, after having understood the correct physical interpretation of the adiabatic results we are obtaining in Eqs. (3.11) and (3.18).

We continue now with our previous analysis for the final Skyrmion background. We know that for the vacuum configuration we have $Q_{ad}(t \to -\infty) = 0$. The value of the adiabatic charge for the Skyrmion can be evaluated in a similar way as in Eq. (3.16), but for $j_0(r,\infty)$, and one finds that $Q_{ad.}(t \to \infty) = 1$. All the time before t=0 the charge density is well defined at every point, so the charge must remain unchanged and equal zero. In the same way one can state that the charge must be 1 for t>0. At t=0 the charge is changing. Analysing Eq.(3.13) one can visualize the charge distribution as in Fig. 5.



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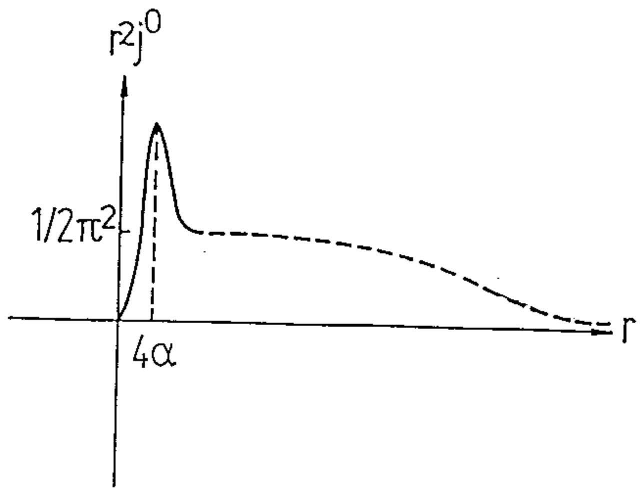


Figure 5: Charge density as a function of r, at times near t = 0. a) Setting $h(t=0) = 1/2$, the total charge is 1/2. Since it goes like a constant for r near the origin (filled curve), then it must decrease to give the calculated finite value (dashed curve). b) Setting $h = 1/2 + \alpha$ with $\alpha < 0$ there is a negative contribution for r near the origin, whose width is proportional to $\sqrt{\alpha}$ and its deepness to $1/\alpha$. Its area must be 1/2 to compensate the positive 1/2 contribution and give the known zero value for the total charge. c) Setting $h = 1/2 + \alpha$ with $\alpha > 0$, the negative contribution at small r turns into a positive one to give the total charge one.

From the above calculation it is pretty clear that to exclude or not the origin is irrelevant for the final mathematical result of the adiabatic charge. It is interesting however to ask what happens if one extrapolates naively the adiabatic method to evaluate the charge in the spirit of Eq. (3.9), but includes the origin? For this purpose we analyse carefully the topological current gotten, for the scalar fields we are working with, near that point [12]. With j^0, $\vec{j}$ as given in Eqs. (3.12), (3.10), respectively, we make a Taylor expansion at t=0, $h(t) = 1/2 + \dot{h}(t=0)\, t + \ldots$, and replace again the radial functions f_1 and f_2 by their asymptotic expressions at small r. Then, near the origin, we have

$$
\begin{aligned}
j^k(r,t) &= \frac{r^k}{r^3}\frac{\dot{h}(0)r^3}{2\pi^2}\frac{\left(1/2+\dot{h}(0)t\right)^2}{\left\{\left[1-\left(\frac{1}{2}+\dot{h}(0)t\right)(2-r^2)\right]^2+r^2\left(\frac{1}{2}+\dot{h}(0)t\right)^2\right\}^2} \\
&= \frac{r^k}{r^3}\frac{\dot{h}(0)r^3}{8\pi^2}\frac{1}{\left(4\dot{h}(0)^2t^2+r^2/4\right)^2} \\
&\equiv \frac{r^k}{r^3}g(r,t). \qquad (3.21)
\end{aligned}
$$

We already obtained a similar expression for j_0, which we rewrite here as

$$j_0 = \frac{1}{8\pi^2}\frac{\dot{h}(0)t + r^2/4}{\left(4\dot{h}(0)^2t^2 + r^2/4\right)^2} \qquad (3.22)$$

Evaluating the total divergence of the current

$$\partial_\mu j^\mu = \partial_0 j^0 + \partial_k j^k \qquad (3.23)$$

we have

$$\partial_0 j^0 = \frac{\dot{h}\left(-12\dot{h}^2t^2 + r^2/4\right)^2}{8\pi^2\left(4\dot{h}^2t^2 + r^2/4\right)^3} \qquad (3.24)$$

and

$$
\begin{aligned}
\partial_k j^k &= \partial_k\left(\frac{r^k}{r^3}\right)g(r,t) + \frac{r^k}{r^3}\partial_k(g(r,t)) \\
&= \partial_k\left(\frac{r^k}{r^3}\right)g(r,t) - \partial_0 j^0. \qquad (3.25)
\end{aligned}
$$

Since

$$g(r\to 0,t) = \begin{cases} 0 & t > 0 \\ 0 & t < 0 \\ \infty & t = 0 \end{cases} \qquad (3.26)$$

while the integral

$$I = \int_{-\infty}^{\infty} dt\, g(r,t) = \frac{1}{2\pi^2}\int_{-\infty}^{\infty}\frac{d\left(\dot{h}(0)t\right) r^3/4}{\left[4\left(\dot{h}(0)t\right)^2 + r^2/4\right]^2} \qquad (3.27)$$

after some work gives $I = 1/4\pi$, we observe that g(r,t) behaves like $\delta(t)/4\pi$ as $r \to 0$. Since it is well known that $\vec{\nabla}\cdot\vec{r}/r^3$ behaves as $4\pi\delta(\vec{r})$, one sees that the ill-definition of the current leads to

$$\partial_\mu j^\mu(\vec{r},t) = \delta(\vec{r})\delta(t). \qquad (3.28)$$

Considering this 'anomaly' one obtains the same result for Eq. (3.9), since by dealing now with the complete configuration space there is no contribution from S_0 but there is one from Eq. (3.28). That is, when no flux is allowed at the origin, the non-trivial divergence replace the above surface contribution. However, this 'anomalous' divergence has no physical significance since it appears while extrapolating the adiabatic method beyond its validity. These considerations suggests that, in this example, the adiabatic method gives only the correct induced charge in a configuration space without the origin. We will show below that, to obtain the real induced charge in the system considering the *complete* configuration space, in reality one must drop the anomaly contribution of Eq. (3.28).

3.2 Zero Energy Modes and the True Induced Charge

To proceed with our analysis let us consider a scalar field configuration which vanishes at the origin, but which gives a non-trivial flux at spatial infinity and is slightly more specific than Eq. (3.7). We take [12]:

$$\varphi = \frac{x^\nu \tau_\nu}{r} \frac{r}{\sqrt{r^2+\rho^2}} \frac{v}{\sqrt{2}} \varphi_0 \ , \tag{3.29}$$

with $\tau_\nu = (1, \imath\sigma_a)$ and φ_0 a constant SU(2) spinor

$$\varphi_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3.30}$$

Observe that if we consider r as the four dimensional Euclidean space radial vector, then this configuration is the scalar field associated to the t'Hooft instanton (ρ is the instanton size). In what follows, we consider the temporal coordinate as the parameter t which connects the initial and final configurations while building up adiabatically the final scalar field. As the field in Eq. (3.29) vanishes at $x = 0$, the same considerations about the ill-definition of the adiabatic current apply here also.

As $\varphi_{|t=\infty} = -\varphi|_{t=-\infty} = (v/\sqrt{2})\varphi_0$, it is clear from the abiabatic current expression from Eq. (3.4), that the initial and final configurations have a zero charge value.

$$\triangle\, Q_{ad.} = Q|_{t=\infty} - Q|_{t=-\infty} = 0. \tag{3.31}$$

Invoking current conservation we can rewrite the change of the adiabatic charge in terms of a surface integral:

$$\triangle\, Q_{ad.} = -\int_{-\infty}^{\infty} dt \oint d\vec{S}^i \cdot \vec{j}_i \ , \tag{3.32}$$

where the i index denotes, as before, the sum over all the outward surfaces of the space under consideration. As we have excluded the origin in order to apply the adiabatic method, then the surface given by a small sphere S_0 surrounding this point must be also considered. Eqs. (3.31), (3.32) imply that the net flux through the outward surfaces vanishes, this means:

$$\int_{-\infty}^{\infty} dt \ \left[\oint_{S_0} d\vec{S}\cdot\vec{j} - \oint_{S_\infty} d\vec{S}\cdot\vec{j}\right] = 0 \ . \tag{3.33}$$

If we now want to evaluate the induced charge, *including* the origin in the configuration space, and we insist on current conservation (so that the spurious anomalous contribution is omitted) we can still use Eq. (3.32). As now the only outward surface is that one at infinity, this leads to (see appendix A)

$$Q_{ind.} = -\int_{-\infty}^{\infty} dt \oint_{S_\infty} d\vec{S}\cdot\vec{j} = 1 \ . \tag{3.34}$$

The value unity follows since the above integral is, besides the minus sign, the same as the one that gives, in Euclidean space, the winding number of the scalar field of Eq.(3.29). Note

that the winding number is here defined as the number of times that the field configuration wraps the configuration space S^3 when x varies on the sphere S^3 of R^4 with $x \to \infty$.

In the above analysis we have not paid attention to possible spectral flow contributions. We will now study the zero energy fermion modes to check that indeed the different charge values we have obtained are perfectly consistent [12]. That is to say, the zero induced charge calculated via the adiabatic technique while performing a hole in the configuration space, Eq. (3.31), and the value one for the induced charge after we include the origin in our space, Eq. (3.34), refer to different physical spaces and can be different.

We recall, from the discussion in chapter 2, that after the transition of $n_+(n_-)$ levels from $E<0$ ($E>0$) to $E>0$ ($E<0$) the system is left in a state which is no longer the ground state and then the charge expectation value in this state, $Q_{ind.}$, is related to the ground state charge as, $Q_{ind.} = Q_{GS} + n_+ - n_-$. The final scalar configuration of Eq. (3.29) at t=∞ is the trivial one, so the final ground state charge is zero and this fact is independent of the way one arrives to the final scalar field. From the above, we conclude that, if the value one for the induced charge is correct, then an energy level crossing must occur at some point and an occupied zero energy fermion state must be found there.

We consider the eigenvalue equations

$$\begin{aligned}
\imath\sigma_i\partial_i\psi_L(\vec{x},t) - g_y\tilde{\varphi}(\vec{x},t)\psi_R^{(1)}(\vec{x},t) - g_y\varphi(\vec{x},t)\psi_R^{(2)}(\vec{x},t) &= -E\psi_L(\vec{x},t) \\
-\imath\sigma_i\partial_i\psi_R^{(1)}(\vec{x},t) - g_y\tilde{\varphi}^+(\vec{x},t)\psi_L(\vec{x},t) &= -E\psi_R^{(1)}(\vec{x},t) \\
-\imath\sigma_i\partial_i\psi_R^{(2)}(\vec{x},t) - g_y\varphi^+(\vec{x},t)\psi_L(\vec{x},t) &= -E\psi_R^{(2)}(\vec{x},t)
\end{aligned} \tag{3.35}$$

where ψ_L , $\psi_R^{(1,2)}$ are Lorentz doublets and $\tilde{\varphi} = \imath\sigma_2\varphi^*$. We set E = 0, take the background field $\varphi(\vec{x}, t=0)$ and look for a normalizable solution to the system of time independent equations.

$$\imath\sigma_i\partial_i\psi_L(\vec{x}) - g_y\tilde{\varphi}(\vec{x})\psi_R^{(1)}(\vec{x}) - g_y\varphi(\vec{x})\psi_R^{(2)}(\vec{x}) = 0 \tag{3.36}$$

$$-\imath\sigma_i\partial_i\psi_R^{(1)}(\vec{x}) - g_y\tilde{\varphi}^+(\vec{x})\psi_L(\vec{x}) = 0 \tag{3.37}$$

$$-\imath\sigma_i\partial_i\psi_R^{(2)}(\vec{x}) - g_y\varphi^+(\vec{x})\psi_L(\vec{x}) = 0 \tag{3.38}$$

The above zero energy Dirac equation is the one studied in Ref. [32], in the limit of neglecting the vector fields. Analogously to Ref. [33] we choose the Ansatz:

$$\begin{aligned}
\psi_{L_{n\alpha}} &= \imath\sigma_{2_{n\alpha}} f(r) \\
\psi_{R_\alpha}^{(1)} &= \delta_{n\alpha}\varphi_{0_n} g(r) \\
\psi_{R_\alpha}^{(2)} &= -\imath\sigma_{2_{\alpha n}}\varphi_{0_n}^* g(r) .
\end{aligned} \tag{3.39}$$

where $\alpha = 1,2$ and n= 1,2 are Lorentz and weak isospin indices, respectively. Explicitly we rewrite the above Ansatz as

$$\psi_L = f(r) \begin{pmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ \begin{bmatrix} -1 \\ 0 \end{bmatrix} \end{pmatrix}$$

$$\psi_R^{(1)} = g(r) \begin{bmatrix} \varphi_{01} \\ \varphi_{02} \end{bmatrix}$$

$$\psi_R^{(2)} = g(r) \begin{bmatrix} -\varphi_{02}^* \\ \varphi_{01}^* \end{bmatrix} \tag{3.40}$$

where we have used square brackets and parentheses to identify the Lorentz and weak isospin doublets, respectively. From now on we define $\mathcal{T}_i$ and $\mathcal{S}_i$ as the Pauli matrices σ_i, but applying to different spaces, weak isospin and Dirac spaces, respectively. We write also explicitly the expressions of φ, φ^+, $\tilde{\varphi}$ and $\tilde{\varphi}^+$ to be used below

$$\varphi = \frac{v}{\sqrt{2}} \frac{\imath\vec{x}\cdot\vec{\mathcal{T}}}{\sqrt{r^2+\rho^2}} \begin{pmatrix} \varphi_{01} \\ \varphi_{02} \end{pmatrix}$$

$$\tilde{\varphi} = \frac{v}{\sqrt{2}} \frac{\imath\vec{x}\cdot\vec{\mathcal{T}}}{\sqrt{r^2+\rho^2}} \begin{pmatrix} \varphi_{02}^* \\ -\varphi_{01}^* \end{pmatrix}$$

$$\varphi^+ = \frac{-v}{\sqrt{2}} (\varphi_{01}^* \quad \varphi_{02}^*) \frac{\imath\vec{x}\cdot\vec{\mathcal{T}}}{\sqrt{r^2+\rho^2}}$$

$$\tilde{\varphi}^+ = \frac{-v}{\sqrt{2}} (\varphi_{02} \quad -\varphi_{01}) \frac{\imath\vec{x}\cdot\vec{\mathcal{T}}}{\sqrt{r^2+\rho^2}} \tag{3.41}$$

Using the definition of φ_0 as given in Eq. (3.30) the set of time independent equations Eqs. (3.36), (3.37) and (3.38) transforms as follows. For Eq. (3.36) one has,

$$\imath\partial_i f(r)\mathcal{S}_i \begin{pmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ \begin{bmatrix} -1 \\ 0 \end{bmatrix} \end{pmatrix} - m_f \frac{\imath x^i \mathcal{T}_i}{\sqrt{r^2+\rho^2}} \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} g(r) = 0 \tag{3.42}$$

Since $r\partial_i f(r) = x^i \partial_r f(r)$ and

$$\mathcal{T}_i \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} = \mathcal{T}_i \begin{pmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ \begin{bmatrix} -1 \\ 0 \end{bmatrix} \end{pmatrix} = -\mathcal{S}_i \begin{pmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ \begin{bmatrix} -1 \\ 0 \end{bmatrix} \end{pmatrix} \tag{3.43}$$

then Eq. (3.42) becomes

$$\imath\vec{x}\cdot\vec{\mathcal{S}} \begin{pmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ \begin{bmatrix} -1 \\ 0 \end{bmatrix} \end{pmatrix} \left(\frac{1}{r}\partial_r f(r) + m_f \frac{g(r)}{\sqrt{r^2+\rho^2}} \right) = 0 \tag{3.44}$$

Proceeding in the same way with Eq. (3.37)

$$-\imath\vec{x}\cdot\vec{\mathcal{S}}\frac{1}{r}\partial_r g(r) \begin{bmatrix} 0 \\ 1 \end{bmatrix} - m_f (1 \quad 0) \frac{-\imath\vec{x}\cdot\vec{\mathcal{T}}}{\sqrt{r^2+\rho^2}} \begin{pmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ \begin{bmatrix} -1 \\ 0 \end{bmatrix} \end{pmatrix} f(r) = 0 \tag{3.45}$$

leads to

$$-\imath\vec{x}\cdot\vec{\mathcal{S}} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \left(\frac{1}{r}\partial_r g(r) + m_f \frac{f(r)}{\sqrt{r^2+\rho^2}} \right) = 0 \tag{3.46}$$

Finally working out Eq. (3.38) in a similar fashion one obtains Eq. (3.46), but with the replacement of $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ by $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$. The preceding calculations have reduced Eqs. (3.36), (3.37) and (3.38) to :

$$\begin{aligned} \partial_r g(r) + m_f \frac{r}{\sqrt{r^2+\rho^2}} f(r) &= 0 \\ \partial_r f(r) + m_f \frac{r}{\sqrt{r^2+\rho^2}} g(r) &= 0 \end{aligned} \tag{3.47}$$

where $m_f = g_y v/\sqrt{2}$ is the fermion mass. It is easy to check that a normalizable solution to these equations exists and is:

$$g(r) = f(r) = exp(-m_f\sqrt{r^2+\rho^2}). \tag{3.48}$$

The scalar field of Eq.(3.29) obeys the relation

$$\varphi(\vec{x},t) = -\varphi(-\vec{x},-t)\ . \tag{3.49}$$

Using the above, the eigenvalue equations, Eq.(3.35), after a redefinition of $\vec{x} \to -\vec{x}$, $t \to -t$, may be rewritten as follows

$$\begin{aligned} -\imath\sigma_i\partial_i\psi_L(-\vec{x},-t) + g_y\tilde{\varphi}(\vec{x},t)\psi_R^{(1)}(-\vec{x},-t) + g_y\varphi(\vec{x},t)\psi_R^{(2)}(-\vec{x},-t) &= -E\psi_L(-\vec{x},-t) \\ \imath\sigma_i\partial_i\psi_R^{(1)}(-\vec{x},-t) + g_y\tilde{\varphi}^+(\vec{x},t)\psi_L(-\vec{x},-t) &= -E\psi_R^{(1)}(-\vec{x},-t) \\ \imath\sigma_i\partial_i\psi_R^{(2)}(-\vec{x},-t) + g_y\varphi^+(\vec{x},t)\psi_L(-\vec{x},-t) &= -E\psi_R^{(2)}(-\vec{x},-t) \end{aligned} \tag{3.50}$$

Thus, one finds that for each solution: $\psi_L(\vec{x},t)$, $\psi_R^{(1,2)}(\vec{x},t)$ of energy E at time t, there is a solution: $\psi_L(-\vec{x},-t)$, $\psi_R^{(1,2)}(-\vec{x},-t)$ of energy -E at time -t. At t=0 there exists a symmetry in the Hamiltonian in the background field, which gives a one to one correspondence between states of positive and negative energy. Thus the spectral asymmetry vanishes and, provided no energy level crossing occurs before, the charge is given by one-half the difference between the occupied and empty zero energy fermion states. This is implied by the relation deduced in chapter 2, $Q_{ind.}(t) = -\frac{1}{2}\eta_{[H_t]} + \frac{1}{2}(N^{occ.}_{t_{E=0}} - N^{emp.}_{t_{E=0}}) + n_+ - n_-$. These considerations and the existence of the zero energy mode, Eq. (3.48), imply that at t=0 the charge must be $+\frac{1}{2}$ or $-\frac{1}{2}$. On the other hand, evaluating the induced charge at t=0, with the prescription of ignoring the anomaly contribution for the adiabatic current, one has $Q_{ind.} = \frac{1}{2}$. This result emerges clearly from Eq. (3.34) after using the relation $\vec{j}(\vec{x},t) = \vec{j}(\vec{x},-t)$ (see appendix A), which implies that $Q_{ind.}|_{t=0} = \frac{1}{2}Q_{ind.}|_{t=\infty}$.

The above discussion is gratifying since it reconfirms the value one for the induced charge to be the right one and it connects it with the zero energy mode found. Next we must understand the zero adiabatic result. We already said the difference is due to the fact of making a hole in the configuration space, but what does this really mean?. Since the ground state charge depends only on the static configuration and as the zero energy mode of Eq. (3.48) is always present, then if the adiabatic result gives the induced charge in a slightly different space, it must be that an extra empty zero energy mode must appear in such space. In order to find another solution to Eqs.(3.36), (3.37) and (3.38), we propose the following Ansatz for the fermion fields:

$$\begin{aligned}
\psi_{L_{n\alpha}} &= \imath\vec{x}\cdot\vec{T}\, p(r)\, (\imath\sigma_{2_{n\alpha}}) \\
\psi^{(1)}_{R_\alpha} &= \imath\vec{x}\cdot\vec{S}\, q(r)\, \delta_{n\alpha}\varphi_{0_n} . \\
\psi^{(2)}_{R_\alpha} &= \imath\vec{x}\cdot\vec{S}\, q(r)\, (-\imath\sigma_{2_{\alpha n}}\varphi^*_{0_n})
\end{aligned} \tag{3.51}$$

Replacing the above in the set of equations we are working with and using, once more, the relation $\mathcal{T}_i\sigma_{2_{n\alpha}} = -\mathcal{S}_i\sigma_{2_{n\alpha}}$ we have, for Eq. (3.36),

$$\begin{aligned}
\imath\mathcal{S}_i\partial_i\left[\imath x^j\mathcal{T}_j p(r)\right]\imath\sigma_{2_{n\alpha}} - m_f\frac{\imath\vec{x}\cdot\vec{\mathcal{T}}}{\sqrt{r^2+\rho^2}}\imath\vec{x}\cdot\vec{\mathcal{S}}q(r)\imath\sigma_{2_{n\alpha}} &= 0 \\
\left\{\mathcal{S}_i\mathcal{S}_j\left[(\partial_i x^j)p(r) + x^i x^j\frac{1}{r}\partial_r p(r)\right] - m_f\frac{q(r)}{\sqrt{r^2+\rho^2}}\left(\vec{x}\cdot\vec{\mathcal{T}}\right)^2\right\}\imath\sigma_{2_{n\alpha}} &= 0
\end{aligned} \tag{3.52}$$

For Eq.(3.37),

$$\begin{aligned}
-\imath\mathcal{S}_i\partial_i\left[\imath x^j\mathcal{S}_j q(r)\right]\begin{bmatrix}0\\1\end{bmatrix} - m_f\frac{1}{\sqrt{r^2+\rho^2}}(1\ \ 0)(-\imath)\vec{x}\cdot\vec{\mathcal{T}}\imath\vec{x}\cdot\vec{\mathcal{T}}p(r)\imath\sigma_{2_{n\alpha}} &= 0 \\
\left\{\mathcal{S}_i\mathcal{S}_j\left[(\partial_i x^j)p(r) + \frac{x^i x^j}{r}\partial_r q(r)\right] - m_f\frac{p(r)}{\sqrt{r^2+\rho^2}}\left(\vec{x}\cdot\vec{\mathcal{T}}\right)^2\right\}\begin{bmatrix}0\\1\end{bmatrix} &= 0
\end{aligned} \tag{3.53}$$

Once more, Eq. (3.38) gives the same as Eq. (3.37) after replacing $\begin{bmatrix}0\\1\end{bmatrix}$ by $\begin{bmatrix}-1\\0\end{bmatrix}$. With this Ansatz, the set of equations we finally obtain is the following:

$$\begin{aligned}
r\partial_r q(r) + 3q(r) - m_f\frac{r^2}{\sqrt{r^2+\rho^2}}p(r) &= 0 \\
r\partial_r p(r) + 3p(r) - m_f\frac{r^2}{\sqrt{r^2+\rho^2}}q(r) &= 0 .
\end{aligned} \tag{3.54}$$

It is easy to check that:

$$p(r) = -q(r) = \frac{1}{r^3}\,exp(-m_f\sqrt{r^2+\rho^2}) . \tag{3.55}$$

is a solution to these equations. As expected, the above time independent solution becomes nonnormalizable as soon as the origin is included in the configuration space. But, in a space with the origin removed, assuming this zero energy fermion state is empty, from the above arguments one deduces a zero fermion charge at t=0. This value is the same as the one one can derive from Eqs. (3.32), (3.33) after using, once more, the relation $\vec{j}(\vec{x},t) = \vec{j}(\vec{x},-t)$.

The present calculations show how the modification of the configuration space used in Ref. [8],[9] leads to non-trivial modification of the fermion energy states and to a different induced charge value from the one derived without the insertion of the hole in the configuration space. One can use the adiabatic method to evaluate the real induced charge even in the presence of somewhere vanishing scalars fields, but then to get the correct answer one must ignore the spurious anomaly or , what is the same, the spurious flux at the singularity.

3.3 Including Gauge Fields in the Background Configuration

The adiabatic method may be also applied with the inclusion of gauge fields as background fields. The current expression may be obtained by a similar procedure as in the scalar case. As a result, the current defined before must now be invariantized and it takes the form [7],[10]

$$j^\mu(x) = \frac{1}{24\pi^2}\epsilon^{\mu\nu\alpha\beta}Tr[\hat{\Phi}^+D_\nu\hat{\Phi}\hat{\Phi}^+D_\alpha\hat{\Phi}\hat{\Phi}^+D_\beta\hat{\Phi} + \frac{3}{2}\imath g\hat{\Phi}^+F_{\nu\alpha_L}D_\beta\hat{\Phi}] , \tag{3.56}$$

where $D_\mu = \partial_\mu - \imath gA_{\mu_L}$ with A_{μ_L} the $SU(2)_L$ gauge potential and $F_{\mu\nu_L} = \partial_\mu A_{\nu_L} - \partial_\nu A_{\mu_L} - \imath g[A_{\mu_L}, A_{\nu_L}]$ its field strength. In what follows we omit the L subindex. The second term in the above current expression is necessary if one wants to obtain the usual fermionic anomaly contribution from the gauge fields. This invariant current may be divided into three terms, as follows. The first term in Eq. (3.56) may be explicitly written as:

$$\begin{aligned}
T_1 = \frac{1}{24\pi^2}\epsilon^{\mu\nu\alpha\beta}Tr\Big[&\hat{\Phi}^+\partial_\nu\hat{\Phi}\hat{\Phi}^+\partial_\alpha\hat{\Phi}\hat{\Phi}^+\partial_\beta\hat{\Phi} + \imath g^3A_\nu A_\alpha A_\beta - \\
&-3\imath g\partial_\nu\hat{\Phi}\hat{\Phi}^+\partial_\alpha\hat{\Phi}\hat{\Phi}^+A_\beta - 3g^2\partial_\nu\hat{\Phi}\hat{\Phi}^+A_\alpha A_\beta\Big]
\end{aligned} \tag{3.57}$$

The second term from Eq.(3.56) leads to:

$$T_2 = \frac{1}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}Tr\left[\imath g\hat{\Phi}^+ F_{\nu\alpha}\partial_\beta\hat{\Phi} + g^2 F_{\nu\alpha}A_\beta\right]$$

$$= \frac{1}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}\left(g^2 Tr\left[F_{\nu\alpha}A_\beta\right] + 2g^2 Tr\left[\hat{\Phi}^+ A_\nu A_\alpha\partial_\beta\hat{\Phi}\right] + 2\imath g Tr\left[\hat{\Phi}^+\partial_\nu A_\alpha\partial_\beta\hat{\Phi}\right]\right) \quad (3.58)$$

Using in the above expression the relation

$$\epsilon^{\mu\nu\alpha\beta}Tr\left[\partial_\beta\hat{\Phi}\hat{\Phi}^+\partial_\nu A_\alpha\right] = \epsilon^{\mu\nu\alpha\beta}Tr\left[\partial_\nu\hat{\Phi}\hat{\Phi}^+\partial_\alpha\hat{\Phi}\hat{\Phi}^+A_\beta + \partial_\nu\left(\partial_\beta\hat{\Phi}\hat{\Phi}^+A_\alpha\right) - \left(\partial_\nu\partial_\beta\hat{\Phi}\right)\Phi^+A_\alpha\right], \quad (3.59)$$

the current finally becomes

$$j^\mu(x) = \frac{1}{24\pi^2}\epsilon^{\mu\nu\alpha\beta}Tr\left[\hat{\Phi}^+\partial_\nu\hat{\Phi}\hat{\Phi}^+\partial_\alpha\hat{\Phi}\hat{\Phi}^+\partial_\beta\hat{\Phi}\right] + \frac{\imath g}{8\pi^2}\epsilon^{\mu\nu\alpha\beta}Tr\left[\partial_\nu\left(\hat{\Phi}\partial_\alpha\left(\hat{\Phi}^+A_\beta\right)\right)\right] + \frac{g^2}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}Tr\left[F_{\nu\alpha}A_\beta + \frac{2}{3}\imath g A_\nu A_\alpha A_\beta\right]. \quad (3.60)$$

Let us define

$$j^\mu(x) \equiv j^\mu_{\hat{\Phi}}(x) + j^\mu_{\hat{\Phi}A}(x) + j^\mu_A(x). \quad (3.61)$$

The first term in Eq. (3.60) is the current $j^\mu_{\hat{\Phi}}(x)$ due to the Higgs field, which is the expression given in Eq. (2.15). The second term is also a conserved current, due to the interaction of the Higgs and gauge fields,

$$j^\mu_{\hat{\Phi}A}(x) = \frac{g}{8\pi^2}\epsilon^{\mu\nu\alpha\beta}Tr\left[\imath\partial_\nu\left(\hat{\Phi}\partial_\alpha\left(\hat{\Phi}^+A_\beta\right)\right)\right]. \quad (3.62)$$

Finally, the third term is the gauge non conserved current, whose divergence is the fermionic anomaly:

$$j^\mu_A(x) = \frac{g^2}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}Tr\left[F_{\nu\alpha}A_\beta + \frac{2}{3}\imath g A_\nu A_\alpha A_\beta\right], \quad (3.63)$$

with

$$\partial_\mu j^\mu_A = \frac{g^2}{32\pi^2}\partial_\mu K^\mu = \frac{g^2}{16\pi^2}Tr[F_{\mu\nu}\tilde{F}^{\mu\nu}],$$

where

$$K^\mu = \epsilon^{\mu\nu\alpha\beta}(F^a_{\nu\alpha}A^a_\beta - \frac{1}{3}g\epsilon_{abc}A^a_\nu A^b_\alpha A^c_\beta). \quad (3.64)$$

We will now use the adiabatic current expression from Eq. (3.56) to evaluate the fermionic induced charge in the background of gauge and Higgs fields, which evolve from the vacuum to a topological non-trivial final configuration [12]. The gauge invariant generalization of the adiabatic current shows clearly that the charge value is now independent of the gauge in which we evaluate it. For simplicity let us work in the gauge $A_0 = 0$. We consider that both background fields give no contribution to the current flow at spatial infinity. Starting with

$$A_i(t=-\infty) = 0, \qquad \varphi(t=-\infty) = \frac{v}{\sqrt{2}}\varphi_0, \quad (3.65)$$

the Higgs field develops a twist while vanishing at some point, as the gauge field goes to the final pure gauge configuration. At t=∞ , the field configuration reads

$$A_i(t=\infty) = -\frac{\imath}{g}\partial_i U\ U^+, \qquad \varphi(t=\infty) = \frac{v}{\sqrt{2}}U\varphi_0, \quad (3.66)$$

with $U^+ = 1$ at spatial infinity and of winding number one.

Applying naively Eq. (3.56) one derives a zero induced charge for the final configuration. This is trivially obtained, since the current density vanishes identically at t=∞ as at t=$-\infty$, as $D_i\hat{\Phi} = 0$ in the background of Eqs. (3.65) and (3.66). Then

$$\triangle Q_{ad.} = Q_{ad.}|_{t=\infty} - Q_{ad.}|_{t=-\infty} = 0 \quad (3.67)$$

However, this result is sensible only if one works in the space where this current is well defined.

The correct induced charge is again obtained after dropping the spurious scalar contribution to the anomaly. In the present gauge it is easy to see that the total flux at spatial infinity vanishes, as each current term, Eqs. (2.15), (3.62) and (3.63) gives no flux contribution there. Of course, due to the gauge invariance of the current expression, this property of vanishing total fermion flux holds in any gauge. Then the total charge is only due to the gauge current contribution in agreement with the general result of D'Hoker and Goldstone [19].

$$\triangle Q_{ind.}|_{t=t_f} = \triangle Q_{ind.}(A)|_{t=t_f} = \int_{-\infty}^{t_f} dt\left(\int d^3x\partial_\mu j^\mu - \int d\vec{S}\cdot\vec{j}\right) = \int_{-\infty}^{t_f} dt\int d^3x\partial_\mu j^\mu_A = \frac{g^2}{32\pi^2}\left(\int_{t=t_f} d^3x\ K_0 + \int_{-\infty}^{t_f} dt\int d\vec{S}\cdot\vec{K}\right). \quad (3.68)$$

In the gauge we are considering the last term in the above equation vanishes and the total charge is given by the gauge current density contribution. Evaluating Eq. (3.68) at $t_f = \infty$ we have $Q_{ind.}|_{t=\infty} = 1$

In the same way as we did for the pure scalar field case, we can now look at the zero energy fermion modes in the background of both gauge and scalar fields. In this context the existence of one zero energy mode was already demostrated in Ref. [32]. There, with the scalar field as in Eq. (3.29) at t=0,

$$\varphi'(\vec{x}) = \frac{v}{\sqrt{2}}\frac{\imath\vec{x}\cdot\vec{T}}{\sqrt{r^2+\rho^2}}\varphi_0 \quad (3.69)$$

and with the gauge field given by:

$$A'_i(\vec{x}) = \frac{1}{g}\frac{r^2}{(r^2+\rho^2)}\frac{\epsilon_{ijk}x^j T_k}{r^2}\,, \tag{3.70}$$

a normalizable solution to the zero energy Dirac equation at t=0 was found. One must only replace $\partial_i\psi_L \to \mathcal{D}_i\psi_L = (\partial_i - \imath\, g\, A'_i)\psi_L$ in Eq. (3.36) and consider again the same Ansatz as that given in Eqs. (3.39), (3.40). One has

$$\left(\imath\mathcal{S}_i\mathcal{D}_i f(r) - m_f\frac{\imath x^i T_i}{\sqrt{r^2+\rho^2}}g(r)\right)\imath\,\sigma_{2_{n\alpha}} = 0 \tag{3.71}$$

The new contribution reads

$$\begin{aligned}\imath\mathcal{S}_i\,(-\imath g A'_i)\,f(r)\,\imath\,\sigma_{2_{n\alpha}} &= \frac{f(r)}{r^2+\rho^2}\,\imath\mathcal{S}_i\left(-\imath\epsilon_{ijk}x^j T_k\right)\imath\,\sigma_{2_{n\alpha}}\\ &= \frac{f(r)}{r^2+\rho^2}\,\imath\,\imath\epsilon_{ijk}\mathcal{S}_i\mathcal{S}_k x^j\,\imath\,\sigma_{2_{n\alpha}}\\ &= \frac{f(r)}{r^2+\rho^2}\,\imath\mathcal{S}_i\,(\mathcal{S}_i\mathcal{S}_j - \delta_{ij})\,x^j\,\imath\,\sigma_{2_{n\alpha}}\\ &= \frac{2f(r)}{r^2+\rho^2}\,\imath\vec{\mathcal{S}}\cdot\vec{x}\;\imath\,\sigma_{2_{n\alpha}}\end{aligned} \tag{3.72}$$

Eqs. (3.37) and (3.38) remain unchanged, and the set of equations to be solved is

$$\begin{aligned}\frac{1}{r}\partial_r f(r) + \frac{2f(r)}{r^2+\rho^2} + m_f\frac{g(r)}{\sqrt{r^2+\rho^2}} &= 0\\ \frac{1}{r}\partial_r g(r) + m_f\frac{f(r)}{\sqrt{r^2+\rho^2}} &= 0\end{aligned} \tag{3.73}$$

The zero energy solution is readily found to be

$$\begin{aligned}f(r) &= \left(\frac{1}{r^2+\rho^2} + m_f\frac{1}{\sqrt{r^2+\rho^2}}\right)\exp\left(-m_f\sqrt{r^2+\rho^2}\right)\\ g(r) &= m_f\frac{1}{\sqrt{r^2+\rho^2}}\exp\left(-m_f\sqrt{r^2+\rho^2}\right)\end{aligned} \tag{3.74}$$

The field configuration in the temporal gauge, that we have previously considered, which has its topology at $t = +\infty$, may be transform into the t'Hooft instanton, which has its topology at spatial infinity, through $\Omega(\vec{x},t)$, with $\Omega(\vec{x},-\infty) = 1$ and $\Omega(\vec{x},\infty)$ of unit topological number.

$$\begin{aligned}A_0^{inst.}(\vec{x},t) &= \frac{\imath}{g}\frac{\vec{x}\cdot\vec{T}}{r^2+\rho^2} = \frac{\imath}{g}\Omega(\vec{x},t)\partial_0\Omega^+(\vec{x},t)\\ A_i^{inst.}(\vec{x},t) &= \frac{\imath}{g}\left(\frac{x_0 T_i}{r^2+\rho^2} - \frac{\imath\epsilon_{ijk}x^j T_k}{r^2+\rho^2}\right) = \Omega(\vec{x},t)A_i(\vec{x},t)\Omega^+(\vec{x},t) + \frac{\imath}{g}\Omega(\vec{x},t)\partial_i\Omega^+(\vec{x},t)\\ \varphi^{inst.}(\vec{x},t) &= \frac{\imath x_0 + \imath\vec{x}\cdot\vec{T}}{\sqrt{r^2+\rho^2}}\frac{v}{\sqrt{2}}\varphi_0 = \Omega(\vec{x},t)\varphi(\vec{x},t)\,.\end{aligned} \tag{3.75}$$

While transforming now the instanton configuration applying $\Omega(\vec{x},0)\,\Omega^+(\vec{x},t)$ one gets

$$\begin{aligned}A_0''(\vec{x},t) &= 0\\ A_i''(\vec{x},t) &= \Omega(\vec{x},0)A_i(\vec{x},t)\Omega^+(\vec{x},0) + \frac{\imath}{g}\Omega(\vec{x},0)\partial_i\Omega^+(\vec{x},0)\\ \varphi''(\vec{x},t) &= \Omega(\vec{x},0)\varphi(\vec{x},t)\end{aligned} \tag{3.76}$$

and evaluating the above fields at t=0 it leads to

$$\begin{aligned}A_0''(\vec{x},0) &= 0\\ A_i''(\vec{x},0) &= A_i^{inst.}(\vec{x},0) = \Omega(\vec{x},0)A_i(\vec{x},0)\Omega^+(\vec{x},0) + \frac{\imath}{g}\Omega(\vec{x},0)\partial_i\Omega^+(\vec{x},0)\\ \varphi''(\vec{x},0) &= \varphi^{inst.}(\vec{x},0) = \Omega(\vec{x},0)\varphi(\vec{x},0)\,.\end{aligned} \tag{3.77}$$

Since

$$A'_i(\vec{x}) = A_i''(\vec{x},0)\,, \qquad \varphi'(\vec{x}) = \varphi''(\vec{x},0)\,, \tag{3.78}$$

the field configuration given in Eqs. (3.69), (3.70) and the one we have first considered while evaluating the induced fermion charge value in Eq. (3.68) are related at t=0 by a static continuous gauge transformation, $\Omega(\vec{x},0)$. Therefore, both energy spectra are the same.

We observe that the gauge field given in Eq. (3.70) obeys the condition,

$$A'_i(\vec{x}) = -A'_i(-\vec{x})\,. \tag{3.79}$$

Therefore, the symmetry between positive and negative energy states at t=0, that one finds for the scalar background fields, Eq. (3.29), still holds when the gauge field is added. Assuming that the zero energy mode is occupied, we have that the fermionic charge induced by the t=0 background fields configuration is 1/2. This assumption is consistent, as the same gauge invariant result may be obtained by evaluating Eq. (3.68) at t=0.

One can prove, that besides the zero energy mode of Ref. [32] there exists an extra mode, in accordance with the zero adiabatic result of Eq. (3.67) in the space with the origin removed. In order to find it we propose the same Ansatz as in Eq. (3.51) for the fermion fields and evaluate the new contributing term to Eq. (3.36) in this case,

$$\begin{aligned}\imath\mathcal{S}_i\,(-\imath g A'_i)\,\imath\vec{x}\cdot\vec{T}\,p(r)\,\imath\,\sigma_{2_{n\alpha}} &= \frac{p(r)}{r^2+\rho^2}\,\imath\mathcal{S}_i\left(-\imath\epsilon_{ijk}x^j T_k\right)\imath\vec{x}\cdot\vec{T}\;\imath\,\sigma_{2_{n\alpha}}\\ &= \frac{p(r)}{r^2+\rho^2}\,\mathcal{S}_i\,(T_i T_j - \delta_{ij})\,x^j\vec{x}\cdot\vec{T}\;\imath\,\sigma_{2_{n\alpha}}\\ &= \frac{-2r^2 p(r)}{r^2+\rho^2}\,\imath\,\sigma_{2_{n\alpha}}\,.\end{aligned} \tag{3.80}$$

The set of equations to be solved now reads,

$$\begin{aligned}r\partial_r q(r) + 3q(r) - m_f\frac{r^2}{\sqrt{r^2+\rho^2}}\,p(r) &= 0\\ r\partial_r p(r) + 3p(r) - \frac{2r^2 p(r)}{r^2+\rho^2} - m_f\frac{r^2}{\sqrt{r^2+\rho^2}}\,q(r) &= 0\,.\end{aligned} \tag{3.81}$$

Rescaling the functions $p(r)$ and $q(r)$ by defining

$$\begin{aligned} p(r) &= K(r)P(r) \\ q(r) &= K(r)Q(r)\,, \end{aligned} \tag{3.82}$$

with $K(r) = \exp\left(-m_f\sqrt{r^2+\rho^2}\right)/r^3$, we have,

$$\begin{aligned} \partial_r p(r) &= K(r)\partial_r P(r) - \left(\frac{3}{r} + \frac{m_f r}{\sqrt{r^2+\rho^2}}\right) K(r)P(r) \\ \partial_r q(r) &= K(r)\partial_r Q(r) - \left(\frac{3}{r} + \frac{m_f r}{\sqrt{r^2+\rho^2}}\right) K(r)Q(r)\,. \end{aligned} \tag{3.83}$$

Thus, in terms of P(r) and Q(r), after using Eqs. (3.82), (3.83), equations (3.81) reduce to

$$\begin{aligned} (P+Q)\,\frac{m_f r^2}{\sqrt{r^2+\rho^2}} - r\partial_r P + \frac{2r^2}{r^2+\rho^2}P &= 0 \\ (P+Q)\,\frac{m_f r^2}{\sqrt{r^2+\rho^2}} - r\partial_r Q &= 0\,. \end{aligned} \tag{3.84}$$

As $r \to \infty$ we have

$$\begin{aligned} (P+Q)\,m_f - \partial_r P &= 0 \\ (P+Q)\,m_f - \partial_r Q &= 0 \end{aligned} \tag{3.85}$$

and $P = -Q = c$, with c an arbitrary constant, is a solution. This gives

$$p(r\to\infty) = -q(r\to\infty) = cK(r) = c\exp\left(-m_f\sqrt{r^2+\rho^2}\right)\,/r^3\,. \tag{3.86}$$

In the limit $r \to 0$ the set of equations in consideration reduce to those given in Eq. (3.54), when taking only the scalar background. From the above it is clear that another zero energy mode appears at t=0. However this mode becomes nonnormalizable as soon as the origin is considered within the configuration space. Therefore, its existence supports the zero charge value obtained from the adiabatic current, as the induced fermion charge in the configuration space with the origin excluded.

3.4 The Sphaleron

As a final point in this chapter, we want to use the above considerations to derive via symmetry arguments the fractional fermionic charge induced by the sphaleron configuration [12]. We recall that in the temporal gauge the sphaleron scalar and gauge fields as first obtained by Manton [15] are:

$$\begin{aligned} \varphi_{sph} &= \frac{v}{\sqrt{2}}\,h(gvr)U^\infty \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ A_{i_{sph.}} &= -\frac{i}{g}\,f(gvr)\partial_i U^\infty\, U^{\infty^+} \end{aligned} \tag{3.87}$$

where

$$U^\infty = \frac{1}{r}\begin{pmatrix} z & x+iy \\ -x+iy & z \end{pmatrix}\,,$$

and the functions $h(gvr)$, $f(gvr)$ have the following asymptotic behaviour:

$$h = \beta\xi\,, \qquad f = \gamma\xi^2 \tag{3.88}$$

near $\xi = 0$ and

$$h = 1 - \frac{\eta}{\xi}exp(-\sqrt{2\lambda/g^2}\xi)\,, \qquad f = 1 - \delta exp(-\xi/2) \tag{3.89}$$

as $\xi \to \infty$ ($\xi = gvr$ is the dimensionless radial distance and β, γ, η, δ are constants of order unity, which can only be determined by finding the complete solution).

A physically equivalent solution may be obtained by replacing U^∞ by

$$(U^\infty)' = U_L U^\infty U_R\,, \tag{3.90}$$

with U_L and U_R being constant SU(2) matrices. U_R has no effect on the sphaleron gauge field. Taking

$$U_L = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \qquad U_R = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{3.91}$$

we get

$$\begin{aligned} \varphi'_{sph} &= \frac{v}{\sqrt{2}}\,h(gvr)(U^\infty)' \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \frac{v}{\sqrt{2}}\,h(gvr)\,i\frac{\vec{x}\cdot\vec{T}}{r}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\,, \end{aligned} \tag{3.92}$$

and

$$\begin{aligned} A'_{i_{sph.}} &= \frac{-i}{g}\,f(gvr)\,\partial_i\left(U_L U^\infty\right)\,U^{\infty^+}U_L^+ \\ &= \frac{1}{g}\,\frac{f(gvr)}{r^2}\,\epsilon_{ijk}x^j T_k \end{aligned} \tag{3.93}$$

We can still make a further transformation of the form

$$(U^\infty)'' = -U(\vec{x})\,(U^\infty)'\,, \tag{3.94}$$

with

$$U(\vec{x}) = \exp\left(\frac{\imath\theta(r)}{2}\frac{\vec{T}\cdot\vec{x}}{r}\right) , \tag{3.95}$$

where $\theta(r)$ increases from 0 to π as r runs from 0 to ∞, so that $U^{\infty''} = 1$ at spatial infinity. After such transformation we obtain for the scalar sphaleron field:

$$\varphi''_{sph.} = \frac{v}{\sqrt{2}} h(gvr) \begin{pmatrix} -(\hat{y}+\imath\hat{x})\cos(\theta(r)/2) \\ (\imath\hat{z})\cos(\theta(r)/2)+\sin(\theta(r)/2) \end{pmatrix} , \tag{3.96}$$

which is the expression given in Eq. (3.19). However, the sphaleron solution obtained after transforming U^{∞} to $(U^{\infty})'$ is the best choice for our purpose, since looking at both configurations, Eqs. (3.69), (3.70) and Eqs. (3.92), (3.93), we observe that they differ only in the radial functions. Nevertheless, these functions have the same boundary conditions. Using these boundary conditions for the radial functions, it is easy to prove, in precise analogy to the preceding case, that a zero energy fermion mode must appear in the presence of these background fields. The explicit calculation, using the variational Ansatz for the sphaleron radial functions [15], is done in Ref. [34].

Furthermore, as we observe from Eqs. (3.92), (3.93), the scalar and vector fields obey the conditions:

$$\varphi'_{sph}(\vec{x}) = -\varphi'_{sph}(-\vec{x}) \quad , \quad A'_{i_{sph}}(\vec{x}) = -A'_{i_{sph}}(-\vec{x}). \tag{3.97}$$

This leads, once more, to the existence of symmetry between positive and negative energy states. Reasoning in the same way as above, we then conclude immediately that the fermionic charge induced by the sphaleron is 1/2. This result is the same as that obtained by Klinkhamer and Manton [18] considering only the gauge current density in the gauge where the vector field has a trivial expression at spatial infinity. That is to say, placing $t_f = 0$ in Eq. (3.68). We have arrived at the same induced charge value in the background of the sphaleron as that obtained after reaching the static configuration of the t'Hooft instanton and its associated scalar field at t=0 in the $A_0 = 0$ gauge. The fact that we obtain the same result for both background configurations is due to the fact that the energy level crossings depend on the radial functions only through their boundary conditions. Of course, the main point in considering the sphaleron as a particular configuration is that the functions $f(gvr)$, $h(gvr)$ are those which minimize the energy functional, but their explict expression is irrelevant for the above considerations.

Chapter 4

Energy Level Crossings Dependence on the Adiabatic Path

In the previous chapter, we have evaluated, using the adiabatic method [7]-[9], the fermionic charge induced by the scalar fields considered as background for the fermions. We proved there, with appropiate examples, that the correct induced fermion charge in the background of the final scalar configuration may differ from the induced charge evaluated naively, applying the adiabatic current expression, depending on the intermediate path one considers. More explicitly, $Q_{ad.} \neq Q_{ind.}$, whenever the intermediate path is such that the adiabatic current is ill defined at some point during the adiabatic process. As we already observed, to obtain the ground state charge of the final scalar fields, while evaluating the charge induced by them, one must take into account the existence of zero energy level crossings, $Q_{GS} = Q_{ind.} - (\eta_+ - \eta_-)$. D'Hoker and Fahri [10] have studied, in the framework of the nonlinear σ model, the appearance of zero energy modes, as a function of the fermion mass and the typical mass scale of the scalar field. These authors build up adiabatically a final Skyrmion of winding number one, starting from the normal vacuum, and find that no energy level crossing occurs if the fermion Compton wavelength, $1/m_f$, is much smaller than the soliton width, ρ_s, while one and only one level crossing occurs for $1/m_f \gg \rho_s$. They conclude that the Skyrmion carries the fermion number of any fermion with mass of the order of, or greater than, its own typical mass scale. This conclusion was done without any consideration at all about the reliability of the adiabatic result to give the correct induced charge. However, since while they turn on adiabatically the Skyrmion the current expression is well behaved everywhere, they are in the case where, indeed, $Q_{ad.} = Q_{ind.}$ and their statement is correct. Kahana and Ripka [11], arrived at a similar conclusion, by evaluating explicitly, in a static background, the relation between the fermion number and the winding number of the soliton.

In the above chapter, while considering the Skyrmion as the final background configuration, we obtained that the true induced charge may be zero, even when the adiabatic result, or equivalently the topological charge, gives one for any intermediate background. The point is that the induced charge value depends on the intermediate path. Since the ground state charge is independent of the way one arrives at the final configuration, it follows from the relation between the induced and ground state charges and the number of zero energy level crossings n_+ (n_-) in the positive (negative) direction of the energy axis, that n_+, n_- must also depend on the intermediate path. In this chapter we will show this dependence explicitly. We will show also, for completeness, that the different number of zero energy modes which appear, depending on the intermediate stages, is consistent with the physical fact that the ground state charge of a soliton can be identified with its topological charge, whenever heavy fermions or, equivalently, a wide soliton are considered.

In section 4.1 we give, as a preliminary for the coming more rigorous computations, a qualitative argument supporting the dependence of the zero energy modes on the adiabatic path and the scale of the soliton. In section 4.2 we obtain the eigenvalue equations for a general scalar configuration and analyze some of their properties for the case of zero orbital momentum, where the zero energy modes are expected. In section 4.3 we propose two different intermediate paths to build up adiabatically a final Skyrmion, in order to analyze afterwards the energy eigenvalues in each evolving background. In section 4.4 an iterative method to solve numerically the eigenvalue equations is shortly described. In section 4.5 we obtain explicitly the dependence of the zero energy level crossings on the intermediate background configuration. We evaluate the ground state charge for both intermediate configurations of section 4.3, in terms of path dependent quantities and the soliton scale, to verify that they give the expected concordant results. Finally, in section 4.6 we analyse the same issue for two different scalar configurations proposed to build up adiabatically a final soliton of winding number two.

4.1 Zero Energy Modes and the Soliton Scale: a First Approach

We have already mentioned that, the fermionic ground state charge of a sufficiently wide soliton is associated to its topological number. Within the adiabatic approach, this result is deduced based on the fact, that while building up a sufficiently wide soliton no crossings between positive and negative energy levels occur, and assuming the accuracy of the adiabatic technique to give the induced charge [8]-[10]. However, we have infered above that both points may be only correct for the case where the change of the topological charge during the adiabatic procedure is due to the existence of current flow at spatial infinity. When the change of the topological charge comes while the configuration goes over the energy barrier, vanishing at some point and leading to an ill definition in the current expression, the adiabatic result is no longer reliable and the condition for having no zero energy level crossings is exactly the opposite one, i.e. then one must consider a narrow soliton. We present here a qualitative proof of this statement.

We recall first here, very succinctly, the standard case, already exhaustively discussed by many other authors. In the frame of the nonlinear σ model, consider a solitonic scalar background of winding number 2 defined as,

$$\left(\phi_0, \vec{\phi}\right) = \left(\cos\theta(r), \frac{\vec{r}}{r}\sin\theta(r)\right) , \tag{4.1}$$

with $\theta(0) = 0$ and $\theta(\infty) = 2\pi$. One may interpolate from the vacuum to the above final configuration by relaxing the finite energy conditions at spatial infinity, for example, through the following evolving scalar fields:

$$\left(\phi_0, \vec{\phi}\right) = \left(\cos F(r,t), \frac{\vec{r}}{r}\sin F(r,t)\right) , \tag{4.2}$$

where

$$F(r,t) = \frac{\theta(r) f(t)}{2\pi} \tag{4.3}$$

with $f(t)$ varying from 0 to 2π as t goes from $-\infty$ to ∞. Thus, $F(\infty, t) = f(t)$ and $F(r, \infty) = \theta(r)$. Applying the adiabatic method as the functions vary infinitely slow, one obtains $Q_{ad.} = 2$. Since the adiabatic current is well defined everywhere, the equivalence $Q_{ad.} = Q_{ind.}$ holds in this case. However if $\theta(r)$ varies rapidly (in the extreme it becomes a step function) then, at $t \to \infty$, the scalar field background is sensed by the fermions as the normal vacuum, and $Q_{GS} = 0$. It must happen therefore that two level crossings occur as one sharpens the soliton.

As we said earlier, D'Hoker and Fahri [10] have proved explicitly, for the nonlinear σ model, the existence of a zero energy mode while building up adiabatically a sufficiently narrow soliton of winding number one. At the same time they also give an argument to infer the absence of zero energy modes for a final wide soliton. These considerations and other explicit computations done in the literature in the nonlinear limit [8],[9], support the above rough reasoning.

We consider now a linear σ model, where we may interpolate, for example, from the vacuum to a final scalar soliton of winding number two by going over the energy potential as the fields vanish at some point. Let us take, for example, the configuration

$$\left(\phi_0, \vec{\phi}\right) = v\left(1, \vec{0}\right) \qquad r \neq r_0$$

$$\left(\phi_0, \vec{\phi}\right) = v\left(1 - 2h(t), \vec{0}\right) \qquad for \;\; r = r_0 \tag{4.4}$$

with $h(t)$ varying monotonically from 0 to 1, as before. The fermionic energy spectrum may not be altered due to a finite change in the background fields at a sphere of radius r_0. Since the fermions sensed this configuration as the vacuum, no zero energy level crossing may occur in the frame of this evolving background. However, the configuration given in Eq.(4.4) is the limiting case of one which interpolates between the vacuum and a soliton of winding number two, remaining the vacuum at spatial infinity, when the soliton width goes to zero. At $h(t) = 0$ we have the normal vacuum, at $h(t) = \frac{1}{2}$ the scalar fields vanish allowing the change of the topological charge from 0 to 2 and at $h(t) = 1$ we have an infinitely thin soliton. Therefore, building up a sufficiently narrow (in the limit step soliton), no zero energy level crossings occur, thus, $Q_{GS} = Q_{ind.} = 0$. This statement differs from the one obtained for the case with nonvanishing fermion flux considered in the literature. Although the above derivation is not rigorous, it is offered here as an intuitive way of anticipating the exact results of the next sections

4.2 Energy Eigenvalue Equations in a General Scalar Background

We take, as in the above chapter, a σ model coupled to an SU(2) doublet fermion ψ. We will consider below two different expressions for the scalar fields, to build up adiabatically the final Skyrmion of winding number one, starting from the normal vacuum. In order to look for zero energy modes in both backgrounds and study the dependence of these modes on the soliton width , we first analyse here the Dirac equation in the background of a general scalar quartet

$$\imath\partial^\mu\gamma_\mu\psi - \frac{g_y}{\sqrt{2}}\left(\phi_0 + \imath\vec{\phi}\cdot\vec{\mathcal{T}}\gamma_5\right)\psi = 0\,. \tag{4.5}$$

The isospinor components, ψ_n, can be decomposed into upper and lower components: $\psi_n = \begin{pmatrix}\chi_n^+\\ \chi_n^-\end{pmatrix}$, and in the Dirac representation

$$\gamma_D^a = \begin{pmatrix} 0 & S_a \\ -S_a & 0\end{pmatrix} \qquad \gamma_D^0 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \tag{4.6}$$

Eq.(4.5) becomes,[1]

$$\begin{aligned} \imath\partial_a S_a\chi_n^- - \frac{g_y}{\sqrt{2}}\phi_0\chi_n^+ - \frac{g_y}{\sqrt{2}}\imath\vec{\phi}\cdot\vec{\mathcal{T}}\chi_n^- &= -\imath\partial_0\chi_n^+ \\ -\imath\partial_a S_a\chi_n^+ - \frac{g_y}{\sqrt{2}}\phi_0\chi_n^- - \frac{g_y}{\sqrt{2}}\imath\vec{\phi}\cdot\vec{\mathcal{T}}\chi_n^+ &= \imath\partial_0\chi_n^- \end{aligned} \tag{4.7}$$

or more explicitly, specifying components,

$$\left((\vec{\sigma}\cdot\vec{p})_{ij}\,\delta_{nm} \pm \imath\,\frac{g_y}{\sqrt{2}}\,\delta_{ij}\,(\vec{\phi}\cdot\vec{\sigma})_{nm}\right)\chi_{jm}^{\mp} = \left(E \mp \frac{g_y}{\sqrt{2}}\,\phi_0\right)\chi_{in}^{\pm}\,, \tag{4.8}$$

with n,m = 1,2 and i,j = 1,2 being the isospin and Lorentz indices, respectively. For our case, since $\phi_0 = \phi_0(r)$, $\phi_a = \phi(r)\hat{r}_a$, we observe that,

$$[S_a p_a, j_b] = 0\,, \qquad [\mathcal{T}_a r_a, I_b + l_b] = 0\,, \tag{4.9}$$

with $\vec{j} = \vec{l} + \vec{s}$ being the ordinary angular momentum and $\vec{I}$ being the isospin. Then, the grand momentum operator, defined as $\vec{M} = \vec{j} + \vec{I}$, commutes with Eq. (4.8). So, following Ref.[1], it is useful to expand the $\chi^\pm$ components in eigenstates of M^2 and M_3. The solutions of Eq. (4.8) will also have upper and lower components of definite, opposite parity.

We now define 2×2 matrices $\mathcal{M}^\pm$ by

$$\chi_{in}^{\pm} = \mathcal{M}_{im}^{\pm}\sigma_{2\,mn}\,, \tag{4.10}$$

[1] In this section, we use the same convention as before: S_a and $\mathcal{T}_a$ are the Pauli matrices σ_a in the Dirac and weak isospin spaces, respectively; alternatively, we specify the indices denoting them as $\sigma_{a_{ij}}$ and $\sigma_{a_{nm}}$, respectively.

and rewrite Eq. (4.8) as

$$\left((\imath\partial_a\sigma_a)_{ij}\,\delta_{nm} \mp \imath\,\frac{g_y}{\sqrt{2}}\,\delta_{ij}\,\phi_a\sigma_{anm}\right)\mathcal{M}_{jl}^{\mp}\sigma_{2lm} = -\left(E \mp \frac{g_y}{\sqrt{2}}\,\phi_0\right)\mathcal{M}_{il}^{\pm}\sigma_{2ln}\,. \tag{4.11}$$

Using the relation $\sigma_2\,\sigma_a^{tr.} = -\sigma_a\,\sigma_2$ which gives $\sigma_{anm}\,\mathcal{M}_{il}^{\mp}\,\sigma_{2lm} = -\mathcal{M}_{im}^{\mp}\,\sigma_{aml}\,\sigma_{2ln}$ this leads to

$$(\imath\partial_a\sigma_a)_{ij}\;\mathcal{M}_{jl}^{\mp} \pm \imath\,\frac{g_y}{\sqrt{2}}\phi_a\mathcal{M}_{im}^{\mp}\sigma_{aml} = -\left(E \mp \frac{g_y}{\sqrt{2}}\,\phi_0\right)\mathcal{M}_{il}^{\pm}\,. \tag{4.12}$$

We expand $\mathcal{M}^\pm$ in terms of two scalar and two vector functions

$$\mathcal{M}_{im}^{\pm}(\vec{r}) = g^{\pm}(\vec{r})\delta_{im} + g^{a\,\pm}(\vec{r})\sigma_{a\,im}\,, \tag{4.13}$$

which can be themselves expanded in terms of scalar and vector spherical harmonics

$$\begin{aligned} g^{\pm}(\vec{r}) &= \sum_{MM_3} G_M^{\prime\pm}(r)\,Y_{MM_3}(\Omega) \\ g^{a\,\pm}(\vec{r}) &= \sum_{MM_3}\left(P_M^{\pm}(r)\,\mathcal{P}_{MM_3}^{a}(\Omega) + B_M^{\pm}(r)\,\mathcal{B}_{MM_3}^{a}(\Omega) + C_M^{\pm}(r)\mathcal{C}_{MM_3}^{a}(\Omega)\right). \end{aligned} \tag{4.14}$$

Eqs. (4.12) reduce, for M=0, (see appendix B), to a set of coupled equations involving $P^\mp(x)$ and $G^\pm(x) = \mp\imath G^{\prime\pm}(x)$

$$\begin{aligned} \partial_x P^{\mp}(x) &= (\varphi_0(x) \mp E)\;G^{\pm}(x) - \left(\frac{2}{x} \pm \varphi(x)\right)\;P^{\mp}(x) \\ \partial_x G^{\pm}(x) &= (\varphi_0(x) \pm E)\;P^{\mp}(x)\;\pm\;\varphi(x)\;G^{\pm}(x)\,, \end{aligned} \tag{4.15}$$

In Eq. (4.15), we have redefined variables in terms of the fermion mass, $m_f = g_y v/\sqrt{2}$, as $E \to E/m_f$, $x = r\;m_f$, [2] and dimensionless fields, with $\varphi_0 = \phi_0/v$, $\varphi = \phi/v$. We evaluate the energy eigenvalues only in the case M=0, based on the assumption that the zero energy modes, if they exist, are expected to appear for the lowest grand momentum orbitals. This assumption is actually proved in [10], where it is shown analytically, in the framework of the non-linear σ model, that the zero energy mode appears in the M=0 orbital, as one builds up adiabatically a sufficiently narrow Skyrmion of winding number one.

Some very simple relations between solutions of opposite parities, which are going to be useful in the next section, may be observed from the eigenvalue equations we have just obtained in Eq. (4.15). Whenever a solution of defined parity, for example P^-, G^+, in the background φ_0, φ, with energy E exists, then in the background φ_0, $-\varphi$ one has a solution of opposite parity, related to the initial one as $P^+ = P^-$, $G^- = G^+$, with energy -E. Similarly, in the background $-\varphi_0$, φ one has the solution $-P^-$, G^+, with energy -E. Furthermore if both signs of the scalar fields are simultaneously changed one still has a solution of energy E but with the opposite parity, $P^+ = -P^-$, $G^- = G^+$.

[2] This redefinition leads also to $\rho = \rho_s\;m_f$

4.3 Two Different Paths to Build up the Skyrmion

We choose now two expressions for the evolving scalar fields, which give different intermediate paths, one which allows no fermion flux from infinity and the other one which does [13]. As a first possibility, consider

$$\begin{aligned} \phi_0 &= v\,\{1 - h(t)[1 - f_1(r)]\} \\ \vec{\phi} &= v\, h(t) f_2(r)\frac{\vec{r}}{r} \end{aligned} \tag{4.16}$$

with

$$\begin{aligned} f_1(r) &= 1 - 2\exp\left(-\frac{r^2\ln 2}{\rho_s^2}\right), \\ f_2(r) &= -2\sqrt{1-\exp\left(-\frac{r^2\ln 2}{\rho_s^2}\right)}\ \exp\left(-\frac{r^2\ln 2}{2\,\rho_s^2}\right), \end{aligned} \tag{4.17}$$

and with h(t) being a function which varies slowly and monotonically from 0 to 1 . Using the adiabatic current expression, Eq. (3.4), we can, in principle, evaluate the fermionic induced charge in the above background. Eqs. (4.16) give the normal vacuum, $(v,\vec{0})$, at spatial infinity for all time t. Therefore, they give no fermion current flow there. This means, since we are dealing with scalar fields, that no fermionic charge may be induced in this background. However, if one calculates the induced charge using naively Eq. (3.4), a different value is obtained, namely 1. Such incorrect value, as discussed in chapter 3, is due to the vanishing value of the background fields at r=0, h(t=0)= $\frac{1}{2}$. Indeed, the above scalar fields are the same configuration we first considered in the previous chapter, Eq. (3.7), with the same boundary conditions for $f_1(r)$ and $f_2(r)$, but now with the above explicit expressions for these radial functions.

As a second configuration, we want to consider one which interpolates between the vacuum and the Skyrmion, but allows flux at spatial infinity. Note that, in principle, configurations which allow fermion flux at spatial infinity are required, whenever one is working with a nonlinear σ model, since in these models the change in the winding number cannot be achieved through a somewhere vanishing field. As a first try consider the scalar background

$$\begin{aligned} \phi_0 &= v\,\cos[h(t)\arccos f_1(r)] \\ \vec{\phi} &= \begin{cases} -v\,\sin\{h(t)[\pi + \arcsin f_2(r)]\}\ \hat{r} & r \le \rho_s \\ v\,\sin[h(t)\arcsin f_2(r)]\ \hat{r} & r \ge \rho_s\,, \end{cases} \end{aligned} \tag{4.18}$$

with $h(t)$, $f_1(r)$ and $f_2(r)$ being the same as above and arcsin $f_2(r)$ and arccos $f_1(r)$ taking values in the intervals $[-\pi/2\,,\pi/2\,]$ and $[\,0\,,\pi\,]$, respectively. This configuration builds up, effectively, the Skyrmion starting from the vacuum. However, something unexpected happens, since even though $|\phi|^2 = v^2$ for any point in R^4, these fields manage to interpolate

between configurations of different topological numbers without allowing any fermion flux at spatial infinity, $\vec{\phi}(r\to\infty,t) = 0$. The puzzle is easily solved by focusing on the value of the fields at r = 0. In principle, as $\vec{\phi} = \phi(r)\vec{r}/r$, then $\phi(r=0)$ must vanish at r=0 to avoid an ambiguity at that point, but here $\phi(r=0) = -\sin(h\pi)$. This discontinuity is responsible for the change of the topological charge in this case. It follows therefore, as shown in Fig. 6, that trying to consider a configuration with spherical symmetry for ϕ_0, $\vec{\phi}$, which owes the change of its topological number to the flux from infinity, one has to take a configuration which interpolates between the $-vacuum$ to the *Skyrmion* or its adjoint conjugate configuration (one could also take a configuration which interpolates from the *vacuum* to the $-Skyrmion$ or minus its adjoint conjugate). Note that a change in the sign of the configuration - i.e. a global change in the sign of the scalar quartet ϕ_a - leaves unchanged the sign of its associated topological charge. However, the topological charge does change sign, whenever the configuration is replaced by its adjoint conjugate. That is, whenever a change in the sign of ϕ_0 relative to that of $\vec{\phi}$, or vice versa, takes place. For our purposes, the above issue makes no fundamental difference. The relations given at the end of the last section show the correlation between the energy and parity of the eigenfunctions under changes of sign of the scalar fields. Therefore, one must only keep in mind the relation between zero energy level crossings and induced and ground state charges, and be aware of which final soliton has been taken in each case, when comparing the fermion energy spectra.

For simplicity, in order to get rid of any further considerations about change of signs in the energy eigenvalues while comparing them with those obtained for the background configuration of Eq. (4.16), we consider the following expression for the scalar quartet, which interpolates from the $-vacuum$ to the *Skyrmion*.

$$\begin{aligned} \phi_0 &= -v\,\cos\{h(t)[\pi - \arccos f_1(r)]\} \\ \vec{\phi} &= \begin{cases} v\,\sin[h(t)\arcsin f_2(r)]\ \hat{r} & r \le \rho_s \\ -v\,\sin\{h(t)[\pi + \arcsin f_2(r)]\}\ \hat{r} & r \ge \rho_s \end{cases} \end{aligned} \tag{4.19}$$

Eqs. (4.19) give an intermediate configuration which does allow fermion current flow at spatial infinity and is nonvanishing everywhere. In this case, the final induced fermionic charge is obtained directly from the adiabatic current expression, since this is well defined for all points. The result coincides then with the topological charge of the soliton. We will work, from now on, with both configurations, Eq. (4.16) and Eq. (4.19), which reduce to the Skyrmion at $h(t) = 1$.

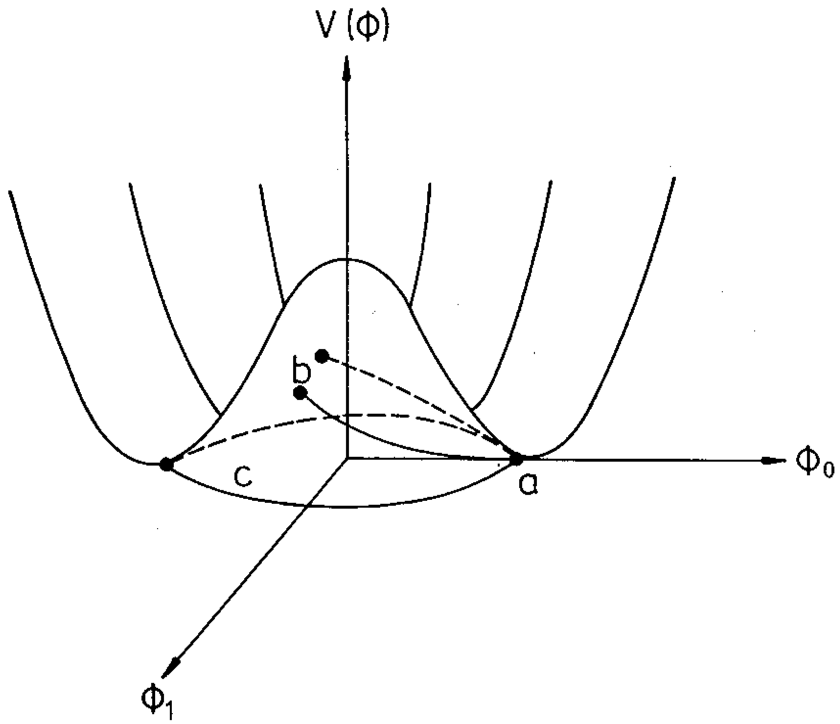


Figure 6: Sequence of field configurations from a) the *vacuum*, to c) the *Skyrmion*, while considering an interpolating configuration with non-trivial asymptotic behaviour as b, in the frame of a linear σ model. In the non linear limit, one has the same development, but with the evolving configurations taking values on the ring $\phi^2 = v^2$. Note that on the curve b $\phi(-\infty) \neq \phi(\infty)$.

4.4 Iterative Method to Evaluate Numerically the Energy Eigenvalues

In order to solve the set of equations (4.15), we invoke an iterative eigenvalue solving method proposed in Ref. [35]. We denote the nth- iteration results by $G^{\mp(n)}(r)$, $P^{\pm(n)}(r)$ for $r \leq R$ and $\hat{G}^{\mp(n)}(r)$, $\hat{P}^{\pm(n)}(r)$ for $r > R$, with R being some matching radius to be considered below. The iterative procedure implies replacing E by $E^{(n)}$ in the initial equations, (4.15), with $E^{(n)}$ being the energy value obtained in the $(n-1)$th- iteration and integrating using a variable-order variable-step Adams technique [36] , from the inside, ($G^{\mp(n)}(0)$ arbitrary and

$P^{\pm(n)}(0) = 0$), to the matching radius $R = \rho$. The equations are also integrated from the outside ($\hat{G}^{\mp(n)}(R_{max.})$ and $\hat{P}^{\pm(n)}(R_{max.})$ are small numbers, in order to obtain a normalized solution) to the same matching radius. Denoting,

$$\hat{G}^{\mp(n)}(R^+) = lim_{r \to R}\, \hat{G}^{\mp(n)}(r) \tag{4.20}$$

we define the normalized solutions $G^{\mp(n)}(r)$, $P^{\pm(n)}(r)$, as those satisfying,

$$\begin{aligned} G^{\mp(n)}(r) &= \begin{cases} G^{\mp(n)}(r) & r \leq R \\ \left[G^{\mp(n)}(R)/\hat{G}^{\mp(n)}(R^+)\right] \hat{G}^{\mp(n)}(r) & r > R, \end{cases} \\ P^{\pm(n)}(r) &= \begin{cases} P^{\pm(n)}(r) & r \leq R \\ \left[G^{\mp(n)}(R)/\hat{G}^{\mp(n)}(R^+)\right] \hat{P}^{\pm(n)}(r) & r > R, \end{cases} \end{aligned} \tag{4.21}$$

with the above definition, $G^{\mp(n)}(r)$ becomes continuous at $r = R$, while $P^{\pm(n)}(r)$ remains discontinuous at the matching radius. Then, as proposed in Ref. [35], we have,

$$E^{(n+1)} - E^{(n)} = \pm \frac{G^{\mp(n)}(R)\left[P^{\pm(n)}(R^+) - P^{\pm(n)}(R^-)\right]}{\int_0^{R_{max.}} r^2\, dr\, \left[G^{\mp(n)^2}(r) + P^{\pm(n)^2}(r)\right]}, \tag{4.22}$$

where

$$P^{\pm(n)}(R^\pm) = lim_{\epsilon \to 0} P^{\pm(n)}(R \pm \epsilon) \tag{4.23}$$

The convergence is rapid, once one is in the vicinity of the true solution. We apply these iterations till an energy value is obtained with the desired accuracy. In this way we can compute the fermion energy eigenvalues corresponding to arbitrary background configurations.

4.5 Soliton Ground State Fermionic Charge

In the present section we evaluate, using the iterative method detailed above, the fermion energy eigenvalues in the background of the scalar configurations proposed in section 4.3. Let us recall the point we want to check in detail. We know that the induced charge differs by one unit from the adiabatic one, whenever one has an intermediate background which vanishes somewhere before the final $n = 1$ Skyrmion is arrived at. Since the final ground state charge of the soliton is independent of the way in which we build it up, from Eq. (2.27) it follows that the number of energy level crossings, depends on the intermediate background fields. Therefore, if the scalar fields evolve from the vacuum to the Skyrmion through a path with no fermion flux, the number of zero energy modes obtained should differ by one unit from the number of modes obtained extrapolating between these configurations, using the path appropiate to the nonlinear sigma model (i.e. along a path with fermion flux at spatial infinity).

To prove the above statement explicitly, we solve the set of equations (4.15) for both the scalar backgrounds from Eqs. (4.16), (4.19) [13]. In Fig. 7 we plot the lowest energy eigenvalues, that correspond to the set of equations (4.15) involving $P^-(x)$ and $G^+(x)$, as a function of the evolving backgrounds, for different values of ρ [3]. From now on we define $\rho = \rho_c \simeq 1.5$ as the value of ρ for which a zero energy mode exists in the Skyrmion background, h(t)= 1. We see that in the no fermion flux case given by the background configuration of Eq. (4.16), one zero energy level crossing from $E > 0$ to $E < 0$ occurs before the final time for any $\rho > \rho_c$ and no level crossing occurs whenever $\rho < \rho_c$. As we already said, in this case we have zero induced charge, $Q_{ind.} = 0$. Then using Eq.(2.27) we obtain:

$$Q_{GS} = Q_{ind.} = 0 \qquad for \quad \rho < \rho_c$$

$$Q_{GS} = Q_{ind.} + 1 = 1 \qquad for \quad \rho > \rho_c \tag{4.24}$$

On the other hand, in the fermion flux case given by the background configuration of Eq. (4.19), we see that the same energy level crossing conditions hold for the opposite relations of the size parameter and in the opposite direction of the energy axis. This means that, as suggested, the number of zero energy modes in this background differs by one unit with respect to that of the no fermion current flow case, for any value of ρ. In this case the induced and topological charges are the same, $Q_{ind.} = Q_{top.} = 1$, and this leads to:

$$Q_{GS} = Q_{ind.} - 1 = 0 \qquad for \quad \rho < \rho_c$$

$$Q_{GS} = Q_{ind.} = 1 \qquad for \quad \rho > \rho_c \tag{4.25}$$

We note, from Eqs. (4.24), (4.25), that the induced charge value and the number of zero energy level crossings arrange themselves, in each case, to give finally the same path independent ground state charge. Furthermore, the soliton fermionic charge can be identified with its topological charge whenever $\rho = \rho_s m_f > \rho_c$. That is, whenever $m_f > 1.5/\rho_s$.

One can obtain the same conclusions in an alternative way, which allows the connection of our results for h(t)=1, (Skyrmion), with those obtained in Ref. [11]. In Fig. 8, we plot the fermion energy as a function of ρ and obtain different curves for the different values of h(t). Let $\rho_{E=0}(t)$, be the value of ρ at which a zero energy mode occurs for any value of h(t), thus $\rho_{E=0}(1) = \rho_c \simeq 1.5$. Analysing the curves obtained for different times, in both the flux and no flux case, we can calculate the change in the ground state charge value carried by each scalar configuration, depending on the value of ρ. For each value of h(t), irrespective of the type of background fields, the M=0 orbital has positive energy for $\rho < \rho_{E=0}(t)$. Thus in the ground state these levels must be empty. For $\rho > \rho_{E=0}(t)$ this orbital has negative energy and in the ground state the levels will be filled, thereby increasing the fermion number by one unit. In the no fermion flux case, we have that for $\rho = 0$, the scalar fields give the trivial background for any value of h(t), and carry then zero fermion number. This means we have $Q_{GS} = 1$ whenever $\rho > \rho_{E=0}(t)$. In the fermion flux case, we cannot know a priori the ground state

[3]The set of Eqs. (4.15) which involves $P^+(x)$ and $G^-(x)$ has no zero energy level crossing in these backgrounds.

charge of each intermediate configuration, since we have no trivial background for any value of the size parameter. However, since each curve in Fig. 8 is at a fixed value of h(t), there is a fixed induced charge for each curve. Let us call $Q_{ind.} = \alpha(t)$, with $\alpha(t)$ varying from 0 to 1 as h(t) varies also from 0 to 1. From Fig. 7 we have that $Q_{GS} = Q_{ind.}$ for $\rho > \rho_{E=0}(t)$ for any value of h(t). Since the fermion number is increased by one unit for $\rho > \rho_{E=0}(t)$, then $Q_{GS} = Q_{ind.} - 1 = \alpha(t) - 1$ whenever $\rho < \rho_{E=0}(t)$.

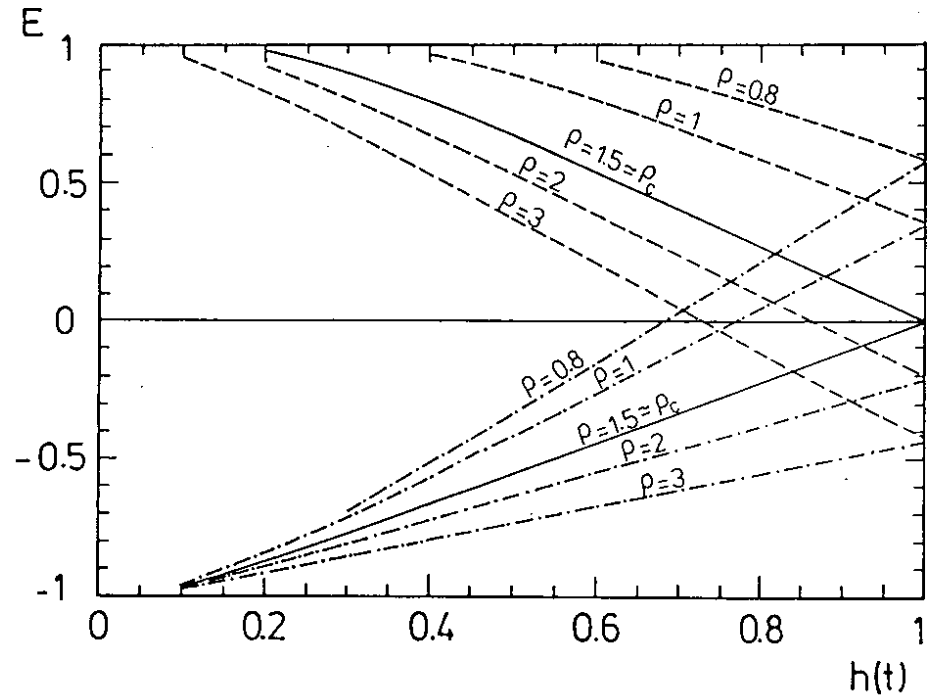


Figure 7: Fermion energy for M=0 as a function of the evolving scalar backgrounds, for different values of the size parameter $\rho = m_f\ \rho_s$. The energy values corresponding to an intermediate path with fermion flux at spatial infinity are drawn with dot-dashed lines, and those belonging to an intermediate path without fermion flux are drawn with dashed lines. The full lines correspond to the critical value $\rho = 1.5 \simeq \rho_c$. Note that at h(t) = 1 the energy eigenvalues coincide, since they correspond to the same background for each ρ.

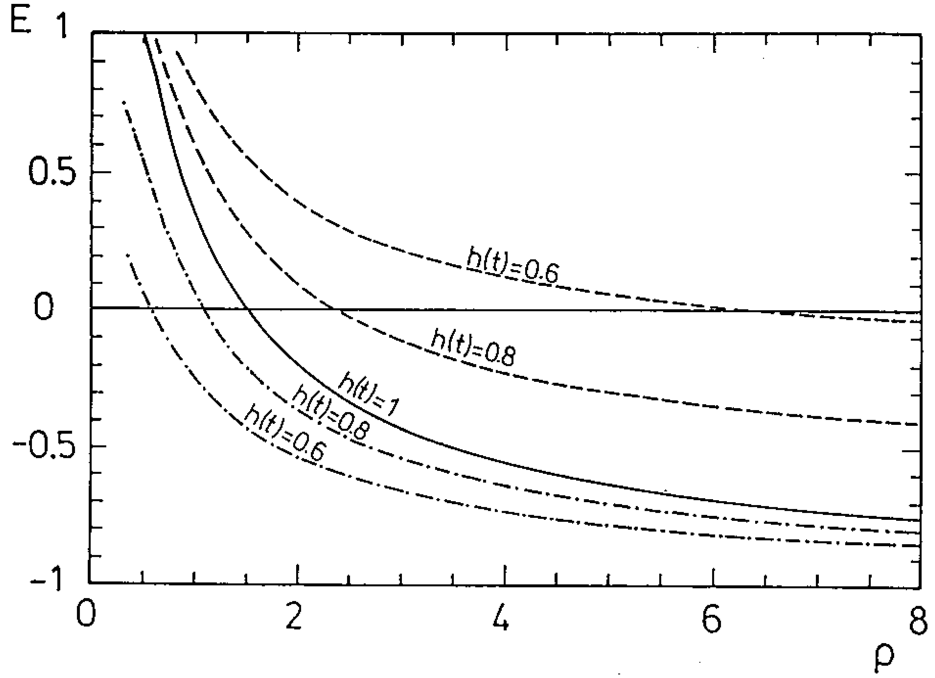
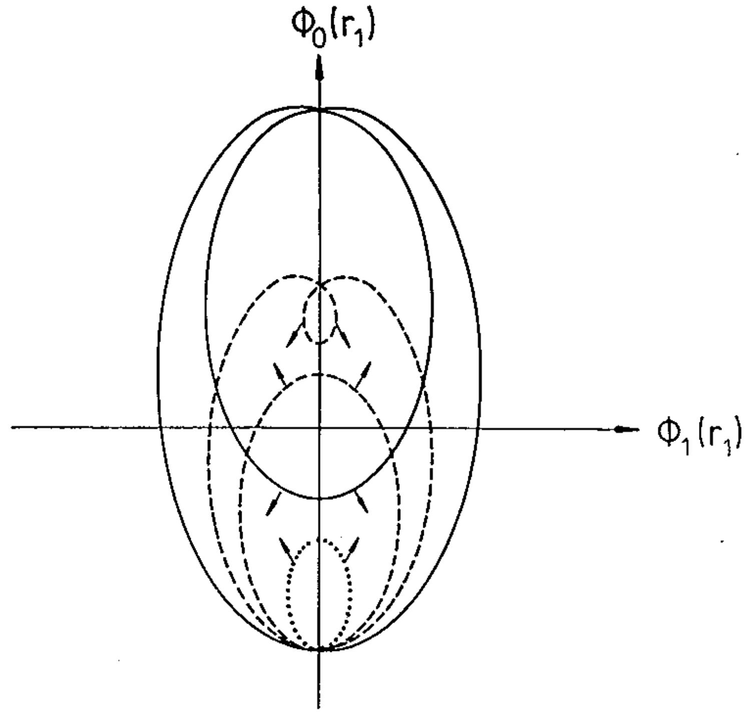


Figure 8: Fermion energy for M=0 as a function of the size parameter ρ. Different curves correspond to different scalar backgrounds, depending on h(t). The energy values corresponding to the static backgrounds with fermion flux at spatial infinity are given by dot-dashed lines and those of configurations without fermion flux, are given by dashed lines. The energy eigenvalues for the Skyrmion background of winding number one (h(t) = 1) are plotted as a full line.

4.6 Extension to Solitons of Topological Charge 2

We can now extend all the above considerations to scalar field configurations which interpolate, once more through different intermediate paths, between the vacuum and a final soliton of winding number 2 [13]. In the no fermion flux case, with scalar fields $(\phi_0(r,t), \phi(r,t)\hat{r})$, there are two different ways of building up the interpolating backgrounds. The first option, is to construct a configuration which vanishes twice for r=0, at two different values of t, going to the normal vacuum as r→ ∞. We illustrate the above option in Fig. 9a, in the dimensionally reduced version of the σ model: an O(2) theory in 1+1 dimensions. This is only because it is easier to visualize circles than three spheres. Using this way to build up the n=2 soliton, the results of chapter 3 can be easily extrapolated. This means that, since the correct induced charge differs by one unit from the adiabatic charge, whenever a zero of the scalar background appears at one intermediate time t at r=0, then with the present background, we have $Q_{ind.} = 0$, even though the adiabatic charge is $Q_{ad.} = Q_{top.} = 2$. The second possibility, which of course must be equivalent to the first one concerning the results for the induced charge, is the one in which the scalar fields vanish only once at a fixed value of t, but at r ≠ 0, and are the normal vacuum at r → ∞ and r=0. We illustrate this second option in Fig. 9b, also for 1+1 dimensions.

9a



9b

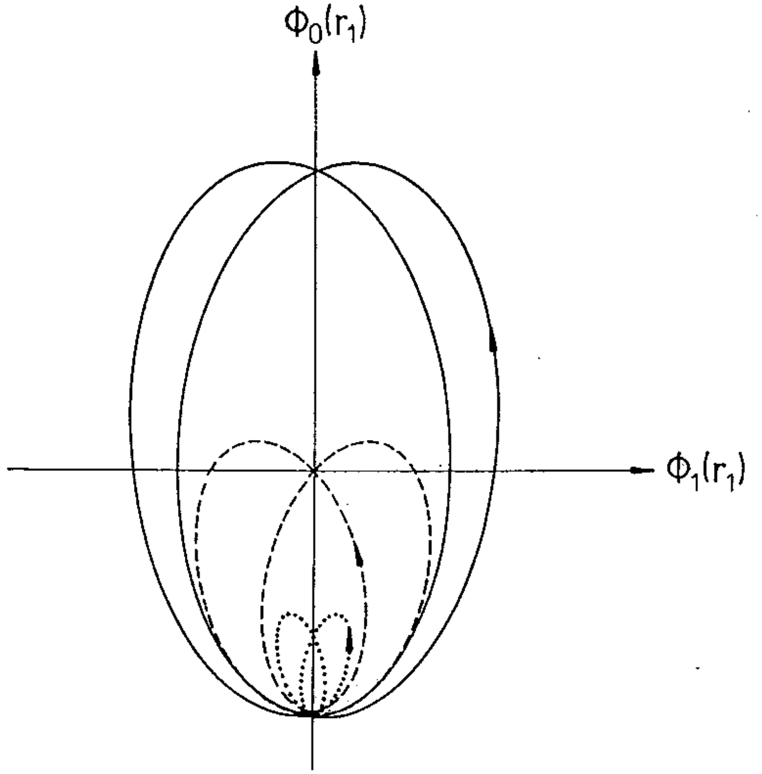


Figure 9: Parametric curves $\phi_0 = \phi_0(|r_1|)$, $\phi_1 = \phi(|r_1|)\, sgn(r_1)$, in 1+1 dimensions, for different intermediate scalar fields which evolve adiabatically from the vacuum to a final soliton of winding number two, with fixed conditions at spatial infinity: a)scalar configuration with two zero values at r=0 at different intermediate times. The dotted, dashed and full lines represent the backgrounds of winding number zero, one and two, respectively, at a fixed time. The arrows show how these backgrounds evolve into each other with increasing time. b) scalar configuration with a zero value at r $\neq$ 0 and at a fixed time t. The dotted and full lines are the static backgrounds of winding number zero and two, respectively. The dashed line is the static configuration when the winding number changes. The arrows show the direction of increasing r_1.

To study the number of zero energy modes in the no fermion flux case, consider for simplicity a scalar field configuration of the second type mentioned above which can be taken as

$$\phi_0 = v\ \{-1 + h(t)\,[1 + F_1(r)]\}$$
$$\vec{\phi} = v\ h(t)\, F_2(r)\ . \qquad (4.26)$$

where

$$F_1(r) = -\cos[2\ \arccos\ f_1(r)]$$
$$F_2(r) = \begin{cases} \sin[2\ \arcsin f_2(r)]\ \hat{r} & r \le \rho_s \\ -\sin[2\ \arcsin f_2(r)]\ \hat{r} & r \ge \rho_s \end{cases} \qquad (4.27)$$

For the fermion flux case, on the other hand, the scalar quartet reads:

$$\phi_0 = -v\ \cos\{2\,h(t)\,[\pi - \arccos\ f_1(r)]\}$$
$$\vec{\phi} = \begin{cases} v\ \sin[2\,h(t)\ \arcsin\ f_2(r)]\ \hat{r} & r \le \rho_s \\ -v\ \sin\{2\,h(t)\,[\pi + \arcsin\ f_2(r)]\}\ \hat{r} & r \ge \rho_s \end{cases} \qquad (4.28)$$

In both set of equations $f_1(r)$ and $f_2(r)$ have the same expressions as in Eq. (4.17) and h(t) varies monotonically from 0 to 1, as before. Eqs. (4.26), (4.28) reduce, for h(t) = 1, to the same soliton configuration of topological charge 2.

We solve Eq. (4.15) for each of the above backgrounds. We find [13], as shown in Fig. 10, two different solutions for P^+, G^- and P^-, G^+, with zero grand momentum, which have zero energy modes for certain values of ρ. This leads to two different critical values for the size parameter, $\rho_{c_1} \simeq 1.1$, for the set of equations involving P^+, G^- and $\rho_{c_2} \simeq 3.35$, for the set of equations involving P^-, G^+. In Figs. 11 and 12, we plot the energy eigenvalues as a function of the evolving scalar fields, for M = 0, with solutions for P^+, G^- and P^-, G^+, respectively. Analysing our results, we have again that the number of zero energy modes depend on the intermediate path, and in this example, in a more complicated way than before. For $\rho < \rho_{c_1}$, there exists two energy level crossings from $E < 0$ to $E > 0$, while building up the final n = 2, soliton through the path with fermion flux at spatial infinity, while no zero energy mode appears if the intermediate path allows no flux there. For $\rho > \rho_{c_2}$, instead, no energy level crossing occurs when the path allows fermion flux at spatial infinity and two zero energy level crossings from $E > 0$ to $E < 0$ exist when the path gives no fermion flux there. For $\rho > \rho_{c_1}$ and $\rho < \rho_{c_2}$, both intermediate configurations give, finally, one zero energy level crossing, but these appear in different M=0 orbitals and cross in opposite directions of the energy axis. On the other hand, doing an analogous analysis as in chapter 3 we have that, if we evaluate the adiabatic fermionic charge, $Q_{ad.}$, using Eq. (3.4) and consider the scalar fields given in Eq. (4.28), the result reads $Q_{ad.} = Q_{top.} = 2$. Since, with this background, the current expression is perfectly well defined at any point, the above result gives the correct induced charge, $Q_{ind.} = Q_{ad.} = 2$. However, if we use Eq. (4.26) the adiabatic result is still two but, as we already remarked, the correct induced charge is zero. From the above results it is possible to deduce that the number and direction of the energy level crossings and the value of the induced fermionic charge behave, in both cases, so as to give:

$$
\begin{aligned}
Q_{GS} &= 0 \qquad for\ \ \rho < \rho_{c_1}, \\
Q_{GS} &= 1 \qquad for\ \ \rho > \rho_{c_1}\ \ and\ \ \rho < \rho_{c_2}, \\
Q_{GS} &= 2 \qquad for\ \ \rho > \rho_{c_2}.
\end{aligned} \tag{4.29}
$$

This implies that the soliton fermionic number can be identified with its topological charge whenever $m_f > 3.35/\rho_s$. The same analysis is possible for solitons of winding number greater than 2.

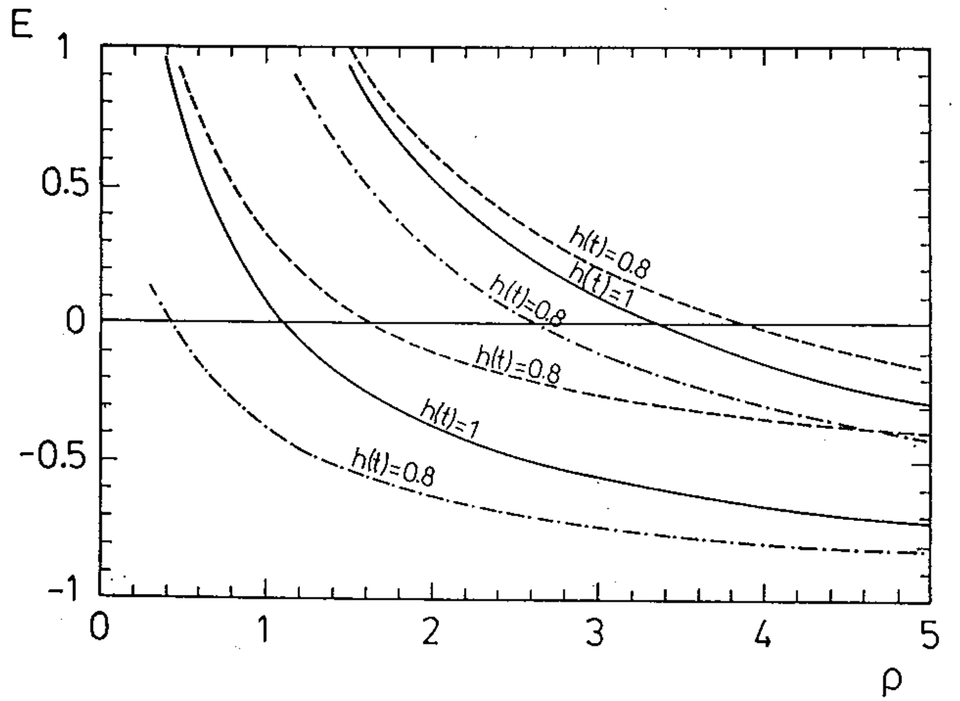


Figure 10: The same as Fig. 8 but with the scalar background evolving from the vacuum to a soliton of winding number 2. Two M=0 orbitals corresponding to solutions of opposite parity give zero energy modes. At h(t) = 1 (n=2 soliton) we have two critical values of the size parameter $\rho_{c_1} \simeq 1.1$ and $\rho_{c_2} \simeq 3.35$.

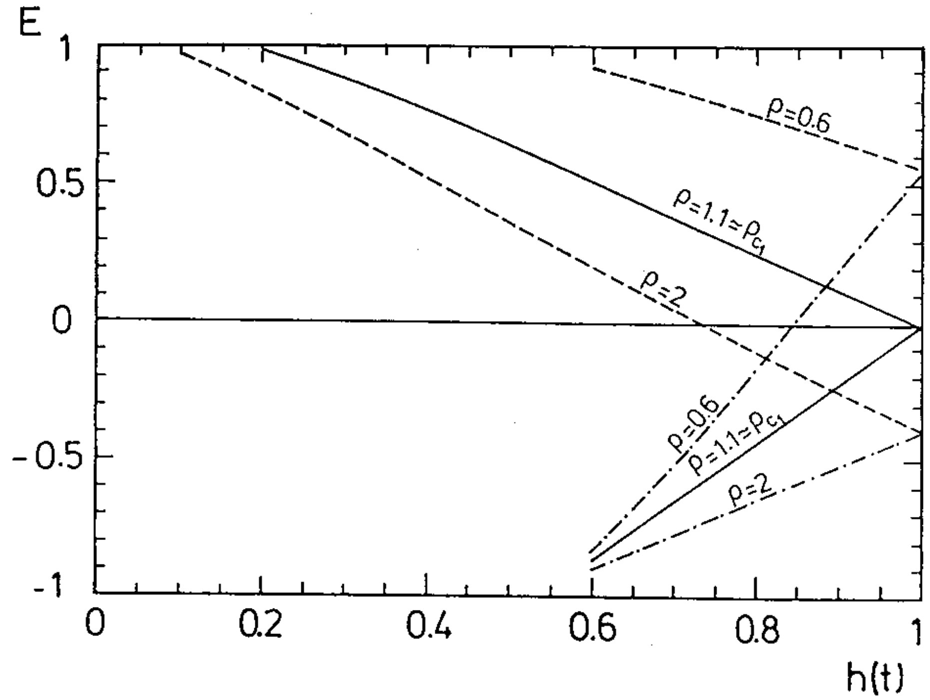


Figure 11: The same as Fig. 7 but for the final n=2 soliton and the M=0 orbital that gives the critical value ρ_{c_1}. It corresponds to the set of equations involving P^+, G^-.

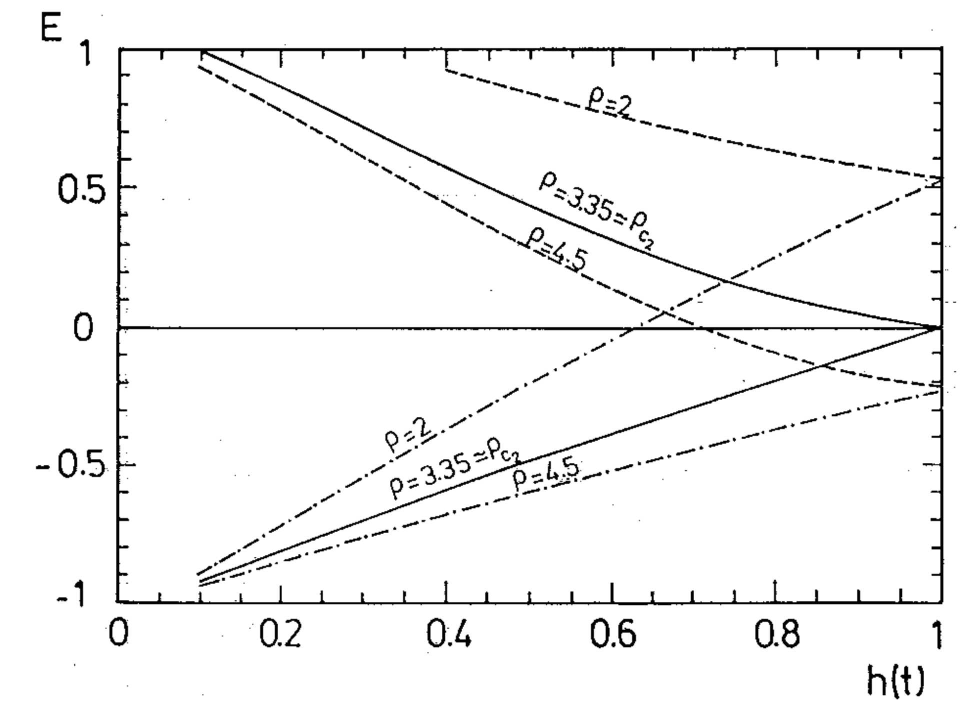


Figure 12: The same as Fig. 11 but for the M=0 orbital that gives ρ_{c_2}. It corresponds to the set of equations involving P^-, G^+.

Chapter 5

Fermionic Charge and Statistics in 2+1 dimensions

All the above considerations have been done in 3+1 dimensions. In the first section of this chapter, we still shall remain in 4 dimensions to demostrate the self consistency of the fermionic charge of the soliton with its statistics. The relation between fermion charge and spin in this case has been already analysed in the literature [29], [37]-[40]. We emphasize it here, not only for completeness, but particularly because we want to analyse below the analogous issue more carefully in 3 dimensions. In fact, due to the recent revived interest in 2+1 dimensional gauge theories, particularly concerning the influence of the Chern-Simons term on the spin and statistics of charged particles [41]-[48], we devote the rest of this last chapter to consider, quite in general, vacuum polarization effects of fermions interacting with abelian gauge fields in three dimensions. The Chern-Simons term has many analogies with the Wess-Zumino term in 3+1 dimensions. It may be radiatively generated, at large fermion masses, when calculating the effective gauge field action due to fermions minimally coupled to gauge fields [21],[22]. We will evaluate the induced charge and angular momentum in the frame of the effective gauge theory, or directly considering the polarization effects in the Dirac sea, which are due to the presence of the gauge background configuration. Our aim is to show that in 2+1 dimensions the relation between fermionic charge and spin is not always as expected in principle. Actually, we will show that one can associate a bosonic configuration, which induces fermion number one, with a fermion only for restricted values of the factor in the Chern-Simons term, since only for such values the charge and statistics are correlated.

5.1 Fermion Charge and Statistics: Self Consistency in 3+1 Dimensions

In all the above considerations we have never asked about the statistics induced whenever a fermionic number has appeared. Actually, we have tacitly assumed that, for example, whenever a scalar soliton carries the fermion number of a fermion, then the statistics associated to that soliton must be fermionic. This fact is, indeed, already proven in the literature. D'Hoker and Fahri [37],[38] have evaluated, both in general and in particular for the Standard model the full low energy effective action remaining after the decoupling of a fermion, which is rendered infinitely heavy by making the Yukawa coupling large, while keeping the expectation value of the Higgs field fixed. A Wess-Zumino term as a function of the Higgs and gauge fields is induced in the theory in such a context. D'Hoker and Fahri further show that the decoupling of one fermion changes the statistics of the soliton, which may be present in the scalar sector. Since their demostration is based on generating heavy fermions through a large Yukawa coupling, it may in principle fail due to the fact that even in the limit of a large bare Yukawa coupling the triviality of the Yukawa theory is suspected [49]. For example, calculations on the lattice have shown [50], that in the case of the standard Higgs model, the renormalized coupling constant, λ_R, remains small for any reasonable large value of the cutoff. Such study is done for all values of the bare coupling λ, including the limit $\lambda \to \infty$, where, as we said before, the theory reduces to the nonlinear σ model. However, until now, no similar rigorous analysis has been done for the case of the Yukawa coupling and there is no overwhelming argument to affirm that the above approach must be unphysical. At any rate, even though the absolute decoupling of the fermion may be questioned, the Wess-Zumino term may be considered as arising naturally for low energies with respect to the mass of the fermion fields.

We discuss explicilty here, but very shortly, the case when the fermion is only coupled to a scalar field. Starting with the Lagrangian of Eqs. (2.4), (2.6)

$$\begin{aligned}\mathcal{L}_F &= i\bar{\psi}\partial_\mu\gamma^\mu\psi - \frac{g_y}{\sqrt{2}}\bar{\psi}\,(\phi_0 + i\gamma_5\vec{\phi}\cdot\vec{\sigma})\,\psi \\ &= i\bar{\psi}_L\partial_\mu\gamma^\mu\psi_L + i\bar{\psi}_R\partial_\mu\gamma^\mu\psi_R - g_y\left(\bar{\psi}_L\Phi\psi_R + \bar{\psi}_R\Phi^+\psi_L\right) \qquad (5.1)\end{aligned}$$

and assuming that $<\phi_0>= v$, the fermions acquire a mass $m_f = g_y v/\sqrt{2}$. One can always compute the full effective action of the scalar fields

$$W(\Phi) = -\imath \ln\left[\int [d\psi_L][d\psi_R]\exp\left(\imath\int d^4x\,\mathcal{L}_F\right)\right] \qquad (5.2)$$

as the mass of the fermion is sent to infinity by increasing the Yukawa coupling. This computation was done by D'Hoker and Fahri [37], using dimensional regularization, and they obtain

$$W(\Phi) = W'(\Phi) + \Gamma(\hat{\Phi}), \qquad (5.3)$$

with $\Phi = |\phi|\hat{\Phi}$. The first term, $W'(\Phi)$, is given as a function of the scalar fields, an arbitrary regularization scale and quantum corrections to the scalar field mass and coupling, but its exact expression is not of particular interest for our present purpose. The second term , $\Gamma(\hat{\Phi})$, is the Wess-Zumino effective action when no gauge field is present, and is the one which leads to important consequences for the statistics in the system. Its evaluation is more conveniently done in terms of a continuously differentiable interpolating function :

$$\tilde{\Phi}(y, x^\mu) = \begin{cases} 1 & y = 0 \\ \hat{\Phi} & y = 1 \end{cases} \qquad (5.4)$$

Calling Q the domain of $[0,1]\times$ *space time*, where at y=0 (as $\tilde{\Phi} = 1$) all points of space time can be identified, one gets

$$\Gamma(\hat{\Phi}) = \frac{-\imath}{240\pi^2}\int_Q d^5x\,\epsilon^{\alpha\beta\gamma\delta\epsilon}Tr[\tilde{\Phi}^+\partial_\alpha\tilde{\Phi}\;\tilde{\Phi}^+\partial_\beta\tilde{\Phi}\;\tilde{\Phi}^+\partial_\gamma\tilde{\Phi}\;\tilde{\Phi}^+\partial_\delta\tilde{\Phi}\;\tilde{\Phi}^+\partial_\epsilon\tilde{\Phi}], \qquad (5.5)$$

with $\alpha,\beta,\gamma,\delta,\epsilon = 0, 1, 2, 3, 4$ and $x^4 = y$. The only condition required of $\tilde{\Phi}$ is that of Eq. (5.4). So, in general, we have assumed that the interpolating function $\tilde{\Phi}$ exists. However, the existence of $\tilde{\Phi}$ is dependent on whether or not $\hat{\Phi}$ can be smoothly connected to the identity. To study this point it is convenient to make a Wick rotation, working in Euclidean space time. If the configuration $\tilde{\Phi}$ goes to a constant at space time infinity for all values of y, space time (R^4) may be compactified to S^4. In particular, any configuration $\hat{\Phi}$ defined within the group $SU(n)$, may be characterized by the homotopy class in $\pi_4(SU(n))$ to which the map $\hat{\Phi}(x)$ belongs. The question about the possibility of defining the interpolating function reduces to whether $\pi_4(SU(n))$ has more than one element. Since for $n > 2$ one has $\pi_4(SU(n)) = 0$, any configuration $\hat{\Phi}$ is homotopically equivalent to the identity, implying that the interpolating function exists, since one can always find a $\tilde{\Phi}$ which belongs to $SU(n)$ and interpolates between the identity and $\hat{\Phi}$. For $n = 2$, instead, which is our case of interest, $\pi_4(SU(2)) = Z_2$, and thus it has two elements. A configuration $\hat{\Phi}$ belonging to the same homotopy class as the identity may be smoothly connected to it. Therefore, $\tilde{\Phi}$ is chosen as an element of SU(2) and, since there is no rank 5 antisymmetric tensor in $SU(2)$, $\Gamma(\hat{\Phi})$ is identically zero. On the other hand if the configuration $\hat{\Phi}$ is in the non-trivial homotopy class in $\pi_4(SU(2))$, $\tilde{\Phi}$ cannot be defined within SU(2) and must be chosen as belonging to a larger group like SU(3), which has trivial fourth homotopy group (at y = 0,1, $\tilde{\Phi}$ is an element of SU(2), trivially embedded in SU(3)). Thus the Wess-Zumino term must not necessarily vanish in this case. Actually, a laborious explicit calculation, done by Witten [39], has given,

$$\Gamma(\hat{\Phi}) = \begin{cases} 0 & \hat{\Phi}\ trivial\ in\ \pi_4(SU(2)) \\ \pi\ mod\ 2\pi & \hat{\Phi}\ non-trivial\ in\ \pi_4(SU(2)) \end{cases} \qquad (5.6)$$

The above result was also obtained in Ref. [40]. The appearence of the Wess-Zumino term not only changes the statistics of the solitons that may be contained in the scalar theory, but also reproduces the anomaly content of the decoupled fermion, when gauge fields are included [37],[38]. To analyse if the soliton of winding number one is a fermion or a boson, one can rotate it adiabatically through a 2π angle in the course of a long time period T, and evaluate the action in this process. The only nonvanishing contribution comes from the Wess Zumino functional Γ, and it has been already shown [39],[40], that the Γ term of a 2π spatial rotation of the soliton is equal to π. The effective action is shifted by π and this leads to a factor $\exp(\imath\pi) = -1$, which implies odd statistics for the soliton. In fact, in Ref. [39] Witten realized that the non-trivial homotopy class in $\pi_4(SU(2))$ differs from the trivial class by a 2π rotation of a soliton, or equivalently, that the field $\tilde{\Phi}_S$, which defines the adiabatically rotating soliton belongs to the non-trivial homotopy class of $\pi_4(SU(2))$.

From the above considerations we conclude that the soliton, which was a boson for low Yukawa couplings, becomes a fermion for large fermion masses. Moreover, the adiabatic method gives us that the soliton configuration, considered as background in the Dirac sea, carries the fermion number of any fermion with mass m_f greater than the typical mass scale of the soliton, $1/\rho_s$, or, equivalently in the effective theory, it takes the fermion number of the decoupled fermion. In theories with Yukawa couplings in 3+1 dimensions the picture obtained from the adiabatic results and those coming from the effective action approach are complementary. The soliton carries the fermion number of heavy fermions, and from the effective action one obtains that it behaves like a fermion. However, there is, so far, no exact proof of an equivalence between the purely bosonic approach and a local fermionic model in 3+1 dimensions, as it exists in 1+1 dimensions [51], [52], where the bosonization transformation has been developed many years ago.

5.2 QED in 2+1 Dimensions: The Gauge Field Effective Theory Approach

We start considering the action of a two component fermion doublet field $\psi(x)$, minimally coupled to the gauge field $A_\nu(x)$,

$$S_F = \int d^3x\, \bar{\psi}\, (\imath \mathcal{D}^\nu \gamma_\nu - m_f)\, \psi \tag{5.7}$$

with $\mathcal{D}^\nu = \partial^\nu + \imath e A^\nu$ and we define $S_F^0 = S_F(A^\nu = 0)$. The effective action may be written as

$$W[A] \;=\; -\imath \ln \left(\frac{\int \mathcal{D}\bar{\psi}\, \mathcal{D}\psi\, \exp\left(\imath S_F[\bar{\psi}\psi A]\right)}{\int \mathcal{D}\bar{\psi}\, \mathcal{D}\psi\, \exp\left(\imath S_F^0[\bar{\psi}\psi A]\right)} \right) \tag{5.8}$$

Furthermore, functionally differentiating the effective action expression with respect to the gauge field, one gets

$$\frac{\delta W[A]}{\delta A^\nu} = -e < \bar{\psi}\gamma_\nu\psi > = -e < j_\nu(x) > \tag{5.9}$$

One can evaluate the effective action using different regularization schemes, and show [21]-[23] that, in the large mass limit ($m_f \to \infty$) one gets,

$$lim_{m_f \to \infty} W[A] = \frac{e^2\mu}{2} \int d^3x\; \epsilon^{\nu\rho\delta}\, A_\nu(x) \partial_\rho A_\delta(x) \tag{5.10}$$

Thus,

$$< j^\nu(x) >_{m_f \to \infty} = -e\mu\epsilon^{\nu\rho\delta}\partial_\rho A_\delta(x) \tag{5.11}$$

The parameter μ involves the regularization ambiguities that appear in 2+1 dimensions [20], [23],[24] and its value is, therefore, fixed depending on the regularization scheme used. Eq. (5.11) gives the perturbative adiabatic result. This means, when integrating the current density j_0 in space, it gives accurately, at least, the fractional part of the fermion charge.

Adding also the gauge action

$$S_G = -\frac{1}{4} \int d^3x\, F_{\mu\nu} F^{\mu\nu} \tag{5.12}$$

one works with the theory defined through the total action,

$$S = W[A] + S_G\,. \tag{5.13}$$

As it is well-known [20], [53], the appearence of the Chern-Simons term makes the gauge field massive, giving a field strength which vanishes at infinity. However, the vector gauge potential, $\vec{A}$, has non-trivial long range effects, and allows a non vanishing magnetic flux, ϕ_B. We analyse the gauge field equations of motion corresponding to the effective theory from Eq. (5.13),

$$\partial_\rho F^{\rho\nu} + e^2\mu\epsilon^{\nu\rho\alpha}\partial_\rho A_\alpha = 0\,, \tag{5.14}$$

to show that the parameter μ (multiplied by e^2, since it is a dimensionless quantity) may be interpreted as the mass of the gauge fields. For that purpose, we define $B_\nu = \epsilon_{\nu\rho\lambda}F^{\rho\lambda}$, then $F^{\rho\lambda} = \epsilon^{\rho\lambda\nu}B_\nu/2$, and multiplying the equations of motion by $(e^2\mu g_{\sigma\nu} + \epsilon_{\sigma\rho\nu}\partial^\rho)$ we have,

$$\begin{aligned} \left(e^2\mu g_{\sigma\nu} + \epsilon_{\sigma\rho\nu}\partial^\rho\right)\left(e^2\mu B^\nu + \partial_\alpha \epsilon^{\alpha\nu\lambda} B_\lambda\right) &= 0 \\ \left(e^4\mu^2 + \Box\right) B_\sigma &= 0 \end{aligned} \tag{5.15}$$

where the identity $\partial_\nu B^\nu = 0$ has been used in the above. As noted before, Eq. (5.15) shows that the field strength tensor propagates as a free field of mass $e^2\mu$.

To get a qualitative description of the physics involved in the above model, it is adequate to consider the equations of motion of the gauge field in the presence of a static point source of electric charge $q \equiv eQ_{ext.}$. They read,

$$\partial_\rho F^{\rho\nu} + e^2\mu\epsilon^{\nu\rho\lambda}\partial_\rho A_\lambda - q g^{0\nu}\delta^2(x) = 0 \tag{5.16}$$

One can solve these equations exactly and obtain the static solution [53],

$$\begin{aligned} B &= -\frac{qe^2\mu}{2\pi} K_0(e^2\mu r) \\ \vec{E} &= -\frac{q}{2\pi}\vec{\nabla} K_0(e^2\mu r)\,, \end{aligned} \tag{5.17}$$

with K_0 a modified Bessel function and where we have identified: $E^i = -F^{0i}$, $B = -F^{12} = -\epsilon_{ij}\partial_i A_j$. For our particular purposes, it is useful to evaluate the $\nu = 0$ component of Eq. (5.16), and integrate over all the two dimensional space:

$$\int d^2x\, \left(\vec{\nabla}\cdot\vec{E} - e^2\mu B - q\delta^2(x)\right) = 0 \tag{5.18}$$

Because of the result gotten in Eq (5.17), the field $\vec{E}$ behaves for large r as $\exp(-\mu r)$. Thus the first term in the above equation may be dropped and we have a relation between the external charge and the magnetic flux, as follows [20],

$$Q_{ext.} = -e\mu \int d^2x\, B = -e\mu\phi_B \tag{5.19}$$

Furthermore, since from above the induced current is obtained to be $j^{\nu}_{ind.}(x) = -e\mu\epsilon^{\nu\rho\lambda}\partial_\rho A_\lambda(x)$, the fermionic induced charge is given as,

$$Q_{ind.} = \int d^2x\, j^0_{ind.}(x) = -e\mu \int d^2x\, \epsilon^{0ij}\partial_i A_j(x) = e\mu\phi_B \tag{5.20}$$

We get a fermionic charge $Q_{ind.}$ and an electric charge $eQ_{ind.}$ induced by the magnetic flux, which appears in the presence of the external electric charge q. We note that the value of the induced electric charge is exactly the one which makes the total electric charge vanish. The external electric charge is screened, as expected, due to the asymptotic behaviour of the electric field.

We may compute the angular momentum carried by the electromagnetic fields, given by:

$$J_G = \int d^2x\, \epsilon_{ij}\, x^i\, \theta^{0j} \tag{5.21}$$

where the energy momentum tensor expression is:

$$\theta^{\nu\rho} = \frac{1}{4}g^{\nu\rho}F^2 + F^{\nu\sigma}F_\sigma{}^\rho \tag{5.22}$$

The above leads to

$$J_G = -\int d^2x\, B\left(\vec{x}\cdot\vec{E}\right) . \tag{5.23}$$

With the expressions of the electric and magnetic fields given in Eq. (5.17), a simple calculation, using the properties of the modified Bessel functions, $K_n(z)$, gives:

$$\begin{aligned} J_G &= -\frac{q^2\epsilon^2}{2\pi}\mu \int_0^\infty r^2\, dr\, K_0(\epsilon^2\mu r)\partial_r K_0(\epsilon^2\mu r) \\ &= \frac{Q^2_{ext.}}{2\pi\mu}\int_0^\infty dz\, z^2 K_0(z)K_1(z) \\ &= \frac{Q^2_{ext.}}{4\pi\mu} \end{aligned} \tag{5.24}$$

Using Eqs. (5.19), (5.20), we write the angular momentum J_G in terms of the induced fermionic charge, or equivalently, in terms of the magnetic flux,

$$J_G = \frac{Q^2_{ind.}}{4\pi\mu} = \frac{\epsilon^2\mu}{4\pi}\phi_B^2 \tag{5.25}$$

Although the above definition of the angular momentum, Eq. (5.21), seems to be natural, we must point out that other choices have been made in the literature [53]. In what follows, we restrict our discussion only to the expression considered in Eq. (5.21) to calculate the angular momentum of the Coulomb field. The result of Eqs. (5.24), (5.25) agree with those obtained previously in the literature [47], [53], with μ undetermined.

5.3 Vacuum Polarization Effects from Gauge Background Fields

Above we have considered the heavy mass fermion theory through its effects on the gauge theory. However, to demostrate that the results obtained within that approach are self consistent, it will be useful to show that the fermionic vacuum has been polarized, so as to give zero angular momentum. At the same time we compute the fermion charge value by using counting arguments directly on the energy spectrum. In this way we obtain in a very simple way the value of μ, while using a Pauli-Villars cutoff, in the case where only one extra heavy fermion is introduced in the theory.

In the framework of heavy fermions coupled to gauge background fields, one can compute the total angular momentum as,

$$\begin{aligned} J &= \int d^2x\, \psi^+(x)\left(\vec{x}\times\left(\vec{p}-e\vec{A}\right)+\frac{1}{2}\sigma_3\right)\psi(x) \;+\int d^2x\;\vec{x}\times\left(\vec{E}\times\vec{B}\right) \\ &\equiv J_F + J_{int.} + J_G , \end{aligned} \tag{5.26}$$

where J_G is the electromagnetic angular momentum, already defined and computed in Eqs. (5.21), (5.23), (5.24) and (5.25). The other terms are going to be computed below. Within the present context, a discussion about considering an extra term in the expression of Eq. (5.26) has been also done in the literature [54],[55]. However, for the gauge field configuration we are considering, the extra pure surface term proposed there gives no contribution.

To evaluate the interaction term in Eq. (5.26), it is useful to give the explicit expression of the vector potential $\vec{A}$, corresponding to the magnetic field B from Eq. (5.17). We define a function $F(\epsilon^2\mu r)$ as,

$$B = \frac{1}{r}\frac{d}{dr}F(\epsilon^2\mu r) \tag{5.27}$$

Using the explicit expression for the magnetic field, we have to find $F(z\equiv\epsilon^2\mu r)$ satisfying,

$$-\frac{q\epsilon^2\mu}{2\pi}K_0(z) = \frac{\epsilon^4\mu^2}{z}\frac{d}{dz}F(z), \tag{5.28}$$

which implies,

$$F(z) = -\frac{q}{2\pi\epsilon^2\mu}\int z\,dz\,K_0(z). \tag{5.29}$$

Using, once more, the properties of $K_0(z)$ and $K_1(z)$, we finally get,

$$F(z) = -\frac{q}{2\pi\epsilon^2\mu}\left[C - zK_1(z)\right] , \tag{5.30}$$

with C a constant to be determined below. From the previous section, in which we consider the effective theory for QED in the limit of heavy fermions, we see that we have a theory of massive photons and charged particles, whose static gauge invariant field strength has finite range electric and magnetic components. The magnetic field has, in the presence of a point charged particle, a non vanishing total magnetic flux, and therefore, at large distances, Stokes' theorem requires that,

$$\phi_B = \int d^2x\, B = \oint_{C_\infty} A^i\, dx^i . \tag{5.31}$$

In terms of the radial function $F(\epsilon^2\mu r)$, the vector potential may be written as:

$$A_i = -\epsilon_{ij}\frac{x_j}{r^2}F(\epsilon^2\mu r) . \tag{5.32}$$

Taking from Eq. (5.32) the expression of the vector potential at spatial infinity and evaluating the integral in Eq. (5.31) the value of the constant C turns out trivially to be one. The modified Bessel function $K_1(z)$ decays exponentially for very large z and for $z \to 0$ one has $K_1(z \to 0) \to 1/z$. Thus, the asymptotic behaviour of the radial function $F(\epsilon^2\mu r)$ reads,

$$\begin{aligned} F(0) &= 0 \\ F(\infty) &\equiv \frac{F}{\epsilon} \to -\frac{Q_{ext.}}{2\pi\epsilon\mu} \end{aligned} \tag{5.33}$$

For completness one observes that, in terms of this function, the magnetic flux is $\phi_B = 2\pi F(\infty) = -Q_{ext}/e\mu$, and the vector potential A_i falls off like $1/r$ at infinity. We can now evaluate, explicitly, the piece of the angular momentum coming from the interaction, as follows,

$$\begin{aligned} J_{int.} &= -\epsilon \int_0^\infty d^2x\ \psi^+ \left(\vec{x} \times \vec{A}\right)\ \psi \\ &= -e \int_0^\infty d^2x\ \epsilon_{ij}\ x_i\ A_j\ j_0 \\ &= \epsilon^2 \int_0^\infty d^2x\ \epsilon_{ij}\ x_i\ \epsilon_{jk}\ \frac{x_k}{r^2}\ F(\epsilon^2\mu r)\ \mu\ B \\ &= -2\pi\epsilon^2 \int_0^\infty r\, dr\ F(\epsilon^2\mu r)\ \mu\ \frac{1}{r}\ \frac{d}{dr}F(\epsilon^2\mu r) \\ &= -\pi\mu\epsilon^2 F(\infty)^2 \\ &= -\frac{Q_{ext}^2}{4\pi\mu} \end{aligned} \tag{5.34}$$

To compute J_F it is necessary to know explicitly the fermionic energy spectrum. However, to solve the Dirac equation exactly in the self consistent fields given in Eq. (5.17) would be very complicated. Since we want to evaluate spectral asymmetries in order to compute induced quantum numbers, and it is already shown extensively in the literature [22],[47],[54],[56], [57] and follows from the above that these quantum numbers only depend on the magnetic flux, we proceed here by taking directly the static vector potential, Eq.

(5.32), as background. Moreover, we will use some very simple counting arguments to compute the induced quantum numbers, assuming that $\epsilon F(\infty) \equiv F$ is an integer. Otherwise the counting arguments become obscure. However the final result is expected to remain the same for arbitrary F, as suspected from the general calculations done by other authors [47], where the inclusion of a short range electrostatic potential is considered. The calculations that follows are based on those done in Ref [55].

Considering then A_i as given in Eq. (5.32) and setting $A_0 = 0$, one has that the Dirac Hamiltonian

$$H = -\imath\partial_i\gamma^0\gamma^i + eA_i\gamma^0\gamma^i - \gamma^0 m_f , \tag{5.35}$$

is rotationally invariant, $[H, \hat{J}_F] = 0$, with $\hat{J}_F = -\imath\epsilon_{ij}x^i\partial_j + \sigma_3/2 = L + S$ the canonical angular momentum. The eigenstates of H, $H\psi = E\psi$, may be written as

$$\psi(r,\theta) = \frac{\exp(\imath J\theta)}{\sqrt{r}} \begin{bmatrix} \exp(-\imath\theta/2)\ f(r) \\ \exp(\imath\theta/2\ )\ g(r) \end{bmatrix} \tag{5.36}$$

with $J = \pm 1/2, \pm 3/2, \ldots$ and $f(r)$ and $g(r)$ radial functions. The Dirac equation reduces to

$$\begin{aligned} \left\{\frac{d}{dr} + \frac{1}{r}\left[J - eF(\epsilon^2\mu r)\right]\right\} g(r) &= (E - m_f)\, f(r) \\ \left\{-\frac{d}{dr} + \frac{1}{r}\left[J - eF(\epsilon^2\mu r)\right]\right\} f(r) &= (E + m_f)\, g(r) \end{aligned} \tag{5.37}$$

From the above equations it is easy to find the solutions

$$\begin{aligned} &f(r) = \exp\left\{\int^r \frac{dr'}{r'}\left[J - eF(\epsilon^2\mu r')\right]\right\} , \quad g(r) = 0 && for\ E = m_f \\ &f(r) = 0\ , \quad g(r) = \exp\left\{-\int^r \frac{dr'}{r'}\left[J - eF(\epsilon^2\mu r')\right]\right\} && for\ E = -m_f \end{aligned} \tag{5.38}$$

Thus considering the asymptotic behaviour of $F(\epsilon^2\mu r)$ given in Eq. (5.33) and requiring normalizability and regularity at the origin for the proposed solutions , one has:

For $E = m_f$,

$$\psi(r,\theta) = \frac{\exp(\imath(2J-1)\theta/2)}{\sqrt{r}} \begin{bmatrix} f(r) \\ 0 \end{bmatrix}$$

and

$$f(r) \simeq \begin{cases} r^J & \text{as} \quad r \to 0 \\ r^{J-F} & \text{as} \quad r \to \infty \end{cases} , \tag{5.39}$$

The above requires for the angular momentum the condition $-1/2 < J \leq -1/2 + F$. Since our analysis restricts to the case $F = integer$, this implies that F must be positive, $F = 1, 2, \ldots N$.

We observe that

$$F=1 \;\rightarrow\; J=\frac{1}{2}$$
$$F=2 \;\rightarrow\; J=\frac{1}{2},\frac{3}{2}$$
$$\vdots$$
$$F=N \;\rightarrow\; J=\frac{1}{2},\frac{3}{2},\ldots\frac{2N-1}{2}$$

There are in general F threshold states.

On the other hand for $E=-m_f$,

$$\psi(r,\theta)=\frac{\exp\left(\imath(2J+1)\theta/2\right)}{\sqrt{r}}\begin{bmatrix}0\\ g(r)\end{bmatrix}$$

and

$$g(r)\simeq\begin{cases} r^{-J} & \text{as} \quad r\rightarrow 0\\ r^{-(J-F)} & \text{as} \quad r\rightarrow\infty\end{cases}, \tag{5.40}$$

The above gives the condition $1/2 > J \geq 1/2+F$ and implies that F must be negative, $F=-1,-2,\ldots-N$. Analogously, as before, one gets,

$$F=-1 \;\rightarrow\; J=-\frac{1}{2}$$
$$F=-2 \;\rightarrow\; J=-\frac{1}{2},-\frac{3}{2}$$
$$\vdots$$
$$F=-N \;\rightarrow\; J=-\frac{1}{2},-\frac{3}{2},\ldots-\frac{2N-1}{2}$$

and there are $|F|$ states with energy $E=-m_f$.

In the following, we will assume that there are no other bound states apart from the threshold states at $E=\pm m_f$, and that if F is an integer only these states contribute to the spectral asymmetry. We base this consideration on several papers which give support to this issue [58]-[61]. Since the above picture is obtained by working in a fixed gauge, a gauge invariant regularization is required to assure that the correct physical results are obtained while evaluating the fermion charge and angular momentum within this frame. One possibility is to use the Pauli-Villars regularization, which basically consists on cancelling the ultraviolet divergences by substracting the contributions of a heavy fermion to those of the fundamental fermion of the theory. Furthermore, with the above assumption we determine the coefficient

μ in agreement with the results obtained in detailed rigorous studies [21], [23], [24].

It is our purpose to evaluate the fermionic angular momentum, J_F, but first we will use the above counting arguments to compute the fermionic charge, which in this case is the ground state charge, since the only bound states are at the threshold. Thus, we have only to *count* the spectral asymmetry. We assume first we are in the case with a positive fermion mass, $m_f>0$, and from the above we have that, for $F>0$ there exist F states with positive energy. The fermionic charge, obtained from the contribution of the fundamental fermion is then given by,

$$\begin{aligned} Q_{ind._f}=Q_{GS_f} &= -\frac{1}{2}\eta_{[H]}\\ &= -\frac{1}{2}\left(\sum_{E>0}1-\sum_{E<0}1\right)\\ &= -\frac{1}{2}\sum_{1}^{F}1=-\frac{F}{2} \end{aligned} \tag{5.41}$$

For $F<0$ there are $|F|$ states of negative energy, however, they imply the same value for this part of the fermionic charge, since in this case one has,

$$Q_{ind._f}=Q_{GS_f}=-\frac{1}{2}\left(-\sum_{1}^{|F|}1\right)=\frac{|F|}{2}=-\frac{F}{2} \tag{5.42}$$

If we assume a negative value for m_f then the sign of the energy is reversed implying that for $F>0$ there are F states, but now of negative energy, and for $F<0$ there are still $|F|$ threshold states, but of positive energy. A trivial counting as above leads in this case to

$$Q_{ind._f}=Q_{GS_f}=\frac{F}{2} \tag{5.43}$$

From these results one concludes that the general expression for the fermionic charge due to the polarization effects in the Dirac sea of the fundamental fermion is

$$Q_{ind._f}=Q_{GS_f}=-\frac{m_f}{|m_f|}\frac{F}{2}=-\frac{m_f}{|m_f|}\frac{e}{4\pi}\phi_B \tag{5.44}$$

We have now to evaluate the contribution of the extra heavy particle (ghost) to be considered within the Pauli-Villars regularization. Basically the ghost must have associated an energy spectrum equivalent to that of the fermion. Thus it must give a contribution to the spectral asymmetry to be substracted from that one of the fundamental spectrum. Therefore, it is straightforward to compute the fermion charge as

$$Q_{GS} \;=\; -\left(\frac{m_f}{|m_f|}-\frac{m_{ghost}}{|m|_{ghost}}\right)\frac{F}{2}, \tag{5.45}$$

where m_{ghost} is the mass of the extra heavy fermion. Depending on the sign of the ghost mass relative to that of the fundamental fermion mass, we have,

$$Q_{GS}=\begin{cases} 0 & \text{if } sg(m_f)=sg(m_{ghost})\\ -2(m_f/|m_f|)(F/2)=-(m_f/|m_f|)(e/2\pi)\phi_B & \text{if } sg(m_f)=-sg(m_{ghost})\end{cases} \tag{5.46}$$

We can now compare this result with the one obtained before, Eq. (5.20), and get the values for μ as

$$\mu = \begin{cases} 0 & \text{if } sg(m_f) = sg(m_{ghost}) \\ -(m_f/|m_f|)(1/2\pi) & \text{if } sg(m_f) = -sg(m_{ghost}) \end{cases} \tag{5.47}$$

In the following, we restrict to the case $sg(m_f) = -sg(m_{ghost})$, since to consider the same relative sign for the fundamental fermion and ghost masses corresponds to the case when no Chern-Simons term is generated at large m_f.

We want now to evaluate the canonical angular momentum using the same counting arguments as before, based on the fact that whenever $|F|$ (integer) threshold states are present, one in each partial wave up to $|J_F| = |F| - 1/2$, then there has been a depletion of $|F|/2$ states in the $E < -|m_f|$ continuum. As such asymmetry in the spectrum occurs, J_F remains constant for each state, and there is also an asymmetry in the angular momentum. For $m_f > 0$ we can again consider the case with $F > 0$, which means $E > 0$ and we have

$$\begin{aligned} J_{F_f} = -\frac{1}{2}\sum_{J=1/2}^{-1/2+F} J &= -\frac{1}{2}\sum_{i=1}^{F}\frac{2i-1}{2} \\ &= -\frac{1}{2}\left(-\frac{F}{2}+\frac{F(F+1)}{2}\right) \\ &= -\frac{F^2}{4} \end{aligned} \tag{5.48}$$

or the case with $F < 0$, which means $E < 0$, and gives,

$$\begin{aligned} J_{F_f} = -\frac{1}{2}\left(-\sum_{J=-1/2}^{1/2+F} J\right) &= \frac{1}{2}\sum_{i=1}^{|F|}\frac{-2i+1}{2} \\ &= \frac{1}{2}\left(\frac{|F|}{2}-\frac{|F|(|F|+1)}{2}\right) \\ &= -\frac{F^2}{4} \end{aligned} \tag{5.49}$$

Once more, to take a negative value for the mass m_f associates the positive and negative energy states to the reversed conditions of F and then, $J_{F_f} = F^2/4$. The fermionic angular momentum due to the fundamental fermion contribution can be written in the general case as

$$J_{F_f} = -\frac{m_f}{|m_f|}\frac{F^2}{4} = -\frac{m_f}{|m_f|}\frac{e^2\phi_B^2}{(2\pi)^2\,4} \tag{5.50}$$

Considering now the contribution of the additional fermion due to Pauli Villars regularization, one gets

$$J_F = -\left(\frac{m_f}{|m_f|} - \frac{m_{ghost}}{|m|_{ghost}}\right)\frac{F^2}{4} = -\frac{m_f}{|m_f|}\frac{e^2\phi_B^2}{(2\pi)^2\,2} \tag{5.51}$$

Thus, we easily obtain the general expression in terms of μ

$$J_F = \frac{e^2\mu}{4\pi}\phi_B^2 = \frac{Q_{GS}^2}{4\pi\mu}, \tag{5.52}$$

We have demostrated that the canonical angular momentum carried by the fermions is compensated with the one due to their interaction with the gauge fields, $J_F + J_{int.} = 0$. The total angular momentum reduces to the one computed for the gauge fields $J = J_G$

The above calculations, although far from being general, illustrate the polarization effects on the fermions in the presence of a magnetic flux and allow us to obtain the correct induced quantum numbers in this frame. In a recent paper Coste and Lüscher [23], have shown, using lattice regularization techniques and in agreement with the Pauli-Villars method, that the value of μ obtained while integrating out the fermions, has the general result $\mu = n/2\pi$, with n any integer. In particular, as these authors observe, $n = -1$ corresponds to considering the Pauli-Villars regularization scheme with only one extra heavy particle and such that the mass of the ghost has the opposite sign relative to that of the mass of the fundamental fermion in the theory. This result $\mu = -1/2\pi$ is the one obtained in Eq. (5.47) with our simple counting computation.

To analyse the issue of interest, let us define a general value of μ to be called $\hat{\mu}$ and which is undetermined. We write the total angular momentum in term of it as

$$J = \frac{e^2\hat{\mu}}{4\pi}\phi_B^2 = \frac{Q_{GS}^2}{4\pi\hat{\mu}} \tag{5.53}$$

In general, one would expect a relation between the fermion number and the spin (and statistics) associated to the ground state of the system. For simplicity, we consider the external particle to be a boson , then, the fermion number and spin reduce to those induced by the gauge field configuration. Thus, under a 2π rotation, the ground state wave function is supposed to change by a factor $(-1)^{Q_{GS}} = \exp(\imath 2\pi J)$. Therefore if the fermionic charge is, for example, one, Eq. (5.53) gives $\hat{\mu}$ as $2\pi J = 1/2\hat{\mu}$. In general, to fulfill the desired relation between the induced quantum numbers, one must have,

$$\hat{\mu} = \frac{1}{2\pi(2\hat{n}+1)}, \tag{5.54}$$

with $\hat{n}$ any integer. The above set of values for $\hat{\mu}$, even though derived for a unit fermion number, are quite general and they reproduce the expected relation between fermion number and statistics. In other words, any value of μ, which does not coincide with $\hat{\mu}$ for one value of $\hat{n}$ implies that the correct statistic of the charged particle will be missed, if derived directly from its fermion number. Obviously, within the context we are working, the obtained values make sense only if they appear as a factor in the Chern-Simons term while integrating out the fermions within some regularization scheme. In this sense, the only values with significance are those given for $\hat{n} = 0, -1$, as they are gotten from the general result for $n = \pm 1$:

$$\hat{\mu}(\hat{n} = 0, -1) = \pm 1/2\pi = \mu(n = \pm 1). \tag{5.55}$$

It is worth remarking that many authors [22],[57], when using different regularization schemes from the ones mentioned above, have obtained a value of $\mu = -1/4\pi$ and therefore half values for the fermionic charge and the angular momentum [55],[56]. It is not our purpose to really question the validity of all these results. We only want to notice that for such values of μ, which cannot be obtained for any $\hat{n}$ from Eq. (5.54), there is no correlation between fermion charge and statistics. For example, for $F = 2$ one would have a unit fermionic induced charge value, but which is associated with $|J| = 1$ instead of with $|J| = 1/2$, and this problem extends for any other value of F. The same noncompatibility appears, obviously for any value $n \neq \pm 1$ within the general result $n/2\pi$. Unfortunately, we have, up to now, no physical picture to understand this fact and its possible implications. However the regularization ambiguities in 2+1 dimensions may involve a deeper interpretation, which remains obscure at present.

Appendix A

In this appendix we evaluate the correct fermionic charge induced in the background configuration from Eq. (3.29). This configuration given in terms of the unitary 2×2 matrix $\hat{\Phi}$ reads,

$$\hat{\Phi} = \frac{t + i\vec{x} \cdot \vec{T}}{\sqrt{t^2 + \vec{x}^2}} . \tag{A.1}$$

For that purpose we use the adiabatic current expression derived from Eq. (2.15),

$$j^i = \frac{1}{24\pi^2} 3\epsilon^{ijk0} Tr[\hat{\Phi}^+ \partial_j \hat{\Phi} \hat{\Phi}^+ \partial_k \hat{\Phi} \hat{\Phi}^+ \partial_t \hat{\Phi}] . \tag{A.2}$$

We neglect the spurious anomalous contribution at the origin and, since we want to evaluate the charge in the complete configuration space we have also no fermion flux there. Thus,

$$Q_{ind.} = -\int_{-\infty}^{\infty} dt \int_{S_\infty} d\vec{S} \cdot \vec{j} \tag{A.3}$$

At the same time we will observe the invariance of the current expression under temporal inversion.

We start evaluating the various pieces that go into $\vec{j}$. One has

$$\begin{aligned} \hat{\Phi}^+ \partial_j \hat{\Phi} &= \frac{t - i\vec{x} \cdot \vec{T}}{\sqrt{t^2 + \vec{x}^2}} \left(\frac{iT_j}{\sqrt{t^2 + \vec{x}^2}} - \frac{x^j \left(t + i\vec{x} \cdot \vec{T} \right)}{\sqrt{t^2 + \vec{x}^2}\,(t^2 + \vec{x}^2)} \right) \\ &= \frac{itT_j + ix^p T_q \epsilon_{pjq}}{t^2 + \vec{x}^2} \end{aligned} \tag{A.4}$$

and

$$\begin{aligned} \hat{\Phi}^+ \partial_t \hat{\Phi} &= \frac{t - i\vec{x} \cdot \vec{T}}{\sqrt{t^2 + \vec{x}^2}} \left(\frac{1}{\sqrt{t^2 + \vec{x}^2}} - \frac{t \left(t + i\vec{x} \cdot \vec{T} \right)}{\sqrt{t^2 + \vec{x}^2}\,(t^2 + \vec{x}^2)} \right) \\ &= \frac{-i\vec{x} \cdot \vec{T}}{t^2 + \vec{x}^2} . \end{aligned} \tag{A.5}$$

The above calculations lead to

$$\begin{aligned} j^i &= \frac{\imath}{8\pi^2}\frac{\epsilon^{ijk0}}{(t^2+\vec{x}^2)^3}\, Tr\left[(tT_i + x^pT_q\epsilon_{pjq})(tT_k + x^rT_s\epsilon_{rks})x^lT_l\right] \\ &= \frac{\imath}{8\pi^2}\frac{\epsilon^{ijk0}}{(t^2+\vec{x}^2)^3}\left(t^2x^lTr\,[T_jT_kT_l] + t\left(x^rx^l\epsilon_{rks}Tr\,[T_jT_sT_l] + x^lx^p\epsilon_{pjq}Tr\,[T_qT_kT_l]\right) - \right. \\ &\quad \left. + x^px^rx^l\epsilon_{pjq}\epsilon_{rks}Tr\,[T_qT_sT_l]\right) \\ &\equiv j^i_{(2)}t^2 + j^i_{(1)}t^1 - j^i_{(0)}t^0 . \end{aligned} \tag{A.6}$$

Considering the term linear in t we obtain

$$j^i_{(1)}\, t = \frac{\epsilon^{ijk}}{4\pi^2(t^2+\vec{x}^2)^3}x^lt\left(x^r\epsilon_{rks}\epsilon_{jsl} + x^p\epsilon_{pjq}\epsilon_{qkl}\right) = 0 \tag{A.7}$$

Since Eq. (A.7) gives that the only odd power order in t of the current vanishes, one has the relation $\vec{j}(\vec{x},t) = \vec{j}(\vec{x},-t)$. The term in t^2 from Eq. (A.6) gives

$$j^i_{(2)}\, t^2 = \frac{1}{4\pi^2}\frac{\epsilon^{ijk}\epsilon_{jkl}}{(t^2+\vec{x}^2)^3}x^lt^2 = \frac{-1}{2\pi^2}\frac{x^it^2}{(t^2+\vec{x}^2)^3} . \tag{A.8}$$

and the third term in Eq.(A.6) becomes,

$$j^i_{(0)} = \frac{1}{4\pi^2}\frac{\epsilon^{ijk}}{(t^2+\vec{x}^2)^3}x^px^rx^l\epsilon_{pjq}\epsilon_{rks}\epsilon_{lqs} = \frac{-1}{2\pi^2}\frac{\vec{x}^2x^i}{(t^2+\vec{x}^2)^3} . \tag{A.9}$$

Finally replacing the above results in Eq. (A.3), we obtain,

$$\begin{aligned} Q_{ind.} &= \frac{2}{\pi}\int_{-\infty}^{\infty} dt\frac{r^3}{(r^2+t^2)^2} \\ &= \frac{2}{\pi}\int_{-\infty}^{\infty}\frac{d(t/r)}{\left[(t/r)^2+1\right]^2} \end{aligned} \tag{A.10}$$

The above integral leads trivially to the value 1 for the induced charge.

Appendix B

In the present appendix we derive the eigenvalue equations in the background of a general scalar quartet configuration, $(\phi_0, \bar{\phi})$, coupled to the fermion spinor and isospinor upper and lower components $\chi^{\mp}_{in} \equiv \mathcal{M}^{\pm}_{im}\,\sigma_{2mn}$. Starting from equation (4.10), which we rewrite below,

$$(\imath\partial_a\sigma_a)_{ij}\;\mathcal{M}^{\mp}_{jl} \pm \imath\,\frac{g_y}{\sqrt{2}}\phi_a\mathcal{M}^{\mp}_{im}\sigma_{aml} = -\left(E \mp \frac{g_y}{\sqrt{2}}\,\phi_0\right)\mathcal{M}^{\pm}_{il} . \tag{B.1}$$

and replacing

$$\mathcal{M}^{\pm}_{im}(\vec{r}) = g^{\pm}(\vec{r})\delta_{im} + g^{a\,\pm}(\vec{r})\sigma_{a\,im}. \tag{B.2}$$

we get for the above equation:

$$\begin{aligned} &\imath\left(\partial_a \pm \frac{g_y}{\sqrt{2}}\phi_a\right)g^{\mp}\sigma_{ail} + \imath\partial_a g^{b\mp}\sigma_{aij}\sigma_{bjl} \pm \imath\frac{g_y}{\sqrt{2}}\phi_a g^{b\pm}\sigma_{aml}\sigma_{bim} = \\ &= -\left(E \mp \frac{g_y}{\sqrt{2}}\phi_0\right)\left(g^{\pm}\delta_{il} + g^{b\pm}\sigma_{bil}\right) . \end{aligned} \tag{B.3}$$

Using the algebra of the Pauli matrices we finally obtain the following set of equations for $g^{\pm}$ and $g^{a\pm}$

$$\begin{aligned} \left(\partial_a \pm \frac{g_y}{\sqrt{2}}\phi_a\right)g^{\mp} + \left(\partial_b \mp \frac{g_y}{\sqrt{2}}\phi_b\right)g^{c\mp}\imath\epsilon^{abc} &= \imath\left(E \mp \frac{g_y}{\sqrt{2}}\phi_0\right)g^{a\pm} \\ \left(\partial_a \pm \frac{g_y}{\sqrt{2}}\phi_a\right)g^{a\mp} &= \imath\left(E \mp \frac{g_y}{\sqrt{2}}\phi_0\right)g^{\pm} \end{aligned} \tag{B.4}$$

The scalar and vector functions can be expanded in terms of spherical harmonics, as already indicated in section 4.2,

$$\begin{aligned} g^{\pm}(\vec{r}) &= \sum_{MM_3} G^{\pm}_M(r)\,\mathcal{Y}_{MM_3}(\Omega) \\ g^{a\,\pm}(\vec{r}) &= \sum_{MM_3}\left(P^{\pm}_M(r)\,\mathcal{P}^a_{MM_3}(\Omega) + B^{\pm}_M(r)\,\mathcal{B}^a_{MM_3}(\Omega) + C^{\pm}_M(r)\mathcal{C}^a_{MM_3}(\Omega)\right) , \end{aligned} \tag{B.5}$$

where the explicit expressions for the vector spherical harmonics in term of the scalar ones are:

$$\mathcal{P}^a_{MM_3} = \hat{r}_a \mathcal{Y}_{MM_3}$$

$$B^a_{MM_3} = \left(M^2+M\right)^{-1/2} r\,\partial_a \mathcal{Y}_{MM_3}$$

$$C^a_{MM_3} = \left(M^2+M\right)^{-1/2} \imath\epsilon^{abc}\, r\,\partial_b\left(\hat{r}_c \mathcal{Y}_{MM_3}\right) . \tag{B.6}$$

The first equation from Eq. (B.4), after considering $\phi_a = \phi\hat{r}_a$, with $\hat{r}_a \equiv r_a/r$, becomes,

$$\begin{aligned}
&\mathcal{Y}_{MM_3}\hat{r}_a\left(\partial_r G^{*\mp} \pm \frac{g_y}{\sqrt{2}}\phi G^{*\mp}\right) + \left(\partial_a\mathcal{Y}_{MM_3}\right)G^{*\mp} + \imath\epsilon^{abc}\left\{\hat{r}_b\left[\hat{r}_c\mathcal{Y}_{MM_3}\partial_r P^{\mp} + \right.\right.\\
&+\left(M^2+M\right)^{-1/2} r\left(\partial_c\mathcal{Y}_{MM_3}\right)\partial_r B^{\mp} + \imath\epsilon^{cde}\left(M^2+M\right)^{-1/2} r\hat{r}_e\left(\partial_d\mathcal{Y}_{MM_3}\right)\partial_r C^{\mp} \mp\\
&\mp\frac{g_y}{\sqrt{2}}\phi\left(\hat{r}_c\mathcal{Y}_{MM_3}P^{\mp} + \left(M^2+M\right)^{-1/2} r\left(\partial_c\mathcal{Y}_{MM_3}\right)B^{\mp} + \right.\\
&\left.\left.+\imath\epsilon^{cde}\left(M^2+M\right)^{-1/2} r\hat{r}_e\left(\partial_d\mathcal{Y}_{MM_3}\right)C^{\mp}\right)\right] + \hat{r}_c\left(\partial_b\mathcal{Y}_{MM_3}\right)P^{\mp} +\\
&\left.+\left(M^2+M\right)^{-1/2}\hat{r}_b\left(\partial_c\mathcal{Y}_{MM_3}\right)B^{\mp}\right\} - \left[\left(M^2+M\right)^{-1/2}\partial_a\mathcal{Y}_{MM_3} + \right.\\
&\left.+\frac{\left(M^2+M\right)^{1/2}}{r}\hat{r}_a\mathcal{Y}_{MM_3}\right]C^{\mp} = \imath\left(E \mp \frac{g_y}{\sqrt{2}}\phi_0\right)\left[P^{\pm}\hat{r}_a\mathcal{Y}_{MM_3} + \right.\\
&\left.B^{\pm}\left(M^2+M\right)^{-1/2} r\partial_a\mathcal{Y}_{MM_3} + \imath\epsilon^{abc}\left(M^2+M\right)^{-1/2} r\hat{r}_c\left(\partial_b\mathcal{Y}_{MM_3}\right)C^{\pm}\right]
\end{aligned}\tag{B.7}$$

Decomposing the above equation in independent vector spherical harmonics, we arrive at three independent equations for the radial functions. For all values of M one has that

$$\left(\partial_r \pm \frac{g_y}{\sqrt{2}}\phi\right)G^{*\mp} - \frac{\left(M^2+M\right)^{1/2}}{r}C^{\mp} = \imath\left(E \mp \frac{g_y}{\sqrt{2}}\phi_0\right)P^{\pm} \tag{B.8}$$

while for $M \geq 1$ one gets the other two equations

$$\begin{aligned}
\left(\partial_r + \frac{1}{r} \mp \frac{g_y}{\sqrt{2}}\phi\right)B^{\mp} - \frac{\left(M^2+M\right)^{1/2}}{r}P^{\mp} &= -\imath\left(E \mp \frac{g_y}{\sqrt{2}}\phi_0\right)C^{\pm}\\
\left(\partial_r + \frac{1}{r} \mp \frac{g_y}{\sqrt{2}}\phi\right)C^{\mp} - \frac{\left(M^2+M\right)^{1/2}}{r}G^{*\mp} &= -\imath\left(E \mp \frac{g_y}{\sqrt{2}}\phi_0\right)B^{\pm}
\end{aligned}\tag{B.9}$$

From the second equation from the set of Eqs. (B.4) one gets a fourth independent equation, which is valid for any value of M and reads,

$$\left(\partial_r + \frac{2}{r} \pm \frac{g_y}{\sqrt{2}}\phi\right)P^{\mp} - \frac{\left(M^2+M\right)^{1/2}}{r}B^{\mp} = \imath\left(E \mp \frac{g_y}{\sqrt{2}}\phi_0\right)G^{*\pm} \tag{B.10}$$

For $M = 0$, the set of equations to be consider reduces to those coming from Eqs. (B.8) and (B.10). Redefining

$$G^{*\mp} = \mp\imath G^{\mp} \tag{B.11}$$

we finally have,

$$\begin{aligned}
\left(\partial_r \pm \frac{g_y}{\sqrt{2}}\phi\right)G^{\mp} &= \left(\frac{g_y}{\sqrt{2}}\phi_0 \mp E\right)P^{\pm}\\
\left(\partial_r + \frac{2}{r} \pm \frac{g_y}{\sqrt{2}}\phi\right)P^{\mp} &= \left(\frac{g_y}{\sqrt{2}}\phi_0 \mp E\right)G^{\pm} .
\end{aligned}\tag{B.12}$$

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